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ORIGINAL PAPER

Deviation from proportionality and Lorenz-domination for claims problems

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Abstract

The Lorenz order is commonly used to compare rules for claims problems. In this paper, we incorporate the average of awards rule, the mean value of the set of awards vectors for a claims problem, to the ranking of the standard rules by proving some properties that are satisfied by this rule. We define a pair of coefficients, inspired by the Gini index, aimed at measuring, for any given claims problem, the discrepancy between the awards assigned by a rule and the proportional division. We generalize the proportionality deviation indices by introducing coefficients that measure the deviation between the awards selected by any two division rules. We show how these deviation indices are related to the Lorenz order.

Keywords Claims problems · Division rules · Average of awards rule · Lorenz-domination · Proportionality deviation indices · Generalized deviation indices

1 Introduction

A firm going bankrupt, the division of property among heirs, a government taxing incomes to implement a public project, a rationing problem, the distribution of insufficient supplies such as food or vaccines, or the global carbon budget are just some examples of conflicting claims problems. In all of them, a scarce resource has to be divided or distributed among a group of claimants. The mathematical model used to formally study these problems may look, at first, quite simple: a non-negative real number that represents the endowment, and a finite vector of claims whose coordi-

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nates add up to more than the total amount available. But, in fact, the model is very rich. The book by Thomson (2019) presents a comprehensive review of the fascinating literature on claims problems.

Aristotle is credited to propose sharing the endowment proportional to claims. O'Neill (1982) describes historical instances of different division procedures found in the Talmud and in several medieval texts. In general, a division rule is a way of associating with each claims problem a division among the claimants of the amount available. Therefore, for each claims problem a rule must select an allocation satisfying three basic requirements: no claimant should be asked to pay, no claimant should be awarded more than his claim, and the sum of the awards should be equal to the endowment. The set of all the allocations that meet these basic properties is the set of awards vectors for the claims problem. The inventory of division rules is now large. We consider in this paper nine of the central rules: the proportional, the constrained equal awards, the constrained equal losses, the constrained egalitarian, the Talmud, Piniles', the minimal overlap, the adjusted proportional, and the random arrival rules. In addition, we study the average of awards rule, introduced by Mirás Calvo et al. (2020), that selects for each claims problem the expected value of the (continuous) uniform distribution over its set of awards vectors.

The axiomatic approach has dominated the study of rules. Properties of rules are formulated that one may want to impose because they have some appeal for a particular situation, or because they cover a theoretical or even an ethical aspect. Then, rules are examined, classified, and characterized according to the properties that they satisfy (or violate). Another important issue when evaluating a rule is how differently it treats larger claimants as compared with smaller claimants. The economist Max Otto Lorenz proposed in 1905 a simple method, now called the Lorenz curve, for visualizing distributions of income or wealth (Lorenz 1905). In the context of claims adjudication the closely related Lorenz order is used as a general criterion to rank rules. Basically, an awards vector Lorenz-dominates another if the cumulative sums of ordered awards are bigger for the first vector. The Lorenz order is a partial order. Using different methods, several authors, among others Schummer and Thomson (1997), Chun et al. (2001), Bosmans and Lauwers (2011), and Thomson (2012), study whether or not the division rules are Lorenz-comparable. As a corollary, we have a complete picture of the ranking of the nine central rules.

Our first goal is to rank the average of awards rule. We rely on the characterizations of Piniles' rule and the minimal overlap rule given by Schummer and Thomson (1997) and Bosmans and Lauwers (2011) respectively. We need to show that the average of awards rule satisfies, besides the basic properties already proven by Mirás Calvo et al. (2020), null claims consistency, order preservation under endowment variations, and order preservation under claims variations. We conclude that the average of awards rule Lorenz-dominates the minimal overlap rule and is Lorenz-dominated by Piniles' rule. Naturally, since the average of awards rule is self-dual it is not Lorenz-comparable with the other self-dual rules: the proportional, the adjusted proportional, the Talmud, and the random arrival rules.

Whether or not the recommendations made by two rules for a claims problem are Lorenz-comparable, the corresponding awards vectors can be similar or they can differ greatly. So, our second objective is to define some coefficients aimed at measuring the discrepancy between the awards vectors selected by two rules. Our basic reference is the Gini index, introduced in 1912 by the statistician and sociologist Corrado Gini as a coefficient intended to measure the degree of income inequality within a population. Ceriani and Verme (2012) provide a historical account of Gini's original formulation. Mathematically, the Gini coefficient is based on the Lorenz curve that represents in the horizontal axis the proportion of the population, from lowest to highest income, and in the vertical axis the cumulative percentage of income or wealth owned. A perfectly equal distribution of wealth would have a Lorenz curve equal to the line $y = x$. The Gini coefficient measures how far the actual Lorenz curve for a population's income is from the line of equality.

Given a claims problem, if one plots the cumulative percentage of awards with respect to the proportion of claimants, from lowest to highest claims, the line of equality represents the egalitarian division of the endowment. But, in general, the egalitarian division does not select an awards vector for the problem, so it is not a rule. Therefore, instead of the proportion of population, we represent in the horizontal axis the cumulative percentage of claims, ordered from small to large. Then, since the proportional rule shares the endowment in the same proportion as claims, the line $y = x$ is now the line of proportionality. So, we plot the cumulative percentage of the endowment that is assigned by a rule to the cumulative percentage of claims. The monotonically increasing continuous piecewise linear function thus obtained, whose graph lies in the unit square, is called the cumulative claims-awards curve. Naturally, the line of proportionality corresponds to the cumulative claims-awards curve of the proportional rule. We show that the claims-awards curve fully captures the Lorenz ranking of rules. Then, adapting the definition of the Gini index, we introduce a pair of coefficients, the proportionality deviation index, and the signed proportionality deviation index, that measure the deviation of the claims-awards curve from the line of proportionality as the ratio of the area, and the net signed area respectively, that lies between that line and the curve over the total area under the line of proportionality. In this framework, the proportional rule is the rule of reference: given the initial inequality of the vector of claims, the proportionality deviation indices measure the deviation of the distribution of the endowment with respect to this initial inequality.

Certainly, the proportional rule stands out as the best-known rule, and questionnaire studies on claims problems, such as Bosmans and Schokkaert (2009), show that it performs very well in describing the choices of the respondents. Even Thomson (2019) states that "proportionality is often taken as the definition of fairness for claims problems", only to successfully challenge this view. Lately, several authors have analyzed the preservation in gains and losses (the differences between claims and awards) of the inequality in claims. Order preservation is a basic property, met by our ten rules, that requires that a rule should respect the ordering of claims and that the losses should also be ordered as claims are. Now, fix an endowment and take two Lorenz-comparable awards vectors whose coordinates add up to the same amount. Hougaard and Østerdal (2005) propose the requirement that the awards and losses vectors selected by a rule for those two problems are also Lorenz comparable in the same direction. Kasajima and Velez (2010, 2011) show that, when there are more than three claimants, the only rule that satisfies order preservation and claims-inequality preservation in gains and losses is the proportional rule. These results reinforce the role of the proportional rule as the rule of reference to define the deviation indices. The information provided by the pair of proportionality deviation indices of a rule for a given claims problem not only indicates if the rule and the proportional division are Lorenz-comparable but also gives a clear and simple numerical value that quantifies how far from proportionality is the awards vector selected by the rule. We also show that, for a fixed vector of claims, the graph plotting the corresponding index for a given rule as a function of the endowment, the index path, is a good visual instrument that conveys much information about the rule itself.

We choose, for the reasons explained above, the proportional rule as the base rule to define the pair of deviation coefficients. But, each division rule entails different principles of fairness, equity, or justice. In order to make the best decision when solving a particular claims problem, it could be interesting to measure the degree of discrepancy of the awards vectors selected by two arbitrary rules. Fix a rule as the base for comparison. We define the curve representing the vector of cumulative percentages of the awards selected by any given rule against the vector of cumulative percentages of the awards recommended by our base rule. Now, the identity line represents the distribution of resources given by the rule of reference. Then, we introduce the deviation index (or signed deviation index) of a rule with respect to the rule of reference, by measuring, for each claims problem, the deviation (respectively, signed deviation) between the cumulative proportions of the initial endowment assigned by both rules. Therefore, the corresponding deviation indices quantify how far any rule moves away from the reference rule. Of particular interest are the indices with respect to the constrained equal awards and the constrained equal losses rules, since they are Lorenz-maximal and Lorenz-minimal among the order preserving rules, and the indices with respect to the average of awards rule, because it is the mean value of all the awards vectors.

Bosmans and Lauwers (2011) and Thomson (2012) explicitly emphasize that the fact that a rule Lorenz-dominates another rule should not be interpreted as a sign that the first rule is superior or inferior to the other. Obviously, the same applies to the claims-awards curves and the deviation indices. Given two rules, the relative position of its curves or the value of their indices just reveal how they are related, how they treat large claims in relation to small claims or how they depart from the proportional division or from any other rule of reference. Of course, it is up to the decision maker to use this information depending on the specific real-world context of the claims problem.

In Sect. 2 we introduce the basic definitions, notations, rules and properties and recall the Lorenz-ranking of the basic rules. We compare in Sect. 3 the average of awards and the other rules. We introduce, in Sect. 4, the cumulative claims-awards curve, the proportionality deviation indices, and the index path, three alternative tools to compare rules with the proportional division. Section 5 generalizes the indices to compare any two given rules. Finally, we leave to the Appendix the proofs of the results. The computations and figures in the examples were carried out using the *ClaimsProblems* R package (Núñez Lugilde et al. 2021).

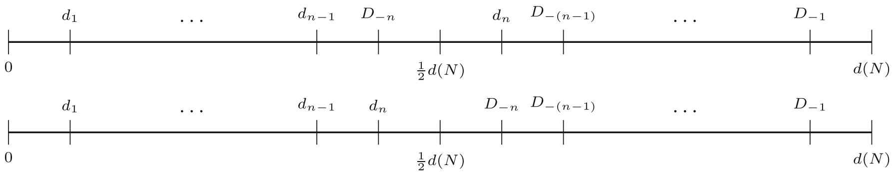

Fig. 1 Claims arranged in ascending order on the interval $[0, d(N)]$

2 Preliminaries

Let $\mathcal{N}$ be the set of all finite non-empty subsets of the natural numbers $\mathbb{N}$. Given $N \in \mathcal{N}$, $x \in \mathbb{R}^N$, and $S \in 2^N$ let $|N|$ be the number of elements of N and $x(S) = \sum_{i \in S} x_i$. Given $x, y \in \mathbb{R}^N$, the notation $x \leq y$ means that $x_i \leq y_i$ for all $i \in N$. If $N' \subset N \in \mathcal{N}$ and $x \in \mathbb{R}^N$, let $x_{N'} = \left(x_i\right)_{i \in N'} \in \mathbb{R}^{N'}$ be the projection of x onto $\mathbb{R}^{N'}$. In particular denote $x_{-i} = x_{N \backslash \{i\}} \in \mathbb{R}^{N \backslash \{i\}}$ the vector obtained by neglecting the ith-coordinate of x, i.e., $x_{-i} = (x_1, \ldots, x_{i-1}, x_{i+1}, \ldots, x_n)$. For simplicity, we write $x = (x_{-i}, x_i)$.

A claims problem with set of claimants $N \in \mathcal{N}$ is a pair (E, d) where $E \geq 0$ is the endowment to be divided and $d \in \mathbb{R}^N$ is the vector of claims satisfying $d_i \geq 0$ for all $i \in N$ and $d(N) \geq E$. We denote the class of claims problems with set of players N by C^N.

For each $(E, d) \in C^N$ and each $i \in N$ let $D_{-i} = d(N) - d_i = d(N \backslash \{i\})$. The minimal right of claimant $i \in N$ in $(E, d) \in C^N$ is the quantity $m_i(E, d) = \max\left\{0, E - D_{-i}\right\}$, what is left of the endowment after all other claimants have been fully compensated if possible, and 0 otherwise. The truncated claim of claimant $i \in N$ in $(E, d) \in C^N$ is $t_i(E, d) = \min\{E, d_i\}$, the minimum of the claim and the endowment. Let $m(E, d) = \left(m_i(E, d)\right)_{i \in N}$ and $t(E, d) = \left(t_i(E, d)\right)_{i \in N}$. To simplify, sometimes we write $m_i = m_i(E, d)$ and $t_i = t_i(E, d)$.

Let $\mathbb{R}^n_{\leq}$ be the set of nonnegative n-dimensional vectors $x = (x_1, \ldots, x_n)$ with coordinates ordered from small to large, i.e., $0 \leq x_1 \leq \ldots \leq x_n$. For simplicity, given $(E, d) \in C^N$ with $|N| = n$, we will assume throughout the paper that $N = \{1, \ldots, n\}$ and that $d \in \mathbb{R}^n_{\leq}$. As a consequence of such an arrangement of the claims we have that $d_i \leq D_{-i}$, $D_{-i} \geq D_{-(i+1)}$ and $m_i(E, d) \leq m_{i+1}(E, d)$ for all $i \in N \backslash \{n\}$. As it is illustrated in Fig. 1, either $d_n \leq D_{-n}$ or $D_{-n} \leq d_n$, but in both cases $\frac{1}{2}d(N)$ is the middle point of the line segment with endpoints d_n and D_{-n}. In fact, $\frac{1}{2}d(N)$ is also the middle point of the intervals $[d_i, D_{-i}]$ for all $i \in N \backslash \{n\}$.

