REVIEW ARTICLE

A Survey of Scattering Theory on Riemann Surfaces with Applications in Global Analysis and Geometry

Eric Schippers[1] · Wolfgang Staubach[2]

Abstract

This paper gives an overview of our work on a scattering theory of one-forms and functions in a system of quasicircles on Riemann surfaces. It is rooted in an "overfare" process which takes a harmonic function on one side of the system of quasicircles to a harmonic function on the other side, with the same boundary values in a certain intrinsic non-tangential sense. This is bounded with respect to Dirichlet energy. If extra cohomological data is specified, one can apply this process to harmonic one-forms, and the resulting "scattering matrix" in terms of the holomorphic and anti-holomorphic components of the one-form is unitary. We describe applications to approximation theory, global analysis of singular integral operators on Riemann surfaces, and a new extension of the classical period map to surfaces of genus g with n boundary curves.

1 Introduction

Our goal is to give a survey of our recent work [24] on a scattering theory of one-forms on Riemann surfaces and highlight some of its applications. These include applications to global analysis (e.g. new index theorems on Riemann surfaces relating conformal and topo-

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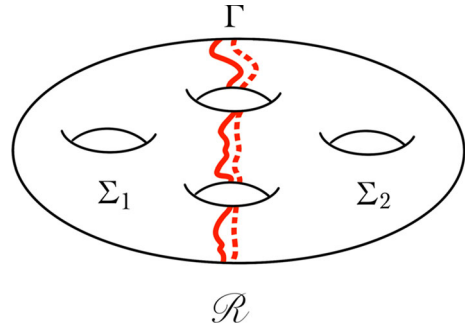To Carlos E. Kenig for the occasion of his 70th birthday.

✉ Wolfgang Staubach
  wulf@math.uu.se

  Eric Schippers
  eric.schippers@umanitoba.ca

[1] Machray Hall, Department of Mathematics, University of Manitoba, R3T 2N2 Winnipeg, MB, Canada

[2] Department of Mathematics, Uppsala University, S-751 06 Uppsala, Sweden

Fig. 1 The Riemann surface with a system of curves

logical invariants on a surface), approximation theory on Riemann surfaces (for functions and one-forms), boundary values of holomorphic one forms and Grunsky operators on Riemann surfaces, a formula $\log \det \left(\dot{\mathbf{J}}^* \dot{\mathbf{J}} \right)$ for the Kähler potential of the Weil–Petersson class universal Teichmüller spaces in terms of a Cauchy-type operator $\dot{\mathbf{J}}$, and algebraic geometry (polarizations of surfaces with boundary and period matrices). Underlying the analytic formulation is a new conformally invariant approach to Sobolev spaces of harmonic functions on Riemann surfaces, which leads to a new characterization of $H^{-1/2}$ spaces on the boundary in terms of equivalence classes of harmonic one-forms.

We now describe the process more precisely. The scattering takes place in a system of curves which separate the Riemann surface into at least two connected components. Let $\mathscr{R}$ be a compact surface divided by a complex Γ of simple closed curves into surfaces Σ_1 and Σ_2, see Fig. 1.

The number of curves is arbitrary, and we allow either Σ_1 or Σ_2 to be disconnected, but not both. Now given a harmonic function h_1 on Σ_1, it has boundary values on Γ, which in turn uniquely determine a harmonic function h_2 on Σ_2 with the same boundary values. We call h_2 the "overfare" of h_1 and write $h_2 = \mathbf{O}_{1,2} h_1$.

For harmonic one-forms, there is a similar overfare procedure. Briefly, one finds an anti-derivative of a form α_1 on Σ_1, applies the overfare $\mathbf{O}_{1,2}$ to the anti-derivative, and differentiates the result to obtain a form α_2 on Σ_2. However, α_1 need not be exact and thus might have no primitive, and furthermore one must also specify the cohomological properties of the form α_2. We deal with this by specifying a harmonic one-form ζ on $\mathscr{R}$ such that $\alpha_1 - \zeta$ is exact on Σ_1, and let α_2 be such that $\alpha_2 - \zeta$ is exact on Σ_2. Thus the extra cohomological data required to specify the overfare of harmonic one-forms is identified with the finite-dimensional vector space of harmonic one-forms on $\mathscr{R}$. In general, neither an overfared function nor a one-form is harmonic on the union.

In our earlier work, we referred to overfare as "harmonic reflection" or "transmission". But these terms are in use with different meanings, some of which appear in related topics (e.g. the "transmission coefficient" in scattering theory). So we were led to coin a different name. We settled on "overfare", derived from Old English "oferferian", which is somewhat suggestive of the concept to modern English speakers. Furthermore, it has cognates in modern related languages (e.g. Plattdüütsch "overföhren", Swedish "överföra", German "überführen") which are also suggestive of "pushing through" or "travelling over".

In analogy with potential-well scattering on the real line, we can regard the aforementioned α_2 as the form obtained from α_1 through scattering. In this scattering process, the curves themselves play the role of the potential well. We assume only that the curves are quasicircles, which generically are non-rectifiable curves arising in quasiconformal Teichmüller theory.

To illustrate the idea and clarify the analogies, we describe the scattering process in the case of the Riemann sphere. Choose a quasicircle Γ separating 0 and ∞, and let Σ_1 and Σ_2 be the components of the complement of Γ in the sphere. We consider harmonic one-forms related by overfare across Γ. As in two-dimensional conformal field theory, we think of the radial direction as time on a logarithmic scale, so that the origin corresponds to time $-\infty$ and the point at infinity corresponds to time ∞.

We can think of the overfared form as solving Laplace's equation with a potential. Free solutions are the harmonic forms, which are harmonic on the domains Σ_1 and Σ_2, but not across the quasicircle "seam". The source has support on the quasicircle itself, that is, the seam is in some sense a thin potential well.

We identify the incoming and outgoing solutions with the harmonic forms $\alpha_1 + \overline{\beta}_1$ (on Σ_1) and $\alpha_2 + \overline{\beta}_2$ (on Σ_2); that is, the free solutions at $t = -\infty \sim z = 0$ and $t = \infty \sim z = \infty$, respectively. We identify the positive and negative modes of the solution with the holomorphic and anti-holomorphic parts of the solution respectively. Thus the solution to the scattering problem is $\alpha_k + \overline{\beta}_k$ on Σ_k. As in the case of scattering on the real line, we can identify a scattering matrix, which in our case relates the positive and negative modes, and is unitary. Finally, we observe that in the higher genus case, we must add the extra cohomological data to this scattering matrix.

This analogy suggests many interesting questions and conjectures in complex geometry and Teichmüller theory, especially in association with solutions to the Beltrami equation. But elaborating on this would place too many demands on a short survey. For now we content ourselves with some tantalizing hints and concrete theorems.

We would also mention that the scattering theory that is developed here is different than the scattering theory of P. Lax and R. Phillips [10] which deals with the spectral decomposition of the hyperbolic Laplacian $\Delta_{\mathbb{H}}$ in $L^2(\mathbb{H}/\Gamma)$ where Γ is a Fuchsian group, using the scattering theory of the automorphic wave equation $\partial_t^2 u = \Delta_{\mathbb{H}} u$.

In the process of overfaring harmonic functions, there are two analytic problems to be resolved. The first is to define the boundary values in preparation for overfare, and the second is to show the existence and continuous dependence of the overfare. The first problem is in a certain sense independent of the boundary regularity, while the second problem is more delicate and sensitive to the regularity of the curve.

In defining the boundary values, the nature of the approach to the boundary could be defined either extrinsically in terms of the geometry of the ambient space containing the curve, or in terms of the intrinsic geometry of the region on which the function is defined. For example, since harmonic functions with L^2 derivatives are in the Sobolev space H^1 for a wide class of curves, one could consider the Sobolev trace to the boundary; in this case, one would need to take into account the regularity of the boundary for this to be defined. The possibility of dealing with boundaries that may not be rectifiable would add additional difficulties. In this context, there are the seminal papers of C. Kenig and D. Jerison on the boundary behavior of harmonic functions in nontangentially accessible (NTA) domains in $\mathbb{R}^n$, see e.g. [6, 7]. Given a bounded open subset Ω in $\mathbb{R}^n$, one defines nontangential balls as those balls for which diameter and distance to the boundary $\partial\Omega$ of Ω are of the same order of magnitude. An NTA domain is a domain Ω such that any point $p \in \partial\Omega$ can be approached in some "suitable" way by a chain of nontangential balls lying in Ω, and also by nontangential balls lying in the complement of Ω. The corresponding regular chains are then used in order to define nontangential notions similar to the one used in the classical case. Now as was shown by P. Jones [8] for a domain in the plane Ω, one has that Ω in NTA if and only if Ω is a quasidisk.

Our approach to boundary values proceeds intrinsically, in such a way that the boundary can be viewed as the ideal boundary of Σ_1, which does not depend on the geometry of the boundary in $\mathscr{R}$. Equivalently, it can be regarded as an analytic Jordan curve in the double of Σ_1. We realize this "radial" approach by defining a kind of conformally non-tangential boundary value (referred to as CNT boundary values), in which non-tangential cones are defined in terms of "collar charts" taking collar neighbourhoods of the boundary to annuli. Then, a classical theorem of A. Beurling applies to show that the boundary values exist except on a Borel set of logarithmic capacity zero in the circle under the chart (we call this a null set). We show that this notion of boundary value is independent of the choice of collar chart; this is essentially because the angle of approach to the ideal boundary is a well-defined conformal invariant. Thus the boundary values are defined along any non-tangentially approaching curve. The independence of the boundary values on the choice of collar chart is a key tool in the application of the cutting and sewing approach to boundary value problems which we have developed in this and other papers [20, 21].

