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ORIGINAL ARTICLE

New Conservation Laws and Energy Cascade for 1d Cubic NLS and the Schrödinger Map

Valeria Banica[1,2] **· Luis Vega**[3,4]

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Abstract
We recall some recent results concerning the Initial Value Problem of 1d-cubic non-linear Schrödinger equation (NLS) and other related systems as the Schrödinger Map. For the latter we prove the existence of a cascade of energy. Finally, some new examples of the Talbot effect at the critical level of regularity are given.

Keywords Non-linear Schrödinger equation · Conservation laws · Cascade of energy

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1 Introduction

In these pages we recall some recent results concerning the 1d-cubic non-linear Schrödinger equation (NLS) and other related systems. One of the main objectives is to explain in which sense

$$u_M(x,t) = c_M \sum_k e^{itk^2+ikx}, \tag{1.1}$$

for some constant c_M, is a "solution" of 1d-cubic NLS and to show the variety of phenomena it induces. Moreover, we will explain that it has a geometrical meaning due to its connection with the Binormal Flow (BF) of curves in three dimensions and the Schrödinger map (SM). Finally,

Dedicated to Carlos Kenig.

✉ Luis Vega
lvega@bcamath.org; luis.vega@ehu.eus

Valeria Banica
Valeria.Banica@sorbonne-universite.fr

1 Sorbonne Université, CNRS, Université de Paris, Laboratoire Jacques-Louis Lions (LJLL), 4 place Jussieu, 75005 Paris, France

2 Institut Universitaire de France (IUF), Paris, France

3 Basque Center for Applied Mathematics, Alameda Mazarredo 14, 48009 Bilbao, Spain

4 Departamento de Matemáticas, Universidad del Pais Vasco/Euskal Herriko Unibertsitatea, Aptdo. 644, 48080 Bilbao, Spain

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we will explain how the so-called Talbot effect in Optics that is mathematically described by u_M is also present in the non-linear setting with data at the critical level of regularity. As it will be explained later on, in this geometrical interpretation the regular polygons of M sides play a crucial role and the choice of the constant c_M of (1.1) is particularly delicate. It is for this reason that we write c_M in (1.1) instead of using a generic complex constant c. Also the notion of solution is quite involved and we will try to give some hints in which sense the solution has to be understood.

Altogether we are speaking about a family of PDE problems. Consider first NLS which is a complex scalar equation with a cubic non-linear potential:

$$\begin{cases} \partial_t u = i\left(\partial_x^2 u + (|u|^2 - \mathcal{M}(t))\right) u, & \mathcal{M}(t) \in \mathbb{R}, \\ u(x,0) = u_0(x), & x \in \mathbb{R}. \end{cases} \tag{1.2}$$

Let us introduce next SM. Calling $T(x,t)$ a unit vector in $\mathbb{R}^3$ the Schrödinger Map onto the sphere is given by

$$\partial_t T = T \wedge \partial_x^2 T. \tag{1.3}$$

Finally, observe that the vector $T(x,t)$ can be seen at any given time as the tangent vector of a 3d-curve $\chi(x,t)$

$$\partial_x \chi = T,$$

with χ a solution of

$$\partial_t \chi = \partial_x \chi \wedge \partial_x^2 \chi. \tag{1.4}$$

Da Rios [16] proposed (1.4) as a simplified model that describes the evolution of vortex filaments. Remember that Frenet equations read

$$\begin{aligned} T_x &= \kappa N \\ N_x &= -\kappa T + \tau B \\ B_x &= -\tau N \end{aligned}$$

with κ the curvature of the curve, τ its torsion, N the normal vector, and B the binormal vector. Hence

$$\partial_t \chi = \partial_x \chi \wedge \partial_x^2 \chi = \kappa B.$$

That is the reason why sometimes the system of PDEs (1.4) is called the Binormal Flow.

The connection of the two systems BF and SM with (1.2) was established by Hasimoto in [26] through a straightforward computation. This computation is slightly simplified if instead of the Frenet frame one uses the parallel one. This is given by vectors (T, e_1, e_2) that satisfy

$$\begin{aligned} T_x &= \alpha e_1 + \beta e_2 \\ e_{1x} &= -\alpha T \\ e_{2x} &= -\beta T. \end{aligned} \tag{1.5}$$

Defining $u = \alpha + i\beta$ it is proved that u solves (1.2) for some given $\mathcal{M}(t)$. From (1.3) and (1.4) we get

$$\partial_t T = \alpha_x e_2 - \beta_x e_1 \tag{1.6}$$

and

$$\partial_t \chi = \alpha e_2 - \beta e_1.$$

These two equations are gauge invariant, namely that given any real φ we can change $e_1 + ie_2$ into $e^{i\varphi}(e_1 + ie_2)$ and $\alpha + i\beta$ into $e^{i\varphi}(\alpha + i\beta)$ and (1.3) and (1.4) remain the same. Therefore, we have the freedom of choosing $\mathcal{M}(t)$.

Observe that u, the solution of (1.2), gives the curvature and the torsion of χ. More concretely, taking $e_1 + ie_2 = e^{i\int_0^x \tau(r)\,dr}(n + ib)$ we get that

$$|u|^2 = \alpha^2 + \beta^2 = \kappa^2$$

and after some calculation (see [26] and (4.2) below)

$$u(x,t) = \kappa e^{i\int_0^x \tau(r)\,dr}, \tag{1.7}$$

with $\boldsymbol{\tau}$ denoting the torsion. (1.7) is usually called Hasimoto transformation.

