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ORIGINAL ARTICLE

Solutions to the Magnetic Ginzburg–Landau Equations Concentrating on Codimension-2 Minimal Submanifolds

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Abstract

We consider the magnetic Ginzburg–Landau equations in a compact manifold N

$$\begin{cases} -\varepsilon^2 \Delta^A u = \frac{1}{2}(1-|u|^2)u, \\ \varepsilon^2 d^* dA = \langle \nabla^A u, iu \rangle. \end{cases}$$

Here $u\colon N \to \mathbb{C}$ and A is a 1-form on N. We discuss some recent results on the construction of solutions exhibiting concentration phenomena near prescribed minimal, codimension 2 submanifolds corresponding to the *vortex set* of the solution. Given a codimension-2 minimal submanifold $M \subset N$ which is also oriented and non-degenerate, we construct a solution $(u_\varepsilon, A_\varepsilon)$ such that u_ε has a zero set consisting of a smooth surface close to M. Away from M we have

$$u_\varepsilon(x) \to \frac{z}{|z|}, \quad A_\varepsilon(x) \to \frac{1}{|z|^2}(-z_2 dz^1 + z_1 dz^2), \quad x = \exp_y(z^\beta \nu_\beta(y)) \tag{1}$$

as $\varepsilon \to 0$, for all sufficiently small $z \neq 0$ and $y \in M$. Here, $\{\nu_1, \nu_2\}$ is a normal frame for M in N. These results improve, by giving precise quantitative information, a recent construction by De Philippis and Pigati (arXiv:2205.12389, 2022) who built solutions for which the concentration phenomenon holds in an energy, measure-theoretical sense. In addition, we consider the non-compact case $N = \mathbb{R}^4$ and the special case of a two-dimensional minimal surface in $\mathbb{R}^3$, regarded as a codimension 2 minimal submanifold in $\mathbb{R}^4$, with finite total curvature and non-degenerate. We construct a solution $(u_\varepsilon, A_\varepsilon)$ which has a zero set consisting of a smooth 2-dimensional surface close to $M \times \{0\} \subset \mathbb{R}^4$. Away from the latter surface we have $|u_\varepsilon| \to 1$ and asymptotic behavior as in (1).

Keywords Minimal submanifolds · Ginzburg–Landau · Yang–mills–higgs

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Dedicated to Carlos Kenig on the occasion of his 70th birthday.

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1 Introduction

Let (N, g) be a closed Riemannian manifold of dimension $n \geq 3$. For $\varepsilon > 0$, the magnetic Ginzburg–Landau energy on N is given by

$$E_\varepsilon(u, A) = \frac{1}{2}\int_N |\nabla^A u|^2 + \varepsilon^2 |dA|^2 + \frac{1}{4\varepsilon^2}(1-|u|^2)^2, \tag{1.1}$$

where A is the magnetic potential, represented as a 1-form $A \in \Omega^1(N)$. In (1.1) we denoted with d the exterior derivative and $\nabla^A := d - iA$. Explicitly, the quantities involved are

$$\begin{aligned} |\nabla^A u|^2 &= \sum_{j,k=1}^{n} g^{ij}(\partial_j u - iA_j u)(\partial_k \bar{u} + iA_k \bar{u}), \\ |dA|^2 &= \frac{1}{2}\sum_{j,k,s,t=1}^{n} g^{ks} g^{jt}(\partial_k A_j - \partial_j A_k)(\partial_s A_t - \partial_t A_s), \end{aligned}$$

where g is the metric of the manifold N. The corresponding equations are given by

$$\begin{cases} -\varepsilon^2 \Delta^A u = \frac{1}{2}(1-|u|^2)u \\ \varepsilon^2 d^* dA = \langle \nabla^A u, iu\rangle \end{cases} \quad \text{on } N, \tag{1.2}$$

where d^* is the L^2-adjoint of d and $\langle z, w\rangle = \mathrm{Re}(z\bar{w})$. Explicitly, the operators in (1.2) read

$$\begin{aligned} -\Delta^A u &= -\frac{1}{\sqrt{\det g}}(\partial_j - iA_j)\left[\sqrt{\det g}\, g^{jk}(\partial_k - iA_k)u\right], \\ d^* dA &= -\frac{1}{\sqrt{\det g}} g_{jk}\partial_i \left(\sqrt{\det g}\, g^{li} g^{tk}(\partial_l A_t - \partial_t A_l)\right) dx^j, \end{aligned}$$

where we used Einstein's summation convention on repeated indices. Energy (1.1) models phenomena of superconductivity in presence of a magnetic field, where the regions in which $|u| \approx 1$ represent portions of the material in superconducting state, while where $|u| \approx 0$ the material is in its normal state. This phase transition shares many similarities to that famously described by the Allen–Cahn equation

$$-\varepsilon^2 \Delta u = (1-u^2)u. \tag{1.3}$$

In particular, both models exhibit *concentration* for solutions as the scaling parameter $\varepsilon \to 0$. This means that the energy densities of solutions concentrate their mass (as measures) around a minimal submanifold (more generally, a rectifiable stationary varifold). Such limiting object has codimension 1 in the Allen–Cahn case and codimension 2 in the Ginzburg–Landau case.

A natural question to ask is wether or not the converse holds true.

Question 1 Given a minimal submanifold M of some ambient space N can we construct a family of solutions concentrating around M?

If N is a compact manifold and $M \subset N$ a separating hypersurface, Pacard and Ritoré [26] proved that the answer is positive for the Allen–Cahn equation (1.3) under the assumption of non-degeneracy of M. In the case $M = \mathbb{R}^3$, a similar result was found in [11] associated to a non-degenerate minimal surface without boundary, complete with final total curvature. In this paper, we review very recent results parallel to those in [11, 26] in the codimension 2 case for the Ginzburg–Landau equation (1.2). We present a detailed summary of the proofs whose full versions can be found in our preprints [1, 2].

Energy (1.1) is invariant under the action of the unitary group $U(1)$, namely

$$E(u, A) = E(\mathrm{G}_\gamma(u, A)), \quad \text{for any } \gamma \in C^\infty(N), \text{ where } \mathrm{G}_\gamma(u, A) := (ue^{i\gamma}, A + d\gamma). \tag{1.4}$$

In the case $N = \mathbb{R}^2$ with $\varepsilon = 1$ it is well known that there is a unique (up to gauge transformations) degree 1 radial solution $U_0 = (u_0, A_0)^T$, with

$$u_0(\zeta) = f(r)e^{i\theta}, \quad A_0(\zeta) = a(r)d\theta, \quad \zeta = re^{i\theta} \in \mathbb{C} \simeq \mathbb{R}^2 \tag{1.5}$$

for which $f(0) = a(0) = 0$. As established in [15, 35], the solution U_0 is linearly stable. The asymptotic profile as $r \to \infty$ is given by

$$f(r) = 1 + O(e^{-r}), \quad a(r) = 1 + O(e^{-r}),$$

see for instance [4, 29].