The paper is organised as follows. In Section 2, we review the definitions of Bergman and Dirichlet spaces of functions and forms, collar charts, CNT boundary values, harmonic measures and Green functions on Riemann surfaces. In Section 3, we give an exposition of our new approach to conformally invariant Sobolev spaces on Riemann surfaces of harmonic functions and forms, which in a sense is the ground on which our overfare paradigm stands. Section 4 is devoted to the definitions and properties of the overfare operators. In Section 5, we introduce the Schiffer and restriction operators that are the building-blocks of our scattering matrix $\mathcal{S}$. We also describe the scattering process and briefly discuss the unitarity of $\mathcal{S}$. Finally in Section 6, we discuss the applications of the scattering theory to approximation theory, global analysis and complex geometry, for example index theorems and explicit formulas for the Kähler potential of the Weil–Petersson metric on universal Teichmüller space. Applications to period matrices are also discussed.

2 Spaces of Functions and Forms

In this paper, we will denote positive constants in the inequalities by C whose value is not crucial to the problem at hand. The value of C may differ from line to line, but in each instance could be estimated if necessary. Moreover, when the values of constants in our estimates are of no significance for our main purpose, then we use the notation $a \lesssim b$ as a shorthand for $a \leq Cb$. If $a \lesssim b$ and $b \lesssim a$ then we write $a \approx b$.

On any Riemann surface, define the dual of the almost complex structure, $*$ in local coordinates $z = x + iy$, by

$$*(a\,dx + b\,dy) = a\,dy - b\,dx.$$

This is independent of the choice of coordinates. It can also be computed in coordinates that for any complex function h

$$2\partial_z h = dh + i * dh.$$

Definition 2.1 We say a complex-valued function f on an open set U is *harmonic* if it is C^2 on U and $d * df = 0$. We say that a complex one-form α is harmonic if it is C^1 and satisfies both $d\alpha = 0$ and $d * \alpha = 0$.

Equivalently, harmonic one-forms are those which can be expressed locally as df for some harmonic function f. Harmonic one-forms and functions must of course be C^∞.

Denote complex conjugation of functions and forms with a bar, e.g. $\bar{\alpha}$. A holomorphic one-form is one which can be written in coordinates as $h(z)\,dz$ for a holomorphic function h, while an anti-holomorphic one-form is one which can be locally written $\overline{h(z)}\,d\bar{z}$ for a holomorphic function h.

Denote by $L^2(U)$ the set of one-forms ω on an open set U which satisfy

$$\iint_U \omega \wedge *\bar{\omega} < \infty$$

(observe that the integrand is positive at every point, as can be seen by writing the expression in local coordinates). This is a Hilbert space with respect to the inner product

$$(\omega_1, \omega_2) = \iint_U \omega_1 \wedge *\overline{\omega_2}. \tag{2.1}$$

Definition 2.2 The *Bergman space of holomorphic one-forms* is

$$\mathcal{A}(U) = \{\alpha \in L^2(U) \,:\, \alpha \text{ holomorphic}\}.$$

The anti-holomorphic Bergman space is denoted $\overline{\mathcal{A}(U)}$. We will also denote

$$\mathcal{A}_{\text{harm}}(U) = \{\alpha \in L^2(U) \,:\, \alpha \text{ harmonic}\}.$$

Observe that $\mathcal{A}(U)$ and $\overline{\mathcal{A}(U)}$ are orthogonal with respect to the inner product (2.1). In fact we have the direct sum decomposition

$$\mathcal{A}_{\text{harm}}(U) = \mathcal{A}(U) \oplus \overline{\mathcal{A}(U)}.$$

If we restrict the inner product to $\alpha, \beta \in \mathcal{A}(U)$ then since $*\bar{\beta} = i\bar{\beta}$, we have

$$(\alpha, \beta) = i \iint_U \alpha \wedge \bar{\beta}.$$

Denote the projections induced by this decomposition by

$$\mathbf{P}_U : \mathcal{A}_{\text{harm}}(U) \to \mathcal{A}(U),$$
$$\overline{\mathbf{P}}_U : \mathcal{A}_{\text{harm}}(U) \to \overline{\mathcal{A}(U)}.$$

Let $f : U \to V$ be a biholomorphism. We denote the pull-back of $\alpha \in \mathcal{A}_{\text{harm}}(V)$ under f by $f^*\alpha$. Explicitly, if α is given in local coordinates w by $a(w)\,dw + \overline{b(w)}\,d\bar{w}$ and $w = f(z)$, then the pull-back is given by

$$f^* \left(a(w)\,dw + \overline{b(w)}\,d\bar{w} \right) = a(f(z))f'(z)\,dz + \overline{b(f(z))f'(z)}\,d\bar{z}. \tag{2.2}$$

The Bergman spaces are all conformally invariant, in the sense that if $f : U \to V$ is a biholomorphism, then $f^*\mathcal{A}(V) = \mathcal{A}(U)$ and this preserves the inner product. The same holds for the anti-holomorphic and harmonic spaces.

Definition 2.3 We define the space $\mathcal{A}^e_{\text{harm}}(U)$ as the subspace of exact elements of $\mathcal{A}_{\text{harm}}(U)$, and similarly for $\mathcal{A}^e(\Sigma)$ and $\overline{\mathcal{A}^e(\Sigma)}$.

By a bordered surface of type (g, n) we mean a surface with g handles and n borders, each of which is homeomorphic to the circle $\mathbb{S}^1$. Here the term "border" is used in the strong sense of a complex analytic boundary of the manifold, see e.g. Ahlfors and Sario [1]. We assume that the surface with the borders included is compact. One-forms which have zero periods along the borders are called *semi-exact*.

Definition 2.4 Let Σ be a bordered surface of type (g, n). We say that an L^2 one-form $\alpha \in \mathcal{A}_{\text{harm}}(\Sigma)$ is semi-exact if for any simple closed curve γ isotopic to a boundary curve $\partial_k \Sigma$,

$$\int_\gamma \alpha = 0.$$

The class of semi-exact one-forms on Σ is denoted $\mathcal{A}_{\text{harm}}^{\text{se}}(\Sigma)$. The holomorphic and anti-holomorphic semi-exact one-forms are denoted by $\mathcal{A}^{\text{se}}(\Sigma)$ and $\overline{\mathcal{A}^{\text{se}}(\Sigma)}$ respectively.

The following spaces also play significant roles in this paper.

Definition 2.5 The *Dirichlet spaces of functions* are defined by

$$\mathcal{D}_{\text{harm}}(U) = \{f : U \to \mathbb{C}, f \in C^2(U) \;:\; df \in L^2(U) \text{ and } d * df = 0\},$$
$$\mathcal{D}(U) = \{f : U \to \mathbb{C} \;:\; df \in \mathcal{A}(U)\}, \quad \text{and}$$
$$\overline{\mathcal{D}(U)} = \{f : U \to \mathbb{C} \;:\; df \in \overline{\mathcal{A}(U)}\}.$$

We can define a degenerate inner product on $\mathcal{D}_{\text{harm}}(U)$ by

$$(f, g)_{\mathcal{D}_{\text{harm}}(U)} = (df, dg)_{\mathcal{A}_{\text{harm}}(U)},$$

where the right hand side is the inner product (2.1) restricted to elements of $\mathcal{A}_{\text{harm}}(U)$. The inner product can be used to define a seminorm on $\mathcal{D}_{\text{harm}}(U)$, by letting

$$\|f\|_{\mathcal{D}_{\text{harm}}(U)}^2 := (df, df)_{\mathcal{A}_{\text{harm}}(U)}.$$

We note that if one defines the *Wirtinger operators* via their local coordinate expressions

$$\partial f = \frac{\partial f}{\partial z} \, dz, \quad \overline{\partial} f = \frac{\partial f}{\partial \overline{z}} \, d\overline{z},$$

then the aforementioned inner product can be written as

$$(f, g)_{\mathcal{D}_{\text{harm}}(U)} = i \iint_U \left[\partial f \wedge \overline{\partial g} - \overline{\partial} f \wedge \partial \overline{g} \right].$$

Although this implies that $\mathcal{D}(U)$ and $\overline{\mathcal{D}(U)}$ are orthogonal, there is no direct sum decomposition into $\mathcal{D}(U)$ and $\overline{\mathcal{D}(U)}$. This is because in general there exist exact harmonic one-forms whose holomorphic and anti-holomorphic parts are not exact.

Observe that the Dirichlet spaces are conformally invariant in the same sense as the Bergman spaces. That is, if $f : U \to V$ is a biholomorphism then

$$\mathbf{C}_f h = h \circ f$$

satisfies

$$\mathbf{C}_f : \mathcal{D}(V) \to \mathcal{D}(U)$$

and this is a seminorm preserving bijection. Similar statements hold for the anti-holomorphic and harmonic spaces.

In general, we define the Sobolev space of a bordered surface Σ in the following way. We treat Σ as a subset of its compact double Σ^d, so that the borders are analytic curves and in particular smooth. By the uniformization theorem, the double has a constant curvature Riemannian metric compatible with the complex structure. The Sobolev space $H^s(\Sigma)$ consists of restrictions of $H^s(\Sigma^d)$ to Σ. We can similarly define $H^s(\partial \Sigma)$ in the standard way.

We define a kind of chart on bordered surfaces near the boundary, which we call a collar chart.

Definition 2.6 Let Σ be a bordered Riemann surface of type (g, n). Let $\mathbb{A}_{r,1}$ denote the annulus $\{z : r < |z| < 1\}$. A biholomorphism $\phi : U \to \mathbb{A}_{r,1}$ is called a *collar chart* of $\partial_k \Sigma$ (for some fixed k) if U is an open set in Σ bounded by two Jordan curves $\partial_k \Sigma$ and Γ, such that Γ is isotopic to $\partial_k \Sigma$ within the closure of U, and such that ϕ extends homeomorphically to the closure. A domain U is called a *collar neighbourhood* of $\partial_k \Sigma$ if it is the domain of some collar chart.

One can show that every boundary curve $\partial_k \Sigma$ has a collar chart.

Next we define a notion of non-tangential limit which in a sense is the natural notion of non-tangential limit on the border of a Riemann surface.

