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Difference-form group contests

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Abstract

This paper is the first to study difference-form group contests, that is, contests fought among groups where their probability of victory depends on the absolute difference of their effective efforts. We show that key equilibrium variables in these contests can be expressed as a function of a modified version of the Watts poverty index. We use the properties of this index to study the impact of heterogeneity, both within and between groups. In the case of homogeneous groups, we show that multiple groups can be active in equilibrium and that more groups are active and aggregate effort is higher the more similar their valuations of victory are. We then characterize equilibria under heterogeneous groups. We show that within-group heterogeneity is typically detrimental to the success of a group in the contest. Groups may have an incentive to become more homogeneous in order to increase their chances of victory.

Keywords Contests · Contest success function · Groups · Heterogeneity

JEL Classification C72 · D63 · D72 · D74

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1 Introduction

Group conflicts are a constant among human and non-human animals, from clans of otters fighting for the control of a river to alliances of countries at war over a resource. Sports, elections, lobbying and R&D races are other contexts where players, political parties or firms pool their efforts to attain a joint objective. Because of the ubiquity and welfare consequences of these group confrontations, economic theory is naturally interested in understanding their outcome and the behavior of contenders.

The literature on group contests has developed along several themes such as the effect of group size, the choice of sharing rules and alliance formation. [1] In this paper we study another relevant issue: how the outcome of group contests depends on the contest technology and on heterogeneity, both within and between groups.

We do this in a novel set up: *difference-form contests*. In these contests, contenders' probability of success depends on the absolute difference between their effective efforts. This type of probabilistic contest, introduced in Hirshleifer (1989, 1991), is well suited to model confrontations where absolute performance is crucial.[2] One example is contests among workers where the chances of obtaining a promotion depend on the margin by which a worker's performance is above their colleagues'. The difference-form contest technology we study in this paper displays two features that arise in many applications. First, a contender who outpowers their rivals by a large enough margin wins the contest with certainty.[3] Analogously, a contender who expends zero effort can enjoy a positive winning probability if their rivals are not too aggressive. Probabilistic contests of the widely-used ratio form (Tullock 1980) display none of these features.[4]

Ours is the first paper to study difference-form group contests, that is, contests among groups whose chances of victory depend on the difference between their effective efforts or *impacts*.[5] These impacts result from the aggregation of individual efforts within groups. This aggregation of efforts admits different technologies, from perfect substitutes to perfect complements.

Because we are interested in contests where groups' winning probabilities depend on the absolute difference between their impacts, a key property we impose on our

[1] See Bloch (2012), Kolmar (2013) and Flamand and Troumpounis (2015) for surveys of these strands of the literature.

[2] Difference-form success functions appear also in rank-order tournaments with random noise, from the seminal Lazear and Rosen (1981) to more recent contributions such as Drugov and Ryvkin (2017). Cubel and Sanchez-Pages (2021) explore the connection between tournaments *à la* Lazear and Rosen (1981) and probabilistic difference-form contests of the type studied here.

[3] This feature relates the group contests we study here with the group contests under all-pay auction format studied in Barbieri and Malueg (2016), Chowdhury et al. (2016) and Barbieri et al. (2019). Che and Gale (2000) derived conditions under which the mixed equilibria of the individual difference-form contest converges to the one of the all-pay auction under complete information.

[4] Probabilistic difference-form contest success functions have been micro-founded using non-cooperative games (Gersbach and Haller 2009; Corchón and Dahm 2010), through mechanism design (Corchón and Dahm 2011; Polishchuk and Tonis 2013; Beviá and Corchón 2019) and in a Bayesian framework (Skaperdas and Vaidya 2012).

[5] Baik (1998) and Che and Gale (2000) were among the first to study two-player probabilistic difference-form contests. Recently, Skaperdas et al. (2016) and Cubel and Sanchez-Pages (2021) generalized these contests to non-linear impact functions. The latter also considered strictly convex costs and more than two players.

difference-form success function is that it must be translation invariant. In other words, if all groups increase their impact by the same fixed amount, so the absolute differences of their impacts remain invariant, their winning probabilities must not change. Translation invariance is the counterpart to homogeneity of degree zero in ratio-form contests.[6]

To ensure translation invariance, we employ a novel impact function, the technology aggregating the efforts of group members into a measure of influence. This function encompasses as particular cases the perfect substitutes and the perfect complements technologies of aggregation. We thus contribute to the growing literature which studies group contests under different impact technologies. This literature has only dealt with ratio-form contests so far.[7] We also introduce a novel functional form for the effort cost, which we assume to be exponential. This function encompasses linear costs as a particular case, the standard assumption in the contest literature.

We start our analysis by studying the case of homogeneous groups, that is, when members' valuations of victory are the same within groups but different across groups. In the literature, these homogeneous group valuations are often called group-specific public good prizes (e.g. Baik 1993). We find that the non-existence of pure strategy equilibria and the *preemption result* observed under linear costs in individual difference-form contests (e.g. Che and Gale 2000) extends to linear group contests. Preemption refers to the feature that in any pure strategy equilibrium all contenders but one expend zero effort. We then show that as soon as the cost function becomes strictly convex more than one group can be active in equilibrium. The number of active groups depends both on absolute and relative considerations: how many groups have valuations above a certain activity threshold and how similar groups are to each other.

We then analyze the case of groups with heterogeneous valuations and study how internal heterogeneity affects the number of active members and the success of groups in the contest. Equilibrium characterization is more complex in this case as group members with valuations below the activity threshold remain inactive. Nevertheless, we are able to explore a number of relevant cases. In these cases, we show that equilibrium variables such as group impacts and winning probabilities can be expressed as a function of an *affluence index*, a modified version of the Watts poverty index (Watts 1968; Zheng 1993). Modified poverty indices arise in this context because they feature a poverty line akin to our activity threshold; groups or individuals are active in the contest only if their valuation of victory is above that threshold. Our paper thus contributes to the recent literature establishing links between the equilibria of conflict and contest models and well-known measures of inequality and polarization.[8]

6 Translation invariance was first studied in group contests by Münster (2009). It can be traced back to the indices of absolute inequality introduced by Kolm (1976a, b) and characterized by Blackorby and Donaldson (1980).

7 Lee (2012) analyzed group contests with perfect complements efforts whereas Chowdhury et al. (2013) studied the other polar case, the best-shot technology. Starting with Kolmar and Rommeswinkel (2013), several papers have allowed for intermediate degrees of complementarity using CES impact functions. Among these are Cubel and Sanchez-Pages (2014), Brookins et al. (2015), Choi et al. (2016), Cheikbossian and Fayat (2018) and Crutzen et al. (2020).

8 Esteban and Ray (2011a), who pioneered this strand of the literature, linked conflict intensity to a combination of inequality, polarization and fractionalization indices. Other papers in this vein include

In the last part of the paper, we study whether more homogeneous or more heterogeneous groups have a relative advantage in difference-form contests. In that comparative statics exercise, we use the properties of the affluence index. We first show that aggregate equilibrium effort in the case of homogeneous groups is an increasing function of the index; in other words, total contest effort increases when group valuations become more similar. For the case of heterogeneous groups, we show that homogeneity within the set of active members increases the chances of a group in the contest. More homogeneous valuations above the activity threshold increase the effort of active members and can make some inactive member become active. However, changes in the distribution of valuations that take place below the activity threshold or that fail to make more members active have either no impact or are detrimental to the chances of victory of the group.

The remainder of the paper is as follows: In Sect. 2 we present the difference-form group contest we study. Section 3 and 4 analyze the cases of homogeneous and heterogeneous group valuations respectively. Section 5 studies the impact of heterogeneity between and within groups on equilibrium total effort and winning probabilities. We conclude and offer some further remarks in Sect. 6. Proofs are relegated to the appendix.

2 The contest game

Let us consider a society exogenously divided into $K \geq 2$ disjoint groups indexed by $k = 1, \dots K$ and populated by $n_k \geq 1$ individuals each. Denote the set of groups by $\mathbb{K}$ and the total number of individuals in society by N. These K groups are engaged in a contest which can have a sole winner. Members of these groups can expend non-negative efforts in order to help their group win the contest. Depending on the specific application, these efforts can be money, time or weapons. Denote by $\mathbf{x}_k = (x_{1k}, \dots, x_{n_k k})$ the vector of individual efforts in group k and by $\mathbf{x}$ the vector $(\mathbf{x}_1, \dots, \mathbf{x}_K)$. We will say that a member $i = 1, \dots, n_k$ is *active* if $x_{ik} > 0$, that a group k is active if at least one of its members is active, and that it is *fully active* if all its members are active.

2.1 The impact function

We assume that the efforts of group members are aggregated according to the *impact function*

$$h_k(\mathbf{x}_k) = \ln\left(\frac{1}{n_k}\sum_{i=1}^{n_k} e^{-\gamma x_{ik}}\right)^{-\frac{\beta}{\gamma}}. \tag{1}$$

Cubel and Sanchez-Pages (2014), Andonie et al. (2019) and Vesperoni and Yıldızparlak (2019), who
Footnote 8 continued
established links between equilibrium outcomes in contests and the Atkinson index of inequality, the family of Generalized Entropy indices and the Generalized Gini index respectively.

This aggregation technology is the natural logarithm of a CES function of exponential efforts. It is a non-decreasing function which satisfies $h_k(\mathbf{0}) = 0$. The parameter $\beta > 0$ measures the sensitivity of the impact function to individual efforts, whereas $\gamma \geq 0$ measures the complementarity of efforts. When $\gamma = 0$ the impact of a group becomes just the sum of its members' efforts, i.e. $h_k(\mathbf{x}_k) = \frac{\beta}{n_k} \sum_{i=1}^{n_k} x_{ik}$. That is, efforts within the group are perfect substitutes (Olson 1965). When $\gamma \to \infty$ then $h_k(\mathbf{x}_k) = \beta \cdot \min\{x_{1k}, \ldots, x_{n_k k}\}$, which corresponds to the weakest-link technology (Hirshleifer 1983). The weakest-link impact function is the only case in which the impact function $h_k(\mathbf{x}_k)$ is not strictly increasing in all its arguments.

The key feature of this impact function is that it is *translatable:* If all members increase their effort by a fixed amount λ, then the group impact increases by a fixed amount $\beta\lambda$. This concept is the additive counterpart of returns to scale, with the parameter $\beta > 0$ corresponding the degree of translatability of the function. This is a key property in difference-form contests, as we argue below.[9]

Let us mention that the impact function (1) satisfies anonymity as well.[10] That is, all members of all groups are equally efficient in transforming their effort into impact. Later on, we will introduce individual heterogeneity in valuations of victory.

2.2 The contest success function

Impacts determine the winning probability of each group according to a Contest Success Function (CSF), a function $p : R_+^N \to \Delta^K$ mapping the vector of efforts $\mathbf{x}$ into a vector of group winning probabilities such that $\sum_{k=1}^K p_k(\mathbf{x}) = 1$. Under risk neutrality $p_k(\mathbf{x})$ can be thought of as the share of the prize associated to victory that group k obtains. That said, we will favor the probability interpretation throughout the paper.

