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ORIGINAL PAPER

Modelling and analysis of heat transfer through 1D complex granular system

E. Szymanek · T. Blaszczyk · M. R. Hall · P. Keikhaei Dehdezi · J. S. Leszczynski

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Abstract This article demonstrates the solution to the problem of the passage of air through the external wall barrier and the influence of the materials type and its layer arrangement on heat conductivity, in respect of building heat losses. It shows how the temperature changes inside the wall barriers and in a room while the external temperature changes. Also, this article presents the mathematical model based on fractional differential equation describing the analysed phenomenon.

Keywords Heat transfer · Granular material · Fractional differential equation

E. Szymanek (✉) · J. S. Leszczynski
Institute of Advanced Energy Technologies, Czestochowa University of Technology, ul. Dabrowskiego 73, 42-200 Czestochowa, Poland
e-mail: ewaszymanek@vp.pl

J. S. Leszczynski
e-mail: jaclesz@gmail.com

T. Blaszczyk
Institute of Mathematics, Czestochowa University of Technology, al. Armii Krajowej 21, 42-200 Czestochowa, Poland
e-mail: tomasz.blaszczyk@im.pcz.pl

M. R. Hall
Division of Materials, Mechanics and Structures, Faculty of Engineering, University of Nottingham, University Park, Nottingham NG7 2RD, UK
e-mail: Matthew.Hall@nottingham.ac.uk

P. K. Dehdezi
URS Infrastructure and Environment UK Limited, 12 Regan Way, Chetwynd Business Park, Nottingham NG9 6RZ, UK
e-mail: pejman.keikhaeidehdezi@urs.com

1 Introduction

The process of heat transfer, being present in many technological disciplines, is very common, as it occurs wherever temperature differences appear. Heat transfer is achieved in three physically-different ways, that is through heat conduction, convection and radiation. Without any doubt, the heat conduction is the most important form of the heat transfer process. Next to the already-mentioned three types of heat transfer there are also combinations of those types in the forms of heat transmission (penetration) and heat permeability. This work will investigate the heat flow through a complex system, understood as a system made of the granulated material, air and water. At the core of the heat transfer phenomenon in a complex system lies temperature difference. Therefore, one should look for solutions which allow properly defining the temperature distribution in its interior. A more precise explanation of the phenomenon of heat transfer between the constituents of a given process may allow the better control of temperature changes. The empty spaces between the granular material may be filled with, for example, gas or water. Additionally, many connected phenomena take place between the granular material. In the conditions of fluctuating pressure and temperature chemical reactions between bodies and the material filling the empty space might occur. The number of phenomena and the multiplicity of parameters cause problems with their description, and because the phenomenon of the heat flow is connected both with everyday life and with many technological disciplines interest in the said subject is still current. In literature we can find models describing heat flow through complex systems. The three-phase porous material consisting of the granule material framework and water and air filling the pores of the material [3,15,18], as well as the system consisting of two materials: solid bodies and fluid or air [10,21], are taken into consideration. In order to describe the heat flow, computational analysis, for example, the finite difference method, is applied. Some problems with the method application are connected with the boundary conditions and with the irregular shape of the boundaries. The other method applied to describe the phenomenon of heat flow is the finite-element method. This method is used to examine heat flow and to investigate other issues described with a differential equation of the first and second order [9]. The common application of the said methods in solving technical problems is a result of the possibility of obtaining an equation result, which cannot be solved in an analytical way, or its solution is too complex and time-consuming [36]. The idea of the method of boundary elements, also used to describe the analysed phenomenon, is to reduce the given boundary element to a simultaneous integral equation, which is defined on the boundary of the given closed and connected set and is equivalent to the reduction of the problem size. The next method applied to the examination of heat flow is a model prepared by Rajagopal and Massoudi [30], where the material density is of great significance. This model was used to examine various problems like the heat flow in a vertical pipe [13] or the heat flow occurring due to natural convection [27]. A very common method used in granular media for prediction of global heat flux, isotherms, and isofluxes is the Lattice Boltzmann Method. LBM is a relatively new simulation technique for complex fluid systems. Due to its particulate nature and local dynamics, LBM has several advantages over other conventional methods, especially in dealing with complex boundaries, incorporating of microscopic interactions, and parallelization of the algorithm [11,14]. The necessity to create a model of the heat flow through complex systems has been already indicated in the former publications. The granular materials were very often treated as systems of barriers separated by air layers [19]. To define the heat flow through granular materials Prasolov considered the granular structure as a system of a solid body and a gas [29], while Smolukovsky [32] examined the granulated materials in lowered-gas-pressure conditions, assuming that the grains of the granular material are round. Kelly and Schwarz [16] in their work analysed models of heat transfer. In these models the physical geometry, which is reproduced in all porous materials, is idealised and shown in a simplified form.