Definition 2.7 Let Γ be a border of Σ with a collar chart ψ of Γ in Σ. The conformally non-tangential limit (denoted henceforth by CNT limit) of a function $h : \Sigma \to \mathbb{C}$ at $p \in \Gamma$ is ζ if $h \circ \psi^{-1}$ has a non-tangential limit of ζ at $\psi(p)$.

The CNT limit has the following three basic properties [24]:

- It is independent of the choice of ψ.
- It is the same as that obtained by treating Γ as the abstract border of the Riemann surface Σ.
- It is conformally invariant, meaning that if $F : \Sigma_1 \to \Sigma$ is a conformal map, then h has a CNT limit of ζ at p if and only if $h \circ F$ has a CNT limit of ζ at $F^{-1}(p)$.

Next, we define a potential-theoretically negligible set on the border which we call a null set.

Definition 2.8 We say that a set $I \subset \Gamma$ is null if it is a Borel set and $\psi(I)$ has logarithmic capacity zero in $\mathbb{S}^1$.

This definition is independent of the choice of ψ, but rather interestingly, not independent of which side of Γ one is at.

Using a classical theorem of Beurling and some surgery, the authors proved in [20] the following result:

Theorem 2.9 *A function $h \in \mathcal{D}_{\mathrm{harm}}(\Sigma)$ has* CNT *boundary values (i.e.* CNT *limit on the boundary) except possibly on a null set.*

A suitable adaptation of the proof of [20, Theorem 3.17] also yields

Theorem 2.10 *Let Σ be a bordered surface of type (g, n) and let $U_k \subseteq \Sigma$ be collar neighbourhoods of $\partial_k \Sigma$ for $k = 1, \ldots, n$. Let $h_k \in \mathcal{D}_{\mathrm{harm}}(U_k)$ for $k = 1, \ldots, n$. There is a function $H \in \mathcal{D}_{\mathrm{harm}}(\Sigma)$ whose* CNT *boundary values agree with those of h_k on $\partial_k \Sigma$ up to a null set for each $k = 1, \ldots, n$.*

We thus make the following definition.

Definition 2.11 Let Σ be a Riemann surface and let $\partial_k \Sigma$ be a component of the border of Σ homeomorphic to $\mathbb{S}^1$. Given functions $h_j : \partial_k \Sigma \backslash I_j \to \mathbb{C}$, $j = 1, 2$, where I_1 and I_2 are null sets, we say that $h_1 \sim h_2$ if h_1 and h_2 are both defined on $\partial_k \Sigma \backslash I$ for some null set I and $h_1 = h_2$ on $\partial_k \Sigma \backslash I$.

In order to define conformally invariant Sobolev H^1-spaces we use harmonic measure on bordered Riemann surfaces. We can also use Green's functions to give an equivalent definition of these spaces. First we recall the notion of harmonic measure in the context of bordered Riemann surfaces.

Definition 2.12 Let Σ be a bordered surface of type (g, n). Let ω_k, $k = 1, \ldots, n$ be the unique harmonic function which is continuous on the closure of Σ and which satisfies

$$\omega_k = \begin{cases} 1 & \text{on } \partial_k \Sigma, \\ 0 & \text{on } \partial_j \Sigma, \ \ j \neq k. \end{cases}$$

The one-forms $d\omega_k$ are the *harmonic measures*.

Another basic notion which is of fundamental importance in our investigations is that of Green's functions.

Definition 2.13 Let Σ be a type (g, n) surface. For fixed $z \in \Sigma$, we define the *Green's function of* Σ to be a function $g(w; z)$ such that

1. for a local coordinate ϕ vanishing at z the function $w \mapsto g(w; z) + \log |\phi(w)|$ is harmonic in an open neighbourhood of z;
2. $\lim_{w \to \zeta} g(w; z) = 0$ for any $\zeta \in \partial \Sigma$.

That such a function exists, follows from [1, II.3 11H, III.1 4D], considering Σ to be a subset of its double Σ^d.

Definition 2.14 For compact surfaces $\mathscr{R}$, one defines the *Green's function* $\mathscr{G}$ (see e.g. [14]) as the unique function $\mathscr{G}(w, w_0; z, q)$ satisfying

1. $\mathscr{G}$ is harmonic in w on $\mathscr{R} \backslash \{z, q\}$;
2. for a local coordinate ϕ on an open set U containing z, $\mathscr{G}(w, w_0; z, q) + \log |\phi(w) - \phi(z)|$ is harmonic for $w \in U$;
3. for a local coordinate ϕ on an open set U containing q, $\mathscr{G}(w, w_0; z, q) - \log |\phi(w) - \phi(q)|$ is harmonic for $w \in U$;
4. $\mathscr{G}(w_0, w_0; z, q) = 0$ for all z, q, w_0.

The existence of such a function is a standard fact about Riemann surfaces, see for example [14]. It can be shown that $\mathscr{G}$ is also harmonic in z everywhere it is non-singular. Green's function is conformally invariant. That is, if Σ is of type (g, n), and $f : \Sigma \to \Sigma'$ is conformal, then

$$g_{\Sigma'}(f(w); f(z)) = g_\Sigma(w; z). \tag{2.3}$$

Similarly, if $\mathscr{R}$ is compact and $f : \mathscr{R} \to \mathscr{R}'$ is a biholomorphism, then

$$\mathscr{G}_{\mathscr{R}'}(f(w), f(w_0); f(z); f(q)) = \mathscr{G}_{\mathscr{R}}(w, w_0; z, q). \tag{2.4}$$

These follow from uniqueness of Green's function; in the case of type (g, n) surfaces, one also needs the fact that a biholomorphism extends to a homeomorphism of the boundary curves.

Remark 2.15 The condition (4) involving the point w_0 simply determines an arbitrary additive constant, and is not of any interest in the paper. This is because $\partial_w \mathscr{G}$ is independent of w_0, and only such derivatives enter in the paper. So we leave w_0 out of the expression for $\mathscr{G}$ from here onwards.

3 Conformally Invariant Approach to Sobolev Spaces of Harmonic Functions

As is well known, every harmonic function with L^2 derivatives is in the Sobolev H^1 space. However, the L^2 norm on the function itself is not conformally invariant. One could perhaps

make the L^2 norm on the function conformally invariant by restricting to conformal maps which are local isometries of the metric, but this is too restrictive. In this section, we give simple ways to define conformally invariant norms, which are equivalent to the Sobolev norm when restricting to harmonic functions. We also develop a new approach to the Sobolev $H^{1/2}$ space on the boundary and its dual. The main new result is a description of the $H^{-1/2}$ space on the boundary in terms of equivalence classes of harmonic one forms and a natural pairing. This makes use of the complex structure in a fundamental way and is conformally invariant. These equivalent descriptions and norms are particularly adapted to quasiconformal surgery of Riemann surfaces.

We begin by defining certain boundary integrals of Dirichlet-bounded harmonic functions. Let $d\omega_k$ be the harmonic measures given in Definition 2.12. For a collar neighbourhood U_k of $\partial_k\Sigma$ and $h_k \in \mathcal{D}_{\text{harm}}(U_k)$, the integral

$$\int_{\partial_k\Sigma} h_k * d\omega_k$$

can be defined in terms of a limit of the integral over curves γ_k approaching the boundary $\partial_k\Sigma$, and this is well-defined [24, Lemmas 3.14 and 3.15]. Equivalently, we can fix a simple closed analytic curve γ_k which is isotopic to $\partial_k\Sigma$, and define

$$\int_{\partial_k\Sigma} h_k * d\omega_k := \iint_{V_k} dh_k \wedge *d\omega_k + \int_{\gamma_k} h_k * d\omega_k,$$

where V_k is the region bounded by $\partial_k\Sigma$ and γ_k. Here $\partial_k\Sigma$ is oriented positively with respect to Σ and γ_k has the same orientation as $\partial_k\Sigma$. This is independent of γ_k. Now given $h_k \in \mathcal{D}_{\text{harm}}(\Sigma)$ we set

$$\mathcal{H}_k := \int_{\partial_k\Sigma} h_k * d\omega_k. \tag{3.1}$$

Definition 3.1 Let Σ be a bordered surface of type (g, n) and let $U_k \subseteq \Sigma$ be collar neighbourhoods of $\partial_k\Sigma$ for $k = 1, \ldots, n$. Set $U = U_1 \cup \cdots \cup U_n$. By $H^1_{\text{conf}}(U)$ we denote the harmonic Dirichlet space $\mathcal{D}_{\text{harm}}(U)$ endowed with the norm

$$\|h\|_{H^1_{\text{conf}}(U)} := \left(\|h\|^2_{\mathcal{D}_{\text{harm}}(U)} + \sum_{k=1}^n |\mathcal{H}_k|^2 \right)^{\frac{1}{2}}$$

for $n > 1$. In the case that $n = 1$, fix a point $p \in \Sigma \setminus U_1$ and define instead

$$\mathcal{H}_1 := \int_{\partial_1\Sigma} h_1 * dg(w, p), \tag{3.2}$$

where $g(w, p)$ is Green's function of Σ.

For the Riemann surface Σ, assuming that Σ is connected, we need only one boundary integral to obtain a norm. If $n > 1$, we can choose any fixed boundary curve $\partial_n\Sigma$ say, and define the norm

$$\|h\|_{H^1_{\text{conf}}(\Sigma)} := \left(\|h\|^2_{\mathcal{D}_{\text{harm}}(\Sigma)} + |\mathcal{H}_n|^2 \right)^{1/2},$$

where any of the $\mathcal{H}_k$ could in fact be used in place of $\mathcal{H}_n$. In the case that $n = 1$ we use (3.2) to define $\mathcal{H}_1$.