A relevant simple example is

$$u_o = c_o \frac{1}{\sqrt{4\pi t}} e^{ix^2/4t}, \qquad \mathcal{M}(t) = \frac{c_o^2}{4\pi t}, \tag{1.8}$$

which is related to the self-similar solutions of SM and BF. Formally $u_o(x,0) = c_o\delta$, and the corresponding χ has a corner at $(x,t) = (0,0)$. Here δ stands for the Dirac-δ function. We will sometimes refer to this solution as either a fundamental brick or a coherent structure [35]. Our main interest is to consider rough initial data as polygonal lines and regular polygons. As we will see, the latter are related to (1.1) and therefore u_M could be understood as a superposition of infinitely many simple solutions $u_o(x-j)$ centered at the integers j. As a consequence, the curve obtained from u_M can be seen as an interaction of these coherent structures (see https://www.youtube.com/watch?v=fpBcwuY57FU).

It is important to stress that to obtain χ from T, besides integrating in the spatial variable the parallel system (1.5), one has to find the trajectory in time followed by one point of χ_o. This is not obvious even for (1.8), see [23]. It turns out that to compute that trajectory of, say, one corner of a regular polygon is rather delicate. The corresponding curve can be as complicated as the graph of the so-called Riemann's non differentiable function:

$$\sum_k \frac{e^{itk^2} - 1}{k^2}.$$

See [5] for more details.

We will review some recent results regarding the IVP for (1.2) in Sections 2 and 3. In particular, we will show the existence of three new conservation laws (2.4), (3.7), and (3.9), valid at the critical level of regularity. As it is well known (1.2), and as a consequence also (1.3) and (1.4), are completely integrable systems with infinitely many conservation laws that start at a subcritical level of regularity, L^2. For the others laws more regularity, measured in the Sobolev class, is needed. In Section 4 we will recall some work done on the transfer of energy for the Schrödinger map (1.3). Finally, in Section 5 we revisit the Talbot effect and modify some examples obtained in [3] to establish a connection with some recent work on Rogue Waves given in [20].

2 The Initial Value Problem

We start with the IVP associated to NLS equation (1.2):

$$\begin{cases} \partial_t u = i\left(\partial_x^2 u + (|u|^2 - \mathcal{M}(t))\right)u, & \mathcal{M}(t) \in \mathbb{R}, \\ u(x,0) = u_0(x), & x \in \mathbb{R}. \end{cases}$$

We are interested in initial data which are at the critical level of regularity. There are two symmetries that leave invariant the set of solutions that we want to consider. One is the scaling invariance: if u is a solution of (1.2), then

$$\lambda > 0, \qquad u_\lambda(x,t) = \lambda u(\lambda x, \lambda^2 t), \tag{2.1}$$

is also a solution of (1.2) with $\lambda^2 \mathcal{M}(t)$ instead of $\mathcal{M}(t)$. The second one is the so-called Galilean invariance: If

$$\nu \in \mathbb{R}, \qquad u^\nu(x,t) = e^{-it\nu^2 + i\nu x} u(x - 2\nu t, t), \tag{2.2}$$

then u^ν is also a solution of (1.2) with the same $\mathcal{M}(t)$. Hence, we want to work in a functional setting where the size of the initial data does not change under the scaling and Galilean transformations.

Let us review the classical results on NLS. The well-posedness of 1D cubic NLS on the full line and on the torus was firstly done in [8, 41] for data in L^2. Observe that the space $L^2(\mathbb{R})$, although is invariant by Galilean symmetry (2.2), misses the scaling (2.1) by a power of $1/2$ in the homogeneous Sobolev class $\dot{H}^s$. In fact, the critical exponet for scaling is $s = -1/2$ which is not invariant under the Galilean symmetry. The first result obtained beyond the $L^2(\mathbb{R})$ theory was given in [42] using some spaces of tempered distributions built on the well known Strichartz estimates. Later on in [21] well-posedness is studied in the Fourier–Lebesgue spaces that we denote by $\mathcal{F}L^p$. These are spaces where the Fourier transform is bounded in $L^p(\mathbb{R})$ and therefore leave invariant (2.2). Moreover, $\mathcal{F}L^\infty$ is also scaling invariant and therefore critical according to our definition. In [22] local well-posedness, also under periodic boundary conditions, was shown in $\mathcal{F}L^p$ with $2 < p < +\infty$.

In the setting of Sobolev spaces of non-homogeneous type the progress has been remarkable. On the one hand, there is ill-posedness in H^s with $s < 0$ in the sense that there is no uniformly continuous data-to-solution map and even some growth of the Sobolev norms has been proved, [11, 14, 29, 31, 32]. On the other hand, there is well-posedness in H^s for $s > -1/2$ as has been shown in [27]. In this case a weaker notion of continuity for the data-to-solution map is used.

In this paper, we will consider solutions of (1.2) such that

$$\omega(\xi,t) := e^{it\xi^2} \widehat{u}(\xi,t) \text{ is } 2\pi\text{-periodic.} \tag{2.3}$$

Here $\widehat{u}$ denotes the Fourier transform of u,

$$\int_{\mathbb{R}} e^{-ix\xi} u(x)\, dx.$$

To prove that this periodicity is preserved by the evolution is not completely obvious and it is a relevant property of (1.2). It can be proved writing the equation for ω, ($\mathcal{M}(t) = 0$) in (1.2):

$$\partial_t \omega(\eta,t) = \frac{i}{8\pi^3} e^{-it\eta^2} \int\int_{\xi_1+\xi_2+\xi_3-\eta=0} e^{it(\xi_1^2-\xi_2^2+\xi_3^2)} \omega(\xi_1)\bar{\omega}(\xi_2)\omega(\xi_3)\, d\xi_1 d\xi_2 d\xi_3.$$

Under the condition $\xi_1 - \xi_2 + \xi_3 - \eta = 0$, we get

$$\xi_1^2 - \xi_2^2 + \xi_3^2 - \eta^2 = 2(\xi_1 - \xi_2)(\xi_1 - \eta).$$

The last quantity is invariant under translations so that the periodicity is formally preserved. Interestingly this calculation does not work for general dispersive systems as for example for modified KdV.