We find solutions $U_\varepsilon = (u_\varepsilon, A_\varepsilon)$ of (1.2) concentrating as $\varepsilon \to 0$ around a 2-codimensional minimal submanifold $M \subset N$. These solutions look like ε-scalings of U_0 in a region close to M in the following sense: let $\{\nu_1, \nu_2\}$ be an orthonormal basis for $T^\perp M$. We describe a neighbourhood of M in N by Fermi coordinates

$$x = X(y, z) = \exp_y(z^1\nu_1(y) + z^2\nu_2(y)), \quad y \in M, \ |z| < \tau \tag{1.6}$$

for some $\tau > 0$. We construct solutions $U_\varepsilon(x) = (u_\varepsilon(x), A_\varepsilon(x))$ with asymptotic behaviour given by

$$u_\varepsilon(x) \approx f\left(\frac{z}{\varepsilon}\right)\frac{z}{|z|}, \quad A_\varepsilon(x) \approx a\left(\frac{z}{\varepsilon}\right)\frac{1}{|z|^2}(-z_2dz^1 + z_1dz^2).$$

We do this in two different settings.

1.1 The Compact Case

We consider first a closed, n-dimensional manifold N and a closed $(n-2)$-dimensional minimal submanifold $M \subset N$. We say that a minimal manifold $M \subset N$ is *admissible* if

(H) M is the boundary of a $(n-1)$-dimensional, oriented, embedded submanifold $B^{n-1} \subset N^n$.

Recall that the Jacobi operator of M is the second variation of the area functional, which explicitly is given by $\mathcal{J}[h] = (\mathcal{J}^1[h], \mathcal{J}^2[h])$, where

$$\mathcal{J}^\gamma[h] = \Delta_M h^\gamma + \sum_{i,j=1}^{n-2}\sum_{\beta=n-1}^{n}\left(R_{i\beta i\gamma} + A^\beta_{ij}A^\gamma_{ij}\right)h^\beta, \quad \gamma = 1, 2.$$

In the above expression R is the curvature tensor and A is the second fundamental form. We require that M is non-degenerate in the sense that the Jacobi operator has trivial bounded kernel, namely

$$h \in L^\infty(M), \quad \mathcal{J}[h] = 0 \implies h = 0. \tag{1.7}$$

Assumption (1.7) of non-degeneracy and Fredholm alternative for elliptic operators imply the following result.

Lemma 1.1 *Let* $f \in C^{0,\gamma}(M, \mathbb{R}^2)$. *Then the system*

$$\mathcal{J}(h) = f \quad \text{on } M$$

admits a solution $h = \mathcal{H}(f)$ *satisfying*

$$\|h\|_{C^{2,\gamma}(M)} \leq C\|f\|_{C^{0,\gamma}(M)}.$$

Our first result is the following.

Theorem 1 ([2]) *Let* (N, g) *be a closed* n*-dimensional Riemannian manifold and let* $M \subset N$ *be an admissible, non-degenrate, codimension-2 minimal submanifold. Then there is* $\delta > 0$ *such that for* $\sigma \in (0, 1)$ *and all sufficiently small* $\varepsilon > 0$ *there exists a solution* $(u_\varepsilon, A_\varepsilon)$ *to* (1.2) *which as* $\varepsilon \to 0$ *satisfies*

$$\begin{aligned} u_\varepsilon(x) &= u_0\left(\frac{z - \varepsilon^2 h_0(y)}{\varepsilon}\right) + O\left(\varepsilon^2 e^{-\frac{\sigma|z|}{\varepsilon}}\right), \\ A_\varepsilon(x) &= A_0\left(\frac{z - \varepsilon^2 h_0(y)}{\varepsilon}\right) + O\left(\varepsilon^2 e^{-\frac{\sigma|z|}{\varepsilon}}\right), \end{aligned} \qquad |z| < \delta, \tag{1.8}$$

for all points $x = X(y, z)$ *of the form* (1.6) *and where* h_0 *is a smooth function on* M. *Moreover,* $|u_\varepsilon| \to 1$ *uniformly on compact subsets of* $N \setminus M$.

1.2 The Non-compact Case

Consider the class of complete, minimal surfaces embedded in $\mathbb{R}^3$ and with finite total curvature, that is

$$\int_M |K| < \infty,$$

where K is the Gaussian curvature of M. It is known, see [18, 25, 33], that outside a large cylinder a general manifold M in this class decomposes into the disjoint union of m unbounded connected components $M_1, \ldots, M_m$, called its ends, which are asymptotic to either catenoids or planes with parallel axes. After a rotation, we can choose coordinates $x = (x_1, x_2, x_3) = (x', x_3)$ in $\mathbb{R}^3$ and a large number R_0 such that

$$M \setminus \{|x'| < R_0\} = \bigcup_{k=1}^m M_k.$$

Each end M_k can be represented by

$$M_k = \{x \in \mathbb{R}^3 : |x'| \geq R_0, x_3 = F_k(x')\},$$

where

$$F_k(x') = a_k \log|x'| + b_k + b_{ik}\frac{x_i}{|x'|^2} + O(|x'|^{-2}), \quad |x'| \geq R_0, \tag{1.9}$$

for some constants a_k, b_k, b_{ik}, such that the coefficients a_k are ordered and balanced, in the sense that

$$a_1 \leq a_2 \leq \cdots \leq a_m, \quad \sum_{k=1}^m a_k = 0.$$

Given M in this class we find solutions to (1.2) when $N = \mathbb{R}^4$ concentrating around $M \times \{0\}$. As for the compact case a form of non-degeneracy is needed. However, the symmetries of the immersion of $M \times \{0\}$ in $\mathbb{R}^4$ automatically generate bounded Jacobi fields, given by

$$z_j = \begin{pmatrix} z_j \\ 0 \end{pmatrix}, \quad j = 0, 1, 2, 3, \quad z_4 = \begin{pmatrix} 0 \\ 1 \end{pmatrix}, \tag{1.10}$$

where

$$z_0(y) = \nu(y) \cdot (-y_2, y_1, 0, 0), \quad z_i(y) = \nu(y) \cdot e_i \quad i = 1, 2, 3,$$

being ν a choice of unit normal vector field on M and $y \in M$. Thus, the non-degeneracy condition in the non-compact case becomes

$$h \in L^\infty(M) \text{ and } \mathcal{J}(h) = 0 \implies h \in \text{span}\{z_0, z_1, z_2, z_3, z_4\}.$$

We need a further geometrical condition before stating the theorem. Let $(\lambda_1, \ldots, \lambda_m)$ be a balanced, ordered vector of real numbers

$$\lambda_1 \leq \lambda_2 \leq \cdots \leq \lambda_m, \quad \sum_{i=1}^m \lambda_i = 0 \tag{1.11}$$

and assume that, for some $\sigma \in (0, 1)$ to be determined

$$\lambda_{k+1} - \lambda_k > 4/\sigma \quad \text{if} \quad a_{k+1} = a_k. \tag{1.12}$$

We are now ready to state the main result in the non-compact case.