We can also use Green's function to define the norm of harmonic functions on Σ in the case that $n \geq 1$. Indeed as was shown in [24, Lemma 3.17], one has

Lemma 3.2 *Let Σ be a connected Riemann surface of type (g, n). For any fixed point $p \in \Sigma$, the norms given by*

$$
\begin{aligned}
\|h\|^2_{\mathcal{D}_{\mathrm{harm}}(\Sigma), p} &= \|h\|^2_{\mathcal{D}_{\mathrm{harm}}(\Sigma)} + |h(p)|^2 \\
&= \|h\|^2_{\mathcal{D}_{\mathrm{harm}}(\Sigma)} + \left| \lim_{\varepsilon \searrow 0} \frac{1}{\pi i} \int_{\Gamma_\varepsilon} \partial_w g(w; p) h(w) \right|^2,
\end{aligned}
$$

where g is Green's function of Σ and Γ_ε are the level curves of Green's function based at p, and the $H^1_{\mathrm{conf}}(\Sigma)$ norm are equivalent.

As a class of functions, the $H^1_{\mathrm{conf}}(\Sigma)$ space and the Sobolev space H^1 are the same. It is also not hard to show that the norms are equivalent (see Theorem 3.19 in [24]).

Theorem 3.3 *Let Σ be a Riemann surface of type (g, n). Then the restriction of the Sobolev $H^1(\Sigma)$ norm to $H^1_{\mathrm{conf}}(\Sigma)$ is equivalent to the $H^1_{\mathrm{conf}}(\Sigma)$ norm. Similarly, for a collar neighbourhood U of a boundary $\partial_k \Sigma$, the restriction of the Sobolev norm $H^1(U)$ to $H^1_{\mathrm{conf}}(U)$ is equivalent to the $H^1_{\mathrm{conf}}(U)$ norm.*

We now give an alternate description of the $H^{1/2}(\partial_k \Sigma)$ space and its dual on a border $\partial_k \Sigma$.

Definition 3.4 (Boundary values of harmonic functions) Let Σ be a bordered Riemann surface.

- $\mathcal{H}(\partial_k \Sigma)$ is the set of equivalence classes of functions in $\mathcal{D}_{\mathrm{harm}}(U)$ on collars U of $\partial_k \Sigma$, under the following equivalence relation:
  For $h_1 \in \mathcal{D}_{\mathrm{harm}}(U_1)$ and $h_2 \in \mathcal{D}_{\mathrm{harm}}(U_2)$, with U_1 and U_2 collars of $\partial_k \Sigma$, $h_1 \sim h_2$ if the CNT boundary values of h_1 and h_2 agree up to a null set.
- $\dot{\mathcal{H}}(\partial_k \Sigma)$ is the set of equivalence classes of functions in $\mathcal{D}_{\mathrm{harm}}(U)$ on collars U of $\partial_k \Sigma$ under the following equivalence relation:
  For $h_1 \in \mathcal{D}_{\mathrm{harm}}(U_1)$ and $h_2 \in \mathcal{D}_{\mathrm{harm}}(U_2)$, with U_1 and U_2 collars of $\partial_k \Sigma$, $h_1 \sim h_2$ if there is a constant $c \in \mathbb{C}$ such that $h_1 = h_2 + c$ up to a null set.

$\mathcal{H}(\partial_k \Sigma)$ can be identified with $H^{1/2}(\partial_k \Sigma)$ in the following way. Given $h \in \mathcal{H}(\partial_k \Sigma)$, any representative $H \in \mathcal{D}_{\mathrm{harm}}(U)$ has CNT boundary values except possibly on a null set. This agrees with an element $\hat{h} \in H^{1/2}(\partial_k \Sigma)$ almost everywhere. It can be shown that $\hat{h}$ is independent of the representative. Conversely, any element of $H^{1/2}(\partial_k \Sigma)$ (non-uniquely) extends to a harmonic function on a collar neighbourhood U. A similar argument identifies $\dot{H}^{1/2}(\partial_k \Sigma)$ with $\dot{\mathcal{H}}(\partial_k \Sigma)$.

We therefore can endow $\mathcal{H}(\partial_k \Sigma)$ with the norm on $H^{1/2}(\partial_k \Sigma)$. It follows from Schwarz reflection and Carathéodory's theorem that any collar chart ψ on a collar neighbourhood U of $\partial_k \Sigma$ extends to a conformal map of a neighbourhood of $\partial_k \Sigma$ in the double. Thus by invariance of Sobolev spaces under diffeomorphisms, we have the equivalences

$$
\|f\|_{\mathcal{H}(\partial_k \Sigma)} \approx \|f \circ \psi^{-1}\|_{H^{1/2}(\mathbb{S}^1)}, \quad \|f\|_{\dot{\mathcal{H}}(\partial_k \Sigma)} \approx \|f \circ \psi^{-1}\|_{\dot{H}^{1/2}(\mathbb{S}^1)},
$$

for any such collar chart. Recall that $\dot{H}^{1/2}(\mathbb{S}^1)$ and $H^{1/2}(\mathbb{S}^1)$ denote the classical homogeneous and inhomogeneous Sobolev spaces on the unit circle.

Definition 3.5 (Boundary values of harmonic one-forms) Let Σ be a bordered Riemann surface.

- $\mathcal{H}'(\partial_k \Sigma)$ is the set of equivalence classes of harmonic forms in $\mathcal{A}_{\mathrm{harm}}(U)$ on collars U of $\partial_k \Sigma$, under the following equivalence relation:
  For $\alpha_1 \in \mathcal{A}_{\mathrm{harm}}(U_1)$ and $\alpha_2 \in \mathcal{A}_{\mathrm{harm}}(U_2)$, with U_1 and U_2 collars of $\partial_k \Sigma$, $\alpha_1 \sim \alpha_2$ if there exists collar neighbourhood $U_3 \subset U_1 \cap U_2$ and $\beta \in \mathcal{A}_{\mathrm{harm}}(U_3)$ such that $\alpha_1 - \beta$ and $\alpha_2 - \beta$ are exact on U_3 and the primitives h_k of $\alpha_l - \beta$, $k = 1, 2$ are equivalent in $\mathcal{H}(\partial_k \Sigma)$.
- $\dot{\mathcal{H}}'(\partial_k \Sigma) = \left\{ [\alpha] \in \mathcal{H}'(\partial_k \Sigma); \ \int_{\partial_k \Sigma} [\alpha] = 0 \right\}$.

Intuitively, the equivalence relation identifies harmonic one forms which have the same value on any vector tangent to the boundary at any point.

We define the norm in $\dot{\mathcal{H}}'(\mathbb{S}^1)$ to be

$$\|[\alpha]\|_{\dot{\mathcal{H}}'(\mathbb{S}^1)} = \|\alpha\|_{\mathcal{A}_{\mathrm{harm}}(\mathbb{D})},$$

where $\alpha \in \mathcal{A}_{\mathrm{harm}}(\mathbb{D})$ is the unique representative of $[\alpha]$ defined on $\mathbb{D}$. Similarly, the norm on $\mathcal{H}(\mathbb{S}^1)$ is

$$\|[\alpha]\|_{\mathcal{H}(\mathbb{S}^1)}^2 = \|\beta\|_{\mathcal{A}_{\mathrm{harm}}(\mathbb{D})}^2 + |\lambda|^2,$$

where $\lambda \in \mathbb{C}$ and $\beta \in \mathcal{A}_{\mathrm{harm}}(\mathbb{D})$ are uniquely chosen [24, Lemma 5.7] such that

$$\beta + \frac{\lambda}{4\pi i} \left(\frac{dz}{z} - \frac{d\bar{z}}{\bar{z}} \right)$$

is a representative of $[\alpha]$. Given a collar chart ψ on a collar neighbourhood U of $\partial_k \Sigma$, we then define, for any $[\alpha] \in \dot{\mathcal{H}}'(\partial_k \Sigma)$

$$\|[\alpha]\|_{\dot{\mathcal{H}}'(\partial_k \Sigma)} = \|(\psi^{-1})^*[\alpha]\|_{\dot{\mathcal{H}}'(\mathbb{S}^1)}$$

and for any $[\alpha] \in H^{1/2}(\partial_k \Sigma)$

$$\|[\alpha]\|_{\mathcal{H}(\partial_k \Sigma)} = \|(\psi^{-1})^*[\alpha]\|_{\mathcal{H}(\mathbb{S}^1)}.$$

It can be shown that the pull-back preserves the equivalence relation, so that this is well-defined. We will also see momentarily that different collar charts induce equivalent norms.

Given any $[\alpha] \in \mathcal{H}'(\partial_k \Sigma)$ we obtain a linear functional

$$L_{[\alpha]} : H^{1/2}(\partial_k \Sigma) \to \mathbb{C}$$

$$[h] \mapsto \lim_{\epsilon \searrow 0} \int_{\Gamma_\epsilon} \alpha h$$

where the curves Γ_ϵ approach the boundary as $\epsilon \searrow 0$. If $[\alpha] \in \dot{\mathcal{H}}'(\partial_k \Sigma)$ then $L_{[\alpha]}$ is a well-defined functional on $\dot{H}^{1/2}(\partial_k \Sigma)$.

As was shown in [24] (Section 5), the main results are the following two theorems:

Theorem 3.6 *For any $[\alpha] \in \mathcal{H}'(\partial_k \Sigma)$, $L_{[\alpha]}$ is a well-defined bounded linear functional.*

Using the identification of $H^{1/2}(\partial_k \Sigma)$ and $\mathcal{H}(\partial_k \Sigma)$ this defines a linear functional on $H^{1/2}(\partial_k \Sigma)$. In fact, we have (see Theorem 5.11 in [24])

Theorem 3.7 *The map*

$$\mathcal{H}'(\partial_k \Sigma) \to H^{-1/2}(\partial_k \Sigma)$$

$$[\alpha] \mapsto L_{[\alpha]}$$

is a bounded isomorphism. Similarly for $\dot{\mathcal{H}}'(\partial_k \Sigma)$ and $\dot{H}^{-1/2}(\partial_k \Sigma)$.

This also shows that any pair of collar charts give rise to equivalent norms on $\mathcal{H}'(\partial_k \Sigma)$ and $\dot{\mathcal{H}}'(\partial_k \Sigma)$.