Given an effort vector $\mathbf{x}$ and the resulting profile of group impacts, let us order groups in a decreasing manner by their impact, i.e. $h_k(\mathbf{x}_k) \geq h_{k+1}(\mathbf{x}_{k+1})$. This can be done without loss of generality given that our impact function satisfies between-group anonymity as well (Münster 2009).[11] Next, define K^* as the largest integer such that

$$\frac{1}{K^*} + h_{K^*}(\mathbf{x}_{K^*}) - \frac{1}{K^*} \sum_{l=1}^{K^*} h_l(\mathbf{x}_l) > 0. \tag{2}$$

Note that $K^* = K$ when $h_k(\mathbf{x}_k) = 0$ for all k. Observe also that if the above condition holds for one group with impact $h_k(\mathbf{x}_k) = 0$, then it must hold for all other groups with zero impact too, and thus $K^* = K$. This implies that K^* can only take two values: It is either equal to K or to the number of groups with positive impact.

9 Translatability appears in the indices of absolute inequality axiomatized by Blackorby and Donaldson (1980). Our impact function is inspired by these indices (e.g., Pollak 1971; Kolm 1976a, b).

10 Formally, consider to two effort vectors $\mathbf{x}_k$ and $\mathbf{x}'_k$ where $\mathbf{x}'_k$ is a permutation of $\mathbf{x}_k$. The impact function is anonymous if $h_k(\mathbf{x}_k) = h_k(\mathbf{x}'_k)$.

11 Formally, the impact function is between-group anonymous if $h_k(\mathbf{x}_k) = h_l(\mathbf{x}_l)$ for any two disjoint groups $k, l \in \mathbb{K}$ whenever $\mathbf{x}_k = \mathbf{x}_l$.

The group contest success function (CSF henceforth) that we employ here was axiomatized in Cubel and Sanchez-Pages (2016) and it is defined as follows:

$$p_k(\mathbf{x}) = \begin{cases} \frac{1}{K^*} + h_k(\mathbf{x}_k) - \frac{1}{K^*} \sum_{l=1}^{K^*} h_l(\mathbf{x}_l) & \text{for } k = 1, \dots, K^*; \\ 0 & \text{otherwise} \end{cases} \tag{3}$$

This CSF establishes that groups with a positive but sufficiently low impact relative to other groups have a zero winning probability. This is in sharp contrast with the Tullock family of group CSFs where a positive impact ensures a positive winning probability. Within the set of groups with a positive winning probability, the difference-form group CSF in (3) relates chances of victory to the difference between a group's impact and the average impact of other groups in that set. If a group's impact is above (below) that average, its winning probability is above (below) the winning probability the group would be awarded under a fair lottery, i.e. $\frac{1}{K^*}$. It is important to note that a group with zero impact can still enjoy a positive winning probability if the rest of groups have sufficiently low impacts.

Note as well that under the group CSF in (3), the marginal benefit of effort of one individual is independent of efforts in other groups as long as $p_k(\mathbf{x}) \in (0, 1)$. This separability generates individual best response functions that are independent of other groups' efforts as long as the own group impact is not too high or too low

To fix ideas before proceeding any further, the reader may find useful to see how the group CSF in (3) works in the two-group case. Ordering groups such that $h_1(\mathbf{x}_1) \geq h_2(\mathbf{x}_2)$, the two groups enjoy a positive winning probability, i.e. $K^* = 2$, if and only if

$$\frac{1}{2} + h_2(\mathbf{x}_2) - \frac{1}{2} \sum_{l=1}^{2} h_l(\mathbf{x}_l) > 0 \Leftrightarrow h_2(\mathbf{x}_2) > h_1(\mathbf{x}_1) - 1.$$

In that case, the marginal benefit of effort of members in each group is independent of the effort in the other group. Otherwise, group 2 has a zero winning probability and the marginal benefit of effort of members of group 1 drops to zero. Observe also that group 2 enjoys a positive winning probability when its impact is zero if $h_1(\mathbf{x}_1) < 1$.[12]

Finally, note that another key property of the CSF in (3) is that it is *translation invariant:* Group winning probabilities do not change if all contenders increase their effort by the same fixed amount. [13] This property must hold when success in the contest depends on absolute differences in effort, usually because the metric of performance is meaningful, e.g., Elo ratings in chess. As Cubel and Sanchez-Pages (2016) showed, a

[12] The CSF for the two-group case could also be written following Che and Gale (2000) as

$$p_k = \max\left\{\min\left\{\frac{1}{2} + \frac{h_k(\mathbf{x}_k) - h_l(\mathbf{x}_l)}{2}, 1\right\}, 0\right\},$$

where the bounds replace the procedure to find K^* as defined in (2). Whilst the bounds formulation is convenient for $K = 2$, it is not generalizable beyond that case.

[13] Formally, the CSF is translation invariance if $p_k(\mathbf{x}+\lambda\cdot\mathbf{1}) = p_k(\mathbf{x})$ for all $\lambda > 0$ and $k \in \mathbb{K}$.

necessary condition for a group CSF to be invariant is that all group impact functions must be translatable of the same degree. This property is satisfied by the impact function we introduced in (1), which is translatable of degree β for any group size n_k and any degree of complementarity of efforts γ.

2.3 Effort cost

Members' efforts are costly. Their cost is given by the following exponential cost function:

$$c(x_{ik}) = \frac{e^{\phi x_{ik}} - 1}{\phi} \qquad \text{for } \phi \geq 0. \tag{4}$$

The parameter ϕ measures the convexity of the cost function or, in other words, the speed at which the marginal cost of effort increases. Convexity is a natural assumption in contests. Funds to finance a war or lobbying effort are increasingly costly to raise. Similarly in sport contests, fatigue increases the marginal cost of competitive effort. To the best of our knowledge we are the first to employ this particularly family of cost functions in contests. Note that it encompasses as a particular case the linear cost function when $\phi = 0$; this is the assumption employed in most of the difference-form contests studied in the literature. For any value of ϕ, observe that $c'(0) = 1$. As we will see below, this contributes to individuals and groups remaining inactive in the pure-strategy equilibria of the contest.

2.4 The group-contest game

Let us move to the study of the strategic interaction among the K groups engaged in the contest. Victory can be interpreted as providing group members with a prize, a territory, a pool of resources or the right to implement a particular policy. Group members can be heterogeneous in their valuation of victory by their group. Denote by v_{ik} the payoff that a member i of group k obtains in case her group wins the contest. Depending on the interpretation of victory, the profile of valuations $\mathbf{v}_k = (v_{1k}, \ldots, v_{n_k k})$ can be seen as a binding agreement on the distribution of the object being contested or as the intensity of members' feelings about the policy the group will implement in case it prevails. We assume the valuation of defeat in the contest for all individuals is zero.[14]

Summarizing, the payoff of a member i of group k is given by

$$u_{ik}(\mathbf{x}) = p_k(\mathbf{x}) v_{ik} - c(x_{ik}). \tag{5}$$

We look for the Nash Equilibrium in pure strategies of this group-contest game where members decide how much effort to contribute to the success of their group

[14] This is without loss of generality since v_{ik} can also be interpreted as the difference between the valuations of victory and defeat.

whilst taking as given the effort of outsiders and of their fellow group members. We will refer to this simply as the equilibrium.

2.5 Note on exponential efforts

Before proceeding any further, let us mention that the contest with exponential efforts presented above is equivalent to a contest with linear efforts where $\varkappa_{ik} = e^{x_{ik}}$ after imposing the restriction of $\varkappa_{ik} \geq 1$. [15] The impact function in (1) would then become the log of a standard CES function under the requirement that contributions to the contest require a minimum investment of $\varkappa_{ik} = 1$. This would correspond for instance to armed conflicts in the Antiquity and the Middle Ages, when combatants were supposed to show up in the battlefield with their own equipment. That of course meant that, typically, only wealthier members of society could afford to fight in a war (Finer 1975). Note however that under $\varkappa_{ik} = e^{x_{ik}}$, the impact function (1) would no longer be translatable and the CSF no longer translation invariant but scale invariant, i.e., homogeneous of degree zero (Cubel and Sanchez-Pages 2016).

2.6 Note on mixed strategies

In this paper, we study pure strategy equilibria only. But the group contest game admits mixed strategy equilibria too. This is the case, for instance, when members of a group value victory not enough to obtain a positive winning probability against another active group, but highly enough for its members to want to become active when that other group is the only active one.

Mixed strategy equilibria remain outside the scope of this paper, though. This is for two reasons. First, because mixed strategy equilibria are extremely cumbersome to characterize in difference-form contests. Ewerhart (2021) and Ewerhart and Sun (2018, 2020) have characterized the mixed equilibria of difference-form individual contests with smooth noise *à la* Hirshleifer. Unfortunately, the techniques they employ there cannot be applied to the difference-form contests studied here. We have been able to make only some small progress on that front for the n player version of our contest game (Cubel and Sanchez-Pages 2021). Secondly, mixed strategy equilibria are especially hard to characterize in group contests; we are aware of just a few group contest papers investigating this (Barbieri et al. 2014; Chowdhury and Topolyan 2016; Chowdhury et al. 2016), although they study all-pay auctions rather than probabilistic contests like the one we explore here.

3 Homogeneous groups

As a first step in our analysis, let us start by exploring the case of homogeneous valuations within groups. All members of group k have the same valuation of victory v_k. In the literature, homogeneous group valuations have been called group-specific pure public goods (e.g. Katz et al. 1990). This case encompasses as a particular case

15 We thank Alberto Vesperoni for pointing this equivalence to us.

the individual contest when $n_k = 1$ for all $k \in K$ which we analyzed for more general impact functions in Cubel and Sanchez-Pages (2021). With this exercise, we can gain some key intuitions on how difference-form group contests work and generalize some results obtained in the contest literature.

In particular, we show next that when the cost of effort is linear, the *preemption effect* observed in Baik (1998) and Che and Gale (2000) arises: At most one group is active in any pure-strategy equilibrium. However, when the cost of effort is strictly convex, more than one group can be active in equilibrium; even all of them, provided their valuations of victory are not too heterogeneous.

Let us first write down the FOC resulting from maximizing the payoff function (5) for member i of group k assuming that in equilibrium K^* groups enjoy a winning positive probability:

$$\frac{\partial u_{ik}}{\partial x_{ik}} = \beta \frac{K^* - 1}{K^*} \frac{e^{-\gamma x_{ik}}}{\sum_{j=1}^{n_k} e^{-\gamma x_{jk}}} v_k - e^{\phi x_{ik}} \leq 0. \tag{6}$$

If the inequality is strict for member i, she will be inactive. Note that for any K^*, the efforts of outsiders have no direct effect on members' best responses. This is because our CSF is separable in group impacts. Therefore, the impact of a group enjoying a positive winning probability in equilibrium is the one resulting from the equilibrium of the internal game played among its members. The effort of outsiders has an indirect effect, though. If the rest of groups have such a high impact that the group has a zero wining probability by playing its internal equilibrium, all members would prefer to remain inactive.

The SOC of the problem shows it is strictly concave:

$$\frac{\partial^2 u_{ik}}{\partial^2 x_{ik}} = -\beta \frac{K^* - 1}{K^*} \frac{\gamma e^{-\gamma x_{ik}} \sum_{j \neq i} e^{-\gamma x_{jk}}}{(\sum_{j=1}^{n_k} e^{-\gamma x_{jk}})^2} v_k - \phi e^{\phi x_{ik}} < 0.$$

Therefore members' best responses are uniquely defined. Note that the marginal benefit of exerting effort, the first term in (6), is decreasing in the own effort and increasing in the efforts of other members. This will play an important role in our characterization of the equilibrium below.