Theorem 2 ([1]) *Let M be a complete, minimal surface embedded in $\mathbb{R}^3$ and with finite total curvature and whose ends are represented by (1.9). Then there is a number $\delta > 0$ such that for all sufficiently small $\varepsilon > 0$, $\sigma \in (0, 1)$ and $\lambda = (\lambda_1, \ldots, \lambda_m)$ satisfying conditions (1.11) and (1.12) there exists a solution $(u_\varepsilon, A_\varepsilon)$ to (1.2) with $N = \mathbb{R}^4$ which as $\varepsilon \to 0$ satisfies*

$$\left.\begin{aligned} u_\varepsilon(x) &= u_0\left(\frac{z - \varepsilon h_0(y)}{\varepsilon}\right) + O\left(\varepsilon^2 e^{-\frac{\sigma|z|}{\varepsilon}}\right), \\ A_\varepsilon(x) &= A_0\left(\frac{z - \varepsilon h_0(y)}{\varepsilon}\right) + O\left(\varepsilon^2 e^{-\frac{\sigma|z|}{\varepsilon}}\right), \end{aligned}\right\} \quad |z| < \delta,$$

for all points

$$x = y + z_1 \nu(y) + z_2 \mathbf{e}_4, \quad y \in M, \quad |z| < \delta.$$

Here, the smooth function $h_0 : M \to \mathbb{R}^2$ satisfies

$$h_0(y) = \left((-1)^k \lambda_k \log|y'|,\ 0\right) + O(\varepsilon), \quad y = (y', y^3, 0) \in M_k \times \{0\}. \tag{1.13}$$

Besides, $|u_\varepsilon| \to 1$ uniformly in compact subsets of $\mathbb{R}^4 \setminus M$.

Remark 1 Consider the family $u_\varepsilon(x)$ predicted by Theorem 1. Using (1.8) we see that locally around M the equation $u_\varepsilon = 0$ has the form

$$u_0\left(\frac{z - \varepsilon^2 h_0(y)}{\varepsilon}\right) + \theta\left(\frac{z}{\varepsilon}, y\right) = 0,$$

where θ is a regular function. By the implicit function theorem and the fact that $\partial_z u_\varepsilon(0) \neq 0$ we find that locally around M the zero level set of u_ε can be parametrized by

$$z = \varepsilon^2 h_0(y) + O(\varepsilon^3), \quad y \in M. \tag{1.14}$$

Also, using that u_ε doesn't vanish away from M we find that the parametrization (1.14) defines the entire 0 level set. The same argument can also be applied in the setting of Theorem 2.

There is a rich literature devoted to the connection between critical points of Allen–Cahn (resp. Ginzburg–Landau) energy and minimal submanifolds of codimension 1 (resp. 2). Concentration phenomena on minimal hypersurfaces for local minimizers of the Allen–Cahn energy have been studied in [19, 23, 24, 34], then generalized to the case of general critical points in [16] with the limiting object being a stationary varifold (a measure theoretic, non-regular generalization of minimal manfiold). The connection between solutions to the inhomogeneus Allen–Cahn equation and constant-mean-curvature hypersurfaces has been studied in [31]. The reverse problem, namely the construction of concentrating families of solution has been explored, among other works, in [1, 2, 10–12, 26]. This concentration phenomenon has also been used as a PDE alternative to the Almgren–Pitts min-max approach [22, 28, 32] for the construction of minimal (resp. CMC) submanifolds of codimension 1, see [3, 6, 13, 14].

Similar results in the context of complex-valued Ginzburg–Landau equations have been obtained, among others, by [5, 7, 8, 17, 20, 21, 27, 30].

Recently, De Philippis and Pigati [9] established a result that complements the findings in [27] for the scenario of a non-degenerate codimension 2 minimal submanifold. Their method, based on variational techniques, does not provide detailed asymptotic information. However, they have successfully resolved the more challenging case of Ginzburg–Landau equations where no induced magnetic field is present. Our techniques do not extend to cover that particular case.

1.3 The Linearized Operator

We start by defining an inner product on the pairs $W = (u, A)$

$$\begin{aligned}\langle W_1, W_2\rangle &:= \int_N \begin{pmatrix} u_1 \\ A_1 \end{pmatrix} \cdot \begin{pmatrix} u_2 \\ A_2 \end{pmatrix} \\ &= \int_N \langle u_1, u_2\rangle + \varepsilon^2 A_1 \cdot A_2,\end{aligned}$$

where $\langle u_1, u_2\rangle = \mathrm{Re}(u_1 \bar{u}_2)$ and $A_1 \cdot A_2 = g^{ij}(A_1)_i(A_2)_j$, being g the metric on N. Let

$$S(W) = \begin{pmatrix} -\varepsilon^2 \Delta^A u - \frac{1}{2}(1 - |u|^2)u \\ \varepsilon^2 d^* dA - \langle \nabla^A u, iu\rangle \end{pmatrix}.$$

If W is a solution to (1.2), i.e. $S(W) = 0$, gauge-invariance (1.4) implies the existence of an infinite dimensional subspace of the kernel of the linearised operator $S'(W)$ around a pair W. It's easy to check that the gauge-kernel is given by the range of Θ_W, where

$$\Theta_W[\gamma] = (iu\gamma, d\gamma).$$

It is also direct to check that L^2-orthogonality with the space generated by $\Theta_W[\gamma]$ is characterised by it's adjoint: if $\Phi = (\phi, \omega)$

$$\Phi \perp \Theta_W[\gamma] \quad \forall \gamma \iff \Theta_W^*[\Phi] := \varepsilon^2 d^* \omega + \langle \phi, iu\rangle = 0.$$

We recall the decomposition of the linearised operator, given by

$$S'(W) = L_W - \Theta_W \Theta_W^*,$$

where

$$L_W[\Phi] = \begin{pmatrix} -\varepsilon^2 \Delta^A \phi - \frac{1}{2}(1 - 3|u|^2)\phi + 2i\varepsilon^2 \nabla^A u \cdot \omega \\ -\varepsilon^2 \Delta\omega + |u|^2\omega - 2\langle \nabla^A u, i\phi \rangle \end{pmatrix}$$

is an elliptic operator which is well defined in the space of pairs (ϕ, ω) for which

$$\|\Phi\|_{H^1_W(N)} := \|\nabla^A \phi\|_{L^2(N)} + \|\phi\|_{L^2(N)} + \|\nabla\omega\|_{L^2(N)} + \|\omega\|_{L^2(N)} < \infty,$$

where $\nabla\omega$ is the Levi-Civita connection applied to the 1-form ω. We call L_W the "gauge-corrected linearised". Define the operator ∇_W and $-\Delta_W = \nabla_W^* \nabla_W$ as

$$\nabla_W \begin{pmatrix} \phi \\ \omega \end{pmatrix} = \begin{pmatrix} \nabla^A \phi \\ d\omega + d^*\omega \end{pmatrix}, \quad -\Delta_W \begin{pmatrix} \phi \\ \omega \end{pmatrix} = \begin{pmatrix} -\Delta^A \phi \\ -\Delta\omega \end{pmatrix}.$$

By doing so, we can write

$$L_W[\Phi] = -\varepsilon^2 \Delta_W \Phi + \Phi + T_W \Phi,$$

where

$$T_W \begin{pmatrix} \phi \\ \omega \end{pmatrix} = \begin{pmatrix} -\frac{3}{2}(1 - |u|^2)\phi + 2i\varepsilon^2 \nabla^A u \cdot \omega \\ -(1 - |u|^2)\omega - 2\langle \nabla^A u, i\phi \rangle \end{pmatrix}. \tag{1.15}$$