A vector $x \in \mathbb{R}^N$ is an awards vector of $(E, d) \in C^N$ if $0 \leq x \leq d$ and $x(N) = E$. Let $X(E, d)$ be the set of awards vectors for $(E, d) \in C^N$. O'Neill (1982) associates to each claims problem $(E, d) \in C^N$ a coalitional game with set of players N and characteristic function $v(S) = \max\left\{0, \ E - d(N \backslash S)\right\}$, $S \in 2^N$. Thomson (2019) shows that the set of awards vectors for a claims problem coincides with the core of the associated coalitional game, that is, $X(E, d)$ is the set of allocations satisfying the balance requirement that are bounded from below by the minimal rights and are bounded from above by the truncated claims:

$$X(E,d)=\left\{x\in\mathbb{R}^N: x(N)=E,\ m(E,d)\leq x\leq t(E,d)\right\}.$$

Then, $X(E,d)$ is a nonempty convex polytope that has, at most, dimension $n-1$.

A rule is a function $\mathcal{R}\colon C^N\to\mathbb{R}^N$ assigning to each claims problem $(E,d)\in C^N$ an awards vector $\mathcal{R}(E,d)\in X(E,d)$. The following rules have been discussed in the literature and will be used throughout the paper.

- Proportional rule (PRO): For each $(E,d)\in C^N$ and each $i\in N$, $\mathrm{PRO}_i(E,d)=\frac{d_i}{d(N)}E$.
- Adjusted proportional rule (APRO): For each $(E,d)\in C^N$ and each $i\in N$,

$$\mathrm{APRO}_i(E,d)=m_i+\mathrm{PRO}_i\Big(E-\sum_{j\in N}m_j,\big(\min\big\{d_j-m_j,E-\sum_{j\in N}m_j\big\}\big)_{j\in N}\Big).$$

- Constrained equal awards rule (CEA): For each $(E,d)\in C^N$ and each $i\in N$, $\mathrm{CEA}_i(E,d)=\min\{\alpha,d_i\}$, where $\alpha\geq 0$ is chosen such that $E=\sum_{j\in N}\mathrm{CEA}_j(E,d)$.
- Constrained equal losses rule (CEL): For each $(E,d)\in C^N$ and each $i\in N$, $\mathrm{CEL}_i(E,d)=\max\{0,d_i-\beta\}$, where $\beta\geq 0$ is chosen such that $E=\sum_{j\in N}\mathrm{CEL}_j(E,d)$.
- Talmud rule (T): For each $(E,d)\in C^N$ and each $i\in N$,

$$\mathrm{T}_i(E,d)=\begin{cases}\mathrm{CEA}_i(E,\frac{d}{2}) & \text{if } E\leq\frac{1}{2}d(N)\\ d_i-\mathrm{CEA}_i(d(N)-E,\frac{d}{2}) & \text{if } E\geq\frac{1}{2}d(N)\end{cases}.$$

- Piniles' rule (PIN): For each $(E,d)\in C^N$ and each $i\in N$,

$$\mathrm{PIN}_i(E,d)=\begin{cases}\mathrm{CEA}_i(E,\frac{d}{2}) & \text{if } E\leq\frac{1}{2}d(N)\\ \frac{d_i}{2}+\mathrm{CEA}_i\big(E-\frac{1}{2}d(N),\frac{d}{2}\big) & \text{if } E\geq\frac{1}{2}d(N)\end{cases}.$$

- Constrained egalitarian rule (CE): For each $(E,d)\in C^N$ and each $i\in N$,

$$\mathrm{CE}_i(E,d)=\begin{cases}\mathrm{CEA}_i(E,\frac{d}{2}) & \text{if } E\leq\frac{1}{2}d(N)\\ \max\big\{\frac{d_i}{2},\min\{d_i,\lambda\}\big\} & \text{if } E\geq\frac{1}{2}d(N)\end{cases},$$

  where $\lambda\geq 0$ is chosen such that $\sum_{j\in N}\max\big\{\frac{d_j}{2},\min\{d_j,\lambda\}\big\}=E$.
- Random arrival rule (RA): For each $(E,d)\in C^N$ and each $i\in N$,

$$\mathrm{RA}_i(E,d)=\frac{1}{|N|!}\sum_{\pi\in\Pi^N}\min\big\{d_i,\max\{0,E-d(P_\pi(i))\}\big\},$$

where Π^N is the set of strict orders on N and $P_\pi(i) = \{j \in N : \pi(j) < \pi(i)\}$ for $\pi \in \Pi^N$.

- Minimal overlap rule (MO): Let $d_0 = 0$. For each $(E, d) \in C^N$ and each $i \in N$,

  i) If $E \leq d_n$ then $\mathrm{MO}_i(E, d) = \frac{t_1}{n} + \frac{t_2 - t_1}{n-1} + \cdots + \frac{t_i - t_{i-1}}{n-i+1}$.
  ii) If $E > d_n$, let $s^* \in (d_{k*}, d_{k^*+1}]$, with $k^* \in \{0, 1, \ldots, n-2\}$, be the unique solution to the equation $\sum\limits_{j \in N} \max\{d_j - s, 0\} = E - s$. Then,

$$\mathrm{MO}_i(E, d) = \begin{cases} \frac{d_1}{n} + \frac{d_2 - d_1}{n-1} + \cdots + \frac{d_i - d_{i-1}}{n-i+1} & \text{if } i \in \{1, \ldots, k^*\} \\ \mathrm{MO}_i(s^*, d) + d_i - s^* & \text{if } i \in \{k^* + 1, \ldots, n\} \end{cases}.$$

Recently, Mirás Calvo et al. (2020) introduce the average of awards rule. For each $(E, d) \in C^N$ the average of awards rule, $\mathrm{AA}(E, d)$, selects the centroid of the set of awards vectors $X(E, d)$. Let μ be the $(n-1)$-dimensional Lebesgue measure and denote $V(E, d) = \mu(X(E, d))$ the volume (measure) of the set of awards vectors. If $V(E, d) > 0$ then for each $i \in N$,

$$\mathrm{AA}_i(E, d) = \frac{1}{V(E, d)} \int_{X(E,d)} x_i d\mu.$$

The core-center solution was introduced by González-Díaz and Sánchez-Rodríguez (2007) for the class of balanced games as the centroid of the core. Since the set of awards vectors for a claims problem coincides with the core of the associated coalitional game, the average of awards rule corresponds to core-center solution.

We focus now on properties of division rules. We say that a rule $\mathcal{R}$ satisfies:

- *anonymity*, if for each $(E, d) \in C^N$, each $\pi \in \Pi^N$, and each $i \in N$, we have $\mathcal{R}_{\pi(i)}\big(E, (d_{\pi(i)})\big) = \mathcal{R}_i(E, d)$, where Π^N is the class of bijections from N into itself.
- *continuity*, if for each sequence $(E^\ell, d^\ell) \in C^N$ and each $(E, d) \in C^N$, we have that if $(E^\ell, d^\ell) \to (E, d)$ then $\mathcal{R}(E^\ell, d^\ell) \to \mathcal{R}(E, d)$.
- *$\frac{1}{|N|}$-truncated-claims lower bounds on awards*, if for each $(E, d) \in C^N$ we have $\mathcal{R}(E, d) \geq \frac{1}{|N|} t(E, d)$.
- *minimal rights first*, if for each $(E, d) \in C^N$ then $\mathcal{R}(E, d) = m(E, d) + \mathcal{R}\big(E - \sum\limits_{i \in N} m_i(E, d), d - m(E, d)\big)$.
- *claims truncation invariance*, if for each $(E, d) \in C^N$ we have $\mathcal{R}(E, d) = \mathcal{R}(E, t(E, d))$.
- *order preservation in awards*, if for each $(E, d) \in C^N$ and each $\{i, j\} \subset N$, if $d_i \leq d_j$ then $\mathcal{R}_i(E, d) \leq \mathcal{R}_j(E, d)$.
- *order preservation in losses*, if for each $(E, d) \in C^N$ and each $\{i, j\} \subset N$, if $d_i \leq d_j$ then $d_i - \mathcal{R}_i(E, d) \leq d_j - \mathcal{R}_j(E, d)$.
- *midpoint property*, if for each $(E, d) \in C^N$ such that $E = \frac{1}{2} d(N)$, then $\mathcal{R}(E, d) = \frac{d}{2}$.
- *self-duality*, if for each $(E, d) \in C^N$ we have $\mathcal{R}(E, d) = d - \mathcal{R}\big(d(N) - E, d\big)$.

Table 1 Main properties satisfied by the ten rules

	PRO	APRO	MO	CEA	CEL	CE	PIN	T	RA	AA
Anonymity	✓	✓	✓	✓	✓	✓	✓	✓	✓	✓
Continuity	✓	✓	✓	✓	✓	✓	✓	✓	✓	✓
$\frac{1}{\vert N\vert}$-truncated-claims lower bounds	–	✓	✓	✓	–	✓	✓	✓	✓	✓
Minimal rights first	–	✓	✓	–	✓	–	–	✓	✓	✓
Claims truncation invariance	–	✓	✓	✓	–	–	–	✓	✓	✓
Order preservation	✓	✓	✓	✓	✓	✓	✓	✓	✓	✓
Midpoint	✓	✓	–	–	–	✓	✓	✓	✓	✓
Self-duality	✓	✓	–	–	–	–	–	✓	✓	✓
Endowment monotonicity	✓	✓	✓	✓	✓	✓	✓	✓	✓	✓
Claim monotonicity	✓	✓	✓	✓	✓	✓	✓	✓	✓	✓

- *endowment monotonicity*, if for each $(E, d) \in C^N$ and each $E' \geq 0$, if $d(N) \geq E' \geq E$ then $\mathcal{R}(E', d) \geq \mathcal{R}(E, d)$.
- *claim monotonicity*, if for each $(E, d) \in C^N$, each $i \in N$, and each $d_i' \geq d_i$, then $\mathcal{R}_i(E, (d_{-i}, d_i')) \geq \mathcal{R}_i(E, d)$.

A rule satisfies order preservation if it satisfies both order preservation in awards and in losses. Observe that self-duality implies the midpoint property. The weaker version of continuity obtained by considering small changes only in the endowment is called endowment continuity.

With each rule $\mathcal{R}$ we can associate a unique dual rule $\mathcal{R}^*$, defined by $\mathcal{R}^*(E, d) = d - \mathcal{R}\big(d(N) - E, d\big)$. A rule $\mathcal{R}$ is self-dual if $\mathcal{R} = \mathcal{R}^*$. Of the rules listed above, PRO, APRO, T, RA, and AA are self-dual. The CEA and CEL rules are dual. Two properties are dual if, whenever a rule satisfies one of them, its dual satisfies the other. A property is self-dual if it coincides with its dual. The following are pairs of dual properties: order preservations in awards and order preservation in losses; and minimal rights first and claims truncation invariance. The problems $(E, d) \in C^N$ and $(d(N) - E, d) \in C^N$ are dual claims problems. Table 1, adapted from Thomson (2019) and Mirás Calvo et al. (2020), summarizes which of the above properties are satisfied by the basic rules. A check mark, ✓, in a cell means that the property in the row is satisfied by the rule indexing the column. A minus sign, –, means the opposite.

One of the most commonly used criteria to rank rules is the Lorenz order. Let $x, y \in \mathbb{R}^n_{\leq}$. We say that x Lorenz-dominates y, and write $x \succeq y$, if for each $k = 1, \ldots, n-1$,

$$\sum_{j=1}^{k} x_j \geq \sum_{j=1}^{k} y_j \text{ and } \sum_{j=1}^{n} x_j = \sum_{j=1}^{n} y_j.$$

The Lorenz order is a partial order in $\mathbb{R}^n_{\leq}$, so it is a binary relation that is reflexive, antisymmetric, and transitive. If x Lorenz-dominates y and $x \neq y$, then at least one of the $n-1$ inequalities is strict.

$$
\begin{tikzcd}
 & & & \mathrm{PRO} \arrow[ddrr] & & \\
 & & & \mathrm{RA} \arrow[dr] & & \\
\mathrm{CEA} \arrow[r] & \mathrm{CE} \arrow[r] & \mathrm{PIN} \arrow[uur] \arrow[ur] \arrow[r] \arrow[dr] \arrow[ddr] & \mathrm{APRO} \arrow[r] & \mathrm{MO} \arrow[r] & \mathrm{CEL} \\
 & & & \mathrm{T} \arrow[ur] & & \\
 & & & \mathrm{AA} \arrow[uur] & &
\end{tikzcd}
$$

Fig. 2 Ranking of the ten rules

We have assumed that given a claims problem $(E, d) \in C^N$ the vector of claims $d \in \mathbb{R}^N$ has its coordinates ordered from small to large, that is, $d \in \mathbb{R}^n_{\leq}$. Moreover, the ten rules satisfy order preservation in awards. So if $\mathcal{R}$ is any of these rules then $\mathcal{R}(E, d) \in \mathbb{R}^n_{\leq}$. Therefore, we can use the Lorenz criterion to check whether a rule is more favorable to smaller claimants relative to larger claimants than other. Let $\mathcal{R}$ and $\mathcal{R}'$ be two rules that satisfy order preservation in awards. We say that $\mathcal{R}$ Lorenz-dominates $\mathcal{R}'$, and we write $\mathcal{R} \succeq \mathcal{R}'$, if $\mathcal{R}(E, d) \succeq \mathcal{R}'(E, d)$ for all $(E, d) \in C^N$.