One of the three new conservation laws is precisely

$$\int_0^{2\pi} |\omega(\xi,t)|^2\, d\xi = \text{constant}. \tag{2.4}$$

This can be seen writing $\omega(\xi,t) = \sum_j A_j(t)e^{ij\xi}$ and looking for the ODE system that the Fourier coefficients A_j have to satisfy. Historically, our approach to this question has been different and this is what we explain next.

For solving (1.2) and following Kita in [30], we considered the ansatz

$$u(x,t) = \sum_j \tilde{A}_j(t)e^{it\partial_x^2}\delta(x-j) \tag{2.5}$$

and therefore

$$\widehat{u}(\xi,t) = e^{-it\xi^2}\sum_j \tilde{A}_j(t)e^{ij\xi}.$$

Define

$$V(y,\tau) = \sum_j \hat{V}_\tau(j)(\tau)e^{ijy}, \tag{2.6}$$

with $\hat{V}_\tau(j)$ the j-Fourier coefficient of $V(y,\tau)$. Then

$$\begin{aligned} u(x,t) &= \frac{1}{(4\pi i t)^{1/2}}\sum_j \tilde{A}_j(t)e^{i\frac{(x-j)^2}{4t}} \\ &= \frac{1}{(4\pi i t)^{1/2}}e^{i\frac{|x|^2}{4t}}\sum_j \tilde{A}_j(t)e^{i\frac{j^2}{4t}-i\frac{x}{2t}j} \\ &:= \frac{1}{(4\pi i t)^{1/2}}e^{i\frac{|x|^2}{4t}}\,\overline{V}\left(\frac{x}{2t},\frac{1}{t}\right), \end{aligned}$$

with

$$\hat{V}_\tau(j) = \tilde{B}_j(\tau)e^{-i\frac{\tau}{4}j^2}, \qquad \tilde{B}_j(\tau) = \overline{\tilde{A}_j}\left(\frac{1}{\tau}\right). \tag{2.7}$$

Finally, doing the change of variables

$$y = \frac{x}{2t}, \qquad \tau = 1/t,$$

we easily obtain that V solves

$$\partial_\tau V = i\left(\partial_y^2 + \frac{1}{4\pi\tau}(|V|^2 - m)\right)V; \qquad m(\tau) = \frac{4\pi}{\tau}\mathcal{M}\left(\frac{1}{\tau}\right). \tag{2.8}$$

We actually have that V is a pseudo-conformal transformation of u.

Remark 2.1 1. Observe that formally solutions of (2.8) remain periodic if they are periodic at a given time. That means that given the Fourier coefficients $\hat{V}_\tau(j)$ and using (2.7) to define $\tilde{A}_j$ we conclude that the periodicity of $\omega(\xi,t) = e^{it\xi^2}\widehat{u}(\xi,t)$ is also formally preserved.

2. The change of variable transforms $t = 0$ into $\tau = \infty$. Hence, the initial value problem for u becomes a question about the scattering of the solutions of (2.8).

3. If $V(1) = c_o$, and $m = c_o^2$ then $V(\xi, t) = c_o$ for all t. The corresponding solution is the fundamental brick (1.8)

$$u(x,t) = u_o = \frac{c_o}{(4\pi i t)^{1/2}} e^{i\frac{|x|^2}{4t}},$$

and $u_0 = c_o\delta$. This implies that unless we include the term $\mathcal{M}(t) = c_o^2/(4\pi t)$ in (1.2) the IVP for the Dirac delta is ill–posed, something observed in [29]. It was proved in [1, 2] that even if this term is added and one looks for solutions of the type $V = c_o + z$ with z small with respect to c_o the corresponding u of (1.2) cannot be defined for $t = 0$.
4. It immediately follows from the conservation of mass of (2.8) that

$$m_0 = \int_0^{2\pi} |V(\xi, \tau)|^2 d\xi = \sum_j |\tilde{B}_j(\tau)|^2$$

is formally constant for $\tau > 0$. And from (2.7) we also get that $\sum_j |\tilde{A}_j(t)|^2 = \sum_j |\tilde{B}_j(1/t)|^2 = \int_0^{2\pi} |\omega(\xi, t)|^2 \, d\xi$ remains constant, which is (2.4).

3 Conservation Laws

In [3] a first result on the IVP (1.2) within the functional setting we have just described was obtained. The solution u is written as in (2.5) and for any a_j it is defined R_j by the identity

$$\tilde{A}_j(t) = a_j + R_j(t).$$

Then, an infinite ODE system for R_j can be easily obtained. The corresponding solution is constructed through a fixed point argument in an appropriately chosen space which among other things implies that $R_j(0) = 0$. The condition on the data is that $\sum_j |a_j|$ is finite (i.e. $a_j \in l^1$) but not necessarily small. This condition implies the

$$\lim_{x\to\pm\infty} T(x,t) = A^{\pm}, \qquad t \geq 0.$$

It is proved that $A^{\pm}$ is independent of t and therefore

$$\widehat{T_x}(0,t) = \int_{-\infty}^{+\infty} T_x(x,t)\, dx = A^+ - A^-. \tag{3.1}$$

The result in [3] is local in time. A global result is obtained by assuming the extra condition

$$\sum_j j^2 |a_j|^2 < +\infty, \tag{3.2}$$

whose evolution $\sum_j j^2 |\tilde{A}_j(t)|^2$ is easy to determine as we explain next.