Recall the solution $U_0 = (u_0, A_0)$ defined in (1.5). We denote the gauge-corrected linearised operator around U_0 with $\varepsilon = 1$ with

$$\mathsf{L} := S'(U_0) - \Theta_{U_0} \Theta_{U_0}^*. \tag{1.16}$$

It is known that $Z_{U_0} := \operatorname{span}\{\mathsf{V}_1, \mathsf{V}_2\} \subset \ker \mathsf{L}$, where

$$\mathsf{V}_1 = \begin{pmatrix} f' \\ \frac{a'}{r} dt^2 \end{pmatrix}, \quad \mathsf{V}_2 = \begin{pmatrix} if' \\ -\frac{a'}{r} dt^1 \end{pmatrix}. \tag{1.17}$$

Lastly, recall that the coercivity estimate (proved in [35])

$$\langle \mathsf{L}[\Phi], \Phi \rangle_{L^2} \geq c \|\Phi\|^2_{H^1_{U_0}}, \quad \forall \Phi \in Z_{U_0}^\perp, \tag{1.18}$$

for some $c > 0$, and Lax–Milgram theorem imply the validity of the following existence result.

Lemma 1.2 *For any $\Psi \in L^2(\mathbb{R}^2) \cap Z_{U_0}^\perp$ there exists a unique solution $\Phi \in H^1_{U_0}(\mathbb{R}^2) \cap Z_{U_0}^\perp$ to*

$$\mathsf{L}[\Phi] = \Psi$$

satisfying

$$\|\Phi\|_{H^1_{U_0}(\mathbb{R}^2)} \leq C \|\Psi\|_{L^2(\mathbb{R}^2)}$$

for some $C > 0$.

2 Sketch of the Proof of Theorem 1

2.1 The First Approximation

In what follows we use indices $i, j, k \dots$ and $\alpha, \beta, \gamma, \dots$ respectively for coordinates tangential and normal to M, while we use $a, b, c, \dots$ to indicate all coordinates at once. More precisely

$$1 \le i, j, k, \dots \le n-2, \quad n-1 \le \alpha, \beta, \gamma, \dots \le n, \quad 1 \le a, b, c, \dots \le n.$$

The admissibility hypothesis (H) allows us to choose canonically a basis $\{\nu_1, \nu_2\}$ for $T^{\perp}M$ by setting ν_2 as the normal to B in N and ν_1 as the vector field in TB restricted to M which is normal to TM (we can assume that ν_1 is directed towards B).

We describe a neighbourhood of M in N through Fermi coordinates, that is we consider the points $x \in N$ such that

$$x = X(y, z) = \exp_y(z^\beta \nu_\beta(y)), \quad (y, z) \in M \times B(0, \tau),$$

where $B(0, \tau) \subset \mathbb{R}^2$ and τ is sufficiently small. Given a smooth function $h = (h^1, h^2) \colon M \to \mathbb{R}^2$ satisfying

$$\|h\|_{C^{2,\gamma}(M)} \le K\varepsilon \tag{2.1}$$

consider the change of coordinates

$$z^\beta = \varepsilon(t^\beta + h^\beta(y)), \quad \beta = 1, 2.$$

Then, the neighbourhood of M can be described as the set $\mathcal{N} = X_h(\mathcal{O}_h)$, where $\mathcal{O}_h = \{(y, t) \;:\; y \in M,\ |t + h(y)| < \tau/\varepsilon\}$ and

$$X_h(y, t) = \exp_y\left(\varepsilon(t^\beta + h^\beta(y))\nu_\beta(y)\right). \tag{2.2}$$

On $\mathcal{N}$ we define the first local approximation W_0 by setting

$$W_0(x) = U_0(t), \quad x = X_h(y, t),$$

where $U_0 = (u_0, A_0)^T$ is the degree 1 solution in $\mathbb{R}^2$, given by (1.5).

This is a good first approximation if the error of W_0

$$S(W_0) = \begin{pmatrix} -\varepsilon^2 \Delta^{A_0} u_0 - \frac{1}{2}(1 - |u_0|^2)u_0 \\ \varepsilon^2 d^* dA_0 - \langle \nabla^{A_0} u_0, iu_0 \rangle \end{pmatrix} \tag{2.3}$$

is small. The computation of (2.3) relies on expressing the differential operators $-\Delta^A$ and d^*dA in coordinates (y, t). Using the fact that U_0 is a solution of the corresponding system in $\mathbb{R}^2$ (see [2]), we find

$$\begin{aligned} S(W_0) = {} & \varepsilon^2 t^\gamma \left(R_{i\beta i\gamma}(y, 0) + A^\beta_{ij}(y)A^\gamma_{ij}(y)\right)\mathsf{V}_\beta(t) + \tfrac{1}{3}\varepsilon^2 R_{\alpha\gamma\beta\delta}(y, 0)t^\gamma t^\delta \nabla_{\gamma\delta, U_0} U_0 \\ & -\varepsilon^2\left((\Delta_M h^\beta)(y) + R_{i\beta i\gamma}(y, 0)h^\gamma(y) + A^\beta_{ij}(y)A^\gamma_{ij}(y)h^\gamma(y)\right)\mathsf{V}_\beta(t) \\ & +\varepsilon^3 A^\alpha_{ij}(y)A^\beta_{jk}(y)A^\gamma_{ki}(y)t^\gamma t^\delta \mathsf{V}_\beta(t) + O(\varepsilon^3), \end{aligned} \tag{2.4}$$

where (2.4) has been broken down into sizes in ε, accounting also (2.1). The terms V_β are as in (1.17).

We look for a better approximation to a solution of (1.2) as a perturbation of W_0, namely of the form $W_1 = W_0 + \Lambda_1$. The error of approximation can be split in the following parts

$$S(W_1) = S(W_0) + \mathsf{L}[\Lambda_1] + (S'(W_0) - \mathsf{L})[\Lambda_1] + N_0(\Lambda_1),$$

where

$$N_0(\Lambda_1) = S(W_0 + \Lambda_1) - S(W_0) - S'(W_0)[\Lambda_1]$$

where we recall that the 2-dimensional linearized operator L, given by (1.16), is an operator in the t-variable only. The largest term of $S(W_0)$, namely that of order ε^2, is locally given by

$$\mathsf{Q}_2(y,t) = \varepsilon^2 t^\gamma \left(R_{i\beta i\gamma}(y,0) + A^\beta_{ij}(y)A^\gamma_{ij}(y)\right)\mathsf{V}_\beta(t) + \tfrac{1}{3}\varepsilon^2 R_{\alpha\gamma\beta\delta}(y,0)t^\gamma t^\delta \nabla_{\gamma\delta,U_0}U_0. \tag{2.5}$$

Therefore, if we solve

$$\mathsf{L}[\Lambda_1] = -\mathsf{Q}_2(y,t)$$

the biggest part of the error in terms of ε is cancelled. Such Λ_1 exists thanks to Lemma 1.2, using that

$$\int_{\mathbb{R}^2} \mathsf{Q}_2(y,t)\cdot \mathsf{V}_\alpha(t)dt = 0, \quad \forall y \in M.$$