On the basis of the literature it can be stated that taking into account the classical approach to heat flow we apply the Fourier–Kirchoff equation. But when we manage a complex system, the classical approach is not the best solution. One of the basic problems which appears while modelling the heat flow in a complex system by classical equations is the multiplicity of interrelations, factors and coefficients necessary to describe the structure of the examined system. Defining general interrelations, which will allow the defining of heat conductivity in complex system, is complicated because of the heat qualities of the particular elements, and porosity as well as the moisture content. Therefore, it is necessary to create new models.

In this paper has been proposed mathematical model based on fractional calculus, which is used in many fields, such as engineering, chemistry, electrical and electromechanical systems, etc. The basic mathematical ideas of fractional calculus were developed by the mathematicians Leibnitz, Liouville, Riemann and others and today is fast-growing part of mathematics (Podlubny, Magin, Mainardi, West). Fractional calculus is particulary useful in describing the dynamics of complex systems. During the last decades of the nineteenth century (1892) Heaviside introduced the idea of fractional derivatives in his study of electric transmission lines. Sebaa [31] used fractional calculus describe the viscous interactions between fluid and solid structure in human cancellous bone. Kulish [20] used fractional calculus to fluid mechanics. Assaleh [2] proposed a novel approach for speech signal modelling using fractional calculus in presented. Fellah and Depollier [12] used application to the sound waves propagation in rigid porous materials. Soczkiewicz [33] fractional calculus used in the theory of viscoelasticity. Podlubny's work [28] contains information about the applications of fractional calculus to various problems of mechanics, physics and engineering. West in [35] describes how to use the fractional operators in the modelling of complex phenomena. After the analysis of this work we have observed that the spatial fractional derivatives have long-range interactions and may have deep physical implications when modelling a complex phenomenon. The application of fractional calculus in bioengineering was presented by Magin in [24,25]. Such fractional order models provide an improved description of the observed bioelectrode behaviour. Mainardi, in work [26], shows how the fractional calculus provides a suitable method of describing dynamical properties of linear viscoelastic media with the problems of wave propagation and diffusion. The book by Uchaikin [34] presents detailed motivation for fractional differential equations in various branches of physics.

The growing number of such applications indicates that there is a significant demand for better mathematical models of real objects, and that the fractional calculus provides one of many possible approaches regarding the way to more adequate mathematical modelling of real objects and processes, especially when we are dealing with complex systems. Some process can describe models approach in classical equation but suggest that additional mathematical tools may be needed to better describe this complex system. Fractional derivatives have many properties in common with the classical ones, but not all the properties are the same. These differences can be used to describe complex phenomena that arise due to nonlocal interactions and system memory. Classical approach requires large number of coefficient and constitutive relations in order to minimize uncertainty in mathematical modelling. However large number of coefficient generate uncertainty in determination of their values. To reduce this uncertainty we decided to apply fractional calculus, which requires low number of coefficients describes the analysed process. We limited our consideration only to steady state where the analysed phenomenon depends only on location. This paper proposes a description which does not investigate the structure but assumes some degree of its heterogeneity. When abandoning the classical equation and substituting it with an ordinary differential equation including the left and right fractional derivatives [4,5,7,8,22] we arrive at a model which has such a quality that it does not investigate the structure and does not include such a number of factors. Such equations are the result of the modification of the principle of least action and the application of the fractional rule of integration by parts [5]. The considered problems faced in the process of heat flow during the works will be analysed from the practical and theoretical points of view. Computer simulations were additionally supported by experimental tests in the research station using a climate chamber in order to simulate the temperature conditions.