First of all, it is much more convenient to work with the solution V of (2.8) which was defined in (2.6). Moreover, we define

$$A_j(t) = \tilde{A}_j(t) e^{i\phi_j(t)}, \qquad \phi_j(t) = e^{i\frac{|a_j|^2}{4\pi}\log t}, \tag{3.3}$$

and correspondingly (cf. (2.7))

$$B_j(\tau) = \bar{A}_j(1/\tau).$$

Let us introduce for each k the non-resonant set

$$NR_k = \{(j_1, j_2, j_3),\ k - j_1 + j_2 - j_3 = 0,\ k^2 - j_1^2 + j_2^2 - j_3^2 \neq 0\}.$$

Similarly the resonant set is given by the solutions of

$$k - j_1 + j_2 - j_3 = 0, \quad k^2 - j_1^2 + j_2^2 - j_3^2 = 0.$$

Define

$$w_{k,j_1,j_2} := k^2 - j_1^2 + j_2^2 - j_3^2.$$

It is immediate to see that if $k - j_1 + j_2 - j_3 = 0$ then

$$w_{k,j_1,j_2} = 2(k - j_1)(j_1 - j_2). \tag{3.4}$$

Taking $m = 2\sum_j |a_j|^2$. After some computation one gets that the ODE system that B_j has to verify is

$$\begin{aligned} i\partial_\tau B_k(\tau) = \frac{1}{8\pi\tau} \sum_{NR_k} & e^{-i\tau(k^2-j_1^2+j_2^2-j_3^2)-\frac{i}{4\pi}(|a_k|^2-|a_{j_1}|^2+|a_{j_2}|^2-|a_{j_3}|^2)\log t} B_{j_1}(\tau)\overline{B_{j_2}(\tau)}B_{j_3}(\tau) \\ & -\frac{1}{8\pi\tau}(|B_k|^2 - |a_k|^2)B_k. \end{aligned} \tag{3.5}$$

Hereafter this is the system we are going to solve. Recall that we have got it choosing $m = 2\sum_j |a_j|^2 < +\infty$. For a regular polygon, an example we will consider below, m is not finite any more and therefore we are doing a renormalization as the Wick renormalization done in [9] and in [13].

For any real function $c(k)$ we want to understand the behavior of

$$\frac{d}{d\tau} \sum_k c(k)|B_k(\tau)|^2 \tag{3.6}$$

as a function of τ. A symmetrization argument makes to appear $c(k) - c(j_1) + c(j_2) - c(j_3)$ that vanishes in the resonant set because of (3.4). Then, we have that

$$\begin{aligned} &\frac{d}{d\tau} \sum_k c(k)|B_k(\tau)|^2 \\ &= \frac{1}{2i\tau} \sum_{k-j_1+j_2-j_3=0} (c(k) - c(j_1) + c(j_2) - c(j_3))e^{-i\tau w_{k,j_1,j_2}} B_{j_1}(\tau)\overline{B_{j_2}(\tau)}B_{j_3}(\tau)\overline{B_k(\tau)} \\ &= \frac{1}{2i\tau} \sum_{k;NR_k} (c(k) - c(j_1) + c(j_2) - c(j_3))e^{-i\tau w_{k,j_1,j_2}} B_{j_1}(\tau)\overline{B_{j_2}(\tau)}B_{j_3}(\tau)\overline{B_k(\tau)}. \end{aligned}$$

Relevant examples are $c(j) = 1$ that gives (2.4), the L^2 conservation law already mentioned, and $c(j) = j$, that yields a second conservation law:

$$\sum_j j|B_j(\tau)|^2 = \sum_j j|A_j(1/\tau)|^2 \ \text{ is constant.} \tag{3.7}$$

The final example is $c(j) = j^2$, cf. (3.2). In this case (3.6) can be written in terms of V as

$$\int_0^{2\pi} |\partial_y V(y,\tau)|^2\, dy.$$

Then, defining

$$E(\tau)=\int|\partial_y V(y,\tau)|^2-\frac{1}{16\pi\tau}(|V(y,\tau)|^2-m)^2\,dy,$$

a direct computation gives that solutions of (2.8) satisfy

$$\frac{d}{d\tau}E(\tau)=\frac{1}{16\pi\tau^2}\int(|V(y,\tau)|^2-m)^2\,dy.$$

The next step about (1.2) with the ansatz (2.5)-(3.3) was given in [5], where the Picard iteration is done measuring more carefully the first iterate. Particular attention is given to the example

$$a_j=1 \quad \text{for } |j|\le N \quad \text{and zero otherwise,} \tag{3.8}$$

see [5].

Finally in [10], Bourgain's approach [8] is followed. This amounts to use the Sobolev spaces of the coefficients $B_j(\tau)$. The results in that paper can be summarized as follows, for initial datum in l^p, $p\in(1,+\infty)$:

1. Local well-posedness with a smallness assumption in l^p for the initial datum: for any $T>0$, there exists $\varepsilon(T)>0$ such that if the l^p norm of the initial datum $\{a_j\}$ is smaller then $\varepsilon(T)$, then there exists a unique solution of (1.2)-(2.5)-(3.3) in $[0,T]$ in an appropriate sense.
2. Local well-posedness with a smallness assumption in l^∞ for the initial datum: if the l^∞ norm of $\{a_j\}$ is small enough then there exists a time $T(\|\alpha\|_{l^\infty},\|\alpha\|_{l^p})$ such that a unique solution of (1.2)-(2.5)-(3.3) exists in $[0,T]$ in an appropriate sense.
3. For $p=2$, global in time well-posedness with a smallness assumption in l^∞ for the initial datum. As it can be expected this result follows from (2) and the l^2 conservation law. The smallness condition comes from the linear term that is treated as a perturbation. We don't know if this smallness condition can be removed.