Several authors contributed to the ranking of rules. To summarize these results, we borrow from Bosmans and Lauwers (2011) and Thomson (2019) a simple diagram, Fig. 2, that illustrates the ranking of rules using the Lorenz order. An arrow (or a sequence of arrows) from a rule $\mathcal{R}$ to a rule $\mathcal{R}'$ indicates that $\mathcal{R}$ Lorenz-dominates $\mathcal{R}'$, and the absence of an arrow (or of a sequence of arrows) indicates that there is no relationship. We have added the average of awards rule to the picture, so, in the next section we justify its place in the diagram of Fig. 2.

3 Ranking the average of awards rule

We have defined the average of awards rule in geometrical terms, as the centroid of the set of awards vectors for a claims problem. Naturally, an alternative and simple way of describing this rule is to assume that all the awards vectors are equally likely and therefore choosing their "average". The average of awards rule assigns to each $(E, d) \in C^N$ the value $\mathrm{AA}(E, d)$ given by the expected value of the (continuous) uniform distribution over the set of awards vectors $X(E, d)$. Besides its intuitive definition, the average of awards rule satisfies a good number of properties, see Table 1. Therefore, if only as a "central" point of reference inside the set of awards vectors, it is worthy to compare it to the basic rules.

Let us see that the ranking of the average of awards rule is, in fact, the one shown in Fig. 2. The absence of arrows connecting the average of awards rule with the Talmud, the random arrival, the adjusted proportional, and the proportional rules is a consequence of the fact that any two self-dual rules are incomparable. Then, we just have to prove that the sequence PIN $\rightarrow$ AA $\rightarrow$ MO holds.

Let us introduce three additional properties of rules. Null claims consistency implies that to compute the recommendation made by a rule we can remove the agents whose claims are 0 and apply the rule to the remaining claims problem. Order preservation under endowment variations implies that, given any two agents, if the endowment increases, the smaller claimant should receive a share of the increment that is at most as large as the share received by the larger claimant. Order preservation under claims variations says that if an agent claim increases, given any two claimants whose claim remains the same, the change in the award to the smaller one should be at most as large as the change in the award to the larger one. Formally, we say that a rule $\mathcal{R}$ satisfies:

- *Null claims consistency*, if for each $N \subset \mathcal{N}$, each $(E, d) \in C^N$, and each $N' \subset N$, if $d(N \backslash N') = 0$ then $\mathcal{R}_{N'}(E, d) = \mathcal{R}(E, d_{N'})$.
- *Order preservation under endowment variations*, if for each $(E, d) \in C^N$ and each pair $\{i, j\} \subseteq N$ and each $E' > E$, if $d(N) \geq E'$ and $d_i \leq d_j$, then $\mathcal{R}_i(E', d) - \mathcal{R}_i(E, d) \leq \mathcal{R}_j(E', d) - \mathcal{R}_j(E, d)$.
- *Order preservation under claims variations*, if for each $(E, d) \in C^N$ with $|N| \geq 3$, each $i \in N$, each $d'_i > d_i$, and each pair $\{j, k\} \subseteq N \backslash \{i\}$, if $d_j \leq d_k$, then $\mathcal{R}_j(E, d) - \mathcal{R}_j(E, (d_{-i}, d'_i)) \leq \mathcal{R}_k(E, d) - \mathcal{R}_k(E, (d_{-i}, d'_i))$.

The following characterizations of Piniles' and the minimal overlap rules as Lorenz-minimal and Lorenz-maximal within some classes of rules were established by Schummer and Thomson (1997) and Bosmans and Lauwers (2011) respectively.

1. Let $\mathcal{S}_1$ be the set of rules that satisfy order preservation in awards, endowment monotonicity, the midpoint property, and order preservation under endowment variations. Piniles' rule is the only rule in $\mathcal{S}_1$ that Lorenz-dominates each rule in $\mathcal{S}_1$.
2. Let $\mathcal{S}_2$ be the set of rules that satisfy $\frac{1}{|N|}$-truncated-claims lower bounds on awards, order preservation, null-claims consistency, and order preservation under claims variations. The minimal overlap rule is the only rule in $\mathcal{S}_2$ that is Lorenz-dominated by each rule in $\mathcal{S}_2$.

Now, according to Table 1, the average of awards rule satisfies the midpoint property, order preservation, endowment monotonicity, and $\frac{1}{|N|}$-truncated-claims lower bounds on awards. Let $(E, d) \in C^N$ and $N' \subset N$ such that $d(N \backslash N') = 0$. It is easy to see that $X(E, d) = 0_{N \backslash N'} \times X(E, d_{N'})$. Then $\mathrm{AA}_j(E, d) = 0$ for all $j \in N \backslash N'$ and $\mathrm{AA}_{N'}(E, d) = \mathrm{AA}(E, d_{N'})$, so the average of awards rule satisfies null claims consistency. We prove in Appendix A that the average of awards rule also satisfies order preservation under endowment variations and order preservation under claims variations. Therefore, as a direct consequence of the Lorenz-based characterizations of the minimal overlap and Piniles' rules we have that, in fact, the average of awards rule Lorenz-dominates the minimal overlap rule and is Lorenz-dominated by Piniles' rule.

4 The proportionality deviation index

In 1912, the statistician and sociologist Corrado Gini introduced a coefficient intended to measure the degree of income inequality within a population. To compute the Gini coefficient, first one has to find the Lorenz curve, developed by the economist Max O. Lorenz in 1905, that represents in the horizontal axis the proportion of the population, from lowest to highest income, and in the vertical axis the cumulative percentage of income or wealth owned. A perfectly equal distribution of wealth would have a Lorenz curve equal to the line $y = x$. The Gini coefficient measures how far the actual Lorenz curve for a population's income is from the line of equality. In our setting, given a claims problem if one plots the cumulative percentage of awards with respect to the proportion of claimants, from lowest to highest claims, the line of equality represents the egalitarian division of the endowment. But, the egalitarian division is not a rule and so this line is not particularly suitable. Therefore, instead of the proportion of population, we represent in the horizontal axis the cumulative percentage of claims, ordered from small to large. As a consequence, since the proportional rule shares the endowment in the same proportion as claims, now the line $y = x$ represents the line of proportionality. In this section, we define a pair of indices aimed at measuring the degree of discrepancy between the division proposed by a rule and the proportional distribution.

Given $d = (d_1, \ldots, d_n) \in \mathbb{R}^n_{\leq}$ let $d_0 = \bar{d}_0 = 0$ and $\bar{d}_i = \frac{1}{d(N)} \sum_{k=0}^{i} d_k$ for $i \in N$. Then $\bar{d} = (\bar{d}_1, \ldots, \bar{d}_n) \in \mathbb{R}^n_{\leq}$ is the vector of the percentages of the cumulative claims with respect to the total sum of claims $d(N)$. Naturally, $0 \leq \bar{d} \leq 1$ and $\bar{d}_n = 1$. For each $i \in N$ denote $\Delta\bar{d}_i = \bar{d}_i - \bar{d}_{i-1} = \frac{1}{d(N)} d_i$.

Let $\mathcal{R}$ be a rule that satisfies order preservation in awards. Then, for each claims problem $(E, d) \in C^N$ we know that $\mathcal{R}(E, d) \in \mathbb{R}^n_{\leq}$. In what follows assume that $E > 0$. As above, let $R_0(E, d) = \bar{R}_0(E, d) = 0$ and $\bar{R}_i(E, d) = \frac{1}{E} \sum_{k=0}^{i} \mathcal{R}_k(E, d)$ for $i \in N$. Then $\bar{R}(E, d) = (\bar{R}_1(E, d), \ldots, \bar{R}_n(E, d)) \in \mathbb{R}^n_{\leq}$ is the vector of the percentages of the cumulative awards assigned by the rule $\mathcal{R}$ with respect to the endowment. Obviously, $0 \leq \bar{R}(E, d) \leq 1$ with $\bar{R}_n(E, d) = 1$. For each $i \in N$ denote $\Delta\bar{R}_i(E, d) = \bar{R}_i(E, d) - \bar{R}_{i-1}(E, d) = \frac{1}{E} R_i(E, d)$.

Definition 4.1 Given a claims problem $(E, d) \in C^N$ with $d \in \mathbb{R}^n_{\leq}$ and a rule $\mathcal{R}$ satisfying order preservation in awards, the polygonal path connecting the $n + 1$ points $\big(\bar{d}_i, \bar{R}_i(E, d)\big)$, $i = 0, \ldots, n$, is called the cumulative claims-awards curve.

The continuous piecewise linear function $L^{\mathcal{R}}_{E,d} \colon [0, 1] \to [0, 1]$ whose graph is the cumulative claims-awards curve is called the cumulative claims-awards function of $\mathcal{R}$ for the problem (E, d):

$$L^{\mathcal{R}}_{E,d}(t) = \bar{R}_{i-1}(E, d) + \frac{\Delta\bar{R}_i(E, d)}{\Delta\bar{d}_i}(t - \bar{d}_{i-1}) \text{ if } t \in [\bar{d}_{i-1}, \bar{d}_i].$$

Clearly, $L^{\mathcal{R}}_{E,d}(0) = 0$ and $L^{\mathcal{R}}_{E,d}(1) = 1$ but, contrary to a conventional Lorenz curve, the graph of $L^{\mathcal{R}}_{E,d}$ does not necessarily lay below the identity line (see Figs. 3, 4, and 5). Nevertheless, from elementary calculus, we have that $L^{\mathcal{R}}_{E,d}$ is a monotonically increasing function so its graph is contained in the unit square, i.e., $0 \leq L^{\mathcal{R}}_{E,d}(t) \leq 1$ for all $t \in [0, 1]$. Note that if $E = d(N)$ then $L^{\mathcal{R}}_{d(N),d}(t) = t$ for all $t \in [0, 1]$. Basically, the function $L^{\mathcal{R}}_{E,d}$ represents the proportion of the initial endowment assigned by the rule $\mathcal{R}$ to each cumulative proportion of claims. Since the proportional rule divides the endowment in the same proportions as claims, that is $\overline{\mathrm{PRO}}(E, d) = \bar{d}$, we have that its claims-awards curve is always the identity, $L^{\mathrm{PRO}}_{E,d}(t) = t$ for all $t \in [0, 1]$. In

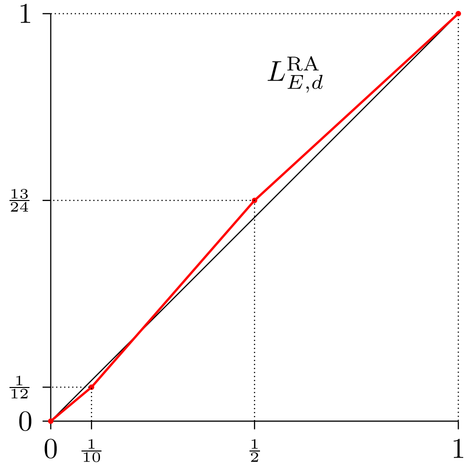

Fig. 3 Cumulative claims-awards curve of the random arrival rule for the problem $\big(4, (1, 4, 5)\big) \in \mathcal{C}^N$

this context, we refer to the diagonal of the unit square connecting the points $(0, 0)$ and $(1, 1)$ as the line of proportionality.

Example 4.2 Let $N = \{1, 2, 3\}$, $d = (1, 4, 5) \in \mathbb{R}^3_{\leq}$, and $E = 4$. Then $d(N) = 10$, $\bar{d} = \big(\frac{1}{10}, \frac{1}{2}, 1\big)$, and $\mathrm{RA}(E, d) = \big(\frac{1}{3}, \frac{11}{6}, \frac{11}{6}\big)$. Therefore, $\overline{\mathrm{RA}}(E, d) = \big(\frac{1}{12}, \frac{13}{24}, 1\big)$. Fig. 3 shows the line of proportionality and the claims-awards curve of the random arrival rule for (E, d).

The claims-awards curve allows us to compare the division recommended by the rule $\mathcal{R}$ with the division that preserves the proportions of the claims, the proportional rule. The claims-awards curve also captures graphically whether or not two rules are Lorenz-comparable.