Also, since the right-hand side (2.5) is $O(e^{-|t|})$ for $|t|$ large a standard barrier argument along with the fact that $\mathsf{L} \sim -\Delta + \mathrm{Id}$ at infinity ensures that

$$\sup_{t\in\mathbb{R}^2} e^{\sigma|t|}|\mathsf{Q}_2(y,t)| < \infty, \quad \forall y \in M$$

for any $0 < \sigma < 1$. Moreover, the error created

$$\mathcal{E}(y,t) = (S'(W_0) - \mathsf{L})[\Lambda_1] + N_0(\Lambda_1)$$

satisfies

$$|\mathcal{E}(y,t)| \le C\varepsilon^4 e^{-\sigma|t|}.$$

To further improve the approximation the non-degeneracy assumption (1.7), and the consequent invertibility Lemma 1.1, are crucial. Indeed if we set $W_2 = W_1 + \Lambda_2$ we can try to cancel the ε^3-terms in (2.4)

$$\mathsf{Q}_3(y,t) = \varepsilon^2 \mathcal{J}^\beta[h]\mathsf{V}_\beta(t) + \varepsilon^3 A^\alpha_{ij}(y)A^\beta_{jk}(y)A^\gamma_{ki}(y)t^\gamma t^\delta \mathsf{V}_\beta(t) + O(\varepsilon^3)$$

in the same manner as before. This can be done by virtue of Lemma 1.2 if the right-hand side satisfies the orthogonality condition with $Z_{U_0,t}$, that is if

$$\int_{\mathbb{R}^2} \mathsf{Q}_3(y,t)\cdot \mathsf{V}_\gamma(t)\,dt = c\varepsilon^2\mathcal{J}^\gamma(h)(y) + \varepsilon^3\mathsf{q}^\gamma(y) = 0, \quad \gamma = 1,2,\ \forall y \in M. \tag{2.6}$$

Here, $c = \int_{\mathbb{R}^2} |\mathsf{V}_\gamma(t)|^2 dt$ (independent of $\gamma = 1, 2$) and

$$\mathsf{q}^\gamma(y) = A^\alpha_{ij}(y)A^\beta_{jk}(y)A^\gamma_{ki}(y)\int_{\mathbb{R}^2} t^\delta t^\beta |\mathsf{V}_\gamma(t)|^2 dt + O(1).$$

Now, Lemma 1.1 guarantees the existence of a bounded $h_0(y)$ with

$$\mathcal{J}(h_0) = -(\mathsf{q}^1, \mathsf{q}^2)^T \quad \text{on } M. \tag{2.7}$$

Choosing $h = \varepsilon h_0$, the right-hand side of (2.6) vanishes for $\gamma = 1, 2$, allowing us to find the sought Λ_2.

The problem with this procedure is that the term in the error created given by $S'(W_1) - \mathsf{L}$ carries second derivatives in y which are not included in the invertibility theory of L, which will create problems in further iterations of the process. Thus, an invertibility theory for the full linearised needs to be made, which is the content of Proposition 2.1 below. We also point out that the approximation $W_1(y,t)$ will be sufficient for our purposes, and the function $\Lambda_2(y,t)$ will be part of the expansion of perturbation in the full solution.

The approximated solution found so far is defined only locally around M. To get a global approximation we extend W_1 to the whole ambient space N. The idea is that of "gluing" W_1 to a pure gauge, namely with a pair of the form

$$\boldsymbol{\Psi} = \begin{pmatrix} \psi \\ \frac{d\psi}{i\psi} \end{pmatrix}, \quad |\psi| = 1$$

where ψ is a smooth S^1-valued function defined away from M and that links well with W_1 in a region close to M. We recall that, being a pure gauge, $\boldsymbol{\Psi}$ satisfies automatically $S(\boldsymbol{\Psi}) = 0$. The existence of such ψ is guaranteed by the admissibility hypothesis (H), by means of the following lemma, proved in [2].

Lemma 2.1 *The admissibility hypothesis (H) guarantees the existence of a $\delta > 0$ and smooth function*

$$\psi : N \setminus M_h \to S^1,$$

where $M_h = \{\exp_y(h^\beta(y)\nu_\beta(y)) : y \in M\}$, such that for every $x = X_h(y,t) \in \operatorname{supp} \zeta_3$

$$\psi(x) = \frac{t}{|t|}, \quad |t + h(y)| < \delta/\varepsilon.$$

Let now $\delta > 0$ and ζ be a smooth cut-off function such that $\zeta(s) = 1$ if $s < 1$ and $\zeta(s) = 0$ if $s > 2$. For $m = 1, 2, \ldots$ consider the cut-off functions defined by

$$\zeta_m(x) = \begin{cases} \zeta(\frac{\varepsilon}{\delta}|t + h(y)| - m) & \text{if } x = X_h(y,t) \in \mathcal{N}, \\ 0 & \text{otherwise.} \end{cases} \tag{2.8}$$

We define the global approximation W to a solution of (1.2) as

$$W = \zeta_3 W_1 + (1 - \zeta_3)\boldsymbol{\Psi}.$$

For $\sigma \in (0,1)$, it holds

$$S(W) = \zeta_3 S(W_1) + (1 - \zeta_3) S(\boldsymbol{\Psi}) + \mathrm{E},$$

where

$$|\mathrm{E}(x)| \le C e^{-\frac{4\sigma\delta}{\varepsilon}} \chi_{\{0 < \zeta_3 < 1\}}(x)$$

and again $S(\boldsymbol{\Psi}) = 0$.

2.2 Proof of Main Result

We look for a solution to (1.2) as a small perturbation of the global approximation W, namely we are looking for a Φ such that (Figs. 1, 2 and 3)

$$S(W + \Phi) = 0 \quad \text{on } N. \tag{2.9}$$

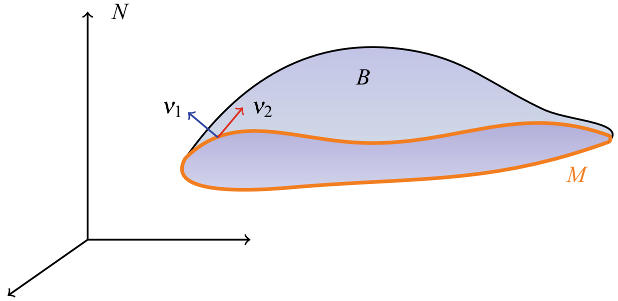

Fig. 1 A representation of M as the boundary of an oriented manifold B in N. Assumption (H) determines the normal fields $\{\nu_1, \nu_2\}$

