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ORIGINAL ARTICLE

Generalized multiplicative stochastic processes arising from one-dimensional maps with noise

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Abstract

We propose a class of generalized multiplicative stochastic processes obtained by introducing an endo-perspective into one-dimensional maps with additive noise. We define an internal state for the noisy dynamics of a given one-dimensional map and study its statistical behavior. We found intermittency characterized by two power-laws in the dynamics of the internal state for the logistic map and the BZ map with noise which exhibit different noise-induced phenomena, namely, noise-induced chaos and noise-induced order, respectively. We show that the power-laws can be explained in a unified way from the theory of generalized multiplicative stochastic processes.

Keywords One-dimensional maps · Noise · Intermittency · Power-law · Endo-perspective

1 Introduction

Power-law distributions are observed ubiquitously in nature and society [2, 19]. It is known that there are many different generating mechanisms for them [17]. Random multiplicative processes [18] are one of such mechanisms with multiple power-law distributions in their dynamical behavior. In this paper, we show that they arise from one-dimensional maps with additive noise under certain conditions and investigate their statistical behavior both by numerical simulation and in terms of the theory of generalized multiplicative stochastic processes.

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Given a one-dimensional map with additive noise and an initial value, we have a sequence of linear maps consisting of the tangent line at each point in the noisy trajectory (x_n) of the map from the initial value. Then, we can generate a sequence (y_n) by applying the linear maps successively starting from the same initial value. The motivation for considering (y_n) is to see what looks like the behavior of one-dimensional maps with noise from the endo-perspective [4, 14] in the sense that it can only use the local information of the map and use it blindly without knowing the external noise. Since the application of the linear maps ignores the additive noise at each time step in (x_n), (y_n) is different from (x_n) in general unless the noise is absent. In addition, if the Lyapunov exponent of (x_n) is positive, then (y_n) can diverge. On the other hand, if the Lyapunov exponent of (x_n) is negative, then y_n can return to a value close to x_n even when it takes a very large value tentatively.

Here, we study the statistical behavior of (y_n) generated by the BZ map [13] and the logistic map with additive noise [3, 15]. We choose the parameters of these maps so that the Lyapunov exponents of the noisy trajectories are negative but close to zero. When (x_n) for the BZ map exhibits noise-induced order (NIO), we observe intermittent bursts in (y_n). For the logistic map, if the parameter of the map and the strength of the additive noise are chosen in the range close to the onset of noise-induced chaos (NIC), then we also observe intermittent bursts in (y_n). We show that the statistics of the intermittent bursts in these different situations can be understood on the same ground. In particular, the distribution of the burst amplitude and the distribution of the duration of the bursts follow power-laws and they can be analyzed in terms of the theory of generalized multiplicative stochastic processes. We apply the theoretical analysis to the numerically obtained sequence (y_n) and confirm the consistency of the theory in terms of the exponents of the power-laws.

2 Model

Let f be a differentiable map from a real interval I to itself. Without loss of generality, we take $I = [0, 1]$. We consider the following discrete-time dynamical system with additive noise

$$x_{n+1} = f(x_n) + \epsilon u_n, \tag{1}$$

where $n = 0, 1, 2, \ldots$, $x_0 \in I$ and $(u_n)_{n\geq 0}$ are an realization of i.i.d. random variables. $\epsilon \geq 0$ is the strength of the additive noise. In the following examples, u_n are sampled from the standard normal distribution with mean 0 and variance 1.

Let $g_z(x) = f(z) + f'(z)(x - z)$, where $f'(z)$ denotes the derivative of f at $z \in I$. The graph of f can be regarded as the envelope of its tangent lines (Fig. 1). In particular, we have $f(x_n) = g_{x_n}(x_n)$. Thus, when $\epsilon = 0$, $x_{n+1} = g_{x_n} \circ g_{x_{n-1}} \circ \ldots \circ g_{x_0}(x_0)$. However, when $\epsilon > 0$, Eq. (1) leads to

$$x_{n+1} = g_{x_n} \circ g_{x_{n-1}} \circ \ldots \circ g_{x_0}(x_0) + \epsilon \sum_{k=0}^{n} \left(\prod_{l=0}^{k-1} f'(x_{n-l}) \right) u_{n-k}. \tag{2}$$