2 The mathematical model

Here, mathematical analysis is used increasingly often for designing wall barriers, such as, for example, fractional differential equation, which requires a significantly-smaller number of coefficients.

Therefore, for describing the phenomenon under consideration, proposed are fractional equations with the left and right Caputo derivatives in the form presented below.

$$ {}^{C}D^{\alpha}_{0+}\,{}^{C}D^{\alpha}_{b-}T(x) + \omega^{2\alpha}T(x) = 0 \qquad (1)$$

where $x \in [0, b]$, α represents a degree of heterogeneity, ω represents a scale parameter and operators ${}^{C}D^{\alpha}_{0+}$, ${}^{C}D^{\alpha}_{b-}$ are defined as [17]

$$ {}^{C}D^{\alpha}_{0+}T(x) = \frac{1}{\Gamma(n-\alpha)}\int_{0}^{x}\frac{T^{(n)}(\tau)}{(x-\tau)^{\alpha-n+1}}d\tau, \quad x>0 \qquad (2)$$

$$ {}^{C}D^{\alpha}_{b-}T(x) = \frac{(-1)^n}{\Gamma(n-\alpha)}\int_{x}^{b}\frac{T^{(n)}(\tau)}{(\tau-x)^{\alpha-n+1}}d\tau, \quad x<b \qquad (3)$$

where $n = [\alpha]+1$. The first of the above derivatives is called the left Caputo derivative and the next, the right derivative respectively.

We assume $\alpha \in (0, 1)$, then Eq. (1) is supplemented by boundary conditions as

$$ T(0) = T_0, \quad T(b) = T_b \qquad (4)$$

2.1 An analysis of the error of approximate solutions

The analytical solution of Eq. (1) has the form of the infinity series of composition of the left and right fractional integrals, see [4,5,23]. Therefore, in our model we will use the discrete form of the considered equation. The resultant numerical scheme should be tested in terms of the errors it generates, as well as in terms of its order of convergence. We write the discrete form of Eq. (1) follows to [4,6] as

$$\begin{cases} T_0 = T(x_0) \\ (\Delta x)^{-2\alpha}\sum_{j=0}^{i}\left[v(i,j)\sum_{k=j}^{N} v(N-j, N-k)T_k\right] + \omega^{2\alpha}T_i = 0 \\ T_N = T(x_N) \end{cases} \qquad (5)$$

for $i = 1, \ldots, N-1$, where

$$ v(i,j) = \frac{1}{\Gamma(2-\alpha)}\begin{cases} (i-1)^{1-\alpha} - i^{1-\alpha} & \text{for } j=0 \\ (i-j+1)^{1-\alpha} - 2(i-j)^{1-\alpha} + (i-j-1)^{1-\alpha} & \text{for } j = 1,\ldots,i-1 \\ 1 & \text{for } j = i \end{cases} \qquad (6)$$

Let $x \in [0, 1]$ and boundary conditions

$$ T(0) = 1, T(1) = 0 \qquad (7)$$

Then, the solution of Eq. (1) with conditions (7) has the following form

$$ T(x) = \frac{\sum_{m=0}^{\infty}\left(-\omega^{2\alpha}\right)^m \left(I^{\alpha}_{1-}I^{\alpha}_{0+}\right)^m (1-x)^{\alpha}}{\sum_{m=0}^{\infty}\left(-\omega^{2\alpha}\right)^m \left(I^{\alpha}_{1-}I^{\alpha}_{0+}\right)^m 1} \qquad (8)$$

where I^{α}_{1-} and I^{α}_{0+} denote the right and left fractional integrals [17]