For establishing the third conservation law we have to observe that w_{k,j_1,j_2} given in (3.4) is invariant under translations. This implies that solutions of (3.5) such that $B_{j+M}=B_j$ at a given time formally preserved this property for all time. Therefore,

$$\sum_{j=1}^{M}|a_j|^2=\sum_{j=1}^{M}|A_j(t)|^2=\sum_{j=1}^{M}|B_j(1/t)|^2=\text{constant.} \tag{3.9}$$

This conservation law is much stronger than (2.4) because it just assumes an l^∞ condition on a_j.

As a consequence, in Proposition 1 of [10] an explicit solution of (3.5) is obtained with

$$c_M=a_j \qquad \text{for all } j \tag{3.10}$$

as in (1.1). For this solution $|B_j(\tau)|=c_M$ and $B_j(\tau)$ is independent of j.

Recall that the ansatz (2.5) has to be modified as in (3.3) to construct a solution of (1.2). The logarithmic correction given by (3.3) implies that even though there is a limit of the amplitudes $|A_j(t)|=|B_j(1/t)|$ when t approaches zero, the limit of the phases does not exist, and therefore the IVP (1.2) is ill posed. As it was proved in [3] this loss is irrelevant when (1.2) is understood in connection to BF and SM. For example, for BF the solutions can be perfectly defined at $t=0$ as a polygonal line. Moreover, the behavior close to a corner is determined by a self-similar solution (1.8). This self-similar solution, and the precise theorem given about them in [23], gives the necessary information at $t = 0$ so that the flow can be continued for $t < 0$. A crucial ingredient in this process is the precise relation established in [23] between c_o and the angle θ_o of the corresponding corner, namely

$$\sin \theta_o = e^{-\frac{\pi c_o^2}{2}}.$$

For a regular polygon with M sides the angle is $\theta_M = 2\pi/M$. Chossing c_M in such a way that

$$\sin\left(\frac{2\pi}{M}\right) = e^{-\frac{\pi c_M^2}{2}},$$

and using it in (3.10), we obtain a solution, once the Wick renormalization is done, for the case of a regular polygon at the level of NLS. This choice is the one conjectured in [17] based on the numerical simulations done in [28].

4 Transfer of Energy

Solutions of (1.2) have associated a natural energy/mass density which is $|u(x,t)|^2\,dx$. For solutions of (1.3), this is the same density associated to $|T_x|^2$ thanks to Hasimoto transformation (1.7). In Section 3 we have constructed solutions of (1.2) whose energy density is also well described in terms of $|\widehat{u}|^2$ as $|\widehat{u}|^2 = |\omega|^2$ with ω given in (2.3). Also remember that ω is related to V through (2.7) and (2.8). Even though there is no reason a priori to think that $\left|\widehat{T_x}(\xi,t)\right|^2 d\xi$ is related to a density energy, in Theorem 1.1 of [4] it is proved that

$$\lim_{n\to\infty} \int_{2\pi n}^{2\pi(n+1)} \left|\widehat{T_x}(\xi,t)\right|^2 d\xi = \int_0^{2\pi} |V(\xi,t)|^2 d\xi. \tag{4.1}$$

This equality suggests that at least in the limit of large frequencies $\left|\widehat{T_x}(\xi,t)\right|^2 d\xi$ is related to an energy density. Recall that it was proved in Sections 2 and 3 that

$$\int_0^{2\pi} |V(\xi,t)|^2 d\xi = \text{constant}$$

and therefore, that there is no flux of energy for $\widehat{u}$ at least for $0 < t < 1$. The situation for T_x is different. It was proved in [6] that there is some cascade of energy. More concretely we have the following result that was motivated by some numerical experiments done in [18].

Theorem 4.1 *Assume*

$$\begin{cases} a_{-1} = a_{+1} \neq 0, \\ \ a_j = 0 \quad \textit{otherwise.} \end{cases}$$

Then there exists $c > 0$ *such that*

$$\sup_{\xi} \left|\widehat{T_x}(\xi,t)\right|^2 \geq \sup_{\xi\in B\left(\pm\frac{1}{t},\sqrt{t}\right)} \left|\widehat{T_x}(\xi,t)\right|^2 \geq c|\log t|, \qquad t > 0.$$

This type of energy cascade is an alternative to the ones in [15, 25], where results about the growth of Sobolev norms are proved. In these papers the authors consider Sobolev spaces H^s with $s > 1$, so that the growth is a consequence of a transfer of energy from low Fourier modes to high Fourier modes. From (4.1) we conclude that in our setting $\widehat{T_x}$ does not tend to zero in the large frequency limit, so that T_x does not belong to L^2. Also observe that the initial data given in Theorem 4.1 satisfies (3.2) that together with (3.1) give that $\widehat{T_x}$ remains bounded for small frequencies for $0 < t < 1$. It is also proved in [6] that $\widehat{T_x}(\cdot, 1)$ is bounded. Nevertheless, the L^∞ norm grows logarithmically when t tends to zero for frequencies ξ such that $|\xi| \sim 1/t$.

Recall that if $u = \alpha + i\beta$ is the solution of (1.2) then T can be obtained from (1.5), and the equation for T_t is (1.6). After a simple calculation one gets that

$$\partial_x(e_1 \cdot e_{2t}) = -\frac{1}{2}\partial_x(\alpha^2 + \beta^2). \tag{4.2}$$

This fact, together with the property that (T, e_1, e_2) is an orthonormal frame for all (x, t) gives

$$\begin{aligned} T_t &= -\beta_x e_1 + \alpha_x e_2 \\ e_{1_t} &= \beta_x T + \frac{1}{2}((\alpha^2 + \beta^2) - \mathcal{M}(t))e_2 \\ e_{2_t} &= -\alpha_x T - \frac{1}{2}((\alpha^2 + \beta^2) - \mathcal{M}(t))e_1, \end{aligned}$$

for some real function $\mathcal{M}(t)$.