Roughly, the strategy to find such Φ is the following. First we write (2.9) as

$$0 = -\Theta_W \Theta_W^*[\Phi] + L_W[\Phi] + S(W) + N(\Phi),$$

where $N(\Phi) = S(W + \Phi) - S(W) - S'(W)[\Phi]$. Secondly, we use a suitable invertibility theory for the gauge-corrected linearised L_W in order to solve

$$L_W[\Phi] + S(W) + N(\Phi) = 0 \tag{2.10}$$

as a fixed point problem. Finally, we use the contraction mapping principle and the small Lipschitz character of N to find a Φ satisfying (2.9). By doing so, we find a solution to

$$S(W + \Phi) = \Theta_W[\gamma] \quad \text{on } N, \;\; \text{where } \gamma = -\Theta_W^*[\Phi].$$

We claim that in this case $\gamma = 0$. Denoting $W = (u, A)$ and $\Phi = (\varphi, \omega)$, we have by gauge invariance (1.4),

$$\begin{aligned} 0 &= \frac{d}{dt}\Big|_{t=0} E(\mathbb{G}_{t\gamma}(W + \Phi)) \\ &= \int_N S(W + \Phi) \cdot \Theta_{W+\Phi}[\gamma] \\ &= \int_N \Theta_W[\gamma] \cdot \Theta_{W+\Phi}[\gamma] \\ &= \int_N \gamma \cdot \left[(-\varepsilon^2 \Delta + |u|^2 + \langle u, \varphi \rangle)\gamma\right] \end{aligned}$$

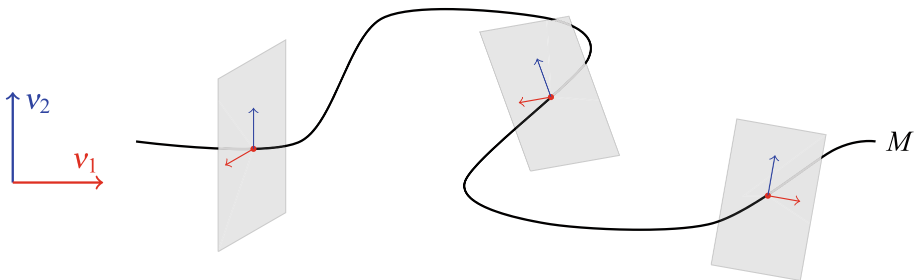

Fig. 2 The normal frame around M

and since $-\varepsilon^2\Delta + |u|^2 + \langle u, \varphi\rangle$ is a positive operator given that φ is small with respect to u, it must be $\gamma = 0$ and the claim is proved. It is thus sufficient to find Φ solving (2.10). However, (2.10) is not solvable, with the appropriate estimates, in absolute terms. Rather than that, we consider the corrected problem

$$L_W[\Phi] = -S(W) - N(\Phi) + \zeta_2 b^\alpha(y)\mathsf{V}_\alpha(t) \quad \text{on } N. \tag{2.11}$$

The adjustment on the right-hand side provides unique solvability in terms of Φ for a precise choice of $b = (b^1, b^2)$, in the sense of the following result.

Proposition 2.1 *Let $0 < \gamma < 1$ and let $\Lambda \in C^{0,\gamma}(N)$. Then, there exists $b \in C^{0,\gamma}(M)$ and a unique solution $\Phi = \mathcal{G}(\Lambda)$ to*

$$L_W[\Phi] = \Lambda + \zeta_2 b^\alpha(y)\mathsf{V}_\alpha(t)$$

satisfying

$$\|\Phi\|_{C^{2,\gamma}(N)} + \|b\|_{C^{0,\gamma}(M)} \le C\|\Lambda\|_{C^{0,\gamma}(N)}$$

for some $C > 0$.

Proposition 2.1 allows us to write (2.11) as a fixed point problem

$$\Phi = -\mathcal{G}\left(S(W) + N(\Phi)\right)$$

on the space

$$X_A = \left\{\Phi \in C^{2,\gamma}(N) \ : \ \|\Phi\|_{C^{2,\gamma}(N)} \le A\varepsilon^3\right\}$$

which admits a solution by the Lipschitz estimate

$$\|N(\Phi_1) - N(\Phi_2)\|_{C^{2,\gamma}(N)} \le C\varepsilon^3\|\Phi_1 - \Phi_2\|_{C^{2,\gamma}(N)}, \quad \Phi_1, \Phi_2 \in X_A,$$

if A is chosen sufficiently large. The final step is to choose h suitably to make the projection $\zeta_2 b^\alpha \mathsf{V}_\alpha$ in (2.11) vanish. We find an expression for b^γ by multiplying (2.11) by $\zeta_4\mathsf{V}_\gamma(t)$, $\gamma = 1, 2$ and integrating on $\mathbb{R}^2$.

$$b^\gamma(y) = \frac{1}{\int_{\mathbb{R}^2}\zeta_2|\mathsf{V}_\gamma|^2}\int_{\mathbb{R}^2}\zeta_4\left[S(W) + N(\Phi) + L_W[\Phi]\right]\cdot\mathsf{V}_\gamma.$$

As previously observed (cfr. (2.6)) the expansion of b^γ yields at first order a $O(\varepsilon^2)$ multiple of the γ-th component of the Jacobi operator. Precisely, letting

$$q_m(y) = \int_{\mathbb{R}^2}\zeta_m(y,t)|\mathsf{V}_\alpha(t)|^2, \quad m = 1, 2, \ldots,$$

the system $b^\gamma = 0$, $\gamma = 1, 2$, can be written as

$$\mathcal{J}(h) = G(h), \tag{2.12}$$

where $G(h) = q_4^{-1}(G_1(h), G_2(h))$ and

$$G_\alpha(h) = q_4\mathcal{J}^\alpha(h) - \varepsilon^{-2}\int_{\mathbb{R}^2}\zeta_4\left[S(W) + N(\Phi) + L_W[\Phi]\right]\cdot\mathsf{V}_\alpha, \quad \alpha = 1, 2.$$

We can use Lemma 1.1 to restate (2.12) as a fixed point problem

$$h = \mathcal{H}(G(h)). \tag{2.13}$$

To conclude the proof we use the following lemma (we refer to our work [2] for a proof).

Lemma 2.2 *The map G satisfies*

$$\|G(0)\|_{C^{0,\gamma}(M)} \leq C\varepsilon$$

and

$$\|G(h_1) - G(h_2)\|_{C^{0,\gamma}(M)} \leq C\varepsilon \|h_1 - h_2\|_{C^{2,\gamma}(M)},$$

for some $C > 0$.

Thanks to Lemma 2.2 we see that by contraction mapping principle equation (2.13) admits a solution in the space

$$\left\{h \in C^{2,\gamma}(M) \ : \ \|h\|_{C^{2,\gamma}(M)} \leq A\varepsilon\right\}$$

for any A sufficiently big. Also, the solution found satisfies $h = \varepsilon h_0 + O(\varepsilon^2)$, where h_0 is the unique solution to (2.7). This concludes the proof.