Put

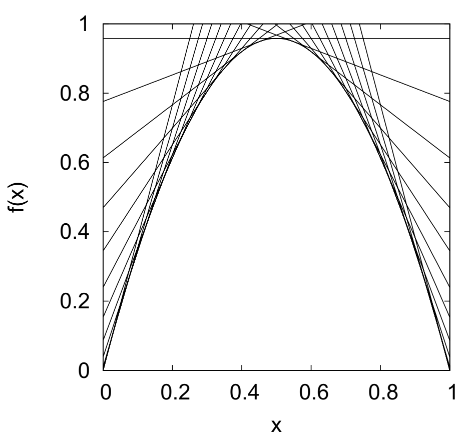

Fig. 1 Tangent lines of the logistic map $f(x) = ax(1 - x)$ with $a = 3.832$. The graph of f is the envelope of its tangent lines

$$y_{n+1} = g_{x_n} \circ g_{x_{n-1}} \circ \ldots \circ g_{x_0}(x_0), \tag{3}$$

and

$$\Delta_n = \sum_{k=0}^{n} \left(\prod_{l=0}^{k-1} f'(x_{n-l}) \right) u_{n-k}. \tag{4}$$

Equation (2) is rewritten as

$$x_{n+1} = y_{n+1} + \epsilon \Delta_n. \tag{5}$$

Equivalently, we can write down a pair of recurrence formulae

$$y_{n+1} = g_{x_n}(y_n), \tag{6}$$

$$\Delta_{n+1} = u_{n+1} + f'(x_{n+1})\Delta_n, \tag{7}$$

where $y_0 = x_0$ and $\Delta_0 = u_0$. Note that we can eliminate x_n in the right-hand side of Eq. (6) and x_{n+1} in the right-hand side of Eq. (7) using Eq. (5). Namely, given a realization of the external noise $(u_n)_{n\geq 0}$, the time evolution of (y_n, Δ_n) is closed. However, the values of y_n or Δ_n can sometimes become very large as we will see in examples in Sect. 3. Hence, cancellation of significant digits can occur when numerically calculating the right-hand side of Eq. (5). To avoid this matter, we use Eqs. (1) and (6) in the numerical simulation below. It is performed in the double precision floating point number. Note also that the noise u_n occasionally bring x_{n+1} outside of I even when ϵ is very small. To orient x_n toward I at the successive time steps, we extend f to a map from $(-\infty, \infty)$ to $(-\infty, \infty)$ by defining $f(x) = -x$ for $x < 0$ and $f(x) = x - 1$ for $x > 1$. However, as long as ϵ is sufficiently small so that x_n rarely goes out of I, this modification of the map has negligible influence on the statistical behavior of y_n studied in Sect. 3.

Equation (5) can be interpreted as follows. y_n is an internal state originating from the endo-perspective in which one only uses the local information of the map f, namely, the tangent line at x_n, to update y_n, and at the same time uses it blindly, namely, one cannot access the value of the external noise u_n when calculating y_{n+1}. Thus, Eq. (6) represents a kind of "leap in the dark". This is the meaning of the endo-perspective which is consistent with that of Matsuno [14] and Gunji [4]. On the other hand, Δ_{n+1} records the accumulated influence of the external noise, which compensates the internal fluctuation represented by y_n and gives rise to the state of the system x_n.

Let us calculate initial a few values of y_n to see how y_n deviates from x_n. By definition, we have $y_0 = x_0$. y_1 is obtained as

$$\begin{aligned} y_1 &= g_{x_0}(y_0) \\ &= f(x_0) + f'(x_0)(y_0 - x_0) \\ &= f(x_0) = x_1 - \epsilon u_0. \end{aligned} \tag{8}$$

For y_2, we have

$$\begin{aligned} y_2 &= g_{x_1}(y_1) \\ &= f(x_1) + f'(x_1)(y_1 - x_1) \\ &= x_2 - \epsilon u_1 - \epsilon f'(x_1) u_0, \end{aligned} \tag{9}$$

where we used Eqs. (8) and (1) in the last equality.

