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ORIGINAL ARTICLE

Real Characters in Nilpotent Blocks

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Abstract
We prove that the number of irreducible real characters in a nilpotent block of a finite group is locally determined. We further conjecture that the Frobenius–Schur indicators of those characters can be computed for $p = 2$ in terms of the extended defect group. We derive this from a more general conjecture on the Frobenius–Schur indicator of projective indecomposable characters of 2-blocks with one simple module. This extends results of Murray on 2-blocks with cyclic and dihedral defect groups.

Keywords Real characters · Frobenius–Schur indicators · Nilpotent blocks

Mathematics Subject Classification (2010) 20C15 · 20C20

1 Introduction

An important task in representation theory is to determine global invariants of a finite group G by means of local subgroups. Dade's conjecture, for instance, predicts the number of irreducible characters $\chi \in \mathrm{Irr}(G)$ such that the p-part $\chi(1)_p$ is a given power of a prime p (see [23, Conjecture 9.25]). Since Gow's work [7], there has been an increasing interest in counting real (i.e. real-valued) characters and more generally characters with a given field of values.

The quaternion group Q_8 testifies that a real irreducible character χ is not always afforded by a representation over the real numbers. The precise behavior is encoded by the *Frobenius–Schur indicator* (F-S indicator, for short)

$$\epsilon(\chi) := \frac{1}{|G|}\sum_{g\in G}\chi(g^2) = \begin{cases} 0 & \text{if } \overline{\chi} \neq \chi, \\ 1 & \text{if } \chi \text{ is realized by a real representation,} \\ -1 & \text{if } \chi \text{ is real, but not realized by a real representation.} \end{cases} \tag{1}$$

Dedicated to Pham Huu Tiep on the occasion of his 60th birthday.

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A new interpretation of the F-S indicator in terms of superalgebras has been given recently in [13]. The case of the dihedral group D_8 shows that $\epsilon(\chi)$ is not determined by the character table of G. The computation of F-S indicators can be a surprisingly difficult task, which has not been fully completed for the simple groups of Lie type, for instance (see [25]). Problem 14 on Brauer's famous list [2] asks for a group-theoretical interpretation of the number of $\chi \in \mathrm{Irr}(G)$ with $\epsilon(\chi) = 1$.

To obtain deeper insights, we fix a prime p and assume that χ lies in a p-block B of G with defect group D. By complex conjugation we obtain another block $\overline{B}$ of G. If $\overline{B} \neq B$, then clearly $\epsilon(\chi) = 0$ for all $\chi \in \mathrm{Irr}(B)$. Hence, we assume that B is real, i.e. $\overline{B} = B$. John Murray [18, 19] has computed the F-S indicators when D is a cyclic 2-group or a dihedral 2-group (including the Klein four-group). His results depend on the fusion system of B, on Erdmann's classification of tame blocks and on the structure of the so-called *extended defect group* E of B (see Definition 7 below). For $p > 2$ and D cyclic, he obtained in [20] partial information on the F-S indicators in terms of the Brauer tree of B.

The starting point of my investigation is the well-known fact that 2-blocks with cyclic defect groups are nilpotent. Assume that B is nilpotent and real. If B is the principal block, then $G = \mathrm{O}_{p'}(G)D$ and $\mathrm{Irr}(B) = \mathrm{Irr}(G/\mathrm{O}_{p'}(G)) = \mathrm{Irr}(D)$. In this case the F-S indicators of B are determined by D alone. Thus, suppose that B is non-principal. By Broué–Puig [4], there exists a height-preserving bijection $\mathrm{Irr}(D) \to \mathrm{Irr}(B)$, $\lambda \mapsto \lambda * \chi_0$, where $\chi_0 \in \mathrm{Irr}(B)$ is a fixed character of height 0 (see also [16, Definition 8.10.2]). However, this bijection does not in general preserve F-S indicators. For instance, the dihedral group D_{24} has a nilpotent 2-block with defect group C_4 and a nilpotent 3-block with defect group C_3, although every character of D_{24} is real. Our main theorem asserts that the number of real characters in a nilpotent block is nevertheless locally determined. To state it, we introduce the *extended inertial group*

$$\mathrm{N}_G(D, b_D)^* := \left\{g \in \mathrm{N}_G(D) : b_D^g \in \{b_D, \overline{b_D}\}\right\},$$

where b_D is a Brauer correspondent of B in $D\mathrm{C}_G(D)$.

Theorem A *Let B be a real, nilpotent p-block of a finite group G with defect group D. Let b_D be a Brauer correspondent of B in $D\mathrm{C}_G(D)$. Then the number of real characters in $\mathrm{Irr}(B)$ of height h coincides with the number of characters $\lambda \in \mathrm{Irr}(D)$ of degree p^h such that $\lambda^t = \overline{\lambda}$, where*

$$\mathrm{N}_G(D, b_D)^*/D\mathrm{C}_G(D) = \langle t\, D\mathrm{C}_G(D)\rangle.$$

If $p > 2$, then all real characters in $\mathrm{Irr}(B)$ have the same F-S indicator.