Proposition 4.3 *Let $\mathcal{R}$ and $\mathcal{R}'$ be two rules satisfying order preservation in awards. For each $(E, d) \in \mathcal{C}^N$ with $d \in \mathbb{R}^n_{\leq}$, $\mathcal{R}(E, d)$ Lorenz-dominates $\mathcal{R}'(E, d)$ if and only if $L^{\mathcal{R}}_{E,d}(t) \geq L^{\mathcal{R}'}_{E,d}(t)$ for all $t \in [0, 1]$.*

Obviously, $\mathcal{R}$ Lorenz-dominates $\mathcal{R}'$ if $L^{\mathcal{R}}_{E,d}$ lies above $L^{\mathcal{R}'}_{E,d}$ for all $(E, d) \in \mathcal{C}^N$. Figure 3 shows the claims-awards curve of the random arrival rule for the claims problem of Example 4.2. The polygonal curve intersects transversally the line of proportionality indicating that the random arrival rule is not Lorenz-comparable to the proportional rule.

Following the idea underlying the definition of the Gini index, we introduce a pair of coefficients that measure the deviation of the claims-awards curve from the line of proportionality. The signed proportionality deviation index is the ratio of the net signed area that lies between the line of proportionality and the claims-awards curve over the total area under the line of proportionality. The proportionality deviation index is the ratio of the area between the line of proportionality and the claims-awards curve over the area under the line of proportionality.

Definition 4.4 Let $(E, d) \in C^N$ with $d \in \mathbb{R}^N_{\leq}$ and let $\mathcal{R}$ be a rule satisfying order preservation in awards. The signed proportionality deviation index of $\mathcal{R}$ for the problem (E, d) is:

$$\mathcal{I}(\mathcal{R}, E, d) = \frac{\int_0^1 \big(t - L^{\mathcal{R}}_{E,d}(t)\big)dt}{\int_0^1 t dt} = \frac{\frac{1}{2} - \int_0^1 L^{\mathcal{R}}_{E,d}(t)dt}{\frac{1}{2}} = 1 - 2\int_0^1 L^{\mathcal{R}}_{E,d}(t)dt.$$

The proportionality deviation index of $\mathcal{R}$ for the problem (E, d) is:

$$\mathcal{I}^+(\mathcal{R}, E, d) = \frac{\int_0^1 |t - L^{\mathcal{R}}_{E,d}(t)|dt}{\int_0^1 t dt} = 2\int_0^1 |t - L^{\mathcal{R}}_{E,d}(t)|dt.$$

Note that, since the Lorenz curve lies below the identity line, the usual Gini coefficient is a value between 0 and 1. In our context, as it is illustrated in Example 4.7 and Fig. 4, the claims-awards curve is not bounded from above by the identity so, as a consequence, the signed proportionality deviation index can take negative values. Obviously $-1 \leq \mathcal{I}(\mathcal{R}, E, d) \leq 1$ and $0 \leq \mathcal{I}^+(\mathcal{R}, E, d) \leq 1$. A proportionality deviation coefficient of zero expresses a distribution equal to the one implied by the vector of claims, that is, the proportional distribution.

In Appendix B we obtain formulae to compute the two proportional deviation indices, $\mathcal{I}(\mathcal{R}, E, d)$ and $\mathcal{I}^+(\mathcal{R}, E, d)$, in terms of the values of the vector of claims $d \in \mathbb{R}^n_{\leq}$ and the recommendation made by the rule $\mathcal{R}(E, d)$. Naturally, $\mathcal{I}(\text{PRO}, E, d) = \mathcal{I}^+(\text{PRO}, E, d) = 0$ for all $(E, d) \in C^N$. Also, for each $d \in \mathbb{R}^n_{\leq}$ and each rule $\mathcal{R}$ we have $\mathcal{I}(\mathcal{R}, d(N), d) = \mathcal{I}^+(\mathcal{R}, d(N), d) = 0$. If $\mathcal{R}$ satisfies the midpoint property then $\mathcal{I}(\mathcal{R}, \frac{1}{2}d(N), d) = \mathcal{I}^+(\mathcal{R}, \frac{1}{2}d(N), d) = 0$ for all $d \in \mathbb{R}^n_{\leq}$.

Example 4.5 Let $N = \{1, 2, 3\}$, $d = (1, 4, 5) \in \mathbb{R}^3_{\leq}$, and $E = 4$ as in Example 4.2. Recall that $\text{RA}(E, d) = \left(\frac{1}{3}, \frac{11}{6}, \frac{11}{6}\right)$ and that, see Fig. 3, the claims-awards curve $L^{\text{RA}}_{E,d}$ crosses the line of proportionality. Then $\text{RA}(E, d)$ and $\text{PRO}(E, d)$ are not Lorenz-comparable so the absolute value of the signed proportionality index and the proportionality index of the random arrival rule for this problem are not equal, in fact, $\mathcal{I}(\text{RA}, E, d) = -\frac{7}{240} = -0.0292$ and $\mathcal{I}^+(\text{RA}, E, d) = \frac{61}{1680} = 0.0363$. Note, that even though $\text{RA}(E, d)$ is not Lorenz-comparable to the proportional division, we know that the corresponding cumulative awards vectors differ by 3.63%.

As a direct consequence of Proposition 4.3 and Definition 4.4, we have the following properties of the proportionality deviation coefficients.

Proposition 4.6 *Let $\mathcal{R}$ and $\mathcal{R}'$ be two rules satisfying order preservation in awards. Let $(E, d) \in C^N$ with $d \in \mathbb{R}^n_{\leq}$. Then:*

1. $\mathcal{I}^+(\mathcal{R}, E, d) = 0$ *if and only if* $\mathcal{R}(E, d) = \text{PRO}(E, d)$.

2. *$|\mathcal{I}(\mathcal{R}, E, d)| < \mathcal{I}^+(\mathcal{R}, E, d)$ if and only if $\mathcal{R}(E, d)$ and* $\mathrm{PRO}(E, d)$ *are not Lorenz-comparable.*
3. *$\mathcal{I}(\mathcal{R}, E, d) = \mathcal{I}^+(\mathcal{R}, E, d)$ if and only if $\mathcal{R}(E, d)$is Lorenz-dominated by* $\mathrm{PRO}(E, d)$.
4. *$\mathcal{I}(\mathcal{R}, E, d) = -\mathcal{I}^+(\mathcal{R}, E, d)$ if and only if $\mathcal{R}(E, d)$ Lorenz-dominates* $\mathrm{PRO}(E, d)$.
5. *If $\mathcal{R}(E, d)$ Lorenz-dominates $\mathcal{R}'(E, d)$ then $\mathcal{I}(\mathcal{R}, E, d) \leq \mathcal{I}(\mathcal{R}', E, d)$.*

Proposition 4.6 shows that when the recommendation made by a rule for a claims problem and the proportional division are Lorenz-comparable, the corresponding proportionality deviation indices reflect the ordering. If a rule $\mathcal{R}$ Lorenz-dominates the proportional rule then its signed proportionality deviation index must be negative, but if it is Lorenz-dominated by the proportional rule it must be positive. But, as we have seen in Example 4.5, even when the awards vectors selected by $\mathcal{R}$ and the proportional rule are incomparable, the indices reveal how far from proportionality is the division proposed by the rule.

Example 4.7 Let $N = \{1, 2, 3, 4\}$ and $d = (3, 4, 5, 6) \in \mathbb{R}^4_{\leq}$ so $d(N) = 18$. Then

(E, d)	$\mathrm{PRO}(E, d)$	$\mathrm{CEA}(E, d)$	$\mathrm{MO}(E, d)$
$\big(9, (3, 4, 5, 6)\big)$	$\left(\frac{3}{2}, 2, \frac{5}{2}, 3\right)$	$\left(\frac{9}{4}, \frac{9}{4}, \frac{9}{4}, \frac{9}{4}\right)$	$\left(\frac{3}{4}, \frac{7}{4}, \frac{11}{4}, \frac{15}{4}\right)$
$\mathcal{I}(\mathcal{R}, E, d)$	0	$-\frac{5}{36}$	$\frac{5}{36}$

The claims-awards curves of the constrained equal awards and the minimal overlap rules for the problem $(9, d) \in C^N$ are depicted in Fig. 4. Neither of these two rules satisfies the midpoint property. We know that CEA $\succeq$ PRO so $L^{\mathrm{CEA}}_{E,d}$ lies above the line of proportionality and $\mathcal{I}^+(\mathrm{CEA}, E, d) = \frac{5}{36}$. Even though, in general, MO and PRO are not comparable, since $L^{\mathrm{MO}}_{E,d}$ lies below the line of proportionality we have that $\mathrm{MO}(E, d)$ is Lorenz-dominated by $\mathrm{PRO}(E, d)$. Therefore, $\mathcal{I}^+(\mathrm{MO}, E, d) = \frac{5}{36}$. Observe that, in Fig. 4, the shadowed area for the CEA rule corresponds to $\int_0^1 \big(L^{\mathrm{CEA}}_{E,d}(t) - t\big)dt$, so the signed proportionality deviation index is negative. On the contrary, the signed proportionality deviation index for the MO rule is positive because the shadowed area for the MO rule corresponds to $\int_0^1 \big(t - L^{\mathrm{MO}}_{E,d}(t)\big)dt$.

Let $(E, d) \in C^N$ and $\mathcal{R}$ be a rule satisfying order preservation in awards. Since the CEA rule Lorenz-dominates each rule that satisfies order preservation in awards, so, by Proposition 4.6, $\mathcal{I}(\mathrm{CEA}, E, d) \leq \mathcal{I}(\mathcal{R}, E, d)$. Moreover, the CEL rule is Lorenz-dominated by each rule that satisfies order preservation in losses, so if $\mathcal{R}$ is such a rule then $\mathcal{I}(\mathcal{R}, E, d) \leq \mathcal{I}(\mathrm{CEL}, E, d)$. As a corollary, if $\mathcal{R}$ satisfies order preservation then $\mathcal{I}(\mathrm{CEA}, E, d) \leq \mathcal{I}(\mathcal{R}, E, d) \leq \mathcal{I}(\mathrm{CEL}, E, d)$.

Let $\mathcal{R}$ be a rule and $\mathcal{R}^*$ its dual. The cumulative claims-awards function $L^{\mathcal{R}^*}_{E,d}$ represents the cumulative proportion of gains with respect to the cumulative proportion of claims of the dual rule $\mathcal{R}^*$ for the problem $(E, d) \in C^N$. But, since $\mathcal{R}^*(E, d) = d - \mathcal{R}(d(N) - E, d)$ then $L^{\mathcal{R}^*}_{E,d}$ can also be interpreted as the cumulative proportion of losses of the rule $\mathcal{R}$ with respect to the cumulative proportion of claims. Naturally,

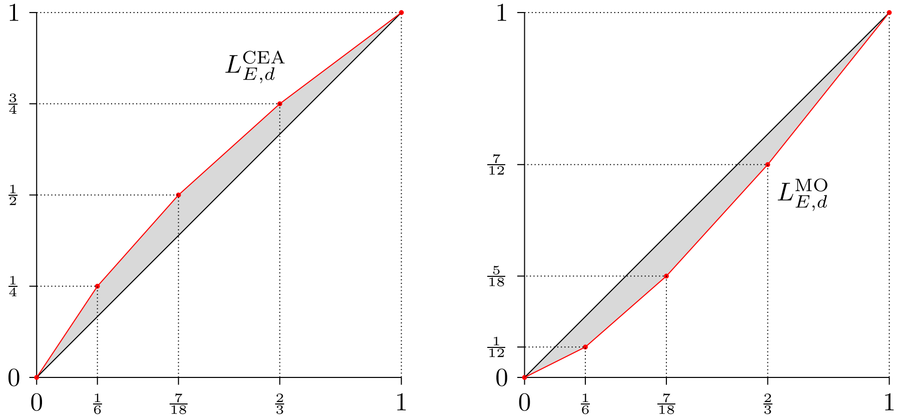

Fig. 4 Curves of the CEA and MO rules for the problem $(9, d) \in C^N$ with $d = (3, 4, 5, 6)$

the cumulative "gains" and "losses" curves of a rule are related and so are the signed proportionality deviation indices of a rule and that of its dual.