3.1 Linear cost

Let us assume, as most of the literature on difference-form contests does, that the cost of effort is linear, i.e. $\phi = 0$. Under cost linearity, expression (6) boils down to

$$\beta \frac{K^* - 1}{K^*} \frac{e^{-\gamma x_{ik}}}{\sum_{j=1}^{n_k} e^{-\gamma x_{jk}}} v_k \leq 1.$$

As mentioned earlier, the left-hand side in this expression is strictly decreasing in x_{ik}, so there cannot exist a generic equilibrium in which $K^* > 1$ and members within a group exert different amounts of effort. Hence, every member is either active or inactive. Either way, all members make the same effort. The other alternative is an equilibrium with $K^* = 1$, so one group wins the contest with certainty.

Knowing that equilibrium effort levels must be symmetric within each group, the FOC above simplifies to

$$\beta \frac{K^* - 1}{K^*} \frac{v_k}{n_k} \leq 1,$$

so for group k to be active when $K^* > 1$ groups enjoy a positive winning probability it must be that

$$\frac{v_k}{n_k} \leq z(K^*) \equiv \frac{1}{\beta} \frac{K^*}{K^* - 1}.$$

It will be very useful to denote $\widetilde{v}_k = \frac{v_k}{n_k}$. Without loss of generality, let us index groups in society decreasingly so $\widetilde{v}_k \geq \widetilde{v}_l$ for $k < l$. Note that, with this, we are anticipating that groups with higher modified valuation $\widetilde{v}_k$ will have higher impact in equilibrium. Since all members of group k will remain inactive if $\widetilde{v}_k < z(K^*)$, we refer to $z(K^*)$ as the *activity threshold.* Note that this threshold is decreasing in K^* and in the sensitivity of impact to effort β, i.e., the degree of translatability of the impact function.

The above notation allows us to rewrite the FOC of any individual member as

$$\frac{\widetilde{v}_k}{z(K^*)} \leq 1.$$

This implies that in any generic pure strategy equilibrium at most one group will be active in equilibrium. For two groups k and l to be active it must be that $\widetilde{v}_k = \widetilde{v}_l$. So, either every group is inactive or only one group is active.[16] On the other hand, the equilibrium with only one active group, i.e., $K^* = 1$, requires that such group wins with certainty. Clearly, the members of that group will not exert more effort than necessary to ensure $p_k(\mathbf{x}_k, \{\mathbf{0}\}_{l \neq k}) = 1$. Two necessary conditions for this equilibrium to exist are 1) that members of the active group do not prefer rather to be inactive and 2) that the valuations of victory of the inactive groups are small enough, so their members do not have an incentive to become active.

The following proposition uses these observations to characterize the existence of the two types of equilibria.

Proposition 1 *Assume valuations of victory are homogeneous within groups and effort cost is linear. If* $\widetilde{v}_1 < z(K)$ *no group is active in equilibrium. Otherwise, at least one pure strategy equilibrium exists if and only if* $\widetilde{v}_2 < z(2)$*. In those equilibria, only one group is active and wins the contest with certainty.*

Proposition 1 generalizes the preemption result observed by Baik (1998) and Che and Gale (2000) in individual difference-form contests to the case of homogeneous

[16] In the non-generic case where $\widetilde{v}_k = \widetilde{v}_l$, members of these groups are willing to supply as much effort as needed to ensure their group wins. No equilibrium in pure strategy can exist then.

group contests: With linear costs, at most one contender -individual or group- is active. The reason is that, under linear effort cost, groups with a sufficiently high valuation of victory would like to contribute as much effort as needed to win the contest with certainty. When there are two or more such groups, an equilibrium in pure strategies cannot exist. This is more likely to be the case the higher the sensitivity of impact to efforts, as $z(2)$ is decreasing in β. As in Che and Gale (2000), when $\beta \to \infty$ the contest becomes an all-pay auction and no equilibrium in pure strategies exists since $z(2) \to 0$.

This result sits in contrast with the equilibrium of Tullock group contests with homogeneous groups and linear cost. There, a pure strategy equilibrium always exists and at least two groups are active; groups with low valuations remain inactive (Hillman and Riley 1989). The difference between the two types of contests is driven by the separability in group impacts of the difference-form CSF. In Tullock contests, the marginal benefit of effort is decreasing in the impact of other groups. This rules out scenarios where a pure strategy equilibrium cannot exist because two groups are willing to provide as much effort as needed to win the contest regardless of the effort of their opponents.

3.2 Strictly convex cost

Despite its prevalence in contest theory, linear cost functions might not describe best the cost of effort in real-world contests. When effort is time or money that must be raised in imperfectly competitive credit markets, its cost is likely to be convex. Moreover, the predictions derived in linear cost contests can be rather non-robust as Esteban and Ray (2001) showed for Tullock ratio-form contests. In Cubel and Sanchez-Pages (2021), we showed that this is also the case in difference-form contests among individuals; the full preemption result no longer holds as soon as cost linearity is assumed away. Next we show this is also the case for difference-form group contests with homogeneous valuations.

Let us first note that $x^*_{ik} = x^*_{jk}$ for any two active members i and j in group k for who (6) holds. If, on the contrary, $x^*_{ik} > x^*_{jk}$ then $\frac{\partial u_{ik}}{\partial x_{ik}} < \frac{\partial u_{jk}}{\partial x_{jk}}$, contradicting that the two members are best responding. Hence, all equilibria must be symmetric within groups. This also implies that 1) if the equilibrium effort of at least one member is strictly positive then it must be so for all other members; and 2) for a given K^*, a necessary condition for a group to be active in equilibrium is $\widetilde{v}_k > z(K^*)$.

Solving (6), we can obtain the equilibrium individual effort for an active group k when K^* groups enjoy a positive winning probability:

$$\widehat{x}_k(K^*) = \frac{1}{\phi} \ln \frac{\widetilde{v}_k}{z(K^*)}. \tag{7}$$

Note that we have dropped the i subindex because this equilibrium must be symmetric as argued earlier. A necessary condition for this candidate equilibrium effort to be positive is $\widetilde{v}_k > z(K^*)$. In what follows, it will become extremely convenient to define the censored group valuation distribution $\widetilde{\mathbf{v}}^*$ as a vector whose elements are

$$\widetilde{v}_k^* = \max\{z(K^*), \widetilde{v}_k\}.$$

Censored valuations allow us to express the vector of candidate equilibrium efforts $\widehat{\mathbf{x}}$ in a compact way for both active and inactive groups:

$$\widehat{x}_k(K^*) = \frac{1}{\phi} \ln \frac{\widetilde{v}_k^*}{z(K^*)},$$

so $\widehat{x}_k(K^*) = 0$ when $\widetilde{v}_k \leq z(K^*)$ as then $\widetilde{v}_k^* = z(K^*)$. We refer to $\widehat{\mathbf{x}}(K^*)$ as the candidate equilibrium profile because this profile is contingent on K^*. It might be that other groups are so aggressive that the group impact under $\widehat{\mathbf{x}}_k(K^*)$ does not secure the group a positive winning probability. To fully characterize the equilibrium, we need to find a number K^* such that when groups play the profile $\widehat{\mathbf{x}}(K^*)$, precisely the K^* groups with the highest censored valuation $\widetilde{v}_k^*$ have a non-zero probability of victory. We do this next.

Proposition 2 *Assume that valuations of victory within groups are homogeneous and the cost of effort is strictly convex, i.e. $\phi > 0$. Then, the difference-form group contest admits an equilibrium where the number of groups with positive winning probability is an integer K^* such that $\widetilde{v}_{K^*+1} < z(K^*)$ and*

$$\widetilde{v}_{K^*}^* > e^{-\frac{\beta}{\phi(K^*-1)}} \widetilde{G}_{K^*-1}^*, \tag{8}$$

where $\widetilde{G}_{K^*-1} = (\prod_{l=1}^{K^*-1} \widetilde{v}_l^*)^{\frac{1}{K^*-1}}$ is the geometric mean of the censored valuation of groups $k = 1, \ldots, K^* - 1$.

Proposition 2 implies that, unlike in the linear cost case, more than one group can enjoy a positive winning probability in equilibrium under strictly convex costs. Recall that a pure strategy equilibrium with more than one active group is not possible under constant marginal cost because when two groups have a sufficiently large valuation of victory, their members want to supply as much effort as needed to win the contest with certainty. An increasing marginal cost of effort rules out that scenario. When a group's valuation of victory is large enough, now the best response effort of their members can be interior, and a pure strategy equilibrium with $K^* \geq 2$ can exist. However, preemption can still take place. It might be, for instance, that group 1 does not win the contest with certainty but it is still the only active group in equilibrium if other groups do not value victory much.

From Proposition 2 we can immediately derive the following condition characterizing an equilibrium with at least two active groups.

Corollary 1 *At least two groups are active in equilibrium if*

$$\widetilde{v}_2 > \max\{e^{-\frac{\phi}{\beta}} \widetilde{v}_1, z(2)\}.$$

The first argument in the maximum function ensures that $K^* \geq 2$, the second that the equilibrium effort in group 2 is positive when $K^* \geq 2$. Let us highlight the stark contrast between the equilibrium under linear costs as described in Proposition 1 and the equilibrium under strictly convex costs characterized in Corollary 1. In the former, at most one group is active in equilibrium whereas in the latter multiple active groups are possible. Moreover, the existence of a pure strategy equilibrium with linear cost fails if $\widetilde{v}_2 > z(2)$, whereas this same condition is a necessary condition for the existence of an equilibrium where at least two groups are active in the strictly convex cost case.

For an equilibrium with multiple active groups to exist, a group's valuation of victory must be above the activity threshold, i.e. $\widetilde{v}_{K^*} > z(K^*)$. This is due to the aforementioned feature of the exponential cost function, namely that $c'(0) = 1$ for any $\phi \geq 0$. In addition, as shown in (8), the valuation of victory must be high enough relative to other groups' valuations. Otherwise, the effort of members may be insufficient to secure a positive winning probability for the group. This should be seen as a realistic feature. Not all groups in society or within an organization engage in confrontation or influence activities; only those with a sufficiently intense preference for victory. This is contrast with the equilibria of Tullock group contests with strictly convex costs and homogeneous groups, where all groups are always active (Esteban and Ray 2001).

Observe finally that the effect of the sensitivity of impact to efforts β on the number of active groups in equilibrium is ambiguous. An increase in β lowers the activity threshold, so members of inactive groups have now more incentives to become active. On the other hand, an increase in the sensitivity of impact to effort also makes groups with high valuations increase their impact relatively more. Groups with lower valuations may find that their impact is insufficient to obtain a positive winning probability, so their members become inactive.

4 Heterogeneous groups

Next, we characterize the equilibria of the contest when valuations are heterogeneous across group members. Unfortunately, the general analysis is too complex, and we must restrict our attention to three particular but relevant cases where closed form solutions can be obtained. Results for these cases show that multiple groups and multiple members may be active in equilibrium; those with low enough valuations, though, remain inactive. We also show that equilibrium variables can be characterized as a function of a modified version of a well-known family of poverty indices. We then exploit the properties of these indices to produce comparative statics.

Let us order members within groups in a decreasing manner such that $v_{ik} > v_{jk}$ for $i < j$. Because members differ in their valuation of victory, it might be the case that some members remain inactive in equilibrium. If this is the case, because the marginal benefit of effort is decreasing in the own effort, it must be that if $x^*_{ik} = 0$ and $x^*_{jk} > 0$ then $v_{jk} > v_{ik}$. In other words, the set of active members in the interior equilibrium of an heterogeneous group is composed by those members with a high enough valuation of victory.