3 Results

In this section, we study the statistical behavior of y_n. Two nonlinear one-dimensional maps are considered. The first one is the logistic map, and the second is the BZ map. For each map, our main concern is the parameter regime where a noise-induced phenomenon is observed.

For the logistic map $f(x) = ax(1 - x)$, we choose $a = 3.832$. This value of a lies in the period three window. Hence, without noise, x_n converges to a periodic solution with period three and has a negative Lyapunov exponent λ. However, when noise is added and its strength is sufficiently large, the behavior of x_n qualitatively changes, namely λ becomes positive. This phenomenon is called Noise-Induced-Chaos (NIC) [3, 15]. It was suggested that the noise makes chaotic aperiodic solutions of Lebesgue measure zero visible. The Lyapunov exponent λ for a given trajectory (x_n) is calculated according to the formula

$$\lambda = \lim_{N\to\infty} \frac{1}{N} \sum_{n=0}^{N-1} \ln \left| f'(x_n) \right|. \tag{10}$$

Figure 2 shows the initial 1000 time steps of the absolute values of x_n and y_n from a random initial condition for two different noise strength: (a) $\epsilon = 0.002$, and (b) $\epsilon = 0.01$. For $\epsilon = 0.002$, $\lambda \approx -0.047$, and x_n is contracting on average. On the other hand, for $\epsilon = 0.01$, $\lambda \approx 0.401$, and x_n is expanding on average. In the latter case, $|y_n|$ diverges as $n \to \infty$ as shown in Fig. 2b corresponding to the positive Lyapunov exponent for (x_n). In the former case, we observe an intermittent behavior for $|y_n|$ (Fig. 2a). $|y_n|$ is also contracting on average corresponding to the negative Lyapunov exponent for (x_n). However, if $|f'(x_n)|$ larger than 1 accumulates locally in time, then a large burst of $|y_n|$ can occur. Here, we define a burst as a maximal subsequence of (y_n) with consecutive indices such that $y_n \notin I$ for all y_n in the subsequence.

Two statistical properties of the bursts in $|y_n|$ are shown in Fig. 3. The first one is the distribution of the maximum value $|y|_{\max}$ of $|y_n|$ in each burst. In Fig. 3a, we plot the probability distribution $P(z)$ of $z = \ln(|y|_{\max})$. It seems that $P(z)$ can be well-approximated by an exponential form $\exp(-\mu z)$ with $\mu = -0.019$ for sufficiently large values of z. The value of μ is calculated by applying the maximum likelihood estimator of the exponential distribution to the numerical data which

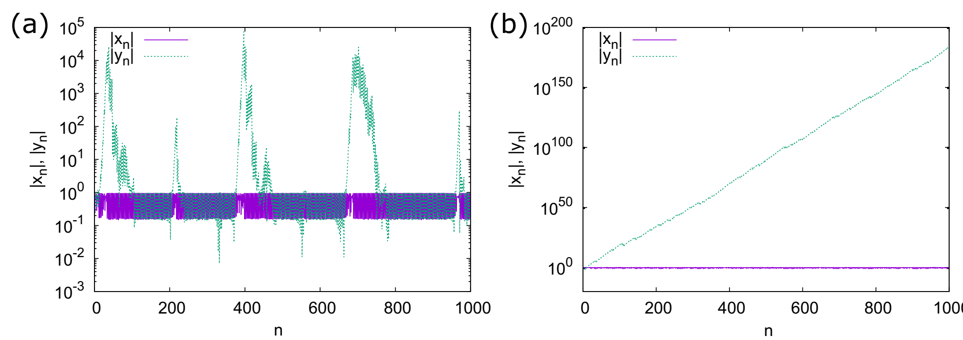

Fig. 2 Time series of $|x_n|$ and $|y_n|$ for the logistic map $f(x) = ax(1 - x)$ with additive noise, where $a = 3.832$. **a** Noise strength $\epsilon = 0.002$. **b** Noise strength $\epsilon = 0.01$. See the main text for details