Theorem 3.7 is the promised description of $H^{-1/2}(\partial_k \Sigma)$, which is new even in the plane. Note that the equivalence classes provide an elegant explicit realization of the distributional derivatives of $H^{1/2}(\partial_k \Sigma)$. Let $\Gamma = \partial_k \Sigma$ be a boundary curve (which can be treated as a regular curve in the double). In the familiar setting, given $f \in H^{1/2}(\Gamma)$, its distributional derivative df is indirectly defined as the extension to $H^{1/2}(\Gamma)$ of the linear functional

$$h \mapsto -\int_\Gamma f \, dh$$

for regular h. Theorem 3.7 shows that one may directly define df to be the boundary values of an actual harmonic one-form α in a collar neighbourhood. Thus, we may *directly* define the linear functional

$$h \mapsto \lim_{\epsilon \searrow 0} \int_{\Gamma_\epsilon} H\alpha,$$

where α is a harmonic extension of df and H is a harmonic extension of h. The sacrifice is that one replaces the direct integral with a limit, and the functions with equivalence classes. Here we see the first glimmers of an approach to propagators for certain PDEs, replacing the distributional calculus with principal value integrals.

As an application, we showed that the possible boundary values of L^2 one-forms on a Riemann surface of type (g, n) can be identified with the Sobolev space $H^{-1/2}(\partial \Sigma)$. Intuitively, by boundary values of a one form, we mean the restriction of the one-form to vectors tangent to the boundary.

Now denote the equivalence relation in $\mathcal{H}'(\partial_k \Sigma)$ by $[\cdot]_k$. By definition, any $\beta \in \mathcal{A}_{\mathrm{harm}}(\Sigma)$ induces boundary values $[\beta]_k$ in $\mathcal{H}'(\partial_k \Sigma)$ for $k = 1, \ldots, n$. The converse is also true, and as was demonstrated in [24, Theorem 5.18], is used to prove the well-posedness of the Dirichlet problem for L^2 one-forms with CNT data.

Theorem 3.8 *Let Σ be a Riemann surface of type (g, n). Given any $[\alpha_k] \in \mathcal{H}'(\partial_k \Sigma)$, $k = 1, \ldots, n$, there is a $\beta \in \mathcal{A}_{\mathrm{harm}}(\Sigma)$, such that $[\beta] = [\alpha_k]$ on each boundary curve. The form β is uniquely determined by its cohomology class and constants*

$$\int_{\partial_k \Sigma} *\beta = \rho_k \tag{3.3}$$

for $k = 1, \ldots, n - 1$.

The cohomology class of β is of course determined by periods around non-trivial closed curves; it is easy to see that the period around a curve isotopic to the boundary is determined by $[\alpha_k]$. This determines β up to an arbitrary exact form with zero boundary values; that is, up to a harmonic form $d\omega$ where ω is constant on the boundary curves. The condition (3.3) removes the remaining ambiguity.

Thus we have completely characterized the analytic class of the possible boundary values of L^2 harmonic one-forms. By Theorem 3.7, the possible boundary values of L^2 harmonic one-forms can be identified with $H^{-1/2}(\partial \Sigma)$.

4 The Overfare of Harmonic Functions and Harmonic Forms

The authors demonstrated existence and boundedness of overfare in the sphere in [19], and on Riemann surfaces with one boundary curve in [20]. Now let Γ be a collection of curves

separating a Riemann surface $\mathscr{R}$ into two components Σ_1 and Σ_2, and consider the following question. Given $h_1 \in \mathcal{D}_{\mathrm{harm}}(\Sigma_1)$, is there an $h_2 \in \mathcal{D}_{\mathrm{harm}}(\Sigma_2)$ with the same boundary values up to a negligible set? We call this the overfare of h_1 to h_2.

For the Dirichlet space, the negligible sets are null sets. However, a null set with respect to Σ_1 need not be null with respect to Σ_2. Thus we must restrict to curves for which this is true, which are the so-called quasicircles.

Definition 4.1 We say that a simple closed curve in the Riemann sphere $\bar{\mathbb{C}}$ is a *quasicircle* if it is the image of $\mathbb{S}^1$ under a quasiconformal map of the plane.

A simple closed curve Γ in a Riemann surface R is a quasicircle if there is an open set U containing Γ and a biholomorphism $\phi : U \to \mathbb{A}$ where $\mathbb{A}$ is an annulus in $\mathbb{C}$, such that $\phi(\Gamma)$ is a quasicircle.

Having the definition of quasicircles at hand, we consider the following situation. Let $\mathscr{R}$ be a compact Riemann surface, and let $\Gamma_1, \ldots, \Gamma_m$ be a collection of quasicircles in $\mathscr{R}$. Denote $\Gamma = \Gamma_1 \cup \cdots \cup \Gamma_m$. We say that Γ *separates* $\mathscr{R}$ into Σ_1 and Σ_2 if

1. there are doubly-connected neighbourhoods U_k of Γ_k for $k = 1, \ldots, n$ such that $U_k \cap U_j$ is empty for all $j \neq k$;
2. one of the two connected components of $U_k \backslash \Gamma_k$ is in Σ_1, while the other is in Σ_2;
3. $\mathscr{R} \backslash \Gamma = \Sigma_1 \cup \Sigma_2$;
4. $\mathscr{R} \backslash \Gamma$ consists of finitely many connected components;
5. Σ_1 and Σ_2 are disjoint.

It turns out that one can identify $\partial \Sigma_1$ and $\partial \Sigma_2$ pointwise with Γ.

Proposition 4.2 *Let $\mathscr{R}$ be a compact Riemann surface and $\Gamma = \Gamma_1 \cup \cdots \cup \Gamma_m$ be a collection of quasicircles separating $\mathscr{R}$ into Σ_1 and Σ_2. Then Σ_1 and Σ_2 are each a finite union of bordered surfaces. For $k = 1, 2$, the inclusion map of Σ_k into $\mathscr{R}$ extends continuously to the border $\partial_k \Sigma$, and this extension is a homeomorphism onto Γ.*

As mentioned above, it is not directly obvious that a null set in $\partial \Sigma_1$ is null in $\partial \Sigma_2$, even though they are the same set. However, as a consequence of Theorem 2.8 in [24], this is indeed the case for quasicircles. It was shown in [24, Theorem 3.41] that:

Theorem 4.3 *The set $I \subset \Gamma$ is null with respect to Σ_1 if and only if it is null with respect to Σ_2. Given any $h_1 \in \mathcal{D}_{\mathrm{harm}}(\Sigma_1)$, there is a unique $h_2 \in \mathcal{D}_{\mathrm{harm}}(\Sigma_2)$ whose boundary values agree with those of h_1 except possibly on a null set.*

This paves the way for the definition of overfare.

Definition 4.4 With the assumption of Theorem 4.3, we define the *overfare operator* $\mathbf{O}_{1,2}$ by

$$\mathbf{O}_{1,2} : \mathcal{D}_{\mathrm{harm}}(\Sigma_1) \to \mathcal{D}_{\mathrm{harm}}(\Sigma_2)$$
$$h_1 \mapsto h_2.$$

Of course one can switch the roles of Σ_1 and Σ_2, and one obviously has that

$$\mathbf{O}_{2,1}\mathbf{O}_{1,2} = \mathrm{Id}.$$

The overfare operator is conformally invariant. That is, if $f : \mathscr{R} \to \mathscr{R}'$ is a biholomorphism and we set $f(\Sigma_k) = \Sigma_k'$ for $k = 1, 2$ then it follows immediately from conformal invariance of CNT limits that

$$\mathbf{O}_{1,2}\mathbf{C}_f = \mathbf{C}_f\mathbf{O}_{1',2'}.$$

Definition 4.5 In the case that Σ is a finite union of connected Riemann surfaces $\Sigma_1, \ldots, \Sigma_N$, we define the Dirichlet seminorm on these components by

$$\|h\|_{\mathcal{D}_{\mathrm{harm}}(\Sigma)} := \sum_{k=1}^{N} \left\| h|_{\Sigma_k} \right\|_{\mathcal{D}_{\mathrm{harm}}(\Sigma_k)}$$

and similarly for the holomorphic and anti-holomorphic Dirichlet spaces, Bergman spaces, etc.

It was shown in [24, Theorem 3.44] that for quasicircles, the overfare exists and is a bounded map with respect to the Dirichlet seminorms, when the originating surface is connected:

Theorem 4.6 *Assume that Σ_1 is connected. The overfare operator*

$$\mathbf{O}_{1,2} : \mathcal{D}_{\mathrm{harm}}(\Sigma_1) \to: \mathcal{D}_{\mathrm{harm}}(\Sigma_2)$$

is bounded with respect to the Dirichlet semi-norm.

Note also that the overfare map is bounded with respect to H^1_{conf} in the general case, if we assume that the quasicircle is more regular, see i.e. [24, Theorem 3.43].

In the plane, the authors showed in [19] that the existence of a bounded overfare characterizes quasicircles. Also recall from the introduction of this paper that in the plane the aforementioned NTA domains are exactly quasidisks, see [8] and [6, Theorem 2.7].

Now we proceed to the overfare of exact one forms. The boundary values of a harmonic one-form do not uniquely specify the one-form, because they do not specify the cohomology class. However, the overfare $\mathbf{O}_{1,2}$ defined above for functions, can be used to uniquely define the overfare of *exact* one-forms from connected surfaces to arbitrary ones.

Definition 4.7 Assume that Σ_1 is connected. The *exact overfare* $\mathbf{O}^{\mathrm{e}}_{1,2}$ is defined by

$$\mathbf{O}^{\mathrm{e}}_{1,2} : \mathcal{A}^{\mathrm{e}}_{\mathrm{harm}}(\Sigma_1) \to \mathcal{A}^{\mathrm{e}}_{\mathrm{harm}}(\Sigma_2)$$
$$\alpha \mapsto d\mathbf{O}_{1,2}d^{-1}\alpha,$$

where d is the exterior differentiation.