Let us turn our attention to the FOC of the problem faced by an active member. Expression (6) becomes:

$$\frac{\partial u_{ik}}{\partial x_{ik}} = \frac{e^{-\gamma x_{ik}}}{n_k - n_k^* + \sum_{j=1}^{n_k^*} e^{-\gamma x_{jk}}} \frac{v_{ik}}{z(K^*)} - e^{\phi x_{ik}} = 0, \tag{9}$$

where n_k^* denotes the number of active members in the candidate equilibrium. Given the discussion above, the set of active members must be composed by the members with the n_k^* highest valuations. Because the full characterization of n_k^* is not feasible in general, we focus on three particular cases. We study them in what follows.

4.1 Linear impact

The case where members' efforts are perfect substitutes, i.e. $\gamma = 0$, is the most straightforward set-up with heterogeneous groups. This is the case most often studied in the Tullock group contests literature (e.g. Katz et al. 1990; Baik 1993, 2008; Ryvkin 2011). Under perfect substitutes efforts, the group impact function is linear and individual payoffs are fully separable in the effort of both outsiders and fellow group members. The impact function becomes

$$h_k(\mathbf{x}_k) = \frac{\beta}{n_k} \sum_{j=1}^{n_k} x_{jk}.$$

The FOC in (9) implies that the candidate optimal effort choice of an active member is

$$\widehat{x}_{ik}(K^*) = \frac{1}{\phi} \ln \frac{\widetilde{v}_{ik}}{z(K^*)},$$

where, as in the homogenous case, we denote $\widetilde{v}_{ik} = \frac{v_{ik}}{n_k}$. Hence, a group member is active in this candidate equilibrium only if $\widetilde{v}_{ik} > z(K^*)$. Let us again use the concept of censored valuations to define the vector of members' censored valuations $\widetilde{\mathbf{v}}_k^*$ as the one whose elements are

$$\widetilde{v}_{ik}^* = \max\{z(K^*), \widetilde{v}_{ik}\}. \tag{10}$$

With this, we can write candidate optimal choices of both active and inactive members succinctly as

$$\widehat{x}_{ik}(K^*) = \frac{1}{\phi} \ln \frac{\widetilde{v}_{ik}^*}{z(K^*)}. \tag{11}$$

For a given K^*, expression (11) determines the candidate set of active members: It consists of the n_k^* members whose censored valuation $\widetilde{v}_{ik}^*$ is above $z(K^*)$. Again, we refer to this as a candidate optimal choice because it might be that when n_k^* members exert effort $\widehat{x}_{ik}(K^*)$ the resulting group impact is not high enough to make the group obtain a positive winning probability given the impact of the other groups. In that case, these members would rather become inactive.

A critical element in the characterization of the equilibrium is thus the distribution of members' valuations across groups with respect to the activity threshold $z(K^*)$. In this endeavor, it will be convenient to exploit the fact that it is possible to express groups' equilibrium impacts and winning probabilities as a function of a modified version of the Watts poverty index (Watts 1968). We thus need to define that index before proceeding with that characterization.

Given a distribution $\mathbf{v}$ in a population of size N and a poverty line z, Watts (1968) defined the poverty index

$$\begin{aligned} W(\mathbf{v},z) &= \frac{1}{N}\sum_{i=1}^{N}[\ln z - \ln v_i^*] \\ &= \ln z - \ln G^*, \end{aligned}$$

where $v_i^* = \min\{z, v_i\}$ is the censored distribution from above (rather than from below, as we have been doing so far), G^* is the geometric mean of that censored distribution and z is the poverty line. This index measures poverty as the absolute welfare loss due to poverty. It has a number of well known properties: It is distribution-sensitive, decomposable and homogeneous of degree zero. Two other key properties of this index are that changes in the distribution above the poverty line z have no effect on its value, whereas increases in income or progressive transfers above the poverty line decrease the index (Zheng 1993).

Let us define the symmetric counterpart of this index, that we will refer to as the *affluence index*.

$$\begin{aligned} W_o(\mathbf{v},z) &= \frac{1}{N}\sum_{i=1}^{N}[\ln v_i^* - \ln z] \\ &= \ln G^* - \ln z, \end{aligned} \tag{12}$$

where $v_i^* = \max\{z, v_i\}$, G^* is the geometric mean of the censored distribution from below and z is the *affluence line* z. This index is bounded below by zero; $W_o(\mathbf{v},z) = 0$ when $v_i = z$ for all i. The properties of the Watts poverty index for changes in $\mathbf{v}$ around the poverty line are reversed in the affluence version. Changes in the distribution below z have no effect on the affluence index, whereas increases in income or progressive transfers above the line increase the index.

Plugging the candidate equilibrium effort (11) into the impact function (1) and using the affluence index defined above, it is possible to write group impacts as

$$h_k(\widehat{\mathbf{x}}_k(K^*)) = \frac{\beta}{\phi}[\ln \widetilde{G}_k^* - \ln z(K^*)] = \frac{\beta}{\phi} W_o(\widetilde{\mathbf{v}}_k^*, z(K^*)), \tag{13}$$

where $\widetilde{G}_k^*$ is the geometric mean of the distribution of censored valuations in group k and $W_o(\widetilde{\mathbf{v}}_k^*, z(K^*))$ denotes the Watts affluence index of that censored valuation distribution $\widetilde{\mathbf{v}}_k^*$ with affluence line $z(K^*)$. After ordering groups decreasingly by their affluence index $W_o(\widetilde{\mathbf{v}}_k^*, z(K^*))$, it is possible to state the following proposition characterizing K^* and the groups' winning probability in equilibrium.

Proposition 3 *Assume group impact is linear, i.e. $\gamma = 0$. Then, the contest admits an equilibrium where the number of groups with positive winning probability is an integer K^* such that*

$$W_o(\widetilde{\mathbf{v}}_{K^*}^*, z(K^*)) > \frac{1}{K^* - 1}\left[\sum_{k=1}^{K^*-1} W_o(\widetilde{\mathbf{v}}_k^*, z(K^*)) - \frac{\phi}{\beta}\right], \tag{14}$$

and $\max\,\{\widetilde{v}_{1k}\}_{k=K^*+1}^{K} < z(K^*)$. *In that equilibrium, winning probabilities are given by*

$$p_k^* = \max\left\{0, \frac{1}{K^*} + \frac{\beta}{\phi} W_o(\widetilde{\mathbf{v}}_k^*, z(K^*)) - \frac{\beta}{\phi}\frac{1}{K^*}\sum_{l=1}^{K^*} W_o(\widetilde{\mathbf{v}}_l^*, z(K^*))\right\}. \tag{15}$$

As in the homogeneous group case, it is important to explicitly characterize the conditions under which equilibria exist where at least two groups are active. The next corollary comes directly from Proposition 3.

Corollary 2 *At least two groups are active in equilibrium if*

$$W_o(\widetilde{\mathbf{v}}_2^*, z(2)) > \max\left\{0, W_o(\widetilde{\mathbf{v}}_1^*, z(2)) - \frac{\phi}{\beta}\right\}.$$

The linear impact case illustrates that the success of the group in a difference-form contest rests on both its absolute affluence, i.e. $W_o(\widetilde{\mathbf{v}}_k^*, z(K^*))$ must be strictly positive, and on its relative affluence with respect to the affluence of the other groups, as defined in (14). Groups with low affluence, that is, whose members have few valuations above the activity threshold or by a small margin, are bound to lose the contest for sure. Note the difference with Tullock group contests under linear impact, where all groups are active and obtain a positive winning probability in equilibrium (Ryvkin 2011).

By expressing equilibrium variables (individual effort, group impact and winning probabilities) as a function of the affluence index, we can use its properties to study comparative statics. For instance, as the index is decreasing in the affluence line, increases in the sensitivity of impact to efforts (which lower the line) have again an ambiguous effect: A larger β can make group 2's affluence large enough to become active, i.e., $W_o(\widetilde{\mathbf{v}}_2^*, z(2)) > 0$, but also makes that group less affluent relative to group 1. In Sect. 5, we exploit the properties of the index to explore the effect of changes in the distribution of valuations.

4.2 All groups are fully active

Because under linear impact some groups and members are inactive, performing comparative statics is difficult: both the number of active members n_k^* and groups K^* change with changes in parameters. Comparative statics can be produced when the within-group distributions of valuations are such that all groups are fully active in equilibrium. This case eases out the equilibrium characterization because then $n_k^* = n_k$ for all $k \in \mathbb{K}$ and $K^* = K$. We analyze this next.

In what follows, we will make extensive use of *generalized means*, also called *means of order* r (Hardy et al. 1934). The mean of order r of the valuations in group k is defined as

$$\mu_r(\mathbf{v}_k) = \begin{cases} [\frac{1}{n_k}\sum_{j=1}^{n_k}(v_{jk})^r]^{\frac{1}{r}} & \text{for } r \in \mathbb{R}, r \neq 0 \\ (\prod_{j=1}^{n_k} v_{jk})^{\frac{1}{n_k}} & \text{for } r = 0 \end{cases}, \tag{16}$$

The case with $r = 1$ corresponds to the arithmetic mean, $r = 0$ to the geometric mean and $r = -1$ to the harmonic mean. Moreover, $\mu_r(\mathbf{v}_k)$ is increasing in r for any valuation vector $\mathbf{v}_k$. Note that our impact function in (1) is a function of the generalized mean of order $r = -\gamma$ of exponential efforts. For what follows, it will also be important to note that means of order $r < 1$ ($r > 1$) are Schur-concave (convex).

Now, note from (9) that for any two group members i and j, their optimal interior effort choices must satisfy

$$x_{ik} - x_{jk} = \frac{1}{\phi + \gamma} \ln\left(\frac{v_{ik}}{v_{jk}}\right). \tag{17}$$

This implies that members' optimal efforts become more similar as efforts become more complementary or the cost function becomes more convex, i.e. γ or ϕ increase.

Assuming all members are active, i.e. $n_k^* = n_k$, and adding up across all $j \in k$ we obtain

$$\sum_{j=1}^{n_k} e^{-\gamma x_{jk}} = \frac{e^{-\gamma x_{ik}}}{(v_{ik})^{\frac{-\gamma}{\phi+\gamma}}} \sum_{j=1}^{n_k} v_{jk}^{\frac{-\gamma}{\phi+\gamma}}.$$

Plugging this into (9) and adding up across all group members yields

$$\sum_{j=1}^{n_k} e^{-\gamma x_{jk}} = \left(\sum_{j=1}^{n_k} v_{jk}^{\frac{-\gamma}{\phi+\gamma}}\right)^{\frac{\phi+\gamma}{\phi}} z(K)^{\frac{\gamma}{\phi}}, \tag{18}$$

where $z(K)$ appears because recall that we are constructing an equilibrium where all K groups are fully active and thus all must have positive winning probabilities. After we plug back (18) into (6), and using again the notation $\widetilde{v}_{ik} = \frac{v_{ik}}{n_k}$, we can obtain the candidate equilibrium effort level for $i \in k$:

$$\widehat{x}_{ik} = \frac{1}{\phi}\left[\frac{\phi}{\phi+\gamma}\ln\widetilde{v}_{ik} - \ln z(K) - \ln\frac{1}{n_k}\sum_{j=1}^{n_k}\widetilde{v}_{jk}^{\frac{-\gamma}{\phi+\gamma}}\right].$$

Note that a necessary condition for this equilibrium to exist is that the cost of effort must be strictly convex, i.e. $\phi > 0$. Denote $\rho = -\frac{\gamma}{\phi+\gamma} \in (-1, 0]$ to rewrite the candidate equilibrium effort as

$$\widehat{x}_{ik} = \frac{1}{\phi}\left[(\rho+1)\ln\widetilde{v}_{ik} - \ln z(K) - \rho\ln\mu_\rho(\widetilde{\mathbf{v}}_k)\right], \tag{19}$$

where $\mu_\rho(\widetilde{\mathbf{v}}_k)$ is the mean of order ρ of the modified valuations in group k.