Notice that this is just a linear system of equations which is hamiltonian and that satisfies the three conservation laws (2.4), (3.7), and (3.9) given in the previous sections for u of type (2.5). Nevertheless, Theorem 4.1 applies and therefore this system also exhibits a cascade of energy.

5 Talbot Effect and Rogue Waves

In this section we want to revisit the examples on the Talbot effect showed in [3]. The Talbot effect was originally observed in linear Optics and has received plenty of attention since the pioneering work of M. Berry and collaborators (see for example [7]). In [17] the fractal properties of this effect is considered, taking in (1.4) regular polygons as initial data. These examples suggest a possible connection with the turbulent dynamics observed in non-circular jets (see for instance [24]). At a more regular level, the Talbot effect for the linear and nonlinear Schrödinger equations with initial data periodic and given by functions with bounded variation was studied in [12, 19, 33, 37, 40]. The fractal behavior was already seen numerically in the setting of the architecture of aortic valve fibers in [36, 38]. These works use (1.4) but with boundary conditions which are not periodic.

The Talbot effect is very well described by (1.1). As it will be shown below in (5.2) and (5.3), the values of (1.1) at times which are rational multiples of the period can be written in a closed form: if the rational is p/q then Dirac deltas appear at all the rationals of $\mathbb{Z}/q$ in space, and the amplitudes are given by a corresponding Gauss sum. Going either backward or forward in time a phenomenon of constructive/destructive interference appears. We think that there is a similitude of this phenomenon with the one exhibited in [20] related to the so-called Rogue Waves.

The example we propose is very similar to (3.8). Recall that the construction of the solution we have sketched in the previous sections is perturbative and therefore, it always implies some smallness condition. This condition is measured in terms of $\sum |a_j|$ that can be small without the corresponding solution u being small. For example, from (2.5) it is immediate that at least for small times the L^∞ norm of u is not small. Something similar can be said for the

$L^1_{\rm loc}$ norm. At this respect it is relevant to notice the definition of u in terms of V given in (2.8). Observe that

$$|u(x,t)| = \frac{1}{\sqrt{4\pi t}}\,|V(x/2t, 1/t)|,$$

and therefore $L^1_{\rm loc}$ grows with t.

We have the following result.

Theorem 5.1 (Appearance of Rogue Waves) *Let $0<\eta<\frac{1}{4}$ and let $p\in\mathbb{N}$ large. There exists u_0 with $\widehat{u_0}$ a 2π-periodic function, supported modulo 2π in $\left[-\eta\frac{2\pi}{p}, \eta\frac{2\pi}{p}\right]$, such that the solution $u(x,t)$ of (1.2) obtained from $a_k=\widehat{u_0}(k)$ satisfies the following properties:*

- *If $t_{\tilde p,\tilde q}=\frac{1}{2\pi}\frac{\tilde p}{\tilde q}$ with $\tilde p\approx\tilde q\approx 1$, coprime, $\tilde p<\tilde q$, and $\tilde q$ odd, then at times $t_{\tilde p,\tilde q}$ on the interval $[-\frac{1}{2\tilde q}, \frac{1}{2\tilde q}]$ the solution $u(x,t)$ is localized and has a large amplitude;*
- *If $t_{p,q}=\frac{1}{2\pi}\frac{p}{q}$ with $q\approx p$, coprime, and q odd, then at times $t_{p,q}$ on the interval $[-\frac{1}{2\tilde q}, \frac{1}{2\tilde q}]$ the solution $u(x,t)$ has a small amplitude.*

We start with a computation for the linear Schrödinger equation on the line, concerning the Talbot effect related to (1.1).

Proposition 5.2 (Talbot effect for linear evolutions) *Let $0<\eta<\frac{1}{4}$, $p\in\mathbb{N}$ and u_0 such that $\widehat{u_0}$ is a 2π-periodic function, supported modulo 2π in $[-\eta\frac{2\pi}{p}, \eta\frac{2\pi}{p}]$. For all $t_{p,q}=\frac{1}{2\pi}\frac{p}{q}$ with q odd and for all $x\in\mathbb{R}$ we define*

$$\xi_x := \frac{\pi q}{p}\, d\left(x, \frac{1}{q}\mathbb{Z}\right) \in \left[0, \frac{\pi}{p}\right).$$

Then, there exists $\theta_{x,p,q}\in\mathbb{R}$ such that

$$e^{it_{p,q}\Delta}u_0(x) = \frac{1}{\sqrt{q}}\,\widehat{u_0}(\xi_x)\, e^{-it_{p,q}\,\xi_x^2+ix\,\xi_x+i\theta_{x,p,q}}. \tag{5.1}$$

In particular $|e^{it_{p,q}\Delta}u_0|$ is $\frac{1}{q}$-periodic and if $d(x,\frac{1}{q}\mathbb{Z})>\frac{2\eta}{q}$ then $e^{it_{p,q}\Delta}u_0(x)$ vanishes.