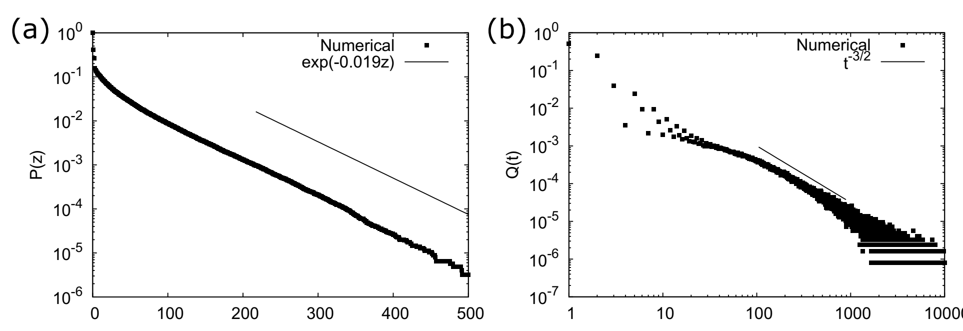

Fig. 3 Two statistical properties of the bursts in $|y_n|$ for the logistic map with additive noise of strength $\epsilon = 0.002$. **a** The probability distribution $P(z)$ of $z = \ln(|y|_{\max})$. **b** The probability distribution of the burst duration. Both numerical distributions are obtained from a single trial of length 10^8

are larger than a give threshold (which is the left-most value of z for the line in Fig. 3a corresponding to $\exp(-\mu z)$). The probability distribution of $|y|_{\max}$ is obtained as

$$P\left(|y|_{\max}\right) \propto P(z)\frac{\mathrm{d}z}{\mathrm{d}|y|_{\max}} = \frac{1}{|y|_{\max}^{1+\mu}}. \tag{11}$$

Thus, the distribution of $|y|_{\max}$ follows a power-law with exponent $-1-\mu$.

The second one is the probability distribution of the burst duration (Fig. 3b). There seems to be a range of the burst duration such that the distribution follows a power-law with exponent $-3/2$.

Thus, we found two statistical properties following a power-law in the intermittent burst behavior of $|y_n|$ for the logistic map with additive noise of strength slightly below the onset of NIC.

The next example is the BZ map given by

$$f(x) = \left(-(0.125-x)^{\frac{1}{3}} + a\right)\exp(-x) + b,$$

for $0 \leq x < 0.125$,

$$f(x) = \left((x-0.125)^{\frac{1}{3}} + a\right)\exp(-x) + b,$$

for $0.125 \leq x < 0.3$, and

$$f(x) = c\left(10x\exp\left(-\frac{10}{3}x\right)\right)^{19} + b,$$

for $0.3 \leq x \leq 1$, where $a = 0.50607357$, $b = 0.023228$, and $c = 0.121205692$. For the chosen values of the parameters, the Lyapunov exponent λ for (x_n) is positive (data not shown). However, if an appropriate level of noise is added, λ becomes negative. This noise-induced phenomenon is known as Noise-Induced Order (NIO) [13]. It was suggested that the uniform coarse graining of the phase space induced by noise destroys the nonuniform Markov partition of the deterministic chaotic dynamics of the BZ map and contributes to contraction of different solutions. Here, we take $\epsilon = 0.001$ and obtain $\lambda \approx -0.049$.

Figure 4 shows the initial 10000 time steps of the absolute values of x_n and y_n from a random initial condition for $\epsilon = 0.001$. As in the case of the logistic map with a negative λ seen above, $|y_n|$ exhibits an intermittent burst behavior.

The distribution of $z = \ln\left(|y|_{\max}\right)$ for the BZ map is shown in Fig. 5a. This indicates that $P(|y|_{\max}) \propto \frac{1}{|y|_{\max}^{1+\mu}}$ with $\mu \approx -0.12$ for sufficiently large values of $|y|_{\max}$. In Fig. 5b, the probability distribution of the burst duration is shown. As in the case of the logistic map above (Fig. 3b), the distribution seems to follow a power-law with exponent $-3/2$ for a moderate range of the burst duration.