In particular, for $\omega = 0$, solution of Eq. (1) simplifies to the form

$$ T(x) = (1-x)^{\alpha} \qquad (9)$$

We determine experimental estimation of the convergence row (EOC) as [1,5]

$$ EOC = log_2\left(\frac{error[N]}{error[2N]}\right) \qquad (10)$$

where

$$ error[N] = \frac{\frac{1}{2}\,|T(x_0) - T_0| + \frac{1}{2}\,|T(x_N) - T_N| + \sum_{i=1}^{N-1}|T(x_i) - T_i|}{N} \qquad (11)$$

In error calculations we take into account boundary conditions (7) and $\omega = 0$. Errors generated by numerical scheme (5) is shown by Table (1).

When the EOC values are put to analysis in Table 1, it may notice that the convergence of our numerical systems (5) equals O(h) and is independent of parameter α.

Table 1 Errors and EOC generated by numerical scheme (5)

	$\alpha = 0.3$		$\alpha = 0.5$		$\alpha = 0.7$	
Δx	Error	EOC	Error	EOC	Error	EOC
$\frac{1}{16}$	1.51e−2		1.46e−2		1.05e−2	
$\frac{1}{32}$	8.47e−3	0.83	8.19e−3	0.83	6.05e−3	0.79
$\frac{1}{64}$	4.60e−3	0.88	4.41e−3	0.89	3.34e−3	0.86
$\frac{1}{128}$	2.44e−3	0.91	2.32e−3	0.93	1.80e−3	0.89

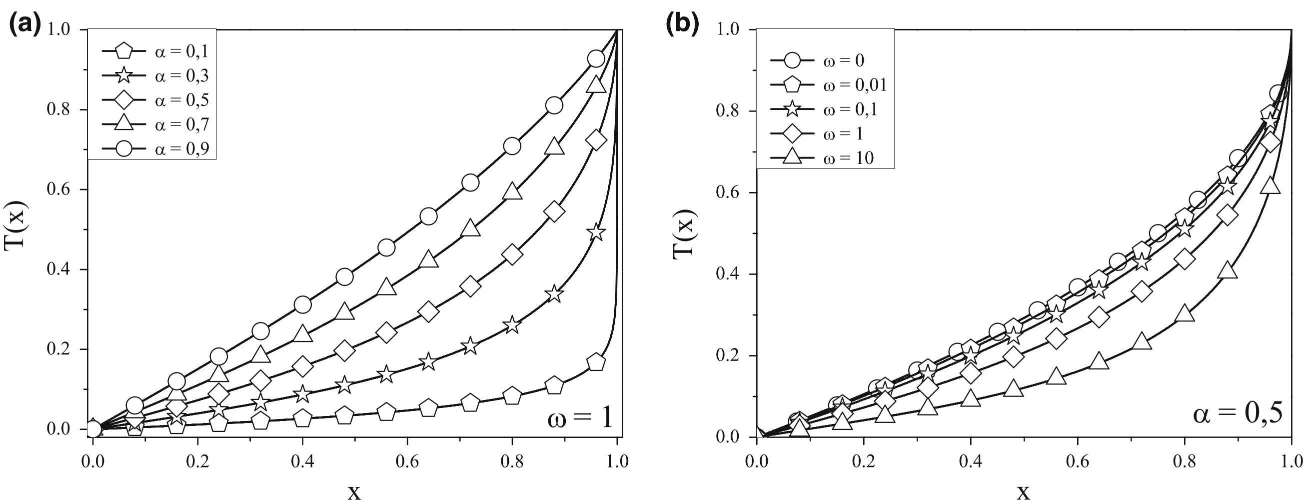

Fig. 1 Numerical solutions of Eq. (1) for boundary conditions (7) for: **a** $\omega = 1$ and $\alpha \in \{0, 1; 0, 3; 0, 5; 0, 7; 0, 9\}$; **b** $\alpha = 0, 5$ and $\omega \in \{0; 0, 01; 0, 1; 1; 10\}$

Subsequently, examples for various values of parameters α and ω are calculated to graphically demonstrate how numerical solutions for the Eq. (1) behave.