We specify the cohomological data as follows:

Definition 4.8 Assume that Σ_1 is connected. Let $\zeta \in \mathcal{A}_{\mathrm{harm}}(\mathscr{R})$ be a one-form such that $\alpha - \zeta$ is exact on Σ_1. We seek a one-form with the same boundary values as α and in the cohomology class of ζ on Σ_2. This form is

$$\mathbf{O}^{\mathrm{e}}_{1,2}\left(\alpha - \zeta|_{\Sigma_1}\right) + \zeta|_{\Sigma_2}.$$

We call ζ def a *catalyzing form*, and forms which are related by overfare via ζ *compatible*.

Now if

$$\alpha_2 = \mathbf{O}^{\mathrm{e}}_{1,2}\left(\alpha_1 - \zeta|_{\Sigma_1}\right) + \zeta|_{\Sigma_2}$$

then $[\alpha_1]_k = [\alpha_2]_k$ for all $k = 1, \ldots, n$ (that is, they have the same boundary values as forms).

Furthermore, if Σ_1 is connected, then given α_1, the conditions that α_k have the same boundary values and that $\alpha_k - \zeta|_{\Sigma_k}$ are exact for $k = 1, 2$, uniquely determine α_2 [24, Corollary 8.4]. In the general case, further conditions are necessary to specify α_2.

5 Schiffer Comparison Operators, Scattering Processes and Scattering Matrices

First, we define the Schiffer operators. To that end, we need to define certain bi-differentials, which will be the integral kernels of the Schiffer operators.

Definition 5.1 For a compact Riemann surface $\mathscr{R}$ with Green's function $\mathscr{G}(w, w_0; z, q)$, the *Schiffer kernel* is defined by

$$L_\mathscr{R}(z, w) = \frac{1}{\pi i} \partial_z \partial_w \mathscr{G}(w, w_0; z, q),$$

and the *Bergman kernel* is given by

$$K_\mathscr{R}(z, w) = -\frac{1}{\pi i} \partial_z \bar{\partial}_w \mathscr{G}(w, w_0; z, q).$$

The kernel functions satisfy the following:

(1) $L_\mathscr{R}$ and $K_\mathscr{R}$ are independent of q and w_0.
(2) $K_\mathscr{R}$ is holomorphic in z for fixed w, and anti-holomorphic in w for fixed z.
(3) $L_\mathscr{R}$ is holomorphic in w and z, except for a pole of order two when $w = z$.
(4) $L_\mathscr{R}(z, w) = L_\mathscr{R}(w, z).$
(5) $K_\mathscr{R}(w, z) = -\overline{K_\mathscr{R}(z, w)}.$

From the conformal invariance of Green's function (given by (2.3), (2.4)), it follows that the kernels above are conformally invariant.

Definition 5.2 For $k = 1, 2$ define the *restriction operators*

$$\mathbf{R}_k : \mathcal{A}(\mathscr{R}) \to \mathcal{A}(\Sigma_k)$$
$$\alpha \mapsto \alpha|_{\Sigma_k}$$

and

$$\mathbf{R}_k^0 : \mathcal{A}(\Sigma_1 \cup \Sigma_2) \to \mathcal{A}(\Sigma_k)$$
$$\alpha \mapsto \alpha|_{\Sigma_k}.$$

In a similar way, we also define the restriction operator

$$\mathbf{R}_k^{\mathrm{h}} : \mathcal{A}_{\mathrm{harm}}(\mathscr{R}) \to \mathcal{A}_{\mathrm{harm}}(\Sigma_k). \tag{5.1}$$

It is obvious that these are bounded operators.

Having the Bergman and Schiffer kernels and the restriction operators at hand, we can now define the Schiffer operators as follows.

Definition 5.3 For $k = 1, 2$, we define the *Schiffer comparison operators* (after Max Schiffer) by

$$\mathbf{T}_k : \quad \overline{\mathcal{A}(\Sigma_k)} \to \mathcal{A}(\Sigma_1 \cup \Sigma_2)$$
$$\bar{\alpha} \mapsto \iint_{\Sigma_k} L_\mathscr{R}(\cdot, w) \wedge \overline{\alpha(w)}$$

and

$$\mathbf{S}_k : \quad \mathcal{A}(\Sigma_k) \to \mathcal{A}(\mathscr{R})$$
$$\alpha \mapsto \iint_{\Sigma_k} K_\mathscr{R}(\cdot, w) \wedge \alpha(w).$$

The integral defining $\mathbf{T}_k$ is interpreted as a principal value integral whenever $z \in \Sigma_k$. Also, we define for $j, k \in \{1, 2\}$

$$\mathbf{T}_{j,k} = \mathbf{R}_k^0 \mathbf{T}_j : \overline{\mathcal{A}(\Sigma_j)} \to \mathcal{A}(\Sigma_k).$$

Theorem 5.4 ([24, Theorem 4.4]) $\mathbf{T}_k$, $\mathbf{T}_{j,k}$, and $\mathbf{S}_k$ are bounded for all $j, k = 1, 2$.

For any operator $\mathbf{M}$, we define the complex conjugate operator by

$$\overline{\mathbf{M}} \alpha = \overline{\mathbf{M} \overline{\alpha}}.$$

So for example

$$\overline{\mathbf{T}}_{1,2} : \mathcal{A}(\Sigma_1) \to \overline{\mathcal{A}(\Sigma_2)}$$

and similarly for $\overline{\mathbf{R}}_k$, etc.

The restriction operator is conformally invariant by conformal invariance of Bergman space of one-forms. By the conformal invariance of the kernel functions defined above, the operators $\mathbf{T}$ and $\mathbf{S}$ are also conformally invariant.

Example 5.5 Let $\mathscr{R} = \overline{\mathbb{C}}$, Σ_1 is a simply-connected domain in the plane, Σ_2 its complement in $\overline{\mathbb{C}}$.

$$g(w, \infty; z, q) = \log \frac{|w - z|}{|w - q|}.$$

Then

$$\mathbf{T}_{1,k} \overline{h(w) \, dw} = -\frac{1}{\pi} \iint_{\Sigma_1} \frac{\overline{h(w)}}{(w - z)^2} \, dA_w, \quad z \in \Sigma_k$$

and since the Bergman kernel is zero in this case, $\mathbf{S}_1 = 0$.

Historically, the operators defined above arose in geometric function theory in the context of conformal maps, Plemelj–Sokhotski jump formula, Grunsky inequalities, and Fredholm eigenvalues (e.g. S. Bergman and M. Schiffer [2]; Schiffer [15, 17]). When the outer domain is not the sphere, they were introduced by Schiffer [15, 17]. Schiffer and D. Spencer [16] adapted the operators to the setting of Riemann surfaces.

Given the operators above we proceed to the definition of the scattering matrix. The scattering operator, acts on the holomorphic parts of the compatible forms together with the anti-holomorphic part of the catalyzing form, and produces the anti-holomorphic parts of the compatible forms and the holomorphic part of the catalyzing form. The anti-holomorphic parts can be thought of as left moving waves, while the holomorphic parts can be thought of as right moving waves. We can explicitly give the scattering matrix $\mathcal{S}$ in terms of the Schiffer operators.

Given $\mu_2 = \alpha_2 + \overline{\beta}_2 \in \mathcal{A}_{\text{harm}}(\Sigma_2)$, let $\zeta = \xi + \overline{\eta} \in \mathcal{A}_{\text{harm}}(\mathscr{R})$ be a catalyzing form. Let $\mu_1 = \alpha_1 + \overline{\beta}_1 \in \mathcal{A}_{\text{harm}}(\Sigma_1)$ be the overfare of μ_2 with respect to this catalyzing form (see the end of Section 4 for the definitions of catalyzing and compatible forms). Then one has

Theorem 5.6 ([24, Theorem 8.7]) *Assume* Σ_2 *is connected, and the genus of* $\mathscr{R}$ *is non-zero, we have*

$$\begin{pmatrix} \overline{\beta}_1 \\ \overline{\beta}_2 \\ \xi \end{pmatrix} = \mathcal{S} \begin{pmatrix} \alpha_1 \\ \alpha_2 \\ \overline{\eta} \end{pmatrix},$$

where

$$\mathcal{S} := \begin{pmatrix} -\overline{\mathbf{T}}_{1,1} & -\overline{\mathbf{T}}_{2,1} & \overline{\mathbf{R}}_1 \\ -\overline{\mathbf{T}}_{1,2} & -\overline{\mathbf{T}}_{2,2} & \overline{\mathbf{R}}_2 \\ \mathbf{S}_1 & \mathbf{S}_2 & 0 \end{pmatrix}$$

is the scattering matrix and it is unitary.

The two main steps of the proof are as follows. The first step is to show that the scattering matrix has the form above, for which one uses the jump formula for quasicircles in conjunction with Stokes formula and a cohomological analysis of the so-called Cauchy–Royden operator (see Definition 4.7 in [24]). This amounts to overcoming an analytic and a geometric hindrance. The analytic hindrance is to establish that the Plemelj–Sokhotski jump formula extends to quasicircles for the Cauchy–Royden operator. This has been established by the authors in a series of papers; for references see [23, 24]. The geometric hindrance is to analyze the effect of the Cauchy–Royden operator on cohomology and the harmonic measures.

The second main step is to establish unitarity of the matrix. This follows from a collection of adjoint identities established by the authors using the functional analysis of the integral operators. One of these identities has roots in a norm identity of Bergman and Schiffer and later work of Schiffer, though interestingly they never discuss the adjoint.

In the genus zero case the scattering matrix takes the following simpler form:

Theorem 5.7 ([24, Theorem 8.8]) *Assuming Σ_2 is connected, and $\mathscr{R}$ is the sphere we have*

$$\begin{pmatrix} \overline{\beta}_1 \\ \overline{\beta}_2 \end{pmatrix} = \begin{pmatrix} -\overline{\mathbf{T}}_{1,1} & -\overline{\mathbf{T}}_{2,1} \\ -\overline{\mathbf{T}}_{1,2} & -\overline{\mathbf{T}}_{2,2} \end{pmatrix} \begin{pmatrix} \alpha_1 \\ \alpha_2 \end{pmatrix}$$

and the matrix is unitary.