Proof We start by recalling the Poisson summation formula $\sum_{k\in\mathbb{Z}} f(k) = \sum_{k\in\mathbb{Z}}\hat f(2\pi k)$ for the Dirac comb:

$$\left(\sum_{k\in\mathbb{Z}}\delta_k\right)(x) = \sum_{k\in\mathbb{Z}}\delta(x-k) = \sum_{k\in\mathbb{Z}} e^{i2\pi kx},$$

as

$$\widehat{\delta(x-\cdot)}(2\pi k) = \int_{-\infty}^{\infty} e^{-i2\pi ky}\delta(x-y)\,dy = e^{-i2\pi kx}.$$

The computation of the free evolution with Dirac comb data is

$$e^{it\Delta}\left(\sum_{k\in\mathbb{Z}}\delta_k\right)(x) = \sum_{k\in\mathbb{Z}} e^{-it(2\pi k)^2+i2\pi kx}. \tag{5.2}$$

For $t=\frac{1}{2\pi}\frac{p}{q}$ we have (choosing $M=2\pi$ in formulas (37) combined with (42) from [17])

$$e^{it\Delta}\left(\sum_{k\in\mathbb{Z}}\delta_k\right)(x) = \frac{1}{q}\sum_{l\in\mathbb{Z}}\sum_{m=0}^{q-1} G(-p,m,q)\delta\left(x-l-\frac{m}{q}\right), \tag{5.3}$$

which describes the linear Talbot effect in the periodic setting. Here $G(-p,m,q)$ stands for the Gauss sum

$$G(-p,m,q)=\sum_{l=0}^{q-1} e^{2\pi i\frac{-pl^2+ml}{q}}.$$

Now we compute the free evolution of data u_0 with $\widehat{u_0}$ a 2π-periodic function, i.e. $\widehat{u_0}(\xi)=\sum_{k\in\mathbb{Z}}\alpha_k e^{-ik\xi}$ and $u_0=\sum_{k\in\mathbb{Z}}\alpha_k\delta_k$:

$$\begin{aligned}
e^{it\Delta}u_0(x)&=\frac{1}{2\pi}\int_{-\infty}^{\infty} e^{ix\xi}e^{-it\xi^2}\widehat{u_0}(\xi)\,d\xi\\
&=\frac{1}{2\pi}\sum_{k\in\mathbb{Z}}\int_{2\pi k}^{2\pi(k+1)} e^{ix\xi-it\xi^2}\widehat{u_0}(\xi)\,d\xi\\
&=\frac{1}{2\pi}\int_0^{2\pi}\widehat{u_0}(\xi)\sum_{k\in\mathbb{Z}} e^{ix(2\pi k+\xi)-it(2\pi k+\xi)^2}\,d\xi\\
&=\frac{1}{2\pi}\int_0^{2\pi}\widehat{u_0}(\xi)e^{-it\xi^2+ix\xi}\sum_{k\in\mathbb{Z}} e^{-it\,(2\pi k)^2+i2\pi k(x-2t\xi)}\,d\xi.
\end{aligned}$$

Therefore, for $t_{p,q}=\frac{1}{2\pi}\frac{p}{q}$ we get using (5.2)–(5.3):

$$e^{it_{p,q}\Delta}u_0(x)=\frac{1}{q}\int_0^{2\pi}\widehat{u_0}(\xi)e^{-it_{p,q}\xi^2+ix\xi}\sum_{l\in\mathbb{Z}}\sum_{m=0}^{q-1}G(-p,m,q)\delta\left(x-2t_{p,q}\xi-l-\frac{m}{q}\right)\,d\xi.$$

For q odd $G(-p,m,q)=\sqrt{q}e^{i\theta_{m,p,q}}$ for some $\theta_{m,p,q}\in\mathbb{R}$ so we get for $t_{p,q}=\frac{1}{2\pi}\frac{p}{q}$

$$e^{it_{p,q}\Delta}u_0(x)=\frac{1}{\sqrt{q}}\int_0^{2\pi}\widehat{u_0}(\xi)e^{-it_{p,q}\xi^2+ix\xi}\sum_{l\in\mathbb{Z}}\sum_{m=0}^{q-1}e^{i\theta_{m,p,q}}\delta\left(x-2t_{p,q}\xi-l-\frac{m}{q}\right)\,d\xi.$$

For a given $x\in\mathbb{R}$ there exists a unique $l_x\in\mathbb{Z}$ and a unique $0\le m_x<q$ such that

$$x-l_x-\frac{m_x}{q}\in\left[0,\frac{1}{q}\right),\quad \xi_x=\frac{\pi q}{p}\left(x-l_x-\frac{m_x}{q}\right)=\frac{\pi q}{p}\,d\left(x,\frac{1}{q}\mathbb{Z}\right)\in\left[0,\frac{\pi}{p}\right).$$

We note that for $0\le\xi<\eta\frac{2\pi}{p}$ we have $0\le 2t\xi<\frac{1}{2q}$. As $\widehat{u_0}$ is supported modulo 2π only in a neighborhood of zero of radius less than $\frac{\eta}{2\pi}p$ then we get the expression (5.1). □

Proof of Theorem 5.1 We shall construct sequences $\{\alpha_k\}$ such that $\sum_{k\in\mathbb{Z}}\alpha_k\delta_k$ concentrates in the Fourier variable near the integers. To this purpose we consider, for $s>\frac{1}{2}$, a positive bounded function $\psi\in H^s$ with support in $[-1,1]$ and maximum at $\psi(0)=1$. We define the 2π-periodic function satisfying

$$f(\xi):=p^\beta\psi\left(\frac{p}{2\pi\eta}\xi\right),\quad\forall\xi\in[-\pi,\pi],$$

with $\beta<\frac{1}{2}-\frac{3}{2}s$, introduce its Fourier coefficients:

$$f(\xi):=\sum_{k\in\mathbb{Z}}\alpha_k e^{ik\xi},$$

and the distribution

$$u_0 := \sum_{k\in\mathbb{Z}} \alpha_k \delta_k.$$

In particular, on $[-\pi,\pi]$, we have $\widehat{u_0} = f$ and the restriction of $\widehat{u_0}$ to $[-\pi,\pi]$ has support included in a neighborhood of zero of radius less than $\eta\frac{2\pi}{p}$. We then get from (5.1):