Expression (19) implies that all members of the group are active in the equilibrium we are constructing only if the lowest valuation within the group, the one for the n_k-th member, satisfies

$$\widetilde{v}_{n_k k} > \widehat{v}_k \equiv \left[z(K)\mu_\rho(\widetilde{\mathbf{v}}_k)^\rho\right]^{\frac{1}{1+\rho}}$$

Assuming this is the case, the group's impact can be written as

$$h_k(\widehat{\mathbf{x}}_k) = \frac{\beta}{\phi}W_\rho(\widetilde{\mathbf{v}}_k, z(K)), \tag{20}$$

where

$$W_\rho(\widetilde{\mathbf{v}}_k, z(K)) = \ln\mu_\rho(\widetilde{\mathbf{v}}_k) - \ln z(K),$$

is the Generalized affluence index. This index is a modified version of the Generalized Watts poverty index where valuations are censored below the affluence line $z(K)$. It encompasses the affluence index we employed in Sect. 4.1 as a particular case when $\rho = 0$, i.e. $\gamma = 0$. This generalized index displays the same properties as that one.

Let us once more order groups in a decreasing manner, by $W_\rho(\widetilde{\mathbf{v}}_k, z(K))$ in this occasion. The following proposition characterizes the conditions under which an equilibrium with all groups being fully active exists and their resulting winning probabilities.

Proposition 4 *Assume $\phi > 0$. An equilibrium in which all groups are fully active exists if and only if $\widetilde{v}_{n_k k} > \widehat{v}_k$ for all groups and*

$$W_\rho(\widetilde{\mathbf{v}}_K, z(K)) > \frac{1}{K-1}\left[\sum_{k=1}^{K-1} W_\rho(\widetilde{\mathbf{v}}_k, z(K)) - \frac{\phi}{\beta}\right],$$

In that equilibrium, winning probabilities are

$$p_k^* = \frac{1}{K} + \frac{\beta}{\phi}W_\rho(\widetilde{\mathbf{v}}_k, z(K)) - \frac{\beta}{\phi}\frac{1}{K}\sum_{l=1}^{K}W_\rho(\widetilde{\mathbf{v}}_l, z(K)) \quad \textit{for } k = 1, \ldots, K. \tag{21}$$

As for Proposition 3, the proof is straightforward; it comes from combining (20), the difference-form CSF and (2).

Proposition 4 shows that when the distributions of valuations within each group are such all groups are fully active, it is possible to express equilibrium variables as a function of the Generalized affluence index. This allows us to explore comparative statics with respect to the degree of complementarity of efforts within groups γ and the sensitivity of impact to effort β.

Corollary 3 *Assume $\phi > 0$ and all groups are fully active in equilibrium. Then, members' efforts equalize within groups as efforts become more complementary, i.e. as γ increases, resulting in lower group impacts. Winning probabilities become more unequal across groups as impact becomes more sensitive to effort, i.e. β increases.*

The first part of the corollary comes from the fact that generalized means are increasing in their order, so given that ρ is decreasing in γ, then $\mu_\rho(\tilde{\mathbf{v}}_k)$ and thus $W_\rho(\tilde{\mathbf{v}}_k, z(K))$ are decreasing in γ too. Differentiating (19) with respect to γ shows that the effort of members with higher (lower) valuations decreases (increases) as efforts become more complementary. The intuition behind why increased complementarity has a net negative effect on group impacts is that the efforts of low valuation members become more critical. These are precisely the members with the lowest incentive to contribute. Consequently, the impact of the group decreases as it relies more heavily on the lowest efforts across members. Note that the effect of more complementary efforts on the winning probability of a specific group is ambiguous as it depends on how much the impacts of other groups decrease.[17] In contrast, the effect of the sensitivity of impact to effort is unambiguous: Groups with above (below) average affluence experience an increase (decrease) in their chances of victory as β increases.

4.3 Linear costs

The last case we explore in this section is cost linearity. This case allows us to establish a comparison with the linear cost case under homogeneous valuations we studied in Sect. 3.1. As we will see next, the preemption result and the lack of pure strategy equilibria that emerged there weaken when groups are heterogeneous.

Consider again an equilibrium in which K^* groups enjoy a positive winning probability and where $n_k^* \leq n_k$ members of group $k = 1, \ldots, K$ are active. Taking expression (9) for $\phi = 0$ and adding across all active members in group k yields

$$\sum_{j=1}^{n_k^*} e^{-\gamma x_{ik}} = \frac{n_k - n_k^*}{\frac{\mu_{-1}^*(\mathbf{v}_k)}{z(K^*) n_k^*} - 1}, \tag{22}$$

where $\mu_{-1}^*(\mathbf{v}_k)$ is the harmonic mean of the individual valuations of the active members in group k.

[17] If impact functions were group-specific with complementarity parameter γ_k, the winning probability of a group would decrease as the efforts of its members became more complementary.

Combining this with expression (9), we can obtain the candidate equilibrium effort for an active member in group k:

$$x_{ik}^*(K^*) = \frac{1}{\gamma}\ln[v_{ik}\frac{\frac{1}{z(K^*)} - \frac{n_k^*}{\mu_{-1}^*(\mathbf{v}_k)}}{n_k - n_k^*}] > 0 \Leftrightarrow v_{ik} > \frac{n_k - n_k^*}{\frac{1}{z(K^*)} - \frac{n_k^*}{\mu_{-1}^*(\mathbf{v}_k)}}. \tag{23}$$

The set of active members is given by all members whose valuation of victory is above the threshold in (23). These must be the n_k^* members with the highest valuation since recall that if $x_{ik}^* = 0$ and $x_{jk}^* > 0$ then $v_{jk} > v_{ik}$. This in turn implies that for n_k^* members to be active, the valuations of the n_k-th and $n_k + 1$-th member should satisfy the following condition:

$$v_{n_k^* k} > \frac{n_k - n_k^*}{\frac{1}{z(K^*)} - \frac{n_k^*}{\mu_{-1}^*(\mathbf{v}_k)}} > v_{n_k^*+1k}. \tag{24}$$

Observe from (23) that it cannot be an equilibrium that $n_k^* = n_k$ in at least two groups. In that case, the marginal benefit of effort would be above its marginal cost for all members of these groups and they would like to supply as much effort as needed to win the contest with certainty. In other words, the non-existence of pure strategy equilibria under linear costs would re-emerge. For this case not to arise, at least one member in each active group must want to remain inactive. Formally, it must be that

$$v_{n_k k} < \frac{1}{\frac{1}{z(K^*)} - \sum_{j=1}^{n_k - 1}\frac{1}{v_{jk}}}. \tag{25}$$

Let us then assume that this condition is satisfied. At this stage, it will be useful to define $u_k = \frac{n_k - n_k^*}{n_k}$ as the *inactivity rate* of group k and $a_k = 1 - n_k^* \frac{z(K^*)}{\mu_{-1}^*(\mathbf{v}_k)}$ as the *activity gap* of group k. The inactivity rate is the proportion of members in group k who remain inactive in the candidate equilibrium. The activity gap is a function of the ratio between the harmonic mean of the valuations of active members and the activity threshold $z(K^*)$. It measures how affluent the group is with respect to the affluence line. With these concepts at hand, we can state the following proposition characterizing the equilibria of contests with heterogeneous groups and linear costs.

Proposition 5 *Assume effort cost is linear. Then, the contest admits an equilibrium where the number of groups with positive winning probability is an integer K^* such that*

$$\frac{1}{K^*} + \frac{\beta}{\gamma}\ln\frac{a_{K^*}}{u_{K^*}} - \frac{1}{K^*}\frac{\beta}{\gamma}\sum_{k=1}^{K^*}\ln\frac{a_k}{u_k} > 0,$$

and there are n_k^* active members in each group if the following conditions hold:

(i) *The lowest valuation in each group $k \le K^*$ satisfies (25);*

(ii) *The number of active members* n_k^* *in each group* $k \leq K^*$ *satisfies (24);*
(iii) *The higher valuation in groups* $k > K^*$ *satisfies* $\max\{\widetilde{v}_{1k}\}_{k=K^*+1}^{K} < z(K^*)$.

In that equilibrium, winning probabilities are

$$p_k^* = \frac{1}{K^*} + \frac{\beta}{\gamma} \ln \frac{a_k}{u_k} - \frac{\beta}{\gamma} \ln \frac{G_a}{G_u} \qquad \textit{for } k = 1, \ldots, K^*,$$

where G_a *and* G_u *are respectively the geometric mean of the activity gaps and the inactivity rates of groups* $k = 1, \ldots, K^*$.

Although the analysis becomes quite intricate at this point, Proposition 5 shows that the lack of pure strategy equilibria and the preemption result we characterized in Proposition 1 no longer holds under heterogeneous groups. Multiple groups can be active. In that regard, difference-form group contests of this kind are closer to their Tullock counterparts; Brookins et al. (2015) find that some groups can remain inactive in the equilibrium of Tullock group contests under some conditions on the complementarity of efforts and the convexity of cost.

Proposition 5 also reveals that groups with higher activity gaps and lower inactivity rates enjoy higher chances of victory. Both dimensions are not independent, of course, but it is easy to see from the expression of the equilibrium impact and winning probability that given two groups with the same activity gap, the one with the lower inactivity rate is more likely to win the contest. Similarly, for two groups with the same number of active members n_k^*, the one more likely to prevail is the most affluent one, that is, the one with the highest harmonic mean of valuations $\mu_{-1}^*(\mathbf{v}_k)$.

The linear cost case allows us to obtain again explicit comparative statics results regarding the complementarity of efforts γ. The following corollary comes directly from the expression for p_k^* in Proposition 5.

Corollary 4 *Under linear costs, group k winning probability increases with the degree of complementarity of efforts* γ *if and only if* $\frac{a_k}{u_k} > \frac{G_a}{G_u}$.

An increase in the complementarity of efforts benefits groups that are relatively more affluent (higher activity gap) and with a lower proportion of inactive members (lower inactivity rate). These are groups where the members with the lowest valuations have nonetheless relatively high valuations of victory. Hence, as γ increases and the impact technology becomes more reliant in the lowest efforts, these groups become relatively more likely to prevail in the contest.

5 Between and within-group heterogeneity

In this last section, we will study the effect of heterogeneity within and across groups on the contest. This is a topic widely studied in the sociology and politics of collective action (e.g. Olson 1965) that has received increased attention in contest theory (e.g. Ryvkin 2011; Brookins et al. 2015; Kolmar and Rommeswinkel 2020). To that end, we will make use of the equilibrium characterizations obtained in the previous sections and of the properties of the affluence index. First, we will explore how aggregate effort in homogeneous contests varies with between-group heterogeneity. Second, we will analyze whether groups with more heterogenous members' valuations are more or less likely to win the contest.