6 Applications

6.1 Index Theorems

An application of the tools described in this survey and developed in [24] determines how the Fredholm indices of the Schiffer comparison operators relate to the topological invariants of the Riemann surfaces. In this connection we have

Theorem 6.1 ([24, Theorems 7.20 and 7.22]) *If Σ_1, Σ_2 are connected, and of genus g_1 and g_2, then* index$(\mathbf{T}_{1,2}) = g_1 - g_2$.
If Σ_2, is connected and of genus g, and Σ_1 consists of n disjoint simply connected regions, then index$(\mathbf{T}_{1,2}) = 1 - n + g$.

In the proof of this theorem, Stokes' theorem relates the integral operator to integrals over the rough boundaries. A central role is played by the non-trivial fact that the homogeneous Sobolev spaces of boundary values attained on either side of the quasicircles agree. That allows one to derive identities for various Schiffer operators and their effect on cohomology, which ultimately makes the computations of the kernels and cokernels possible.

6.2 Approximation Theory for Functions and One-forms

Assume that $\mathscr{R}$ is a compact surface of genus g, $\Sigma_1 = \Omega_1 \cup \cdots \cup \Omega_n$ where Ω_k are simply-connected regions bordered by quasicircles whose closures are disjoint, and Σ_2 is connected. We call this a capped surface.

We define the operator

$$\begin{aligned} \boldsymbol{\Theta} : \quad & \overline{\mathcal{A}(\Sigma_1)} \oplus \mathcal{A}(\mathscr{R}) \to \mathcal{A}^{\mathrm{se}}(\Sigma_2) \\ & (\overline{\gamma}, \tau) \mapsto -\mathbf{T}_{1,2}\overline{\gamma} + \mathbf{R}_2\tau. \end{aligned}$$

We then have

Theorem 6.2 Θ *is a bounded isomorphism.*

Now let $f_k : \mathbb{D} \to \Omega_k$ be conformal maps for $k = 1, \ldots, n$ and set $p_k = f_k(0)$ for $k = 1, \ldots, n$. Recalling the definition of pullback (2.2), define

$$(f^{-1})^* = ((f_1^{-1})^*, \ldots, (f_n^{-1})^*) \ : \ \bigoplus_1^n \overline{\mathcal{A}(\mathbb{D})} \to \overline{\mathcal{A}(\Sigma_1)}.$$

In [18], the first author and M. Shirazi used this theorem to derive approximation theorems for L^2 holomorphic one-forms on Σ_2. Since pull-back is a bounded isometry, we have that

$$(f^{-1})^* \mathbf{T}_{1,2} \ : \ \bigoplus_1^n \overline{\mathcal{A}(\mathbb{D})} \to \mathcal{A}^{\mathrm{se}}(\Sigma_2)$$

is an isomorphism onto its image. The "Faber-Tietz forms" are defined by

$$\alpha_k^m = (f^{-1})^* \mathbf{T}_{1,2} \left(0, 0, \ldots, m\bar{z}^{m-1} d\bar{z}, \ldots, 0 \right),$$

where the non-zero entry appears in the kth place. It can be shown that α_k^m are holomorphic in $\mathscr{R}\backslash\{p_k\}$ with a pole of order $m + 1$ at p_m and no other poles.

The nomenclature is in honour of both G. Faber, who invented the Faber series for holomorphic functions in domains in the plane, and H. Tietz who generalized Faber series to Riemann surfaces. Using Theorem 6.2 it can be shown that

Theorem 6.3 [18] *Let $\mathscr{R}$, Σ_1, and Σ_2 be as above. Let $\{\tau_1, \ldots, \tau_g\}$ be a basis for the space of holomorphic one-forms on $\mathscr{R}$. Every semi-exact holomorphic one-form β on Σ_2 has a unique Faber–Tietz series*

$$\beta = \sum_{k=1}^{g} a_k \tau_k + \sum_{m=1}^{\infty} \sum_{k=1}^{m} b_m^k \alpha_k^m.$$

This series converges in $\mathcal{A}(\Sigma_2)$, and converges uniformly on compact subsets of Σ_2.

Faber–Tietz series for general forms on $\mathcal{A}(\Sigma_2)$ and for L^2 holomorphic functions on Σ_2 were also obtained. In the latter case, certain orthogonality conditions related to the Mittag–Leffler problem on compact surfaces come into play.

In the special case where $\mathscr{R}$ is the Riemann sphere and Σ_2 is a simply-connected domain in $\mathscr{R}$ containing the point at ∞, one obtains the classical Faber series. The approximability of L^2 one-forms by Faber series in the planar setting is due to Y. Shen [25] and A. Çavuş [3] (identifying one-forms $h(z)dz$ with their coefficients $h(z)$ in the Bergman space of functions). Theorem 6.2 is a generalization to Riemann surfaces of one direction of a theorem of V. V. Napalkov and R. S. Yulmukhametov [12]. For references and a survey of the relation between these various theorems in the sphere see [23].

6.3 Boundary Values and the Grunsky Operator

The classical Grunsky operator arose as a matrix associated to locally one-to-one holomorphic functions defined in a neighbourhood of zero. A classical inequality on the matrix coefficients is equivalent to extendability of the function to a conformal map of the disk. It has various formulations. One of these formulations, due to Bergman and Schiffer [2], is as an integral operator on Bergman space. This is appropriate when the map is assumed to be conformal

and its image sufficiently regular. The matrix inequality can be written as a norm inequality, and in general, analytic conditions on the operator or its norm correspond to properties of the conformal map.

Using overfare, one obtains the following interpretation of the Grunsky operator, for the case of conformal maps onto domains bounded by quasicircles. The Grunsky operator is defined to be

$$\mathbf{Gr}_f := -f^* \mathbf{T}_{1,1} (f^{-1})^*.$$

It can be shown that this agrees with the integral operator formulation of Bergman and Schiffer. See Section 9.5 in [24] for details. We also have the following [23]:

Theorem 6.4 *Let* $f : \mathbb{D} \to \overline{\mathbb{C}}$ *be a conformal map onto a quasidisk* Σ_1 *in the sphere. Let* Σ_2 *be the complement of its closure. Then*

$$\overline{\mathbf{P}_{\mathbb{D}}} f^* \mathbf{O}_{2,1}^e \mathbf{T}_{1,2} (f^{-1})^* = \mathbf{Id},$$
$$\mathbf{P}_{\mathbb{D}} f^* \mathbf{O}_{2,1}^e \mathbf{T}_{1,2} (f^{-1})^* = \mathbf{Gr}_f.$$

Thus $f^* \mathbf{O}_{2,1}^e \mathcal{A}(\Sigma_2)$ *is the graph of the Grunsky operator.*

These identities are modifications of classical identities. In the classical identities the overfare does not appear. The bounded overfare result makes it possible to see that the graph of the Grunsky operator is the pull-back to the disk of the boundary values of holomorphic one-forms on Σ_2. One can also easily formulate this result in terms of the boundary values of functions; indeed, since all domains are topologically trivial, all one-forms are exact.

Another interesting fact is that the Grunsky operator generalizes to Riemann surfaces. However, one must then take into account cohomology, since as was pointed out, the boundary values of a one-form do not determine the one-form. Similarly, when working with functions or equivalently exact one-forms, one must add conditions to ensure this exactness. In [26], Shirazi generalized the Grunsky operator for functions to capped surfaces, and showed again that its graph is the boundary values of holomorphic functions on Σ_2.

One may also generalize the Grunsky operator to semi-exact forms on capped surfaces. The formulation necessarily involves the cohomology classes of Σ_2. We use the notation for capped surfaces from the previous section.

By the Hodge theorem, the cohomology classes of semi-exact forms can be identified with restrictions to Σ_2 of elements of $\mathcal{A}_{\text{harm}}(\mathscr{R})$. So we modify the Grunsky operator to include this data. To this end, we define the map

$$\Upsilon : \overline{\mathcal{A}(\Sigma_1)} \oplus \mathcal{A}(\mathscr{R}) \to \mathcal{A}(\Sigma_1) \oplus \overline{\mathcal{A}(\mathscr{R})}$$
$$(\overline{\gamma}, \tau) \mapsto (-\mathbf{T}_{1,1} \overline{\gamma} + \mathbf{R}_1 \tau, \overline{\mathbf{S}}_1 \overline{\gamma})$$

and set

$$\Gamma := (f^*, \mathbf{Id}) \, \Upsilon \, ((f^{-1})^*, \mathbf{Id}) \; : \; \bigoplus^n \overline{\mathcal{A}(\mathbb{D})} \oplus \mathcal{A}(\mathscr{R}) \to \bigoplus^n \mathcal{A}(\mathbb{D}) \oplus \overline{\mathcal{A}(\mathscr{R})}.$$

Now if we let $\beta \in \mathcal{A}^{\text{se}}(\Sigma_2)$, then there is a unique $\sigma \in \mathcal{A}^{\text{se}}(\Sigma_2)$, and furthermore a unique harmonic form

$$\mathbf{O}^e(\beta - \mathbf{R}_2 \sigma) + \mathbf{R}_1 \sigma$$

on Σ_1, compatible with β with respect to the catalyzing form σ. We then define

$$\mathbf{O}^{\text{aug}} : \mathcal{A}_{\text{harm}}^{\text{se}}(\Sigma_2) \to \mathcal{A}_{\text{harm}}(\Sigma_1) \oplus \mathcal{A}_{\text{harm}}(\mathscr{R})$$
$$\beta \mapsto (\mathbf{O}^e(\beta - \mathbf{R}_2 \sigma) + \mathbf{R}_1 \sigma, \sigma),$$

where σ is the unique element of $\mathcal{A}_{\text{harm}}(\mathcal{R})$ such that $\beta - \mathbf{R}_2^{\text{h}}\sigma$ is exact, and $\mathbf{R}^{\text{h}}$ is as in (5.1).