Figure 1a presents plots for the constant value of parameter $\omega = 1$ and variables values of parameter α. In the Fig. 1b we show the influence of parameter $\omega \in \{0; 0, 01; 0, 1; 1; 10\}$ at the constant value of $\alpha = 0, 5$ on the solution.

3 Experimental setup

Heat-flow simulations and schedules of heat distribution for wall barriers allow such a design of wall barriers as to maximally reduce the expenditures on utilisation of buildings. External wall barriers significantly influence the energy performance of a building. The knowledge of heat distribution in these wall barriers will allow the correct assessment of insularity. The paper presents the results and conclusions of research pertaining to the heat flow through layers of granular mass and concrete characterised by the following properties: for granular material—unbound granular materials: $\lambda_g = 1.5\frac{W}{mK}$, $c_{p_g} = 1,000\frac{J}{kgK}$, $\rho_{d_g} = 2,403\frac{kg}{m^3}$, and for concrete-bound granular material: $\lambda_c = 1\frac{W}{mK}$, $c_{p_c} = 840 JkgK$, $\rho_{d_c} = 2,100\frac{kg}{m^3}$, where λ—thermal conductivity, c_p—appropriate heat, ρ_d—density. Temperature profiles in a wall barrier filled with a layer of concrete were presented. These profiles will compare with temperature profiles in wall barriers with added layers of granulate. A well-insulated wall barrier, as shown in Fig. 2, was created with a view to carrying out the tests.

In the research a climate chamber [Series 3 LTCL600 climatic chamber (TAS Ltd., West Sussex, UK)] was used which allows the regulation of temperature in a space which simulates the surroundings. In the chamber a wall barrier was placed, in which layers were changed in consecutive stages of the experiment. These layers were properly insulated, and the air void in the lower part of the wall barrier simulated the room. After setting the parameters, the thermocouple data collected in the course of the experiment were saved on a disc.

The introductory stage of the experiment consisted of entering suitable parameters in the climate chamber and obtaining temperature profiles, in particular wall barriers, with the temperature being maintained at −12 °C, both in the internal, central (of the wall barrier), and external part of the wall barrier. Then the external temperature (in the upper part of the chamber) was gradually being increased. Data collected from thermocouples presented the way in which

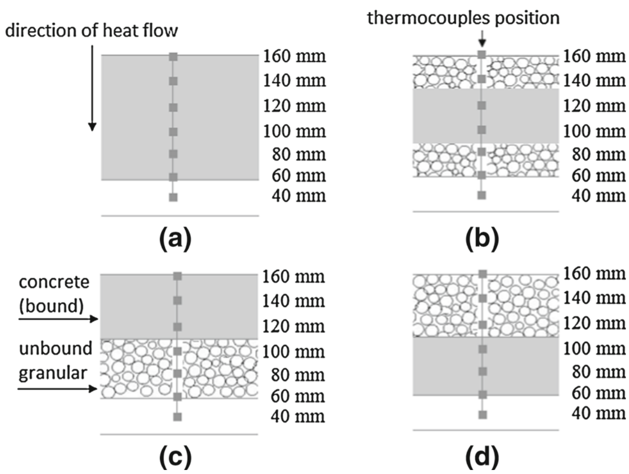

Fig. 2 Schematic arrangement of thermocouples in the research station: **a** in the bulkhead filled with concrete—bulkhead 1, **b** in the bulkhead filled with concrete and two granular layer on the inside and outside—bulkhead 2, **c** in the bulkhead filled with concrete and granular layer on the inside—bulkhead 3, **d** in the bulkhead filled with concrete and granular layer on the outside—bulkhead 4