2.3 Proof of Proposition 2.1

To prove the invertibility theory for L_W we use the fact that, on a region close to M, the gauge-corrected linearised operator can be approximated by L_{U_0}, namely the scaled gauge-corrected linearised operator on $M \times \mathbb{R}^2$ around the canonical profile $U_0(y, t) := U_0(t)$, which in the scaled coordinates $(y, t) = (y, z/\varepsilon)$ reads

$$L_{U_0}[\Phi] = -\Delta_{t,U_0}\Phi - \varepsilon^2 \Delta_M \Phi + \Phi + T_{U_0}(t)\Phi,$$

where T_{U_0} is as in (1.15). We start by considering the cut-off functions introduced in (2.8) and looking for a solution to

$$L_W[\Phi] = -\varepsilon^2 \Delta_W \Phi + \Phi + T_W \Phi = \Lambda + \zeta_2 b^\alpha \mathsf{V}_\alpha \quad \text{on } N \tag{2.14}$$

of the form

$$\Phi(x) = \zeta_2(x)\Phi(y, t) + \Psi(x),$$

where Φ is defined on $M \times \mathbb{R}^2$ and Ψ is defined on N. In terms of the pair (Φ, Ψ) the equation

$$L_W[\Phi] = -\varepsilon^2 \Delta_W \Phi + \Phi + T_W \Phi = \Lambda + \zeta_2 b^\alpha \mathsf{V}_\alpha \quad \text{on } N \tag{2.15}$$

can be broken down in the system

$$L_{U_0}[\Phi] + (L_W - L_{U_0})[\Phi] + \zeta_1 T_W \Psi = \Lambda + b^\alpha \mathsf{V}_\alpha \quad \text{on supp}\, \zeta_2, \tag{2.16}$$

$$-\varepsilon^2 \Delta_W \Psi + \Psi + (1 - \zeta_1) T_W \Psi + \varepsilon^2 \mathcal{R}_W[\zeta_2, \Phi] = (1 - \zeta_2)\Lambda \quad \text{on } N, \tag{2.17}$$

where we denoted

$$\mathcal{R}_W[f, \Phi] = -\Delta_W(f\Phi) + f\Delta_W \Phi. \tag{2.18}$$

Equation (2.17) is solvable directly using the positivity of the operator on the left-hand side on $H^1_W(N)$. For Φ fixed, we find a solution Ψ to (2.17) satisfying

$$\|\Psi\|_{C^{2,\gamma}(N)} \leq C\left(\|(1 - \zeta_1)\Lambda\|_{C^{0,\gamma}(N)} + e^{-\frac{\sigma\delta}{\varepsilon}} \|\Phi\|_{C^{0,\gamma}_\sigma(M \times \mathbb{R}^2)}\right).$$

Plugging such $\Psi = \Psi(\Lambda, \Phi)$ inside of (2.16) we reduce the problem to an equation for Φ only. Define

$$\tilde{\mathsf{B}}[\Phi] = \zeta_4 \mathsf{B}[\Phi] = \zeta_4(L_W - L_{U_0})[\Phi], \quad \tilde{\Lambda} = \zeta_4 \Lambda, \quad (y, t) \in M \times \mathbb{R}^2.$$

Here $\tilde{\mathsf{B}}$ satisfies

$$\|\tilde{\mathsf{B}}[\Phi]\|_{C^{0,\gamma}(M\times\mathbb{R}^2)} \le C\delta\|\Phi\|_{C^{2,\gamma}(M\times\mathbb{R}^2)},$$

where δ is the one from the definition of ζ_4 in (2.8). With this notation equation (2.16) is equivalent to

$$L_{U_0}[\Phi] + \tilde{\mathsf{B}}[\Phi] + \zeta_1 T_W \Psi = \tilde{\Lambda} + b^\alpha \mathsf{V}_\alpha \quad \text{on} \quad M\times\mathbb{R}^2. \tag{2.19}$$

Define the weighted norm $C^{0,\gamma}_\sigma$ on functions $\psi(y,t)$ defined on $M\times\mathbb{R}^2$ as

$$\|\psi\|_{C^{k,\gamma}_\sigma(M\times\mathbb{R}^2)} = \|e^{\sigma|t|}\psi\|_{C^{k,\gamma}(M\times\mathbb{R}^2)},$$

where $k\ge 0$ and $\gamma,\sigma\in(0,1)$. The following result holds.

Proposition 2.2 *Let $\gamma\in(0,1)$ and $\sigma>0$ sufficiently small. Then for every $\tilde{\Lambda}\in C^{0,\gamma}_\sigma(M\times\mathbb{R}^2)$ there exists $b\in C^{0,\gamma}(M)$ such that the problem*

$$L_{U_0}[\Phi] = \tilde{\Lambda} + b^\alpha \mathsf{V}_\alpha \quad \text{on} \quad M\times\mathbb{R}^2$$

admits a unique solution $\Phi=\mathcal{T}(\tilde{\Lambda})$ satisfying

$$\|\Phi\|_{C^{2,\gamma}_\sigma(M\times\mathbb{R}^2)} + \|b\|_{C^{0,\gamma}(M)} \le C\|\tilde{\Lambda}\|_{C^{0,\gamma}_\sigma(M\times\mathbb{R}^2)}$$

for some $C>0$.

To prove this result we restrict to an open cover $\{\mathcal{U}_k\}$ of M and solve the problem locally on $\mathcal{U}_k\times\mathbb{R}^2$ for every k, finding then a global solution by gluing of all the local solutions. See [2] for details. Using Proposition 2.2 we can rephrase (2.19) as

$$\Phi + \mathcal{G}[\Phi] = \mathcal{H}, \tag{2.20}$$

where

$$\mathcal{G}[\Phi] = \mathcal{T}\left(\tilde{\mathsf{B}}[\Phi] + \zeta_1 T^\varepsilon_W \Psi_1[\Phi]\right),$$
$$\mathcal{H} = \mathcal{T}\left(\tilde{\Lambda} - \zeta_1 T^\varepsilon_W \Psi_2[\Lambda]\right).$$

Now, using that

$$\|\tilde{\mathsf{B}}[\Phi]\|_{C^{0,\gamma}_\sigma(M\times\mathbb{R}^2)} \le C\delta\|\Phi\|_{C^{2,\gamma}_\sigma(M\times\mathbb{R}^2)},$$
$$\|\zeta_1 T_W \Psi_1[\Phi]\|_{C^{0,\gamma}_\sigma(M\times\mathbb{R}^2)} \le C\,\|\Psi_1[\Phi]\|_{C^{0,\gamma}(N)} \le Ce^{-\frac{\delta'}{\varepsilon}}\|\Phi\|_{C^{2,\gamma}_\sigma(M\times\mathbb{R}^2)}$$

we find

$$\begin{aligned}\|\mathcal{G}[\Phi]\|_{C^{2,\gamma}_\sigma(M\times\mathbb{R}^2)} &\le C\left(\|\tilde{\mathsf{B}}[\Phi]\|_{C^{0,\gamma}_\sigma(M\times\mathbb{R}^2)} + \|\zeta_1 T_W \Psi_1[\Phi]\|_{C^{0,\gamma}_\sigma(M\times\mathbb{R}^2)}\right)\\ &\le C\left(\delta + e^{-\frac{\delta'}{\varepsilon}}\right)\|\Phi\|_{C^{2,\gamma}_\sigma(M\times\mathbb{R}^2)}\end{aligned}$$

and hence by picking ε,δ sufficiently small we find a unique solution to (2.20), from which we get the existence of a unique solution (Φ,Ψ) to system (2.16)–(2.17). In conclusion, $\Phi=\zeta_2\Phi+\Psi$ solves (2.15) and it follows directly that