5.1 Between-group heterogeneity and aggregate effort

First we study how the level of aggregate effort in the contest varies with heterogeneity in valuations across groups. In Proposition 2 and Corollary 1 we already showed that the number of homogeneous groups active in the contest rests on absolute and relative considerations. A group is active if its valuation is above the relevant activity threshold $z(K^*)$ *and* high enough relative to that of groups who value victory even more.

Next, we show that the aggregate contest effort can be written as a function of the affluence index as defined in Sect. 4.1. This will allow us to use the properties of the index to study the effect of between-group heterogeneity on total effort.

Assume that in equilibrium K^* groups have a positive winning probability. Adding up the individual equilibrium effort as characterized in (7) across group members yields

$$\sum_{i=1}^{n_k} \widehat{x}_k(K^*) = \frac{1}{\phi}[\ln(\widetilde{v}_k)^{n_k} - n_k \ln z(K^*)],$$

and then adding up across groups to get the aggregate conflict effort yields

$$\sum_{k=1}^{K} \sum_{i=1}^{n_k} \widehat{x}_k(K^*) = \frac{N}{\phi}[\ln(\prod_{k=1}^{K}(\widetilde{v}_k)^{n_k})^{\frac{1}{N}} - \ln z(K^*)] = \frac{N}{\phi} W_o(\widetilde{\mathbf{v}}^*, z(K^*)). \quad (26)$$

In words, the equilibrium aggregate effort in an homogeneous difference-form group contest can be written as a function of the affluence index applied to the distribution of censored valuations $\widetilde{\mathbf{v}}^*$ with affluence line $z(K^*)$.

We can now use the properties of the affluence index to perform a comparative statics exercise as valuations across groups become more or less similar. To that end, we first need to define a criterion to judge whether a distribution of group valuations is more or less homogeneous than another one. We will employ the Pigou-Dalton Principle, a widely-used criterion to rank distributions.

Definition (Pigou-Dalton principle for homogeneous groups) *Take two group-homogeneous censored valuation vectors* $\widetilde{\mathbf{v}}$ *and* $\widetilde{\mathbf{v}}'$ *where* $\widetilde{\mathbf{v}}'$ *is obtained by subtracting* $\Delta > 0$ *from each individual valuation in group k and adding* $\Delta \frac{n_k}{n_l}$ *to each individual valuation in group l such that* $\widetilde{v}_l + \Delta \frac{n_k}{n_l} < \widetilde{v}_k - \Delta$. *Then the distribution* $\widetilde{\mathbf{v}}'$ *is more homogeneous/less unequal than* $\widetilde{\mathbf{v}}$.

This principle states that a rank-preserving change in group valuations such that a group with a lower valuation values victory more after the change whilst a group with a higher valuation values victory less makes the distribution of group valuations more homogeneous. The Pigou-Dalton principle is equivalent to Lorenz dominance

(Dasgupta et al. 1973). Because the geometric mean is a Schur-concave function, a Pigou-Dalton change as described above increases that mean.[18] Since the affluence index as defined in (12) is increasing in the geometric mean of the distribution considered, we can employ that property to study the effect of between-group homogeneity on aggregate contest effort.

Proposition 6 (Between-group heterogeneity and aggregate effort) *Consider a contest among homogeneous groups. Then, a Pigou-Dalton change between*

(i) *two active groups which leaves the set of active groups unchanged increases the aggregate contest effort;*

(ii) *an active and an inactive group which makes the lower valuation group active increases aggregate contest effort; the opposite happens if the change leaves the set of active groups unchanged;*

(iii) *two inactive groups has no effect.*

This proposition sheds some light on how a contest organizer should design a difference-form contest to incentivize effort provision. From Proposition 2 we know that the number of active groups in equilibrium is higher when valuations across groups are more homogeneous. We have seen that the same applies to aggregate effort. A group whose valuation is markedly lower than the rest is likely to remain inactive, so more homogeneity across groups increases the number of active groups and total contest effort. This holds for any degree of complementarity of efforts, since the internal equilibrium is symmetric, and for all degrees of cost convexity ϕ. This is in contrast with Ryvkin (2011), who finds that groups becoming more similar increases total effort in Tullock contests only if the cost of effort is not too steep.

If a group or a set of groups have significantly lower valuations than the rest, the contest organizer might better orchestrate separate subcontests or, if possible, change valuations within groups. We explore the latter alternative next.

5.2 Within-group heterogeneity and winning probabilities

We next explore whether groups with more heterogeneous valuations are more or less likely to prevail in the contest. This exercise can help understand better the patterns and outcomes of ethnic or international conflicts (e.g. Galbraith et al. 2007; Esteban and Ray 2011b).

In order to perform this comparative statics exercise, we need to employ some criterion to compare the heterogeneity between distribution of valuations. To that end, let us use the Atkinson index of inequality (Atkinson 1970) which is defined as

$$I_{\xi}(\mathbf{v}_k) = 1 - \frac{\mu_{1-\xi}(\mathbf{v}_k)}{\mu_1(\mathbf{v}_k)},$$

where $\mu_1(\mathbf{v}_k)$ is the average valuation in group k, i.e. its mean of order 1, and $\mu_{1-\xi}(\mathbf{v}_k)$ is the mean of order $1-\xi$ of the valuations in the group. In Atkinson (1970), the

[18] For a discussion of Schur-concavity, see Marshall et al. (1979).

parameter $\xi \geq 0$ is normative and measures the inequality aversion of the social planner. The index equals zero when $\zeta = 0$ as $\mu_{1-\xi}(\mathbf{v}_k) = \mu_1(\mathbf{v}_k)$ and boils down to $1 - \frac{v_{n_k k}}{\mu_1(\mathbf{v}_k)}$ when $\zeta \to \infty$. The Atkinson index ranges between zero and one. Higher values of the index denote more inequality. The ordering of distributions the index generates is equivalent to the one under Lorenz dominance (Dasgupta et al. 1973).

Let us now apply this index to state our next result. Before that, recall we denoted $\rho = -\frac{\gamma}{\gamma+\phi}$ and by $\mu_1(\widetilde{\mathbf{v}}_k)$ the average censored valuation in group k, where censored valuations are defined as in (10).

Proposition 7 (Within-group heterogeneity and success) *Consider a contest among heterogeneous groups with either linear impact or in which all groups are fully active in equilibrium. If $I_{1+\rho}(\widetilde{\mathbf{v}}_k^*) < I_{1+\rho}(\widetilde{\mathbf{v}}_l^*)$ and $\mu_1(\widetilde{\mathbf{v}}_k) \geq \mu_1(\widetilde{\mathbf{v}}_l)$, the equilibrium winning probability of group k is higher than for group l.*

Proposition 7 shows that for two groups with the same average censored valuation, the more homogeneous one according to the Atkinson index of inequality is more likely to prevail in the contest. Let us emphasize that inequality is measured over the distribution of censored valuations. This implies that the index would be the same for two groups with very different distributions of valuations below the activity threshold $z(K^*)$ but identical otherwise. Those qualifications made, we can conclude that there exist strong forces in favor of homogeneity in difference-form group contests. But this homogeneity is defined over the set of active members, that is, the ones with relatively higher valuations. The more homogeneous their active members are, the better the chances of the group winning the contest.

Both the contest organizer and the members of heterogeneous -and thus weaker-groups may be willing to alter the distribution of valuations to their favor. The former, to induce more effort from contestants. The latter, to provide incentives to make more effort to their lower-valuation members. For this to be a meaningful exercise, we will assume that valuations represent a transferable stake in the contest, such as the income that members must defend from outsiders or the claims they have over the object being contested.

We now ask whether groups could benefit from altering the distribution of stakes across their members in order to elicit more effort from them and be more likely to win. To deal with such transfers, we apply again the Pigou-Dalton Principle.

Definition (Pigou-Dalton principle for individual transfers) *Take two valuation vectors $\mathbf{v}_k$ and $\mathbf{v}'_k$, where $\mathbf{v}'_k$ is obtained by subtracting $\Delta > 0$ from v_{jk} and adding it to v_{ik} such that $v_{ik} + \Delta < v_{jk} - \Delta$. Then the distribution $\mathbf{v}'_k$ is more homogeneous/less unequal than $\mathbf{v}_k$.*

In this context, the principle states that a rank-preserving progressive transfer from a member with a higher valuation to a member with a lower valuation cannot increase heterogeneity in the distribution of valuations. Because means of order r are Schur-concave when $r < 1$, a Pigou-Dalton transfer when $1 + \rho > 0$ increases the value of the generalized mean. It is for this reason that generalized means are commonly employed in the inequality measurement literature. As in the previous subsection, we employ this property of generalized means to study the effect of within-group redistribution.

Proposition 8 (Within-group redistribution) *Consider a contest among heterogeneous groups with either linear impact or in which all groups are fully active in equilibrium. Then, a progressive transfer between*

(i) *two active members which leaves the set of active players unchanged increases the group's equilibrium winning probability;*
(ii) *an active and an inactive member which makes the recipient active increases the group's equilibrium winning probability; the opposite happens if the transfer leaves the set of active players unchanged;*
(iii) *two inactive players has no effect.*

Proposition 8 shows that in difference-form group contests, groups may benefit from altering their distribution of valuations through progressive transfers. Reducing the dispersion in valuations among active members makes the group relatively more successful. This is regardless of the complementarity of efforts γ and the convexity of the effort cost ϕ. In contrast, Cubel and Sanchez-Pages (2014) find that progressive changes in valuations in Tullock contests are beneficial for the group only when efforts are sufficiently complementary or the cost of effort convex enough. Otherwise, groups may benefit from regressive transfers. Part of the result here is driven by the convexity of the cost function: High valuation members contribute more and face a higher marginal cost of effort; a transfer to a low valuation member, who contributes less and thus faces a lower marginal cost of effort, induces an increase in impact greater than the decrease in effort of the high valuation member. As a result, the group is more successful. However, if that transfer goes to an inactive member and it is not large enough to lift that member above the affluence line $z(K^*)$, the impact of the group becomes smaller and the transfer is detrimental to its chances of victory. For that same reason, regressive transfers can never be beneficial to the group in difference-form contests; these transfers increase heterogeneity within the group, decreasing the effort of lower-valuation members, even to the point of making them inactive.

6 Conclusion

In this paper we have offered the first study of group contests where winning probabilities depend on the difference between groups' impacts. Whether a group is active or enjoys a positive winning probability depends on absolute and relative considerations. Groups and members must value victory enough to have an incentive to become active. But when opponents have sufficiently high valuations and become very aggressive, groups and individuals with lower valuations prefer to remain inactive. We believe this is a realistic feature that stands in contrast with that of Tullock group contests, where often all groups are active in equilibrium.

We have also shown that key equilibrium variables in these contests can be expressed as a function of inequality and affluence indexes. The properties of these indices have allowed us to study the impact of within and between-group heterogeneity. Aggregate contest effort increases as groups become more homogeneous. Groups whose active members have more homogeneous valuations as measured by the Atkinson index of inequality are more likely to win the contest. Given that result, groups may have an incentive to redistribute stakes internally. Progressive transfers can lead a group to a more likely victory when they make the set of active members more homogeneous or when they induce some inactive members to become active.