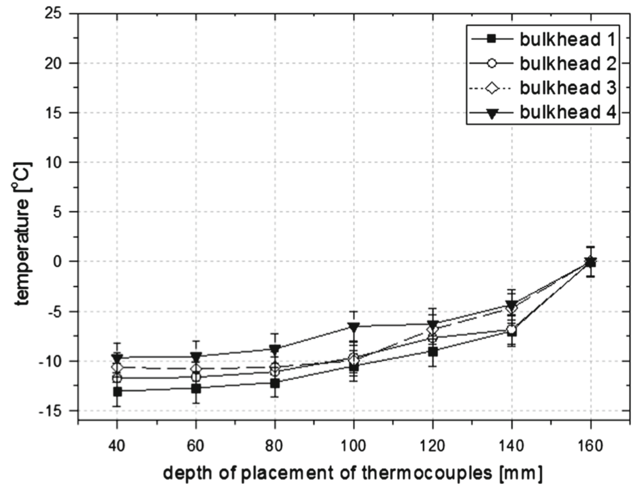

Fig. 3 Temperature profiles for four wall barriers with various layers for the external temperature of 0 °C

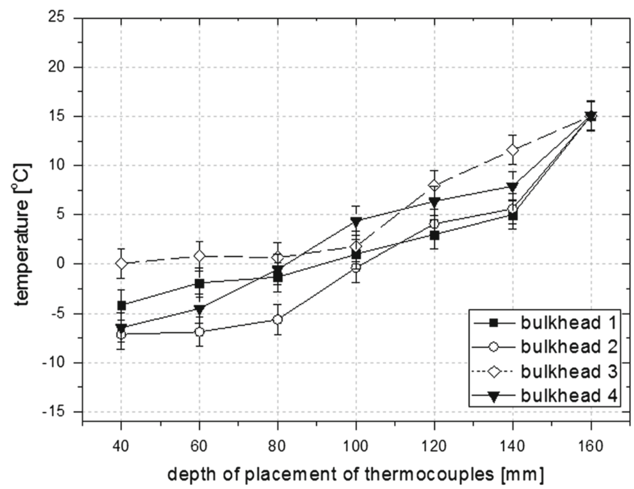

Fig. 4 Temperature profiles for four wall barriers with various layers for the external temperatures of 15 °C

temperature changed in a layer of concrete (Fig. 2a), in a layer of concrete with layers of granular material on both sides (Fig. 2b), in a layer of concrete with layers of granular material on the internal side (Fig. 2c) and on the external side (Fig. 2d).

The four cases presented in Fig. 2 were examined so as to determine the influence of the arrangement of the layer of granular material on the thermal conductivity of a wall barrier.

4 Results

Presented below are the results of the tests of airflow through a wall barrier and temperature profiles for wall barriers constructed of various material layers. Temperature values inside the external wall barriers and the temperatures inside the room at individual time steps for each wall barrier were determined. The research was conducted for four wall barriers as described above and shown in Fig. 2. In each phase the thermocouples were distributed at the same heights (Fig. 2). The thermocouple placed at the height of 40 mm was placed in an air void which simulated the room, whereas the thermocouple placed at the height of 160 mm was placed as designed on the external side of the wall barrier. Before the experiment, the temperature in each layer and in the room was −15 °C. It was gradually increased up to the value of 25 °C. The Figs.3, 4, 5 is the result of research conducted, and present temperature change curves in the examined wall barriers, depending on the type, arrangement and the thickness of materials.
In this paper we consider four bulkheads:

– bulkhead 1 (concrete $\delta = 100\,mm$)
– bulkhead 2 (granular material/concrete/granular material $\delta = 20/40/20\,mm$)
– bulkhead 3 (granular material/concrete $\delta = 50/50\,mm$)
– bulkhead 4 (concrete/granular material $\delta = 50/50\,mm$)

where granular material has parameters $\lambda_g = 1.5\frac{W}{mK}$, $c_{p_g} = 1,000\,\frac{J}{kgK}$, $\rho_{d_g} = 2403\frac{kg}{m^3}$ and concrete has parameters $\lambda_c = 1\frac{W}{mK}$, $c_{p_c} = 840\frac{J}{kgK}$, $\rho_{d_c} = 2100\frac{kg}{m^3}$. The layers were insulated from the external environment and the direction of heat flow could be consider as one-dimensional. The charts presented in Figs. 3, 4, 5 present temperature distribution obtained from various thermocouples in the course of the four stages of the experiment which examined the four types of external walls. The data selected were obtained at the moment when the temperature of the surroundings was −15 °C, and then increased to 0–15 °C and to 25 °C respectively.