Setting $\mathbf{P}_{\mathbb{D}}^n := \mathbf{P}_{\mathbb{D}} \oplus \cdots \oplus \mathbf{P}_{\mathbb{D}}$ and making use of Theorem 6.2, the following result can be shown:

Theorem 6.5

$$\left[\overline{\mathbf{P}_{\mathbb{D}}}^n \oplus \mathbf{P}_{\mathscr{R}}\right](f^*, \mathbf{Id})\mathbf{O}^{\text{aug}}\mathbf{\Theta}(f^{-1})^* = \mathbf{Id},$$

$$\left[\mathbf{P}_{\mathbb{D}}^n \oplus \overline{\mathbf{P}_{\mathscr{R}}}\right](f^*, \mathbf{Id})\mathbf{O}^{\text{aug}}\mathbf{\Theta}(f^{-1})^* = \mathbf{\Gamma}.$$

Thus $(f^*, \mathbf{Id})\mathcal{A}^{\text{se}}(\Sigma_2)$ *is the graph of* $\mathbf{\Gamma}$.

Therefore, we can interpret the graph of $\mathbf{\Gamma}$ as the pull-back to the disks of the boundary values of semi-exact forms under f, together with cohomological data.

6.4 Polarizations on Surfaces with Boundaries

The Grunsky operator has applications in algebraic geometry in connection to the problem of finding a unified interpretation for the period maps of compact and non-compact Riemann surfaces.

The classical period map of a compact Riemann surface $\mathscr{R}$ is an assignment of a matrix of periods of normalized holomorphic one-forms to the Riemann surface. In one model, the period matrix lies in the Siegel disk

$$\{Z \ n \times n \text{ matrices}; \ Z^t = Z, \ \text{Id} - \overline{Z}Z > 0\}.$$

(A more common normalization results in the upper half-plane model, in which Z has positive definite imaginary part. These models are equivalent via a Cayley transform.) One thus associates such a matrix to marked surfaces (and more generally, the Teichmüller space). These periods depend holomorphically on the complex structure.

Interestingly, the period matrices generate a "polarization" (a certain Lagrangian decomposition of a symplectic space). An equivalent picture is the following. Let $\Omega(\mathscr{R})$ denote the cohomology classes of one-forms on $\mathscr{R}$. By the Hodge theorem, any cohomology class on $\mathscr{R}$ has a harmonic representative. The complex structure on $\mathscr{R}$ determines the class $\mathcal{A}_{\text{harm}}(\mathscr{R})$, and thus also a decomposition

$$\Omega(\mathscr{R}) = \mathcal{A}(\mathscr{R}) \oplus \overline{\mathcal{A}(\mathscr{R})}. \tag{6.1}$$

This polarization is an equivalent representation of the period map.

Work of A. Kirillov and D. Yuri'ev [9], and later S. Nag and D. Sullivan [11], found an analogue of the period map for the Teichmüller space of the disk, which is now known as the KYNS period mapping. This maps into the bounded operators on Bergman space of one-forms (equivalently, the Dirichlet space of functions mod constants) which are symmetric in a certain sense and strictly bounded by one in norm. This is the so-called "infinite Siegel disk", whose elements determine a positive polarization in $\dot{H}^{1/2}(\mathbb{S}^1)$.

In contrast to the finite-dimensional Teichmüller space of compact surfaces, the Teichmüller space of the disk is a compact Banach manifold. One model of this is the set of conformal maps $f : \mathbb{D} \to \Sigma_1$ onto quasidisks Σ_1 in the sphere, modulo Möbius transformations. Takhtajan and Teo [27] showed that the KYNS period mapping is explicitly given by the Grunsky operator

$$f \mapsto \mathbf{Gr}_f.$$

The following question then arises: what is the relation between the compact case and the case of the disk, beyond an analogy? In effect, how can the results above be extended to the higher genus non-compact case in a unifying manner? This can be accomplished using the boundary value interpretation of the Grunsky operator, which is made possible by the existence of bounded overfare for quasicircles (achieved in our work).

The idea is as follows. Let Σ_2 be a surface capped by a collection of disks Σ_1 to obtain $\mathscr{R}$. The decomposition

$$\mathcal{A}_{\text{harm}}^{\text{se}}(\Sigma_2) = \mathcal{A}^{\text{se}}(\Sigma_2) \oplus \overline{\mathcal{A}^{\text{se}}(\Sigma_2)}$$

is associated to the complex structure on Σ_2. As we saw, every element of $\mathcal{A}_{\text{harm}}^{\text{se}}(\Sigma_2)$ is uniquely determined by its boundary values and by its cohomology class. Thus it is specified uniquely by an element of $\mathcal{A}_{\text{harm}}(\Sigma_1)$ with the same boundary values, and an element of $\mathcal{A}_{\text{harm}}(\mathscr{R})$ with the same cohomology. This association induces a polarization of

$$\mathcal{A}_{\text{harm}}(\Sigma_1) \oplus \mathcal{A}_{\text{harm}}(\mathscr{R})$$

by

$$\mathcal{A}_{\text{harm}}(\Sigma_1) \oplus \mathcal{A}_{\text{harm}}(\mathscr{R}) = \mathbf{O}^{\text{aug}} \mathcal{A}^{\text{se}}(\Sigma_2) \oplus \mathbf{O}^{\text{aug}} \overline{\mathcal{A}^{\text{se}}(\Sigma_2)}$$
$$=: W \oplus \overline{W}. \tag{6.2}$$

We then have the polarization

$$\mathcal{A}_{\text{harm}}(\mathbb{D})^n \oplus \Omega(\mathscr{R}) = \mathcal{W} \oplus \overline{\mathcal{W}},$$

where

$$\mathcal{W} = (f^*, \mathbf{Id}) W$$

and W is given by (6.2).

By Theorem 6.5 we have that $\mathcal{W}$ is the graph of Γ. In the case that $\mathscr{R}$ is the sphere and $n = 1$, the cohomology of $\mathscr{R}$ is trivial and this reduces to the KYNS period matrix. In the case that Σ_2 is itself compact and Σ_1 is empty, this reduces to the polarization (6.1). Work in progress of the authors embeds the Teichmüller space of bordered type (g, n) into the space of such generalized polarizations.

6.5 Kähler Potential on Weil–Petersson Universal Teichmüller Space

One can also relate the Schiffer operators to the Riemannian geometry of the Teichmüller space. The Teichmüller space of a Riemann surface is a space of quasiconformal deformations of the complex structure, which generically is an infinite-dimensional complex Banach manifold. A refinement of this space with more regular L^2-deformations has arisen over the last two decades, which possesses a Riemannian metric generalizing the Weil–Petersson metric on the Teichmüller space of compact surfaces. For a survey of the development of this "Weil-Petersson" Teichmüller space see [13, 22].

The Teichmüller space of the disk, the so-called universal Teichmüller space, can be identified with the set of quasicircles in the sphere, modulo Möbius transformations. The Weil–Petersson universal Teichmüller space is the subset of equivalence classes of more regular quasicircles known as Weil–Petersson quasicircles.

The paper of G. Cui [4] pioneered the theory of the Weil–Petersson universal Teichmüller space, establishing many of the main theorems. This was followed shortly by the L^p theory of H. Guo [5].

The monograph of Takhtajan and Teo added an extensive study of the geometry and global analysis of the Weil–Petersson universal Teichmüller space. For example, in [27] the Kähler potential was given an explicit form in terms of the Grunsky operator. As a consequence of the identities for the Schiffer operator, we derive here a new formula for this Kähler potential.

Let Γ be a Weil–Petersson quasicircle in the Riemann sphere, whose complement has components Σ_1 and Σ_2. Then we have

Theorem 6.6 *The potential for the Weil–Petersson metric on the Weil–Petersson Teichmüller space of the disk is*

$$\log \det(\mathbf{T}_{1,2}^* \mathbf{T}_{1,2}).$$

Proof As was shown by Takhtajan and Teo [27], one has that $\log \det(\mathbf{I} - \overline{\mathbf{Gr}_f}\mathbf{Gr}_f)$ is a potential for the WP metric. It was shown by the authors in [21] that

$$\overline{\mathbf{Gr}_f}\mathbf{Gr}_f = f^* \mathbf{T}_{1,2}^* \mathbf{T}_{1,2}(f^{-1})^*$$

and therefore

$$\log \det(\mathbf{I} - \overline{\mathbf{Gr}_f}\mathbf{Gr}_f) = \log \det(\mathbf{I} - \mathbf{T}_{1,2}^* \mathbf{T}_{1,2}),$$

which by unitarity of the first row of the scattering matrix in Theorem 5.7 is equal to

$$\log \det(\mathbf{T}_{1,2}^* \mathbf{T}_{1,2}).$$

$\square$

We may also write this in terms of the Cauchy operator on quasicircles. Let $\dot{\mathcal{D}}(\Sigma)$ denote the homogeneous Dirichlet space (that is, mod constants). Let Γ_ϵ approach Γ from within Σ_1. For fixed $q \notin \Gamma$, the Cauchy operator is defined by

$$\dot{\mathbf{J}} : \quad \dot{\mathcal{D}}(\Sigma_1) \to \dot{\mathcal{D}}(\Sigma_2)$$
$$h \mapsto \frac{1}{2\pi i} \lim_{\epsilon \searrow 0} \int_{\Gamma_\epsilon} h(\zeta) \left(\frac{1}{\zeta - z} - \frac{1}{\zeta - q} \right) d\zeta, \quad z \in \Sigma_2,$$

where we discard constants. The integral is in fact independent of q modulo constants, but the presence of q makes Möbius-invariance of the Cauchy integral more transparent [23]. We then have

Theorem 6.7 *The Kähler potential of the Weil–Petersson metric on the universal Weil–Petersson Teichmüller space is*

$$\log \det \left(\dot{\mathbf{J}}^* \dot{\mathbf{J}} \right).$$

Proof The theorem follows immediately from Theorem 6.6 and the fact that

$$d\dot{\mathbf{J}} = \mathbf{T}_{12} d,$$

see e.g. [23]. $\square$

This can be compared with [27, Remark 3.13]. Our theorem involves the complex Cauchy operator in place of the Neumann jump operator, and does not require the assumption that the curve is C^3.

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