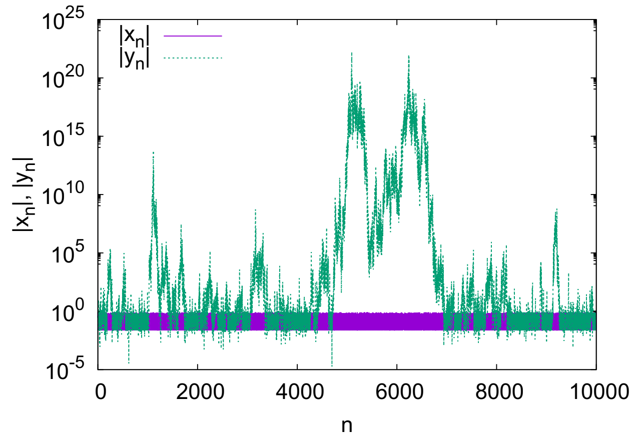

Fig. 4 Time series of $|x_n|$ and $|y_n|$ for the BZ map with additive noise of strength $\epsilon = 0.001$

We have observed that the intermittent burst behaviors of $|y_n|$ for the logistic map and the BZ map have common features. In both cases, the distribution of $|y|_{\max}$ can be approximated by a power-law and the distribution of the burst duration has a range within which it is approximated by the power-law with exponent $-3/2$. This suggests that there is a common mechanism yielding the observed behaviors in the different maps. In the following, we show that the mechanism behind both maps is the generalized multiplicative stochastic processes [18].

A generalized multiplicative stochastic process can be given by

$$w_{n+1} = e^{g\left(w_n, a_n, b_n\right)} a_n w_n, \tag{12}$$

where $a_n > 0$ and b_n are realizations of certain sequences of random variables. Here, we assume that they are realizations of i.i.d. processes. We also assume that the function g satisfies the following two conditions:

$$g(w, a, b) \to 0 \text{ as } w \to \infty, \tag{13}$$

and

$$g(w, a, b) \to \infty \text{ as } w \to 0. \tag{14}$$

Equation (13) implies that the behavior of $\ln w$ can be approximated by a random walk with average drift velocity $\langle \ln a \rangle$ when w is large. On the other hand, Equation (14) means that w is repelled from zero. Thus, the dynamics of $\ln w_n$ can be understood in terms of a constrained biased random walk with a repulsive force from $-\infty$. To proceed further, we need the following three conditions:

$$\partial g/\partial w \to 0 \text{ as } w \to \infty, \tag{15}$$

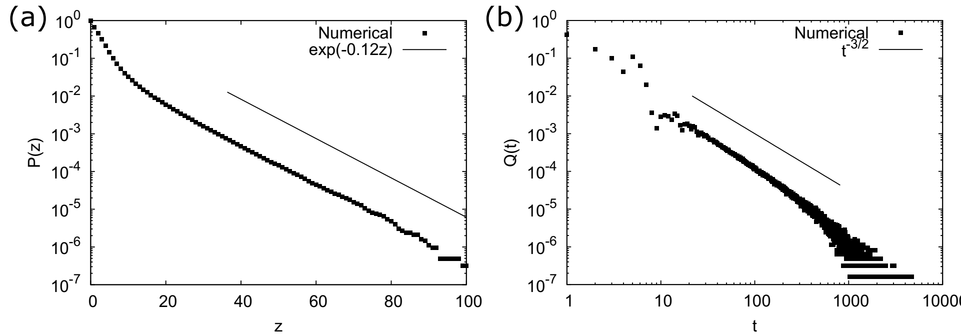

Fig. 5 Two statistical properties of the bursts in $|y_n|$ for the BZ map with additive noise of strength $\epsilon = 0.001$. **a** The probability distribution $P(z)$ of $z = \ln(|y|_{\max})$. **b** The probability distribution of the burst duration. Both numerical distributions are obtained from a single trial of length 10^8

$$\langle \ln a \rangle < 0, \tag{16}$$

and there exists $\mu > 0$ such that

$$\langle a^\mu \rangle = 1. \tag{17}$$

Equation (15) is a regularity condition to derive the Wiener–Hopf integral equation leading to the power-law distribution of w. Equation (16) states that the average drift velocity of $\ln w_n$ is negative, but Eq. (17) implies that the drift velocity can becomes positive with a positive probability. Thus, one can expect that (w_n) exhibits an intermittent burst behavior. A burst can be defined as a maximal time interval such that w_n keeps exceeding a give threshold. When these three additional conditions also hold, it is shown that the probability distribution of w follows a power-law for sufficiently large w [18]

$$P(w) \propto \frac{1}{w^{1+\mu}}. \tag{18}$$

It is known that Eq. (18) also holds when w is replaced by the maximum value of w in each burst.