In contrast to arbitrary blocks, Theorem A implies that nilpotent real blocks have at least one real character (cf. [20, p. 92] and [8, Theorem 5.3]). If $\overline{b_D} = b_D$, then B and D have the same number of real characters, because $\mathrm{N}_G(D, b_D) = D\mathrm{C}_G(D)$. This recovers a result of Murray [18, Lemma 2.2]. As another consequence, we will derive in Proposition 5 a real version of Eaton's conjecture [5] for nilpotent blocks as put forward by Héthelyi–Horváth–Szabó [12].

The F-S indicators of real characters in nilpotent blocks seem to lie somewhat deeper. We still conjecture that they are locally determined by a defect pair (see Definition 6) for $p = 2$ as follows.

Conjecture B *Let B be a real, nilpotent, non-principal 2-block of a finite group G with defect pair (D, E). Then there exists a height preserving bijection $\Gamma : \mathrm{Irr}(D) \to \mathrm{Irr}(B)$ such that*

$$\epsilon(\Gamma(\lambda)) = \frac{1}{|D|} \sum_{e \in E \setminus D} \lambda(e^2) \tag{2}$$

for all $\lambda \in \mathrm{Irr}(D)$.

The right hand side of (2) was introduced and studied by Gow [8, Lemma 2.1] more generally for any groups $D \leq E$ with $|E : D| = 2$. This invariant was later coined the *Gow indicator* by Murray [20, (2)]. For 2-blocks of defect 0, Conjecture B confirms the known fact that real characters of 2-defect 0 have F-S indicator 1 (see [8, Theorem 5.1]). There is no such result for odd primes p. As a matter of fact, every real character has p-defect 0 whenever p does not divide $|G|$. In Theorem 10 we prove Conjecture B for abelian defect groups D. Then it also holds for all quasisimple groups G by work of An–Eaton [1]. Murray's results mentioned above, imply Conjecture B also for dihedral D.

For $p > 2$, the common F-S indicator in the situation of Theorem A is not locally determined. For instance, $G = Q_8 \rtimes C_9 = \texttt{SmallGroup}\,(72, 3)$ has a non-principal real 3-block with $D \cong C_9$ and common F-S indicator -1, while its Brauer correspondent in $\mathrm{N}_G(D) \cong C_{18}$ has common F-S indicator 1. Nevertheless, for cyclic defect groups D we find another way to compute this F-S indicator in Theorem 3 below.

Our second conjecture applies more generally to blocks with only one simple module.

Conjecture C *Let B be a real, non-principal 2-block with defect pair (D, E) and a unique projective indecomposable character Φ. Then*

$$\epsilon(\Phi) = |\{x \in E \setminus D : x^2 = 1\}|.$$

Here $\epsilon(\Phi)$ is defined by extending (1) linearly. If $\epsilon(\Phi) = 0$, then E does not split over D and Conjecture C holds (see Proposition 8 below). Conjecture C implies a stronger, but more technical statement on 2-blocks with a Brauer correspondent with one simple module (see Theorem 13 below). This allows us to prove the following.

Theorem D *Conjecture C implies Conjecture B.*

We remark that our proof of Theorem D does not work block-by-block. For solvable groups we offer a purely group-theoretical version of Conjecture C at the end of Section 4.

Theorem E *Conjectures B and C hold for all nilpotent 2-blocks of solvable groups.*

We have checked Conjectures B and C with GAP [6] in many examples using the libraries of small groups, perfect groups and primitive groups.

2 Theorem A and Its Consequences

Our notation follows closely Navarro's book [22]. In particular, G^0 denotes the set of p-regular elements of a finite group G. Let B be a p-block of G with defect group D. Recall that a B-*subsection* is a pair (u, b), where $u \in D$ and b is a Brauer correspondent of B in $\mathrm{C}_G(u)$. For $\chi \in \mathrm{Irr}(B)$ and $\varphi \in \mathrm{IBr}(b)$ we denote the corresponding generalized decomposition number by $d^u_{\chi\varphi}$. If $u = 1$, we obtain the (ordinary) decomposition number $d_{\chi\varphi} = d^1_{\chi\varphi}$. We put $l(b) = |\mathrm{IBr}(b)|$ as usual.