Proposition 4.8 *Let $\mathcal{R}$ be a rule satisfying order preservation in awards and $\mathcal{R}^*$ its dual rule. Then, for all $(E, d) \in C^N$, $E > 0$, we have that:*

$$E\mathcal{I}(\mathcal{R}, E, d) + (d(N) - E)\mathcal{I}(\mathcal{R}^*, d(N) - E, d) = 0.$$

Proof Let $(E, d) \in C^N$ with $E > 0$. Since $\mathcal{R}(E, d) + \mathcal{R}^*(d(N) - E, d) = d$, some simple algebraic manipulations lead to the following equalities:

1. $E\bar{\mathcal{R}}_j(E, d) + (d(N) - E)\bar{\mathcal{R}}^*_j(d(N) - E, d) = d(N)\bar{d}_j$ for all $j \in N$.
2. $E\Delta\bar{\mathcal{R}}_j(E, d) + (d(N) - E)\Delta\bar{\mathcal{R}}^*_j(d(N) - E, d) = d_j$ for all $j \in N$.
3. $EL^{\mathcal{R}}_{E,d}(t) + (d(N) - E)L^{\mathcal{R}^*}_{d(N)-E,d}(t) = d(N)L^{\mathrm{PRO}}_{E,d}(t)$ for all $t \in [0, 1]$.
4. $E\int_{\bar{d}_{j-1}}^{\bar{d}_j} L^{\mathcal{R}}_{E,d}(t)dt + (d(N) - E)\int_{\bar{d}_{j-1}}^{\bar{d}_j} L^{\mathcal{R}^*}_{d(N)-E,d}(t)dt = \frac{d_j}{2}(\bar{d}_{j-1} + \bar{d}_j)$ for all $j \in N$.
5. $E\int_0^1 L^{\mathcal{R}}_{E,d}(t)dt + (d(N) - E)\int_0^1 L^{\mathcal{R}^*}_{d(N)-E,d}(t)dt = \frac{1}{2}d(N)$.

Now, from the last equality and Definition 4.4, it is straightforward to obtain that $E\mathcal{I}(\mathcal{R}, E, d) + (d(N) - E)\mathcal{I}(\mathcal{R}^*, d(N) - E, d) = 0$. □

We compute in Example 4.9 the proportionality deviation indices of the constrained equal awards and the constrained equal losses rules for some particular claims problems. The example also illustrates that the signed proportionality deviation index can be a number as close to -1 or 1 as wanted.

Example 4.9 Fix $n \in \mathbb{N}$, $n \geq 2$, and let $N = \{1, \ldots, n\}$. For the claims problem $\left(\frac{1}{2}, d'\right) \in C^N$ with $d' = (1, \ldots, 1, n^2 - 2n + 1)$, we have that $\mathrm{CEA}(\frac{1}{2}, d') =$

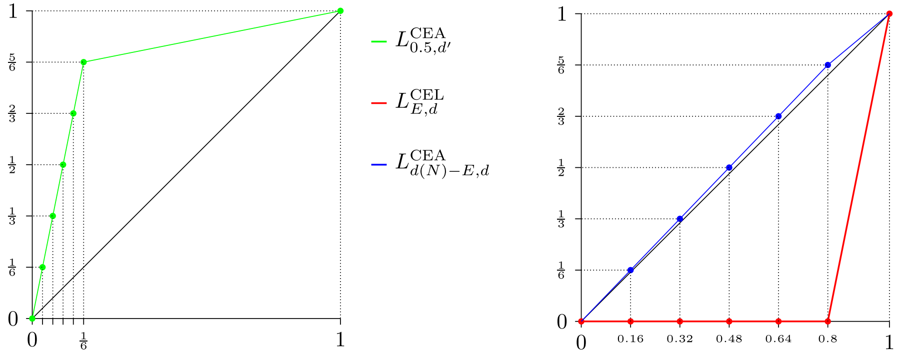

Fig. 5 The curves $L^{\mathrm{CEA}}_{0.5,d'}$, $L^{\mathrm{CEL}}_{E,d}$, and $L^{\mathrm{CEA}}_{d(N)-E,d}$ for $n = 6$

$\left(\frac{1}{2n}, \ldots, \frac{1}{2n}, \frac{1}{2n}\right)$ so $\mathcal{I}^+(\mathrm{CEA}, \frac{1}{2}, d') = -\mathcal{I}(\mathrm{CEA}, \frac{1}{2}, d') = 1 - \frac{2}{n}$. Certainly, the proportionality deviation indices confirm that, for all $n \geq 2$, $\mathrm{CEA}(\frac{1}{2}, d')$ Lorenz-dominates $\mathrm{PRO}(\frac{1}{2}, d')$. But these two coefficients convey more information. When $n = 2$ we know that CEL and PRO coincide (the indices are zero), but as n increases, the CEL rule selects awards vectors that differ more and more from proportionality (the deviation index tends to 1), and we have a precise measure of that discrepancy.

If $n > 2$, consider the claims problem $(E, d) \in C^N$ with $E = \frac{1}{n-2}$ and $d = (1, \ldots, 1, 1 + E)$. Then $\mathrm{CEL}(E, d) = (0, \ldots, 0, E)$. The signed proportionality deviation index of the constrained equal losses rule for this problem is $\mathcal{I}(\mathrm{CEL}, E, d) = 1 - \frac{1}{n-1}$. By Proposition 4.8 we know that $\mathcal{I}(\mathrm{CEA}, n, d) = -\frac{1}{n(n-1)}$. Since $\mathcal{I}^+(\mathrm{CEL}, E, d) = \mathcal{I}(\mathrm{CEL}, E, d)$ and $\mathcal{I}^+(\mathrm{CEA}, \frac{1}{2}, d') = -\mathcal{I}(\mathrm{CEA}, \frac{1}{2}, d')$, we have instances where the proportionality deviation index is very close to 1. The cumulative claims-awards curves for the three problems are depicted in Fig. 5 when $n = 6$.

Fix $d \in \mathbb{R}^N_{\leq}$. Given a rule $\mathcal{R}$ that satisfies order preservation in awards, let us consider the function $\mathcal{I}^{\mathcal{R}}_d\colon (0, d(N)] \to [-1, 1]$ that assigns to each $E \in (0, d(N)]$ the signed proportionality deviation index of rule $\mathcal{R}$ for the problem $(E, d) \in C^N$, that is, $\mathcal{I}^{\mathcal{R}}_d(E) = \mathcal{I}(\mathcal{R}, E, d)$. Let us call $\mathcal{I}^{\mathcal{R}}_d$ the signed index path of $\mathcal{R}$ for the vector of claims d. Observe that $\mathcal{I}^{\mathcal{R}}_d(d(N)) = 0$. Of course, the signed index path of the proportional rule is the zero constant function, i.e., $\mathcal{I}^{\mathrm{PRO}}_d(E) = 0$ for all $E \in (0, d(N)]$. Naturally, if $\mathcal{R}$ is endowment continuous then the signed index path $\mathcal{I}^{\mathcal{R}}_d$ is also continuous. The signed index path is a simple way to visualize the discrepancy of the divisions given by a rule for a fixed vector of claims with respect to the proportional distribution as the endowment increases from zero to the sum o the claims and to compare it with other rules. Similarly, we can define the corresponding index path for the proportionality deviation index, $(\mathcal{I}^+)^{\mathcal{R}}_d\colon (0, d(N)] \to [0, 1]$ that assigns to each $E \in (0, d(N)]$ the proportionality deviation index of rule $\mathcal{R}$ for the problem $(E, d) \in C^N$, that is, $(\mathcal{I}^+)^{\mathcal{R}}_d(E) = \mathcal{I}^+(\mathcal{R}, E, d)$.

Example 4.10 Let $N = \{1, 2, 3, 4\}$ and $d = (3, 4, 5, 6) \in \mathbb{R}^4_{\leq}$. Now, $\mathcal{I}(\mathrm{MO}, 4, d) = -\frac{35}{432}$ and $\mathcal{I}(\mathrm{MO}, 5, d) = \frac{1}{135}$. Since $\mathcal{I}^{\mathcal{R}}_d$ is continuous on $[4, 5]$, applying Bolzano's

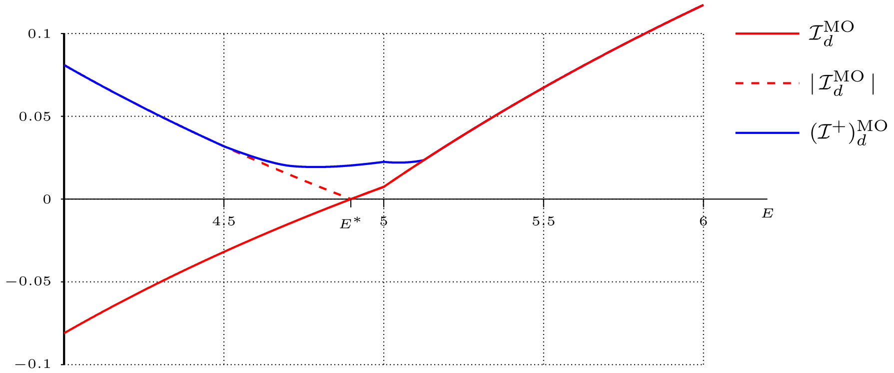

Fig. 6 Detail of the index paths of the MO rule on the interval $[4, 6]$ for $d = (3, 4, 5, 6)$

Theorem we conclude that there is $E^* \in (4, 5)$ such that $\mathcal{I}_d^{\mathcal{R}}(E^*) = 0$. But $\mathrm{MO}(E^*, d) \neq \mathrm{PRO}(E^*, d)$ because $\mathrm{MO}_1(E^*, d) = \frac{3}{4}$, $\mathrm{MO}_2(E^*, d) = \frac{13}{12}$, $\mathrm{PRO}_1(E^*, d) = \frac{1}{6}E^*$, and $\mathrm{PRO}_2(E^*, d) = \frac{2}{9}E^*$. Therefore, we have an instance of a non proportional division with signed deviation index equal to zero. Nevertheless, we know that the proportional deviation index of the minimal overlap rule for this problem must be strictly positive, $\mathcal{I}^+(\mathrm{MO}, E^*, d) > 0$. Figure 6 shows the signed proportionality deviation index path, its absolute value, and the signed proportionality deviation index path of the minimal overlap rule restricted to the interval $[4, 6]$. In the subinterval where the signed proportionality deviation index path and its absolute value differ, we know that the minimal overlap and the proportional rules are not Lorenz-comparable, and that they deviate less that 5%.

The cumulative claims-awards curve, the proportionality deviation indices, and the index path can be useful tools to compare rules beyond the information provided by the Lorenz order. For any given claims problem, they are easy to compute from the values of the vector of claims and the rule and convey much information about the rule and its properties in a clear and simple visual way. Nevertheless, as the proportionality deviation indices (and by extension the index path) comprise all the data from the cumulative claims-awards curve in a pair of numbers, some information must be lost in the process. Nevertheless, the combination of both coefficients solves some shortcomings that each of them has when taken alone. Certainly, as Example 4.7 illustrates, the proportionality index does not capture the Lorenz-ranking of awards vectors that is fully reflected by the signed index. On the other hand, Example 4.10 shows that two different divisions can have the same signed proportionality deviation coefficient, but the corresponding proportionality deviation indices must be different.

Figure 7 portrays the signed proportionality deviation index paths of the ten rules for the vector of claims $d = (3, 4, 5, 6)$. At first sight, one observes that only the proportional rule and the average of awards rule have smooth paths, because they are the only rules that are endowment differentiable.[1] Moreover, according to Proposi-

[1] Since the average of awards rule coincides with the concede-and-divide rule for two-claimant problems, it is an endowment differentiable extension of this rule for $|N| \geq 3$.

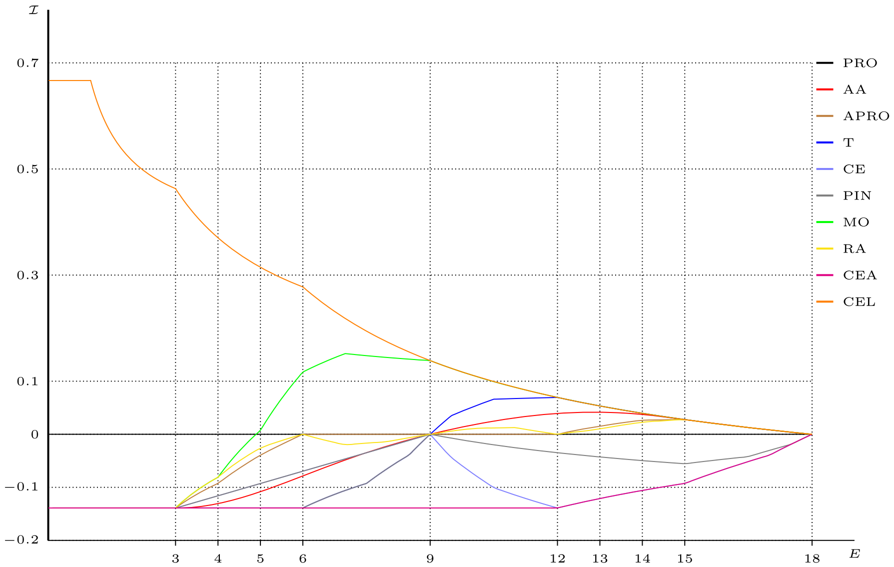

Fig. 7 Signed index path of the ten rules for the claims vector $d = (3, 4, 5, 6)$

tion 4.6, the ranking of rules is reflected in the graph so, for instance, all the paths lie between those of the CEL and the CEA rules. Whether or not a rule satisfies the midpoint property has a clear implication on its index path. Note that, the index paths of the constrained egalitarian, the Talmud, and Piniles' rules coincide in the interval $[0, \frac{1}{2}d(N)]$.[2]

Certainly, for the constrained equal losses rule, both the proportionality deviation index path and the signed proportionality deviation index path coincide. The proportionality deviation index paths of the average of awards, the minimal overlap, and the constrained equal awards rule are compared to the corresponding signed index paths in Fig. 8. For the average of awards and the constrained equal awards rules, the proportionality index path is just the absolute value of the signed proportionality index path. Therefore, according to Proposition 4.6, $\text{CEA}(E, d)$ Lorenz-dominates $\text{PRO}(E, d)$ for all $E \in [0, d(N)]$, while $\text{AA}(E, d)$ Lorenz-dominates $\text{PRO}(E, d)$ if $E \in [0, \frac{1}{2}d(N)]$ but $\text{AA}(E, d)$ is Lorenz-dominated by $\text{PRO}(E, d)$ if $E \in [\frac{1}{2}d(N), d(N)]$. For the minimal overlap rule there is a neighbourhood of $E = 5$ where the proportionality index path is not the absolute value of the signed proportionality index path, so for these values of the endowment $\text{MO}(E, d)$ and $\text{PRO}(E, d)$ are not comparable (see Fig. 6).