A general theoretical calculation of the first return time of general biased random walks [1] suggests that the probability distribution of the burst duration t for sufficiently large t can be given by

$$Q(t) \propto t^{-\frac{3}{2}} e^{-\frac{t}{t_c}}, \tag{19}$$

where t_c is the characteristic duration for which the power-law $t^{-\frac{3}{2}}$ breaks down. Here, we do not pursue how we can obtain the value of t_c.

Now, let us explain the connection between the statistical behavior of $|y_n|$ and the theory of generalized multiplicative processes just described. By Eq. (5) and $|x| \le 1$, $|y| \propto |\Delta|$ for large $|y|$. Hence, the statistical properties of large $|y|$ can be deduced from that of $|\Delta|$. By taking the absolute value of both sides of Eq. (7) and defining

$$g(w, a, b) = \ln \left| 1 + \frac{b}{aw} \right|, \tag{20}$$

we obtain

$$|\Delta_{n+1}| = e^{g(\Delta_n, f'(x_{n+1}), u_{n+1})} |f'(x_{n+1})| |\Delta_n|. \tag{21}$$

We can regard that $|\Delta|$ follows a generalized multiplicative stochastic process Eq. (12) with $a = |f'(x)|$, $b = u$ except the signs of $f'(x)$ and Δ. Indeed, the signs of a and w in the right-hand side of Eq. (20) have negligible influence on the value of g for $|\Delta| \to 0$ and $|\Delta| \to \infty$. In the following, we proceed as if x_n and x_{n+1} are independent. This seems to be valid approximately for the numerical results in this section since the correlation in (x_n) expected to damp rapidly in the vicinity of NIC or NIO due to inherent chaotic dynamics within them. From Eq. (20), one can see that Eqs. (13) and (14) are satisfied for almost all values of x and u in our two examples in this section. Further, checking Eq. (15) is straightforward. Equation (16) is equivalent to having a negative Lyapunov exponent since $\lambda = \langle \ln a \rangle$. For the values of μ in Figs. 3a and 5a, we approximately have Eq. (17), respectively. Indeed, we have $\langle |f'(x)|^\mu \rangle \approx 1.000$ for the former and $\langle |f'(x)|^\mu \rangle \approx 1.045$ for the latter.

In summary, both the intermittent burst behavior for the logistic map and the BZ map shown in this section can be understood from the theory of generalized multiplicative stochastic processes.

4 Discussion

Recently, the notion of endo-perspective was implemented as a probabilistic scheme for a model of consciousness [7] or represented by an algebraic structure to capture the trilemma associated with free will [6]. In this paper, we revisit it in the context of complex dynamical behavior of one-dimensional maps [8]. Given a parameterized family of one-dimensional maps $\{f_a : I \to I\}_{a \in A}$ where A is the set of parameter values, Haruna and Gunji [8] expressed the endo-perspective by a contraction mapping T which gives rise to a time evolutionary rule of the parameter value from a single datum $(x_n, f_a(x_n))$ at time step n, extending the idea of the formal model of internal measurement proposed by Gunji et al. [5]. In this model, the change of the parameter value is driven by the fractal structure of the fixed point of T. Thus, it can be irregular but deterministic in principle. On the other hand, noise plays an essential role in the implementation of the endo-perspective proposed in this paper.

Wang et al. [20] showed that when there exist a periodic attractor and a chaotic saddle in the phase space of a given deterministic dynamical system, adding an appropriate level of noise can lead to an intermittent behavior due to switching between the periodic attractor and the chaotic saddle through noise. It is reported that the duration of the laminar phase follows an exponential distribution. This is called noise-induced on-off intermittency. Such mechanism could be inherent in the two maps studied in this paper. However, we found an intermittent behavior for the internal state y_n not for x_n such that both the distribution of the burst amplitude and the distribution of the burst duration follow power-laws. The explanation in terms of the theory of generalized multiplicative stochastic processes given at the end of Sect. 3 suggests that the intermittency of y_n is not dependent on the fine geometric structure of the phase space trajectories of x_n.