As this paper has shown, the analysis of difference-form group contests can be rather intricate. We still believe that our paper can open new and valuable research avenues. As mentioned at the introduction, the literature on group contests has explored issues such as the impact of group size, endogenous within-group sharing rules and endogenous coalition formation. Another area is the characterization of mixed strategies. Finally, it would be nice to study the optimal design of these contests, which we have only touched very briefly in this paper. These are all areas left open for future research.

A Appendix

Proof or Proposition 1 First, let us characterize the equilibrium where no group is active, i.e., when the strategy profile $\mathbf{x} = \mathbf{0}$ is an equilibrium. When all efforts are zero all group impacts are zero. Then $K^* = K$. The marginal benefit of effort for a member of group k evaluated at that strategy profile is $\frac{\widetilde{v}_k}{z(K)}$. Hence, no member of any group prefers to deviate and become active if and only if $\widetilde{v}_1 < z(K)$.

Assume instead that $\widetilde{v}_1 \geq z(K)$. We know that in any pure strategy equilibrium at most one group will be active. Denote that group by k. Next, we characterize all possible internal equilibria in group k.

The derivative of the payoff function for any member i must satisfy

$$u'_{ik}(\widehat{\mathbf{x}}_k, \{\mathbf{0}\}_{l \neq k}) = \beta \frac{K-1}{K} \frac{e^{-\gamma \widehat{x}_{ik}}}{\sum_{j=1}^{n_k} e^{-\gamma \widehat{x}_{jk}}} v_k - 1 \geq 0. \tag{27}$$

Otherwise that member would like to lower their effort. Note that the sign of the derivative can be strictly positive in the particular case where $p_k(\widehat{\mathbf{x}}_k) = 1$ because further increases of effort do not increase the winning probability at that point. Adding up the above expression across players yields $\widetilde{v}_k \geq z(K)$. Hence, this is a necessary condition for this equilibrium to exist.

The next step is to show that under the internal equilibrium profile $\widehat{\mathbf{x}}_k$ group k wins with certainty. Suppose on the contrary that $p_k(\widehat{\mathbf{x}}_k, \{\mathbf{0}\}_{l \neq k}) < 1$. Then (27) must hold with equality for all i; otherwise at least one member would like to increase their effort. This in turn implies that the equilibrium must be symmetric, i.e., $\widehat{x}_{ik} = \widehat{x}_{jk} = \widehat{x}$ for any $i, j \in k$ so (27) boils down to

$$u'_{ik}(\widehat{\mathbf{x}}_k, \{\mathbf{0}\}_{l\neq k}) = \frac{\widetilde{v}_k}{z(K)} - 1 = 0.$$

Hence, such equilibrium can only exist in the non-generic case where $\widetilde{v}_k = z(K)$. Otherwise $\widehat{\mathbf{x}}_k$ must be such that $h_k(\widehat{\mathbf{x}}_k) = 1$ and $p_k(\widehat{\mathbf{x}}_k) = 1$, which is equivalent to

$$\sum_{i=1}^{n_k} e^{-\gamma \widehat{x}_{ik}} = n_k e^{-\frac{\gamma}{\beta}}. \tag{28}$$

There are multiple strategy profiles that satisfy the above. Substituting (28) into (27) yields that all these profiles must satisfy

$$\widehat{x}_{ik} \leq \frac{1}{\beta} + \frac{1}{\gamma} \ln \frac{\widetilde{v}_k}{z(K)} \qquad \text{for all } i \in k, \tag{29}$$

to ensure no member would prefer to deviate and lower their effort. In sum, all strategy profiles satisfying (28) and (29) constitute a candidate internal equilibrium. One of these profiles is the symmetric one where all members make effort $\widehat{x}_k = \frac{1}{\beta}$.

For these strategy profiles to lead to an overall equilibrium with $K^* = 1$ and $p_k(\widehat{\mathbf{x}}_k, \{\mathbf{0}\}_{l\neq k}) = 1$, we must also make sure that no member of any inactive group wants to become active. Denote as m the group with the highest modified valuation other than $\widetilde{v}_k$. Observe that $m = 2$ if $k = 1$ and $m = 1$ if $k \geq 2$. Members of group m are thus the ones with the most incentive to become active when group k is the only active one. For the equilibrium with only one active group to exist, no member of group m should be willing to become active when no other individual outside group k is active. Note that if a member of group m were to become active then $K^* = 2$, so no member of group m wants to become active if and only if $\widetilde{v}_m < z(2)$. This implies that $\widetilde{v}_2 < z(2)$ is a necessary condition for an equilibrium where the only active group is group 1 and $\widetilde{v}_1 < z(2)$ when that only active group is $k \geq 2$.

To summarize, an equilibrium where group k is the only active group (and wins with certainty) exists only if $\widetilde{v}_1 \geq z(K)$ and $\widetilde{v}_2 < z(2)$ if $k = 1$, and $\widetilde{v}_k \geq z(K)$ and $\widetilde{v}_1 < z(2)$ if $k \geq 2$. When $\widetilde{v}_1 \geq z(K)$ and $\widetilde{v}_2 < z(2)$ there are thus multiple pure strategy equilibria, both in terms of the identity of active group k and of its internal equilibrium profile $\widehat{\mathbf{x}}_k$. □

Proof of Proposition 2 Consider an equilibrium with a given K^*. The fact that the internal equilibrium must be symmetric and the concept of censored valuations allows us to write the impact of a group in this candidate equilibrium succinctly as

$$h_k(\widehat{\mathbf{x}}_k(K^*)) = \frac{\beta}{\phi} \ln \frac{\widetilde{v}_k^*}{z(K^*)}.$$

Ordering the censored valuation distribution across groups $\widetilde{\mathbf{v}}^*$ according to the order of the original valuation distribution $\widetilde{\mathbf{v}}$, it is possible to write group k's winning probability as

$$p_{k^*}(\widehat{\mathbf{x}}(K^*)) = \frac{1}{K^*} + \frac{\beta}{\phi}\ln\frac{\widetilde{v}_k^*}{z(K^*)} - \frac{1}{K^*}\frac{\beta}{\phi}\sum_{l=1}^{K^*}\ln\frac{\widetilde{v}_l^*}{z(K^*)}$$
$$= \frac{1}{K^*} + \frac{\beta}{\phi}\frac{K^*-1}{K^*}\ln\widetilde{v}_k^* - \frac{1}{K^*}\frac{\beta}{\phi}\ln\prod_{l=1,l\neq k}^{K^*}\widetilde{v}_l^*,$$

which is positive for group K^* if and only if

$$\widetilde{v}_{K^*}^* > e^{-\frac{\phi}{\beta(K^*-1)}}\Big(\prod_{l\neq k,l=1}^{K^*}\widetilde{v}_l^*\Big)^{\frac{1}{K^*-1}}.$$

which boils down to the expression stated in the proposition. In addition, it must be that $\widetilde{v}_{K^*+1} < z(K^*)$. Otherwise, members of group K^*+1, the ones with the most incentive to deviate, would like to become active. Recall that K^* is either equal to K or equal to the number of active groups. Hence, this deviation can only take place when the K^* groups with positive winning probability also constitute the set of active groups. Hence, group K^*+1 is inactive.

Let us compute the derivative of the payoff function of a member of group K^*+1 when all her fellow group members remain inactive is

$$\left.\frac{\partial u_{iK^*+1}}{\partial x_{iK^*+1}}\right|_{\mathbf{x}_{K^*+1}=\mathbf{0}} = \beta\frac{K^*-1}{K^*}\frac{v_{K^*+1}}{n_{K^*+1}} - 1 = \frac{\widetilde{v}_{K^*+1}}{z(K^*)} - 1,$$

which is negative if and only if $\widetilde{v}_{K^*+1} < z(K^*)$. In that case no member of any group $k \geq K^*+1$ has an incentive to deviate from the equilibrium. □

Proof of Proposition 3 The condition on K^* follows from plugging the group impact (13) into (3), and then checking that $p_k^*(\widehat{\mathbf{x}}_k(K^*)) > 0$ for groups $k = 1, \ldots, K^*$ using (2). Note that for inactive groups $\widetilde{\mathbf{v}}_k^* = z(K^*)$ and thus $W_o(\widetilde{\mathbf{v}}_k^*, z(K^*)) = 0$.

In addition, we need to ensure that no member of groups $k = K^*+1, \ldots, K$ would like to deviate and become active. Recall that we have ordered members within groups in such a way that the individual indexed as 1 has the highest valuation within the group. Denote $\widetilde{v}_k^{\max} = \max\{\widetilde{v}_{1k}\}_{k=K^*+1}^K$. This is the modified valuation of the individual with the highest incentive to deviate. For the profile characterized so far to be an equilibrium, it must be the case that the derivative of the payoff function of that individual when all members in her group remain inactive is negative, that is,

$$\left.\frac{\partial u_{ik}^{\max}}{\partial x_{ik}^{\max}}\right|_{\mathbf{x}_k=0} = \beta\frac{K^*-1}{K^*}\frac{v_k^{\max}}{n_k} - 1 = \frac{\widetilde{v}_k^{\max}}{z(K^*)} - 1 < 0,$$

which is the condition stated in the text of the Proposition. □

Proof of Proposition 5 From the earlier discussion we know that (24) and (25) are necessary conditions for the equilibrium to exist. These conditions avoid the non-existence of pure strategy equilibria and ensure that n_k^* members are active. In addition, we need to make sure that no member of any group $k > K^*$ would like to become active. To do that, we just need to follow the same procedure as in the proof of Proposition 3 and impose $\max\{\widetilde{v}_{1k}\}_{k=K^*+1}^{K} < z(K^*)$.

We next derive the condition defining K^* and the resulting equilibrium winning probabilities. Using (22) to compute the impact of a group yields:

$$h_k(\mathbf{x}_k^*(K^*)) = \ln\left[\frac{1}{n_k}(n_k - n_k^* + \sum_{j=1}^{n_k^*} e^{-\gamma x_{ik}})\right]^{-\frac{\beta}{\gamma}} = -\frac{\beta}{\gamma}\ln\left[\frac{n_k - n_k^*}{n_k}\frac{1}{1 - n_k^*\frac{z(K^*)}{\mu_{-1}^*(\mathbf{v}_k)}}\right]$$
$$= \frac{\beta}{\gamma}\ln[\frac{1 - n_k^*\frac{z(K^*)}{\mu_{-1}^*(\mathbf{v}_k)}}{1 - \frac{n_k^*}{n_k}}] = \frac{\beta}{\gamma}\ln\frac{a_k}{u_k}.$$

It holds that K^* groups have a positive winning probability if

$$\frac{1}{K^*} + \frac{\beta}{\gamma}\ln\frac{a_{K^*}}{u_{K^*}} - \frac{1}{K^*}\frac{\beta}{\gamma}\sum_{k=1}^{K^*}\ln\frac{a_k}{u_k} > 0.$$

Their equilibrium winning probabilities in this case are

$$p_k^* = \frac{1}{K^*} + \beta\ln\left[\frac{a_k}{u_k}\right]^{\frac{1}{\gamma}} - \frac{\beta}{K^*}\sum_{l=1}^{K^*}\ln\left[\frac{a_l}{u_l}\right]^{\frac{1}{\gamma}} = \frac{1}{K^*} + \beta\ln\frac{a_k^{\frac{1}{\gamma}}}{\left[\prod_{l=1}^{K^*} a_l^{\frac{1}{\gamma}}\right]^{\frac{1}{K^*}}} - \beta\ln\frac{u_k^{\frac{1}{\gamma}}}{\left[\prod_{l=1}^{K^*} u_l^{\frac{1}{\gamma}}\right]^{\frac{1}{K^*}}},$$

which is the expression stated in last part of the Proposition. □

Proof of Proposition 6 The proof of most of the results in the proposition comes directly from the discussion in the text and from expression (26). If the change takes place between two active groups then the geometric average of the censored valuations increases. The same happens if the change lifts the valuation in the lower valuation group above the affluence line $z(K^*)$. Note that then the new affluence line is $z(K^*+1) < z(K^*)$, so the index unambiguously increases. However, if the change in valuations takes place in an active group and an inactive group and the latter remains inactive, the geometric mean unambiguously decreases as the censored valuation of the lower valuation group remains unchanged. Finally, if the transfer takes place between two inactive groups the index remains unchanged as their censored group valuations remain below $z(K^*)$. □

Proof of Proposition 7 Rewriting the Atkinson index yields

$$\mu_\rho(\widetilde{\mathbf{v}}_k) = \mu_1(\widetilde{\mathbf{v}}_k)(1 - I_{1+\rho}(\widetilde{\mathbf{v}}_k)).$$

If $I_{1+\rho}(\widetilde{\mathbf{v}}_k^*) < I_{1+\rho}(\widetilde{\mathbf{v}}_l^*)$ and $\mu_1(\widetilde{\mathbf{v}}_k) \geq \mu_1(\widetilde{\mathbf{v}}_l)$ it must be that $\mu_\rho(\widetilde{\mathbf{v}}_k) > \mu_\rho(\widetilde{\mathbf{v}}_l)$.