$$\|\Phi\|_{C^{2,\gamma}(N)} \le C\|\Lambda\|_{C^{0,\gamma}(N)}.$$

The proof of Proposition 2.1 is complete. □

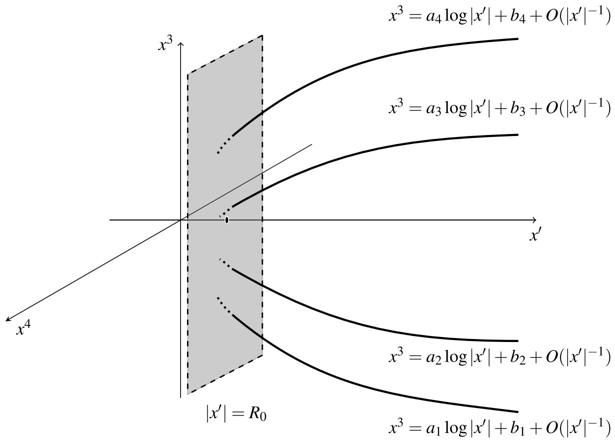

Fig. 3 An outline of the asymptotic behaviour of the ends of the surface M in the case where $m = 4$

3 Sketch of the Proof of Theorem 2

As already mentioned, the proof of Theorem 2 follows the same lines of the proof of Theorem 1. The first main difference is that the solution's 0 level set will actually depart logarithmically from the manifold around the ends, according to (1.13).

This fact prevents the formation of small but non-decaying errors in the space between two consecutive ends, see [11]. This fact is formulated in the following way: we choose $h = h_* + h_1$, with

$$\|h_1\|_* := \|h_1\|_\infty + \|D_M h_1\|_{C_2^{0,\gamma}(M)} + \|D_M^2 h_1\|_{C_4^{0,\gamma}(M)} \le C\varepsilon,$$

where the norms $\|\cdot\|_{C_\mu^{0,\gamma}(M)}$ are defined by

$$\|\phi\|_{C_\mu^{0,\gamma}(M)} = \|r^\mu \phi\|_{C^{0,\gamma}(M)}, \quad r(y) = \sqrt{1 + |y|^2}, \quad y \in M \subset \mathbb{R}^3$$

and account for the decay along M. The function $h_* = (h_*^1, h_*^2)^T$ satisfies $\mathcal{J}(h_*) = 0$ and

$$h_*^1(y) = (-1)^j \lambda_j \log r + \eta \quad \text{on } M_j, \quad h_*^2 = 0,$$

where M_j is the j-th end of M. The local approximation of a solution is then given by

$$W_0(y, z) = U_0(t), \quad t = z/\varepsilon - h_*(y) - h_1(y).$$

The fact that $\mathcal{J}(h_*) = 0$ will allow us to formulate the final, reduced problem as a fixed point involving only the Jacobi operator of h_1, namely of the form

$$\mathcal{J}(h_1) = G(h_1) \quad \text{on } M,$$

similar to (2.12). In this geometrical setting, the Jacobi operator is given by

$$\mathcal{J}\begin{pmatrix} h_1 \\ h_2 \end{pmatrix} = \begin{pmatrix} \Delta_M h^1 + |\mathcal{A}_M|^2 h^1 \\ \Delta_M h^2 \end{pmatrix}$$

and we recall that we assumed non-degeneracy, which here means

$$h \in L^\infty(M) \quad \text{and} \quad \mathcal{J}(h) = 0 \implies h \in \operatorname{span}\{z_0, z_1, z_2, z_3, z_4\},$$

where z_k, $k = 0, \ldots, 4$ are given by (1.10). The presence of this kernel creates an obstruction to a direct invertibility theory for $\mathcal{J}$. Instead, we consider the corrected problem

$$\mathcal{J}(h_1) = G(h_1) - \sum_{k=0}^{4} c^k |\mathcal{A}_M|^2 z_k \tag{3.1}$$

for some constants $c^0, \ldots, c^4$. The correction on the right-hand side provides unique solvability for (3.1) in the sense of the following lemma.

Lemma 3.1 *Let $f = (f^1, f^2)^T$ be a function defined on M such that $\|f\|_{C_4^{0,\gamma}(M)} < +\infty$. Then, there exist constants $c^0, \ldots, c^4$ such that the system*

$$\begin{cases} \Delta_M h^1 + |\mathcal{A}_M|^2 h^1 = f^1 - \sum_{j=0}^{3} c^j |\mathcal{A}_M|^2 \hat{z}_j, \\ \Delta_M h^2 = f^2 - c^4 |\mathcal{A}_M|^2 \end{cases}$$

admits a solution $h = (h^1, h^2)^T = \mathcal{H}(f)$ satisfying

$$\|h\|_* \leq C \|f\|_{C_4^{0,\gamma}(M)}.$$

With the aid of Lemma 3.1 we formulate (3.1) as a fixed point problem

$$h_1 = \mathcal{H}(G(h_1))$$

which admits a solution by contraction mapping principle, using also the Lipschitz character of G (coming from calculations similar to that of the compact case, see [1]). To conclude the proof of Theorem 2 we only need to show that $c^k = 0$ for every k. This is a consequence of the fact that the Jacobi fields z_k are generated by the symmetries of the ambient manifold. In what follows we consider coefficients d_{il} and linear combinations

$$\hat{z}_j = \sum_{i=0}^{4} d_{ij} z_i, \quad j = 0, \ldots, 4,$$

such that

$$\int_M |\mathcal{A}_M|^2 \hat{z}_i \hat{z}_j = \delta_{ij}, \quad i, j = 0, \ldots, 4.$$

At this we have constructed a solution $U = W + \Phi$ of

$$S(U) = q|\mathcal{A}_M|^2 \sum_{j=0}^{3} c^j \hat{z}_j \mathsf{V}_1 + q|\mathcal{A}_M|^2 c^4 \mathsf{V}_2 - \Theta_W \Theta_W^*[\Phi],$$

where we set $q = \varepsilon^2 \zeta_4 \tilde{q}$ and $0 < c \leq \tilde{q} \leq C$. Consider the quantities

$$\begin{aligned} Z_i &:= \nabla_{x_i, U} U, \quad i = 1, 2, 3, 4, \\ Z_0 &:= x_1 \nabla_{x_2, U} U - x_2 \nabla_{x_1, U} U, \\ \gamma &:= -\Theta_W^*[\Phi]. \end{aligned}$$

It holds

$$\int_{\mathbb{R}^4} S(U) \cdot \Theta_U[\gamma] = 0, \quad \int_{\mathbb{R}^4} S(U) \cdot Z_i = 0, \quad i = 0, \ldots, 4. \tag{3.2}$$

This follows from gauge-invariance, the invariances of M under rigid motions and from the balancing condition (1.11). Using (3.2) it is possible to show that the vector $(c^0, c^1, c^2, c^3, c^4,$

γ) is mapped to zero by a linear operator which, for ε small enough, is positive (see [1]). This implies that all coefficients vanish, and thus that we found a true solution. The proof is concluded.

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