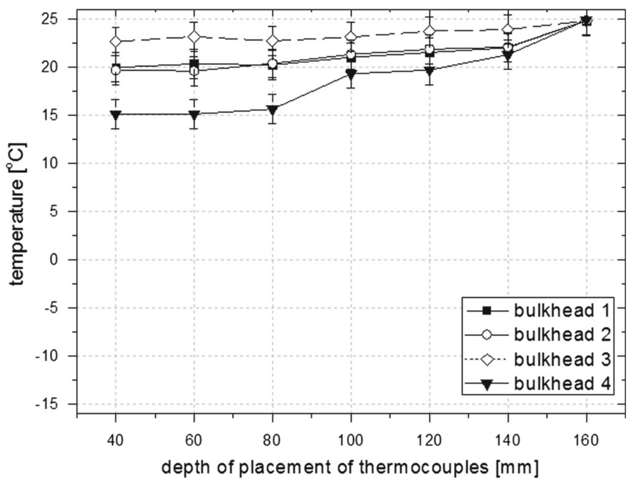

Fig. 5 Temperature profiles for four wall barriers with various layers for the external temperatures of 25 °C

In addition, at the figures was presented measurement uncertainties. These parameters are related to the result of the measurement is characterized by dispersion of results that can be attributed to the measured value. These uncertainties have been calculated and presented in the graphic form. Analyzing deviations of measurements can be seen that did not affect significantly the quality of the results.

Based on Charts 3–5 we observe that the temperature in the combination of concrete and granulated layers increased at the highest rate. Nonetheless, a difference in temperature profiles can be observed in these layers, which are particularly visible at the point of convergence of the layers. The experiment started when the temperature in each wall barrier was −15 °C. When the external temperature increased to 0 °C, the temperature in wall barrier 3, in the first layer (granular material) was decreasing at rather a slow rate, while temperature in the next layer in concrete started decreasing more rapidly. In wall barrier 4 it is evident that in the first (concrete) layer the temperature rapidly decreased, while the temperature profile in the granular material started stabilising.

When the external temperature continued as above zero, the temperature in wall barrier 1 started increasing more rapidly, while the rate of temperature increase was reduced through wall barrier 4. The smallest increase in temperature was noted in wall barrier 2, where it can observe that the temperature profile in layers of granular material in wall barriers was not so sharp as that in concrete layers. Since the heat conductivity of the framework of the material is better in granular material than the heat conductivity of air, air filled inter-granular spaces act as an insulator.

Apart from temperature distribution in the external walls, the air temperatures inside the room were also determined. Figures 3, 4, 5 also presents internal temperature values for the room that were collected from a thermocouple placed 40 mm deep. At external temperature of 0 °C inside the room in both cases of mixed wall barriers (barriers 3 and 4) the temperature was −10 °C. The temperature inside the room measured when the layer of concrete was insulated on both sides with layers of granular material, was −12.5 °C. The temperature inside the room for wall barrier 1 increased at the lowest rate. The temperature inside the room was −13 °C while the external temperature increased to 0 °C. When the external temperature increased to 15 °C, a greater discrepancy can observe in the temperatures inside the room. When the temperature continued at 25 °C, the temperature in the wall barrier 3 was equalising at the highest rate. At the same time, the temperature in the room with wall barrier 4 was increasing at the lowest rate.

In the next part of this paper presents the results of bulkhead 4 filled with granular material and concrete (Fig. 2). During the study collected data from thermocouples placed inside the bulkhead have produced temperature profiles. On their basis selected a steady-state. Presented below is the comparison of experimental results for external temperatures of 0 °C and 15°C respectively with airflow for the wall with the best parameters, i.e. wall barrier 4, with numerical analysis results. Data selected based on the experiment described earlier allowed the determination of temperature profiles in individual wall barriers. Approximate temperature profiles were determined based on Eq. (1). Figures 6 and 7 present the effects of the procedure carried out.