5 Generalized deviation indices

The proportionality deviation indices measure the discrepancy of an awards vector with respect to the proportional division. But, depending on the principles of fairness,

[2] By definition, if $(E, d) \in C^N$ and $0 \leq E \leq \frac{1}{2}d(N)$ then $\text{CE}(E, d) = \text{T}(E, d) = \text{PIN}(E, d)$.

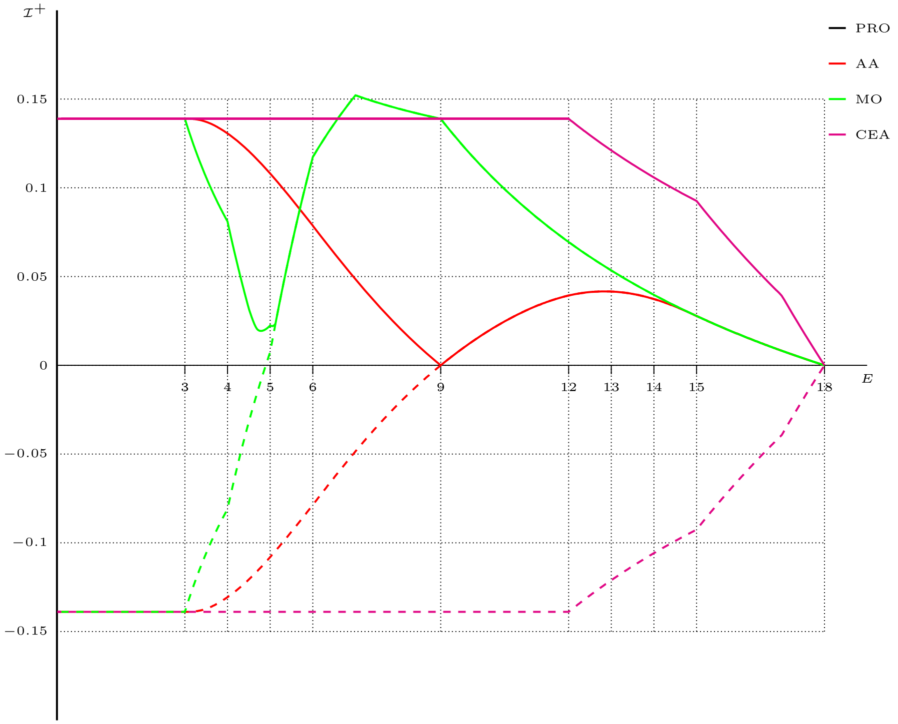

Fig. 8 The index paths $\mathcal{I}^{\mathcal{R}}_d$ (dashed) and $(\mathcal{I}^+)^{\mathcal{R}}_d$ (solid) of some rules for $d = (3, 4, 5, 6)$

equity, or justice, that the decision maker wants to apply when facing a particular claims problem, the proportional division may not be the suitable rule of reference. Therefore, we want to generalize the proportionality indices by providing a way to measure the degree of discrepancy between two arbitrary awards vectors.

Given a pair of vectors $x, y \in \mathbb{R}^n_{\leq}$ it is easy to define a cumulative curve L^y_x representing the vector of cumulative percentages of the coordinates of y against the vector of cumulative percentages of the coordinates of x. If $x = (x_1, \dots, x_n) \in \mathbb{R}^N_{\leq}$ let $\bar{x}_0 = 0$ and $\bar{x}_i = \frac{1}{x(N)} \sum_{k=0}^{i} x_k$ for $i \in N$. Then $\bar{x} = (\bar{x}_1, \dots, \bar{x}_n) \in \mathbb{R}^N_{\leq}$ is the vector of cumulative percentages of the coordinates of x with respect to the total sum $x(N)$. Naturally, $0 \leq \bar{x}_i \leq 1$ for all $i \in N$ and $\bar{x}_n = 1$. Denote $\Delta\bar{x}_i = \bar{x}_i - \bar{x}_{i-1} = \frac{1}{x(N)} x_i$ for $i \in N$. Now, giving a pair $(x, y) \in \mathbb{R}^N_{\leq} \times \mathbb{R}^N_{\leq}$, consider the continuous piecewise linear function $L^y_x \colon [0, 1] \to [0, 1]$ connecting the $n + 1$ points $(\bar{x}_i, \bar{y}_i)$, $i = 0, \dots, n$. Then:

$$L^y_x(t) = \bar{y}_{i-1} + \frac{\Delta\bar{y}_i}{\Delta\bar{x}_i}(t - \bar{x}_{i-1}) \text{ if } t \in [\bar{x}_{i-1}, \bar{x}_i].$$

Clearly, $L_x^y(0) = 0$, $L_x^y(1) = 1$, and L_x^y is monotonically increasing so its graph is contained in the unit square, i.e., $0 \leq L_x^y(t) \leq 1$ for all $t \in [0, 1]$. Now, $L_x^x(t) = t$ or all $t \in [0, 1]$ so the graph of the cumulative curve L_x^x is the diagonal of the unit square connecting the points $(0, 0)$ and $(1, 1)$, the identity line. Moreover, L_y^x is the inverse function of L_x^y and $\int_0^1 L_x^y(t)dt + \int_0^1 L_y^x(t)dt = 1$.

Now, we define the signed deviation index of y with respect to x, $\mathcal{I}(y, x)$, and the deviation index of y with respect to x, $\mathcal{I}^+(y, x)$, as:

$$\mathcal{I}(y, x) = \frac{\int_0^1 \big(t - L_x^y(t)\big)dt}{\int_0^1 tdt} \quad \text{and} \quad \mathcal{I}^+(y, x) = \frac{\int_0^1 |t - L_x^y(t)|dt}{\int_0^1 tdt}.$$

Both indices provide a measure of how far the cumulative percentages of the coordinates of y are from the cumulative percentages of the coordinates of x. The signed deviation index $\mathcal{I}(y, x)$ is the ratio of the net signed area that lies between the line of equality and the cumulative curve L_x^y over the total area under the line of equality. Then $-1 \leq \mathcal{I}(y, x) \leq 1$ and $\mathcal{I}(y, x) = -\mathcal{I}(x, y)$ (see Fig. 9). The deviation index $\mathcal{I}^+(y, x)$ is the ratio of the area between the line of equality and the cumulative curve L_x^y over the area under the identity line. Naturally, $0 \leq \mathcal{I}^+(y, x) \leq 1$.

Obviously, given a rule $\mathcal{R}$ and a claims problem $(E, d) \in C^N$ if we take $x = (1, \ldots, n)$ and $y = \mathcal{R}(E, d)$ then $\mathcal{I}(y, x)$ and $\mathcal{I}^+(y, x)$ give the deviation of $\mathcal{R}(E, d)$ with respect to the egalitarian distribution (the usual Gini index). On the other hand, if we take $x = d$ (or, alternatively, $x = \text{PRO}(E, d)$) and $y = \mathcal{R}(E, d)$, the proportionality deviation indices of $\mathcal{R}$ for (E, d) coincide with the deviation indices of y with respect to x, that is, $\mathcal{I}(\mathcal{R}, E, d) = \mathcal{I}(y, x)$ and $\mathcal{I}^+(\mathcal{R}, E, d) = \mathcal{I}^+(y, x)$.

The role of the proportional rule as the benchmark for comparing awards vectors in the analysis of Sect. 4 can be played by any other rule and a deviation index with respect to this new reference rule can be computed. Given two rules $\mathcal{R}$ and $\mathcal{R}'$ satisfying order preservation in awards and a claims problem $(E, d) \in C^N$ with $d \in \mathbb{R}^n_{\leq}$, take $x = \mathcal{R}(E, d)$ and $y = \mathcal{R}'(E, d)$. Then the cumulative curve L_x^y and the pair of indices $\mathcal{I}(y, x)$ and $\mathcal{I}^+(y, x)$ allow us to compare the awards vector selected by rule $\mathcal{R}'$ for the claims problem (E, d) with respect to the one selected by rule $\mathcal{R}$. The properties stated in Proposition 4.6 for the proportionality deviation indices, and their interpretations, are also valid for the deviation indices of $\mathcal{R}'(E, d)$ with respect to $\mathcal{R}(E, d)$.

Example 5.1 Let $N = \{1, 2, 3\}$, $E = 2$, and $d = (1, 1, 38) \in \mathbb{R}^3_{\leq}$. Then $\text{PRO}(E, d) = \left(\frac{1}{20}, \frac{1}{20}, \frac{19}{10}\right)$. The set of awards vectors has a particular simple structure, in fact,

$$X(E, d) = \{(x_1, x_2, 2 - x_1 - x_2) \in \mathbb{R}^3 : (x_1, x_2) \in [0, 1] \times [0, 1]\}.$$

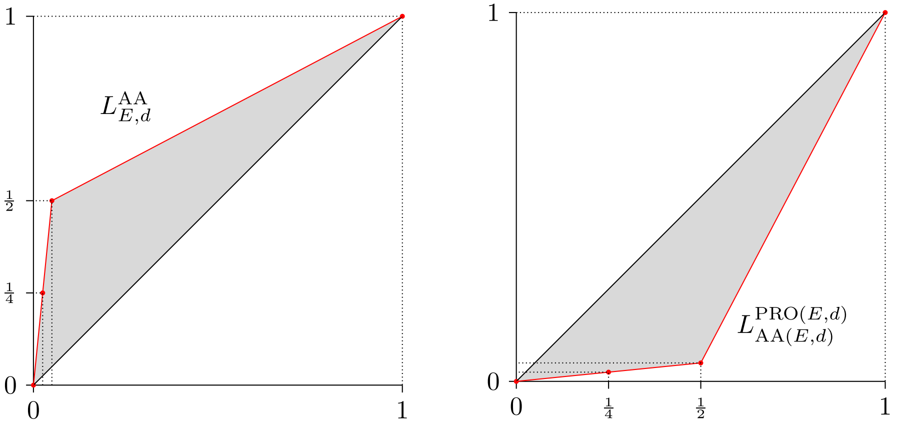

Fig. 9 The curves L^y_x and L^x_y with $x = \mathrm{PRO}(2, (1, 1, 38))$ and $y = \mathrm{AA}(2, (1, 1, 38))$

Therefore $\mathrm{AA}(E, d) = \left(\frac{1}{2}, \frac{1}{2}, 1\right)$. The claims-awards curve $L^{\mathrm{AA}}_{E,d}$ and the curve $L^{\mathrm{PRO}(E,d)}_{\mathrm{AA}(E,d)}$ are depicted in Fig. 9. Clearly, $L^{\mathrm{PRO}(E,d)}_{\mathrm{AA}(E,d)}$ is the inverse function of $L^{\mathrm{AA}}_{E,d}$ so

$$\int_0^1 L^{\mathrm{AA}}_{E,d}(t)dt + \int_0^1 L^{\mathrm{PRO}(E,d)}_{\mathrm{AA}(E,d)}(t)dt = 1.$$

Therefore, $\mathcal{I}\big(\mathrm{PRO}(E, d), \mathrm{AA}(E, d)\big) = -\mathcal{I}(\mathrm{AA}, E, d) = 0.45$. We conclude that $\mathrm{AA}(E, d)$ Lorenz-dominates $\mathrm{PRO}(E, d)$ and that the proportional division deviates by 45% from the average of awards rule, the geometrical center of the set of awards vectors. Note that, for this particular claims problem, the adjusted proportional, the constrained egalitarian, Piniles', the random arrival, and the Talmud rules recommend the same division as the average of awards rule.