$$e^{it_{p,q}\Delta}u_0(x) = \frac{1}{\sqrt{q}}\,\widehat{u_0}\left(\frac{\pi q}{p}\,d\left(x,\frac{1}{q}\mathbb{Z}\right)\right)\,e^{-it_{p,q}\,\xi_x^2+ix\,\xi_x+i\theta_{m_x}},$$

that

$$\left|e^{it_{p,q}\Delta}u_0(0)\right| = \frac{1}{\sqrt{q}}\,|f(0)| = \frac{1}{\sqrt{q}}p^\beta\psi(0) = \frac{p^\beta}{\sqrt{q}}, \tag{5.4}$$

$$\left\|e^{it_{p,q}\Delta}u_0\right\|_{L^\infty} \le \frac{1}{\sqrt{q}}\,\|f\|_{L^\infty} = \frac{p^\beta}{\sqrt{q}}, \tag{5.5}$$

and

$$e^{it_{p,q}\Delta}u_0(x) = 0 \quad \text{if } d\left(x,\frac{1}{q}\mathbb{Z}\right) > \frac{2\eta}{q}. \tag{5.6}$$

We note that

$$\|\alpha_k\|^2_{l^{2,r}} = \sum_k |k|^{2r}|\alpha_k|^2 = \|f\|^2_{\dot{H}^r} = \frac{p^{2(\beta+r-\frac{1}{2})}}{(2\pi\eta)^{2(r-\frac{1}{2})}}\|\psi\|_{\dot{H}^r}.$$

Since $\beta < \frac{1}{2} - s$ and p is large, it follows that $\|\alpha_k\|_{l^{2,s}}$ is small enough so that we can use the results in [3] to construct a solution up to time $t = \frac{1}{2\pi}$ for (1.2) of type

$$u(x,t) = \sum_{k\in\mathbb{Z}} e^{i(|\alpha_k|^2-2\sum_j|\alpha_j|^2)\log t}(\alpha_k + R_k(t))e^{it\Delta}\delta_k(x).$$

Hence

$$\begin{aligned}
&\left|u(t,x) - e^{it\Delta}\left(\sum_{k\in\mathbb{Z}}\alpha_k\delta_k\right)(x)\right| \\
&\le \sum_{k\in\mathbb{Z}}\left(1 - e^{i(|\alpha_k|^2-2\sum_j|\alpha_j|^2)\log t}\right)\alpha_k e^{it\Delta}\delta_k(x) + \sum_{k\in\mathbb{Z}} e^{i(|\alpha_k|^2-2\sum_j|\alpha_j|^2)\log t}R_k(t)e^{it\Delta}\delta_k(x) \\
&\le \frac{C}{\sqrt{t}}\|\alpha_k\|^2_{l^2}\|\alpha_k\|_{l^{2,s}} + \frac{Ct^\gamma}{\sqrt{t}}\|\alpha_k\|^3_{l^{2,s}} = C(\eta)\frac{p^{3(\beta+s-\frac{1}{2})}}{\sqrt{t}}(p^{-s}+t^\gamma),
\end{aligned}$$

for $\gamma < 1$.

Therefore, in view of (5.4), (5.5), and (5.6) we have for times $t_{p,q}$ and $t_{\tilde{p},\tilde{q}}$ both of size $\frac{1}{2\pi}$, but with rational representation of type $q \approx p$ which is fixed to be large, and $\tilde{p} \approx \tilde{q} \approx 1$ with $\tilde{p} < \tilde{q}$, that:

- at time $t_{p,q}$ the modulus $|u(t_{p,q},x)|$ is a $\frac{1}{q}$-periodic function of maximal amplitude $\frac{1}{p^{\frac{1}{2}-\beta}}$ plus a remainder term of size $\frac{1}{p^{3(-\beta-s+\frac{1}{2})}}$, that is negligible provided that $\beta < \frac{1}{2} - \frac{3}{2}s$. So modulo negligible terms $|u(t_{p,q},x)|$ has plenty of $\frac{1}{p}$-period waves of small amplitude $\frac{1}{p^{\frac{1}{2}-\beta}}$,

– at time $t_{\tilde{p},\tilde{q}}$ the modulus $|u(t_{\tilde{p},\tilde{q}}, x)|$ is a $\frac{1}{\tilde{q}}$-periodic function of maximal amplitude $\frac{1}{\tilde{p}^{\frac{1}{2}-\beta}}$ plus a remainder term of size $\frac{1}{\tilde{p}^{3(-\beta-s+\frac{1}{2})}}$, that is again negligible provided that $\beta < \frac{1}{2} - \frac{3}{2}s$. So modulo negligible terms $|u(t_{\tilde{p},\tilde{q}}, x)|$ has in the interval $I := [-\frac{1}{2\tilde{q}}, \frac{1}{2\tilde{q}}]$ a wave of amplitude $\frac{1}{\tilde{p}^{\frac{1}{2}-\beta}}$, and is upper-bounded by a smaller value on $I \setminus [-\frac{2\eta}{\tilde{q}}, \frac{2\eta}{\tilde{q}}]$.

Therefore, observing what happens in the interval I we have at time $t_{p,q}$ small waves while at time $t_{\tilde{p},\tilde{q}}$ a localized large-amplitude (with respect to η) structure emerges. □

Remark 5.3 1. In the above argument we need η to be small. As a consequence, the L^∞ norm and therefore the L^1_{loc} norm of the solution is small. This can be avoided by considering $u_\lambda = \frac{1}{\lambda}u(x/\lambda, t/\lambda^2)$, where u is any of the solutions constructed above. If $\lambda > 1$ the L^∞ norm grows, while for $\lambda < 1$ the L^1_{loc} norm around the corresponding bump grows.

2. The size of the error can be made smaller following the ideas developed in [5]. This is due to the type of data (3.8) we are using. In this case the size of the first Picard iterate is indeed much smaller than the l^1 norm we are using in the above argument.

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