Following [22, p. 114], we define a class function $\chi^{(u,b)}$ by

$$\chi^{(u,b)}(us) := \sum_{\varphi \in \mathrm{IBr}(b)} d^u_{\chi\varphi}\varphi(s)$$

for $s \in \mathrm{C}_G(u)^0$ and $\chi^{(u,b)}(x) = 0$ whenever x is outside the p-section of u. If $\mathcal{R}$ is a set of representatives for the G-conjugacy classes of B-subsections, then $\chi = \sum_{(u,b)\in\mathcal{R}} \chi^{(u,b)}$ by

Brauer's second main theorem (see [22, Problem 5.3]). Now suppose that B is nilpotent and $\lambda \in \operatorname{Irr}(D)$. By [16, Proposition 8.11.4], each Brauer correspondent b of B is nilpotent and in particular $l(b) = 1$. Broué–Puig [4] have shown that, if χ has height 0, then

$$\lambda * \chi := \sum_{(u,b)\in\mathcal{R}} \lambda(u)\chi^{(u,b)} \in \operatorname{Irr}(B)$$

and $(\lambda * \chi)(1) = \lambda(1)\chi(1)$. Note also that $d^u_{\lambda*\chi,\varphi} = \lambda(u)d^u_{\chi\varphi}$.

Proof of Theorem A Let $\mathcal{R}$ be a set of representatives for the G-conjugacy classes of B-subsections $(u, b_u) \le (D, b_B)$ (see [22, p. 219]). Since B is nilpotent, we have $\operatorname{IBr}(b_u) = \{\varphi_u\}$ for all $(u, b_u) \in \mathcal{R}$. Since the Brauer correspondence is compatible with complex conjugation, $(u, \overline{b_u})^t \le (D, \overline{b_D})^t = (D, b_D)$, where $\mathrm{N}_G(D, b_D)^*/D\mathrm{C}_G(D) = \langle tD\mathrm{C}_G(D)\rangle$. Thus, $(u, \overline{b_u})^t$ is D-conjugate to some $(u', b_{u'}) \in \mathcal{R}$.

If $p > 2$, there exists a unique p-rational character $\chi_0 \in \operatorname{Irr}(B)$ of height 0, which must be real by uniqueness (see [4, Remark after Theorem 1.2]). If $p = 2$, there is a 2-rational real character $\chi_0 \in \operatorname{Irr}(B)$ of height 0 by [8, Theorem 5.1]. Then $d^u_{\chi_0,\varphi_u} = d^u_{\chi_0,\overline{\varphi_u}} \in \mathbb{Z}$ and

$$\overline{\chi_0^{(u,b_u)}} = \chi_0^{(u,\overline{b_u})} = \chi_0^{(u,\overline{b_u})^t} = \chi_0^{(u',b_{u'})}.$$

Now let $\lambda \in \operatorname{Irr}(D)$. Then

$$\overline{\lambda * \chi_0} = \sum_{(u,b_u)\in\mathcal{R}} \overline{\lambda}(u)\overline{\chi_0^{(u,b_u)}} = \sum_{(u,b_u)\in\mathcal{R}} \overline{\lambda}(u)\chi_0^{(u',b_{u'})}.$$

Since the class functions $\chi_0^{(u,b)}$ have disjoint support, they are linearly independent. Therefore, $\lambda * \chi_0$ is real if and only if $\lambda(u^t) = \lambda(u') = \overline{\lambda}(u)$ for all $(u, b_u) \in \mathcal{R}$. Since every conjugacy class of D is represented by some u with $(u, b_u) \in \mathcal{R}$, we conclude that $\lambda * \chi_0$ is real if and only $\lambda^t = \overline{\lambda}$. Moreover, if $\lambda(1) = p^h$, then $\lambda * \chi_0$ has height h. This proves the first claim.

To prove the second claim, let $p > 2$ and $\operatorname{IBr}(B) = \{\varphi\}$. Then the decomposition numbers $d_{\lambda*\chi_0,\varphi} = \lambda(1)$ are powers of p; in particular they are odd. A theorem of Thompson and Willems (see [26, Theorem 2.8]) states that all real characters χ with $d_{\chi,\varphi}$ odd have the same F-S indicator. So in our situation all real characters in $\operatorname{Irr}(B)$ have the same F-S indicator. □

Since the automorphism group of a p-group is "almost always" a p-group (see [11]), the following consequence is of interest.

Corollary 1 *Let B be a real, nilpotent p-block with defect group D such that p and $|\operatorname{Aut}(D)|$ are odd. Then B has a unique real character.*

Proof The hypothesis on $\operatorname{Aut}(D)$ implies that $\mathrm{N}_G(D, b_D)^* = D\mathrm{C}_G(D)$. Hence by Theorem A, the number of real characters in $\operatorname{Irr}(B)$ is the number of real characters in D. Since $p > 2$, the trivial character is the only real character of D. □

The next lemma is a consequence of Brauer's second main theorem and the fact that $|\{g \in G : g^2 = x\}| = |\{g \in \mathrm{C}_G(x) : g^2 = x\}|$ is locally determined for $g, x \in G$.