The framework proposed in this paper could be used in reservoir computing [10, 12, 16], particularly for investigating the memory capacity of nonlinear dynamical systems [11]. To calculate the memory capacity of a given dynamical system, one adds an i.i.d. input to the system and estimates the correlations between the past inputs and the present state of the system. Haruna and Nakajima [9] showed that the memory capacity of random recurrent neural networks is optimized slightly below the edge of chaos, where the intermittent burst behavior would be expected to be observed when our framework is suitably extended to higher-dimensional systems and applied. In addition to the maximum Lyapunov exponent, the exponent μ could be useful for characterizing the behavior of the system around optimality.

In conclusion, the embedding of the endo-perspective into the one-dimensional maps with additive noise proposed in this paper reveals a universal mechanism based on generalized multiplicative stochastic processes for observing power-laws under the condition that the Lyapunov exponent of noisy trajectories is negative but deterministic chaotic dynamics is latent.

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References

1. Artime O, Khalil N, Toral R, San Miguel M (2018) First-passage distributions for the one-dimensional Fokker-Planck equation. Phys Rev E 98:042143
2. Clauset A, Shalizi CR, Newman MEJ (2009) Power-law distributions in empirical data. SIAM Rev 51:661–703
3. Crutchfield JP, Farmer JD, Huberman BA (1982) Fluctuations and simple chaotic dynamics. Phys Rep 92:45–82
4. Gunji YP (2004) Protocomputing and ontological measurement. University of Tokyo Press, Tokyo (**in Japanese**)
5. Gunji YP, Ito K, Kusunoki Y (1997) Formal model of internal measurement: Alternate changing between recursive definition and domain equation. Phys D 110:289–312
6. Gunji YP, Nakamura K, Minoura M, Adamatzky A (2020) Three types of logical structure resulting from the trilemma of free will, determinism and locality. Biosystems 195:104151
7. Gunji YP, Shinohara S, Haruna T, Basios V (2017) Inverse Bayesian inference as a key of consciousness featuring a macroscopic quantum logical structure. Biosystems 152:44–65
8. Haruna T, Gunji YP (2006) A generalization of formal model of internal measurement: a construction on one-dimensional maps. Int J Unconv Comput 2:231–248
9. Haruna T, Nakajima K (2019) Optimal short-term memory before the edge of chaos in driven random recurrent networks. Phys Rev E 100:062312
10. Jaeger H (2001) The "echo state" approach to analysing and training recurrent neural networks. GMD-Report 148, GMD-German National Research Institute for Computer Science
11. Jaeger H (2002) Short term memory in echo state networks (2002). GMD-Report 152, GMD-German National Research Institute for Computer Science
12. Maass W, Natschläger T, Markram H (2002) Real-time computing without stable states: a new framework for neural computation based on perturbations. Neural Comput 14:2531–2560
13. Matsumoto K, Tsuda I (1983) Noise-induced order. J Stat Phys 31:87–106
14. Matsuno K (1989) Protobiology: physical basis for biology. CRC Press, Boca Raton
15. Mayer-Kress G, Haken H (1981) The influence of noise on the logistic model. J Stat Phys 26:149–171
16. Nakajima K, Fischer I (eds) (2021) Reservoir computing: theory, physical implementations, and applications. Springer Singapore, Singapore
17. Sornette D (2006) Critical phenomena in natural sciences, 2nd edn. Springer, Berlin
18. Sornette D, Cont R (1997) Convergent multiplicative processes repelled from zero: power laws and truncated power laws. J Phys I Fr 7:431–444
19. Thurner S, Hanel R, Klimek P (2018) Introduction to the theory of complex systems. Oxford University Press, Oxford
20. Wang X, Lai YC, Lai CH (2006) Characterization of noise-induced strange nonchaotic attractors. Phys Rev E 74:016203