For a rule $\mathcal{R}$ that satisfies order preservation, in addition to the proportionality deviation indices, the coefficients $\mathcal{I}\big(\mathrm{CEA}(E, d), \mathcal{R}(E, d)\big), \mathcal{I}\big(\mathrm{CEL}(E, d), \mathcal{R}(E, d)\big)$, and $\mathcal{I}\big(\mathrm{AA}(E, d), \mathcal{R}(E, d)\big)$ are particularly interesting. Since the constrained equal awards and the constrained equal losses rules are Lorenz-maximal and Lorenz-minimal respectively among the rules satisfying order preservation, the signed deviation indices of rule $\mathcal{R}$ with respect to the CEA and CEL rules indicate the variation of rule $\mathcal{R}$ compared to two extreme rules. The signed deviation index of rule $\mathcal{R}$ with respect to the average of awards rule, $\mathcal{I}\big(\mathrm{AA}(E, d), \mathcal{R}(E, d)\big)$, measures the degree of discrepancy of rule $\mathcal{R}$ from a central rule, the geometrical center of the set of awards vectors.

Example 5.2 Let $N = \{1, 2, 3, 4\}$, $E = 16$, and $d = (3, 10, 12, 13) \in \mathbb{R}^4_{\leq}$. We have that:

$\mathrm{AA}(E, d)$	$\mathrm{RA}(E, d)$	$\mathrm{CEA}(E, d)$	$\mathrm{CEL}(E, d)$	$\mathrm{PRO}(E, d)$
$(\frac{29}{20}, \frac{43}{10}, 5, \frac{21}{4})$	$(\frac{3}{2}, 4, 5, \frac{11}{2})$	$(3, \frac{13}{3}, \frac{13}{3}, \frac{13}{3})$	$(0, \frac{11}{3}, \frac{17}{3}, \frac{20}{3})$	$(\frac{24}{19}, \frac{80}{19}, \frac{96}{19}, \frac{104}{19})$

Since $\mathcal{I}\big(\text{AA}(E,d), \text{RA}(E,d)\big) = -0.0180$ and $\mathcal{I}^+\big(\text{AA}(E,d), \text{RA}(E,d)\big) = 0.0188$ we conclude that the awards vectors $\text{AA}(E,d)$ and $\text{RA}(E,d)$ are not Lorenz-comparable. Nevertheless, the deviation index of the random arrival rule with respect to the average of awards rule is not very high, so the cumulative percentages of the awards vectors selected by both rules are close. Among the other basic rules the biggest deviation coefficients with respect to $\text{AA}(E,d)$ correspond to the CEA and CEL rules. In fact, $\mathcal{I}\big(\text{AA}(E,d), \text{CEA}(E,d)\big) = \mathcal{I}^+\big(\text{AA}(E,d), \text{CEA}(E,d)\big) = 0.1290$ and $\mathcal{I}\big(\text{AA}(E,d), \text{CEL}(E,d)\big) = -\mathcal{I}^+\big(\text{AA}(E,d), \text{CEL}(E,d)\big) = -0.1650$. Finally, $\mathcal{I}\big(\text{AA}(E,d), \text{PRO}(E,d)\big) = -\mathcal{I}^+\big(\text{AA}(E,d), \text{PRO}(E,d)\big) = -0.0232$ which implies that the awards vector $\text{AA}(E,d)$ Lorenz-dominates $\text{PRO}(E,d)$. Naturally, the proportionality deviation indices of the average of awards rule for (E,d) are $\mathcal{I}(\text{AA}, E, d) = -\mathcal{I}^+(\text{AA}, E, d) = -0.0232$.

The coefficients that we introduce summarize in a couple of numbers the relative distribution of the endowment recommended by two rules. Therefore, we know not only if they are Lorenz-comparable but also by how much the corresponding awards vectors differ from each other, thus helping the decision maker to select one over the other. Depending on the values of the endowment and the claims, the indices between a giving pair of rules can be very small or very large. If the deviation index is small, both distributions are very similar. However, if the deviation index takes high values then the recommendations made by both rules diverge, and factors like the axiomatics of the rules would play a more significant role.

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A Properties of the average of awards rule

Let $N = \{1, 2\}$ and $(E, d) \in C^N$ with $d = (d_1, d_2) \in \mathbb{R}^N$ such that $0 \leq d_1 \leq d_2$. Then, $X(E, d)$ is the line segment with endpoints $(m_1, E - m_1)$ and $(E - m_2, m_2)$, where $m_1 = \max\{0, E - d_2\}$ and $m_2 = \max\{0, E - d_1\}$. The average of awards rule selects the middle point of this segment:

$$\mathrm{AA}(E,d)=\begin{cases}\left(\frac{E}{2},\frac{E}{2}\right) & \text{if } 0\le E\le d_1\\ \left(\frac{d_1}{2},E-\frac{d_1}{2}\right) & \text{if } d_1\le E\le d_2\\ \left(\frac{E+d_1-d_2}{2},\frac{E-d_1+d_2}{2}\right) & \text{if } d_2\le E\le d_1+d_2\end{cases}. \tag{1}$$

Therefore, for two-claimant problems, the average of awards rule coincides with the concede-and-divide rule.

Let $N\in\mathcal{N}$ such that $|N|\ge 3$ and $i\in N$. Consider the function $g_i(E,u)=\frac{\sqrt{n}}{\sqrt{n-1}}\frac{V(u,d_{-i})}{V(E,d)}$, $(E,u)\in(0,d(N))\times[0,D_{-i}]$. Mirás Calvo et al. (2020) show that if $(E,d)\in C^N$, with $d\in\mathbb{R}^n_{\le}$ such that $0<d_1$, then, for all $j\in N\backslash\{i\}$,

$$\mathrm{AA}_j(E,d)=\int_{r_i(E,d)}^{R_i(E,d)}\mathrm{AA}_j(u,d_{-i})g_i(E,u)du, \tag{2}$$

where $r_i(E,d)=\max\{0,E-d_i\}$ and $R_i(E,d)=\min\{E,D_{-i}\}$. Moreover, the function $\mathrm{AA}(\cdot,d)\colon[0,d(N)]\to\mathbb{R}^N$ that assigns to each $E\in[0,d(N)]$ the awards vector $\mathrm{AA}(E,d)$ is a continuously differentiable function on $[0,d(N)]$. For each $j\in N$ let $\chi_j(E,d)=0$ if $E<d_j$ and $\chi_j(E,d)=1$ otherwise. The derivative function $\frac{\partial\,\mathrm{AA}}{\partial E}(\cdot,d)$ is given by:

1. If $E\in[0,d_1]$ then $\frac{\partial\,\mathrm{AA}_j}{\partial E}(E,d)=\frac{1}{n}$ for all $j\in N$.
2. If $E\in[D_{-n},d_n]$ then $\frac{\partial\,\mathrm{AA}_j}{\partial E}(E,d)=0$ for all $j\in N\backslash\{n\}$ and $\frac{\partial\,\mathrm{AA}_n}{\partial E}(E,d)=1$.
3. If $E\in\left[d_1,\min\left\{\frac{1}{2}d(N),D_{-n}\right\}\right]$ then for each $j\in N$ and $i\ne j$,

$$\begin{aligned}\frac{\partial\,\mathrm{AA}_j}{\partial E}(E,d)&=g_i(E,E)\big(\mathrm{AA}_j(E,d_{-i})-\mathrm{AA}_j(E,d)\big)\\&\quad+\chi_i(E,d)g_i(E,E-d_i)\big(\mathrm{AA}_j(E,d)-\mathrm{AA}_j(E-d_i,d_{-i})\big).\end{aligned}$$

4. If $E\in[\frac{1}{2}d(N),d(N)]$ then $\frac{\partial\,\mathrm{AA}_j}{\partial E}(E,d)=\frac{\partial\,\mathrm{AA}_j}{\partial E}(d(N)-E,d)$ for all $j\in N$.

Observe that, according to Fig. 1, if $E\in[0,d(N)]$ then E has to belong to one of the intervals given above.

Proposition A.1 *The average of awards rule satisfies order preservation under endowment variations.*

Proof The proof is by induction on the number of claimants. When $N=\{1,2\}$, from (1), given $(E,d)\in C^N$ we have that:

$$\mathrm{AA}_2(E,d)-\mathrm{AA}_1(E,d)=\begin{cases}0 & \text{if } 0\le E\le d_1\\ E-d_1 & \text{if } d_1<E<d_2\\ d_2-d_1 & \text{if } d_2\le E\le d_1+d_2\end{cases}.$$

Trivially, if $E<E'\le d(N)$ then $\mathrm{AA}_2(E',d)-\mathrm{AA}_1(E',d)\ge\mathrm{AA}_2(E,d)-\mathrm{AA}_1(E,d)$.

Now, by the induction hypothesis, suppose that the average of awards rule satisfies order preservation under endowment variations for any problem with $n-1 \geq 2$ claimants, and let us show that then it must satisfy the property for problems with n claimants. So, let $|N| = n \geq 3$. Since the average of awards satisfies endowment differentiability, given $d \in \mathbb{R}^n_{\leq}$ and $i \in N\backslash\{n\}$, it suffices to prove that $\frac{\partial(\mathrm{AA}_{i+1} - \mathrm{AA}_i)}{\partial E}(E,d) \geq 0$ for all $E \in [0, d(N)]$, that is $(\mathrm{AA}_{i+1} - \mathrm{AA}_i)(\cdot, d)$ is an increasing function. Now, using the derivative expressions given above, we obtain:

$$\frac{\partial(\mathrm{AA}_{i+1} - \mathrm{AA}_i)}{\partial E}(E,d) = \begin{cases} 0 & \text{if } 0 \leq E < d_1 \\ 0 & \text{if } D_{-n} \leq E \leq d_n \text{ and } i < n-1 \\ 1 & \text{if } D_{-n} \leq E \leq d_n \text{ and } i = n-1 \\ \frac{\partial(\mathrm{AA}_{i+1} - \mathrm{AA}_i)}{\partial E}(d(N)-E,d) & \text{if } \frac{1}{2}d(N) \leq E \leq d(N) \end{cases}.$$

It suffices to establish the result for $E \in \left[d_1, \min\left\{\frac{1}{2}d(N), D_{-n}\right\}\right]$. First, assume that $i < n-1$. Then,

$$\begin{aligned}\frac{\partial(\mathrm{AA}_{i+1} - \mathrm{AA}_i)}{\partial E}(E,d) = g_n(E,E)\big(&\mathrm{AA}_{i+1}(E,d_{-n}) - \mathrm{AA}_{i+1}(E,d) \\ &-(\mathrm{AA}_i(E,d_{-n}) - \mathrm{AA}_i(E,d))\big) \\ &+\chi_n(E,d)g_n(E,E-d_n)\big(\mathrm{AA}_{i+1}(E,d) \\ &-\mathrm{AA}_{i+1}(E-d_n,d_{-n}) \\ &-(\mathrm{AA}_i(E,d) - \mathrm{AA}_i(E-d_n,d_{-n}))\big).\end{aligned} \tag{3}$$

Since $E \leq D_{-n}$, we have that $R_n(E,d) = E$. Then, applying expression (2) and by the induction hypothesis:

$$\begin{aligned}\mathrm{AA}_{i+1}(E,d) - \mathrm{AA}_i(E,d) &= \int_{r_n(E,d)}^{E} (\mathrm{AA}_{i+1}(u,d_{-n}) - \mathrm{AA}_i(u,d_{-n}))g_n(E,u)du \\ &\leq \int_{r_n(E,d)}^{E} (\mathrm{AA}_{i+1}(E,d_{-n}) - \mathrm{AA}_i(E,d_{-n}))g_n(E,u)du \\ &= (\mathrm{AA}_{i+1}(E,d_{-n}) - \mathrm{AA}_i(E,d_{-n}))\int_{r_n(E,d)}^{E} g_n(E,u)du \\ &= \mathrm{AA}_{i+1}(E,d_{-n}) - \mathrm{AA}_i(E,d_{-n}).\end{aligned}$$

On the other hand, $\chi_n(E,d) = 1$ only if $E \geq d_n$ and then $r_n(E,d) = E - d_n$. In that case:

$$\begin{aligned}\mathrm{AA}_{i+1}(E,d) - \mathrm{AA}_i(E,d) &= \int_{E-d_n}^{E} (\mathrm{AA}_{i+1}(u,d_{-n}) - \mathrm{AA}_i(u,d_{-n}))g_n(E,u)du \\ &\geq \int_{E-d_n}^{E} (\mathrm{AA}_{i+1}(E-d_n,d_{-n}) \\ &\quad - \mathrm{AA}_i(E-d_n,d_{-n}))g_n(E,u)du\end{aligned}$$

$$= \mathrm{AA}_{i+1}(E - d_n, d_{-n}) - \mathrm{AA}_i(E - d_n, d_{-n}).$$

Clearly, $g_n(E, E) \geq 0$ and $g_n(E, E - d_n) \geq 0$, so from (3) we conclude that indeed $\frac{\partial(\mathrm{AA}_{i+1} - \mathrm{AA}_i)}{\partial E}(E, d) \geq 0$.