In numerical simulations was used the discrete form of Eq. (1) for the following parameters $\alpha \in \{0, 4; 0, 6\}$, $\omega \in \{0, 003; 0, 022\}$ and boundary conditions:

– T(40) = −11.5 °C, T(160) = 0 °C for (Fig.6)
– T(40) = 0 °C , T(160) = 16 °C for (Fig.7).

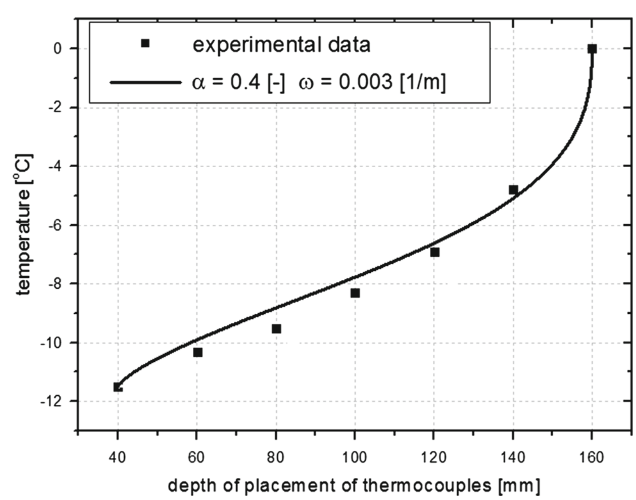

Fig. 6 A comparison of the numerical solution of Eq. (1) with data obtained in the experiment for an external temperature of 0 °C at wall barrier 4

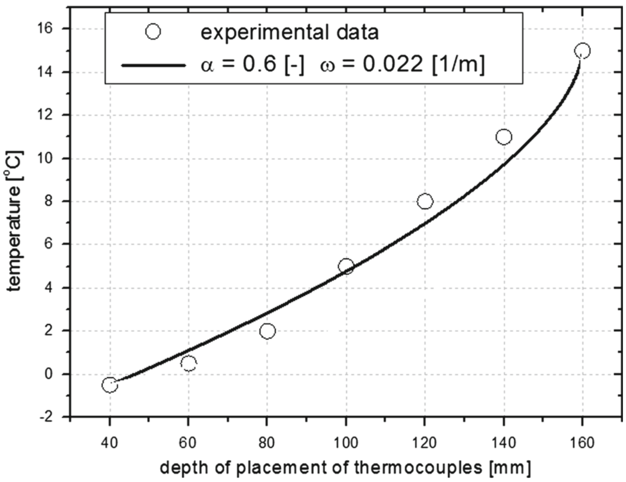

Fig. 7 A comparison of the numerical solution of Eq. (1) with data obtained in the experiment for an external temperature of 15 °C at wall barrier 4

Experimental data come from heat-flow simulations distribution for bulkhead allow such a design of wall barriers as to maximally reduce the expenditures on utilisation of buildings.

5 Concluding remarks

The article discusses temperature profiles for wall barriers composed with various layers of concrete and a granular material. The research was conducted in variable temperature and humidity conditions. Granules were used as insulators for layers made of concrete. In the obtained temperature distributions, as compared to humidity distribution, the temperature differences were in accordance with the results estimated with the use of a mathematical model. Adding an excessive amount of insulator did not result in enhanced conditions. Moreover, the results suggest that a wall barrier consisting of a layer of concrete and an insulation layer on its external side has the best properties. Because this type of complex systems in classic model requires to determine a large number of factors resulting from the structure of the studied system. Using fractional differential equations as mathematical description of the complex system gives an advantage that it does not penetrate into the structure of the analyzed system. This approach assumes only a certain degree of heterogeneity. The great advantage of this model reveals a significant reduction of number of coefficients which are necessary to describe the studied complex phenomena. The results presented in this study show that the proposed model describes well the temperature profile in the bulkhead, in steady regime.

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