By examining (13) and (20) it is straightforward to see that group k enjoys a higher probability of success than group l if and only if $\mu_\rho(\widetilde{\mathbf{v}}_k) > \mu_\rho(\widetilde{\mathbf{v}}_l)$ and $\widetilde{G}_k^* > \widetilde{G}_l^*$ in each respective case. The former entails a comparison between means of order $\rho = -\frac{\gamma}{\phi+\gamma}$, whereas the latter entails a comparison between means of order 0, as it is assumed that $\gamma = 0$. Hence, the ranking $\mu_\rho(\widetilde{\mathbf{v}}_k) > \mu_\rho(\widetilde{\mathbf{v}}_l)$ is preserved in both cases, allowing us to conclude that $p_k^* > p_l^*$. □

Proof of Proposition 8 Consider first the case of linear impacts in Sect. 4.1. There, winning probabilities are given by (15). Note that a group's winning probability is increasing in its Watts affluence index $W_o(\widetilde{\mathbf{v}}_{K^*}^*, z(K^*))$ which in turn is increasing in the geometric mean of the censored valuations $\widetilde{G}_k^*$. Because the geometric mean is a mean of order 0, Pigou-Dalton progressive transfers would increase the average (recall means of order $r < 1$ are Schur-concave). If that transfer takes place between two members with valuations above $z(K^*)$ then the geometric average of the censored valuation does indeed increase. The same happens if the transfer lifts the recipient above the affluence line $z(K^*)$. However, if it takes place from an active member to an inactive member who remains inactive, the geometric mean of the censored distribution decreases as that transfer is "lost" below the affluence line. Finally, if the transfer takes place between two inactive members $\widetilde{G}_k^*$ remains unchanged as the censored valuations for both of them is still $z(K^*)$.

The same logic applies to the case in Sect. s4.2. There, the winning probability of group k is given by (21), which is increasing in its Generalized affluence index $W_\rho(\widetilde{\mathbf{v}}_k, z(K))$. This index is increasing in $\mu_\rho(\widetilde{\mathbf{v}}_k)$. Because this is a mean of order $\rho = -\frac{\gamma}{\phi+\gamma} \in (-1, 0]$ a progressive transfer among any two members (because all members are assumed to be active before and after the transfer) increases the index and thus p_k. □

References

Andonie C, Kuzmics C, Rogers BW (2019) Efficiency-based Measures of Inequality. J Math Econ 85:60–69

Atkinson AB (1970) On the measurement of economic inequality. J Econ Theory 2(3):244–263

Baik KH (1993) Effort levels in contests: the public-good prize case. Econ Lett 41:363–367

Baik KH (1998) Difference-form contest success functions and effort levels in contests. Eur J Polit Econ 14(4):685–701

Baik KH (2008) Contests with group-specific public-good prizes. Soc Choice Welfare 30(1):103–117

Barbieri S, Kovenock D, Malueg D, Topolyan I (2019) Group contests with private information and the "weakest link." Games Econ Behav 118:382–411

Barbieri S, Malueg D (2016) Private-information group contests: best-shot competition. Games Econom Behav 98:219–234

Barbieri S, Malueg D, Topolyan I (2014) The best-shot all-pay (group) auction with complete information. Econ Theor 57(3):603–640

Beviá C, Corchón L (2019) Contests with Dominant Strategies. Econ Theory (forthcoming)

Blackorby C, Donaldson D (1980) A theoretical treatment of indices of absolute inequality. Int Econ Rev 21:107–136

Bloch F (2012) Endogenous formation of alliances in contests. In: Garfinkel M, Skaperdas S (eds) The oxford handbook of the economics of peace and conflict. Oxford University Press, Oxford

Brookins P, Lightle JP, Ryvkin D (2015) Optimal sorting in group contests with complementarities. J Econ Behav Organ 112:311–323

Che Y-K, Gale I (2000) Difference-form contests and the robustness of all-pay auctions. Games Econom Behav 30:22–43

Cheikbossian G, Fayat R (2018) Group size, collective action and complementarities in efforts. Econ Lett 168:77–81
Choi JP, Chowdhury SM, Kim J (2016) Group contests with internal conflict and power asymmetry. Scand J Econ 118(4):816–840
Chowdhury SM, Lee D, Sheremeta RM (2013) Top guns may not fire: best-shot group contests with group-specific public good prizes. J Econ Behav Organ 92:94–103
Chowdhury SM, Lee D, Topolyan I (2016) The max–min group contest: weakest-link (group) all-pay auction. South Econ J 83:105–125
Chowdhury SM, Topolyan I (2016) Best-shot versus weakest-link in political lobbying: an application of group all-pay auction. Soc Choice Welfare 47:959–971
Corchón L, Dahm M (2010) Foundations for contest success functions. Econ Theor 43(1):81–98
Corchón L, Dahm M (2011) Welfare maximizing contest success functions when the planner cannot commit. J Math Econ 47(3):309–317
Crutzen B, Flamand S, Sahuguet N (2020) A model of a team contest, with an application to incentives under list proportional representation. J Public Econ 182:104–109
Cubel M, Sanchez-Pages S (2014) Inequality in conflicts. Working Paper
Cubel M, Sanchez-Pages S (2016) An axiomatization of difference-form contest success functions. J Econ Behav Organ 131:92–105
Cubel M, Sanchez-Pages S (2021) Generalized difference-form contests. Bath Econ Res Paper 79/21
Dasgupta P, Sen A, Starrett D (1973) Notes on the measurement of inequality. J Econ Theory 6(2):180–187
Drugov M, Ryvkin D (2017) Biased contests for symmetric players. Games Econom Behav 103:116–144
Esteban J, Ray D (2001) Collective action and the group size paradox. Am Polit Sci Rev 95(3):663–672
Esteban J, Ray D (2011) Linking conflict to inequality and polarization. Am Econ Rev 101:1345–1374
Esteban J, Ray D (2011) A model of ethnic conflict. J Eur Econ Assoc 9(3):496–521
Ewerhart C (2021) A typology of military conflict based on the Hirshleifer contest. University of Zurich Working Paper 400
Ewerhart C, Sun G-Z (2018) Equilibrium in the symmetric two-player Hirshleifer contest: uniqueness and characterization. Econ Lett 169:51–54
Ewerhart C, Sun G-Z (2020) The n-player Hirshleifer Contest. University of Zurich Working Paper 361
Finer S (1975) State and nation building in Europe: the role of the military. In: Tilly C (ed) The formation of national states in Western Europe. University Princeton Press, Princeton
Flamand S, Troumpounis O (2015) Prize-sharing rules in collective rent seeking. In: Congleton RD, Hillman AL (ed) Companion to the political economy of rent seeking. Elgar Edward Publishing, Cheltenham
Galbraith JK, Priest C, Purcell G (2007) Economic equality and victory in war: an empirical investigation. Defense Peace Econ 18(5):431–449
Gersbach H, Haller H (2009) The affectionate society: does competition for partners promote friendliness? Econ Theor 40(3):389–403
Hardy G, Littlewood JL, Pólya G (1934) Inequalities. Cambridge University Press, Cambridge
Hillman AL, Riley JG (1989) Politically contestable rents and transfers. Econ Polit 1(1):17–39
Hirshleifer J (1983) From weakest-link to best-shot: the voluntary provision of public goods. Public Choice 41:371–86
Hirshleifer J (1989) Conflict and rent-seeking success functions: ratio vs. difference models of relative success. Public Choice 63:101–112
Hirshleifer J (1991) The technology of conflict as an economic activity. Am Econ Rev 81(2):130–134
Katz E, Nitzan S, Rosenberg J (1990) Rent-seeking for pure public goods. Public Choice 65(1):49–60
Kolm S-C (1976) Unequal Inequalities I. J Econ Theory 12:416–442
Kolm S-C (1976) Unequal inequalities II. J Econ Theory 13:82–111
Kolmar M (2013) Group conflicts. Where do we stand? Working Paper
Kolmar M, Rommeswinkel H (2013) Contests with group-specific public goods and complementarities in efforts. J Econ Behav Organ 89:9–22
Kolmar M, Rommeswinkel H (2020) Group size and group success in conflicts. Soc Choice Welfare 55:777–822
Lazear EP, Rosen S (1981) Rank-order tournaments as optimum labor contracts. J Polit Econ 89(5):841–864
Lee D (2012) Weakest-link contests with group-specific public good prizes. Eur J Polit Econ 28:238–248
Marshall AW, Olkin I, Arnold BC (1979) Inequalities: theory of majorization and its applications. Springer, New York
Münster J (2009) Group contest success functions. Econ Theor 41(2):345–357

Olson M (1965) The logic of collective action. Harvard University Press, Cambridge, MA
Polishchuk L, Tonis A (2013) Endogenous contest success functions: a mechanism design approach. Econ Theor 52(1):271–297
Pollak RA (1971) Additive utility functions and linear Engel curves. Rev Econ Stud 38(4):401–414
Ryvkin D (2011) The optimal sorting of players in contests between groups. Games Econom Behav 73(2):564–572
Skaperdas S, Toukan A, Vaidya S (2016) Difference-form Persuasion Contests. J Public Econ Theory 28(6):882–909
Skaperdas S, Vaidya S (2012) Persuasion as a contest. Econ Theor 51:465–486
Tullock G (1980) Efficient rent seeking. In: Buchanan J, Tollison R, Tullock G (eds) Toward a Theory of the Rent-Seeking Society. Texas A&M University Press, College Station, TX
Vesperoni A, Yıldızparlak A (2019) Inequality and Conflict Outbreak. Soc Choice Welfare 53:135–173
Watts H (1968) An economic definition of poverty. In: Moynihan DP (ed) On understanding poverty. Basic Books, New York
Zheng B (1993) An axiomatic characterization of the watts poverty index. Econ Lett 42(1):81–86