Finally, if $i = n-1$, we show that $\frac{\partial(\mathrm{AA}_n - \mathrm{AA}_{n-1})}{\partial E}(E, d) \geq 0$ by repeating the same arguments as above but applied to the integral representations of AA_n and AA_{n-1} given by (2) in terms of the function g_1. □

A rule $\mathcal{R}$ satisfies order preservation under population variation if for each $(E, d) \in B^N$, each $i \in N$ with $E < D_{-i}$ and each pair $\{j, k\} \subseteq N \backslash \{i\}$, if $d_j \leq d_k$, then $\mathcal{R}_k(E, d) - \mathcal{R}_j(E, d) \leq \mathcal{R}_k(E, d_{-i}) - \mathcal{R}_j(E, d_{-i})$. A rule $\mathcal{R}$ satisfies order preservation under the reduction operation if for each $(E, d) \in B^N$, each $i \in N$ with $d_i < E$, and each pair $\{j, k\} \subseteq N \backslash \{i\}$, if $d_j \leq d_k$, then $\mathcal{R}_k(E, d) - \mathcal{R}_j(E, d) \geq \mathcal{R}_k(E - d_i, d_{-i}) - \mathcal{R}_j(E - d_i, d_{-i})$. Let us show that the average of awards rule satisfies order preservation under the reduction operation and order preservation under population variation. Indeed, let $(E, d) \in B^N$, $i \in N$ with $d_i < E < D_{-i}$, and $\{j, k\} \subseteq N \backslash \{i\}$, with $d_j \leq d_k$. We need to prove that

$$\begin{aligned} \mathrm{AA}_k(E - d_i, d_{-i}) - \mathrm{AA}_j(E - d_i, d_{-i}) \leq \mathrm{AA}_k(E, d) \\ - \mathrm{AA}_j(E, d) \leq \mathrm{AA}_k(E, d_{-i}) - \mathrm{AA}_j(E, d_{-i}). \end{aligned}$$

But, $r_i(E, d) = E - d_i$ and $R_i(E, d) = E$, so by equality (2),

$$\mathrm{AA}_k(E, d) - \mathrm{AA}_j(E, d) = \int_{E-d_i}^{E} (\mathrm{AA}_k(u, d_{-i}) - \mathrm{AA}_j(u, d_{-i}))g_i(E, u)du.$$

We show in Proposition A.1 that $\mathrm{AA}_k(., d_{-i}) - \mathrm{AA}_j(., d_{-i})$ is increasing. Then, both properties hold.

Proposition A.2 *The average of awards rule satisfies order preservation under claims variations.*

Proof Let $(E, d) \in C^N$ be a claims problem, $i \in N \backslash \{n\}$ and $d_i < d'_i \leq d_{i+1}$. Denote $d' = (d_{-i}, d'_i)$. It suffices to prove that for each $\{j, k\} \subset N \backslash \{i\}$ with $d_j \leq d_k$ then

$$\mathrm{AA}_j(E, d) - \mathrm{AA}_j(E, d') - \mathrm{AA}_k(E, d) + \mathrm{AA}_k(E, d') \leq 0.$$

Observe that if $E \leq d_i$ then $X(E, d) = X(E, d')$ and the property follows at once. Therefore, assume that $d_i < E$. Let $b = d_i e^i$ and $c = d' - b = (d_{-i}, d'_i - d_i)$, where $e^i \in \mathbb{R}^N$ is the vector with 1 in the ith-coordinate and 0's elsewhere. Then, one can check that the set of awards vectors for (E, d') can be decomposed as the union of two pieces $X(E, d') = X(E, d) \cup \big(b + X(E - d_i, c)\big)$, and the intersection of the two pieces has null Lebesgue measure, i.e. $\mu\Big(X(E, d) \cap \big(b + X(E - d_i, c)\big)\Big) = 0$. The centroid of $X(E, d')$ is the average of the centroids of each part weighted by its relative measure. Therefore, for each $r \in N\backslash\{i\}$, we have that,

$$\mathrm{AA}_r(E, d') = \frac{V(E, d)}{V(E, d')}\,\mathrm{AA}_r(E, d) + \frac{V(E - d_i, c)}{V(E, d')}\,\mathrm{AA}_r(E - d_i, c).$$

Now, taking into account that $V(E, d') = V(E, d) + V(E - d_i, c)$, we obtain that

$$\begin{aligned}\mathrm{AA}_r(E, d) - \mathrm{AA}_r(E, d') &= \mathrm{AA}_r(E, d) - \frac{V(E, d)}{V(E, d')}\,\mathrm{AA}_r(E, d)\\ &\quad - \frac{V(E - d_i, c)}{V(E, d')}\,\mathrm{AA}_r(E - d_i, c)\\ &= \frac{V(E - d_i, c)}{V(E, d')}\Big(\mathrm{AA}_r(E, d) - \mathrm{AA}_r(E - d_i, c)\Big).\end{aligned}$$

Applying the above equality to the pair $\{j, k\} \subset N\backslash\{i\}$, we conclude that $\mathrm{AA}_j(E, d) - \mathrm{AA}_j(E, d') - \mathrm{AA}_k(E, d) + \mathrm{AA}_k(E, d') \leq 0$ if and only if

$$H = \mathrm{AA}_j(E, d) - \mathrm{AA}_j(E - d_i, c) - \mathrm{AA}_k(E, d) + \mathrm{AA}_k(E - d_i, c) \leq 0.$$

Since the average of awards rule satisfies order preservation under population variations and order preservation under the reduction operation,

$$\begin{aligned}&\mathrm{AA}_k(E - d_i, c) - \mathrm{AA}_j(E - d_i, c) \leq \mathrm{AA}_k(E - d_i, d_{-i}) - \mathrm{AA}_j(E - d_i, d_{-i})\\ &\quad \leq \mathrm{AA}_k(E, d) - \mathrm{AA}_j(E, d),\end{aligned}$$

and $H \leq \mathrm{AA}_j(E, d) - \mathrm{AA}_k(E, d) + \mathrm{AA}_k(E - d_i, d_{-i}) - \mathrm{AA}_j(E - d_i, d_{-i}) \leq 0.$ □

B Areas below the cumulative claims-awards curve

Let $x = (x_1, \ldots, x_n) \in \mathbb{R}^N_{\leq}$ and $y = (y_1, \ldots, y_n) \in \mathbb{R}^N_{\leq}$. Define

$$\bar{x}_i = \begin{cases} 0 & \text{if } i = 0 \\ \frac{1}{x(N)} \sum_{k=0}^{i} x_k & \text{if } i \in N \end{cases}, \qquad \bar{y}_i = \begin{cases} 0 & \text{if } i = 0 \\ \frac{1}{y(N)} \sum_{k=0}^{i} y_k & \text{if } i \in N \end{cases}.$$

For each $i \in N$ denote $\Delta\bar{x}_i = \bar{x}_i - \bar{x}_{i-1} = \frac{1}{x(N)} x_i$ and $\Delta\bar{y}_i = \bar{y}_i - \bar{y}_{i-1} = \frac{1}{y(N)} y_i$. Consider the continuous piecewise linear function $L^y_x\colon [0, 1] \to [0, 1]$ connecting the $n + 1$ points $(\bar{x}_i, \bar{y}_i)$, $i = 0, \ldots, n$. that is,

$$L^y_x(t) = \bar{y}_{i-1} + \frac{\Delta\bar{y}_i}{\Delta\bar{x}_i}(t - \bar{x}_{i-1}) \text{ if } t \in [\bar{x}_{i-1}, \bar{x}_i].$$

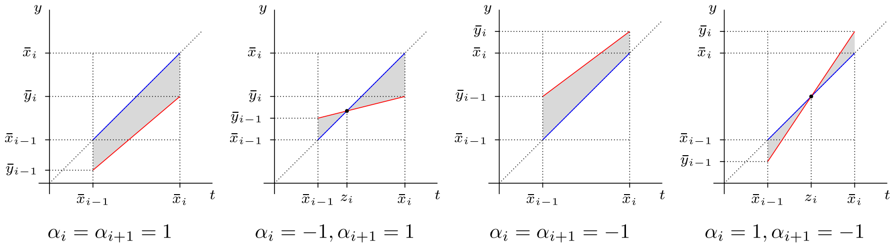

Fig. 10 Relative position of the claims-awards curve and the line of proportionality on $[\bar{x}_{i-1}, \bar{x}_i]$

The difference between the area inside the unit square below the proportionality line and the area below L_x^y is:

$$\int_0^1 \big(t - L_x^y(t)\big)dt = \int_0^1 t\,dt - \sum_{i\in N}\int_{\bar{x}_{i-1}}^{\bar{x}_i} L_x^y(t)dt$$
$$= \tfrac{1}{2}\Big(1 - \sum_{i\in N}\Delta\bar{x}_i(\bar{y}_{i-1}+\bar{y}_i)\Big).$$

In particular, given a claims problem $(E, d) \in C^N$ with $d \in \mathbb{R}^N_{\leq}$ and a rule satisfying order preservation in awards $\mathcal{R}$, taking $x = d$ and $y = \mathcal{R}(E, d)$, we have that the signed proportionality deviation index of $\mathcal{R}$ for the problem (E, d) is given by:

$$\mathcal{I}(\mathcal{R}, E, d) = 1 - \sum_{i\in N}\Delta\bar{d}_i(\bar{\mathcal{R}}_{i-1}(E, d) + \bar{\mathcal{R}}_i(E, d)).$$

Now, the area between the line of proportionality and the piecewise polygonal curve L_x^y is given by the integral

$$\int_0^1 |t - L_x^y(t)|dt = \sum_{i\in N}\int_{\bar{x}_{i-1}}^{\bar{x}_i} |t - L_x^y(t)|dt.$$

For each $i \in \{1, \ldots, n+1\}$, let $\alpha_i = \begin{cases} 1 & \text{if } \bar{x}_{i-1} > \bar{y}_{i-1} \\ 0 & \text{if } \bar{x}_{i-1} = \bar{y}_{i-1} \\ -1 & \text{if } \bar{x}_{i-1} < \bar{y}_{i-1} \end{cases}$. Fix $i \in N$. If $\alpha_i\alpha_{i+1} = -1$ then the line of proportionality and the curve L_x^y intersect at the point $z_i = \frac{\bar{x}_i\bar{y}_{i-1} - \bar{x}_{i-1}\bar{y}_i}{\Delta\bar{x}_i - \Delta\bar{y}_i}$ with $z_i \in (\bar{x}_{i-1}, \bar{x}_i)$. The curve L_x^y lies above the line of proportionality whenever $\alpha_i, \alpha_{i+1} \leq 0$ and lies below the line of proportionality whenever $\alpha_i, \alpha_{i+1} \geq 0$. Clearly, if $\alpha_i = \alpha_{i+1} = 0$ then $L_x^y(t) = t$ for all $t \in [\bar{x}_{i-1}, \bar{x}_i]$. The cases when $\alpha_i\alpha_{i+1} \neq 0$ are depicted in Fig. 10. Therefore, from elementary calculus, we have:

- If $\alpha_i \geq 0$ and $\alpha_{i+1} \geq 0$ then

$$2\int_{\bar{x}_{i-1}}^{\bar{x}_i} |t - L_x^y(t)|dt = \Delta\bar{x}_i(\bar{x}_{i-1} - \bar{y}_{i-1} + \bar{x}_i - \bar{y}_i).$$

- If $\alpha_i \leq 0$ and $\alpha_{i+1} \leq 0$ then

$$2\int_{\bar{x}_{i-1}}^{\bar{x}_i} |t - L_x^y(t)|dt = \Delta\bar{x}_i(\bar{y}_{i-1} - \bar{x}_{i-1} + \bar{y}_i - \bar{x}_i).$$

- If $\alpha_i = 1$ and $\alpha_{i+1} = -1$ then

$$2\int_{\bar{x}_{i-1}}^{\bar{x}_i} |t - L_x^y(t)|dt = (z_i - \bar{x}_{i-1})(\bar{x}_{i-1} - \bar{y}_{i-1}) + (\bar{x}_i - z_i)(\bar{y}_i - \bar{x}_i).$$

- If $\alpha_i = -1$ and $\alpha_{i+1} = 1$ then

$$2\int_{\bar{x}_{i-1}}^{\bar{x}_i} |t - L_x^y(t)|dt = (z_i - \bar{x}_{i-1})(\bar{y}_{i-1} - \bar{x}_{i-1}) + (\bar{x}_i - z_i)(\bar{x}_i - \bar{y}_i).$$

In particular, given a claims problem $(E, d) \in C^N$ with $d \in \mathbb{R}^N_{\leq}$ and a rule satisfying order preservation in awards $\mathcal{R}$, taking $x = d$ and $y = \bar{\mathcal{R}}(E, d)$, we have that the proportionality deviation index of $\mathcal{R}$ for the problem (E, d) is given by: $\mathcal{I}^+(\mathcal{R}, E, d) = 2\int_0^1 |t - L_{E,d}^{\mathcal{R}}(t)|dt.$

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