Lemma 2 (Brauer) *For every p-block B of G and every B-subsection (u, b) with $\varphi \in \operatorname{IBr}(b)$ we have*

$$\sum_{\chi\in\operatorname{Irr}(B)} \epsilon(\chi)d^u_{\chi\varphi} = \sum_{\psi\in\operatorname{Irr}(b)} \epsilon(\psi)d^u_{\psi\varphi} = \sum_{\psi\in\operatorname{Irr}(b)} \epsilon(\psi)\frac{\psi(u)}{\psi(1)}d_{\psi\varphi}.$$

If $l(b) = 1$, *then*

$$\sum_{\chi \in \mathrm{Irr}(B)} \epsilon(\chi) d^u_{\chi\varphi} = \frac{1}{\varphi(1)} \sum_{\psi \in \mathrm{Irr}(b)} \epsilon(\psi)\psi(u).$$

Proof The first equality is [3, Theorem 4A]. The second follows from $u \in \mathrm{Z}(\mathrm{C}_G(u))$. If $l(b) = 1$, then $\psi(1) = d_{\psi\varphi}\varphi(1)$ for $\psi \in \mathrm{Irr}(b)$ and the last claim follows. □

Recall that a *canonical* character of B is a character $\theta \in \mathrm{Irr}(D\mathrm{C}_G(D))$ lying in a Brauer correspondent of B such that $D \leq \mathrm{Ker}(\theta)$ (see [22, Theorem 9.12]). We define the *extended stabilizer*

$$\mathrm{N}_G(D)^*_\theta := \left\{ g \in \mathrm{N}_G(D) : \theta^g \in \{\theta, \overline{\theta}\} \right\}.$$

The following results adds some detail to the nilpotent case of [20, Theorem 1].

Theorem 3 *Let B be a real, nilpotent p-block with cyclic defect group $D = \langle u \rangle$* and $p > 2$. Let $\theta \in \mathrm{Irr}(\mathrm{C}_G(D))$ *be a canonical character of B and set $T := \mathrm{N}_G(D)^*_\theta$. Then one of the following holds:*

1) $\overline{\theta} \neq \theta$. *All characters in* $\mathrm{Irr}(B)$ *are real with F-S indicator* $\epsilon(\theta^T)$.
2) $\overline{\theta} = \theta$. *The unique non-exceptional character* $\chi_0 \in \mathrm{Irr}(B)$ *is the only real character in* $\mathrm{Irr}(B)$ *and* $\epsilon(\chi_0) = \mathrm{sgn}(\chi_0(u))\epsilon(\theta)$, *where* $\mathrm{sgn}(\chi_0(u))$ *is the sign of* $\chi_0(u)$.

Proof Let b_D be a Brauer correspondent of B in $\mathrm{C}_G(D)$ containing θ. Then $T = \mathrm{N}_G(D, b_D)^*$. If $\overline{\theta} \neq \theta$, then T inverts the elements of D since $p > 2$. Thus, Theorem A implies that all characters in $\mathrm{Irr}(B)$ are real. By [20, Theorem 1(v)], the common F-S indicator is the Gow indicator of θ with respect to T. This is easily seen to be $\epsilon(\theta^T)$ (see [20, after (2)]).

Now assume that $\overline{\theta} = \theta$. Here Theorem A implies that the unique p-rational character $\chi_0 \in \mathrm{Irr}(B)$ is the only real character. In particular, χ_0 must be the unique non-exceptional character. Note that (u, b_D) is a B-subsection and $\mathrm{IBr}(b_D) = \{\varphi\}$. Since χ_0 is p-rational, $d^u_{\chi_0\varphi} = \pm 1$. Since all Brauer correspondents of B in $\mathrm{C}_G(u)$ are conjugate under $\mathrm{N}_G(D)$, the generalized decomposition numbers are Galois conjugate, in particular $d^u_{\chi_0\varphi}$ does not depend on the choice of b_D. Hence,

$$\chi_0(u) = |\mathrm{N}_G(D) : \mathrm{N}_G(D)_\theta| d^u_{\chi_0\varphi}\varphi(1)$$

and $d^u_{\chi_0\varphi} = \mathrm{sgn}(\chi_0(u))$. Moreover, θ is the unique non-exceptional character of b_D and $\theta(u) = \theta(1)$. By Lemma 2, we obtain

$$\begin{aligned}\epsilon(\chi_0) &= \mathrm{sgn}(\chi_0(u)) \sum_{\chi \in \mathrm{Irr}(B)} \epsilon(\chi) d^u_{\chi\varphi} \\ &= \frac{\mathrm{sgn}(\chi_0(u))}{\varphi(1)} \sum_{\psi \in \mathrm{Irr}(b_D)} \epsilon(\psi)\psi(u) = \mathrm{sgn}(\chi_0(u))\epsilon(\theta).\end{aligned}$$

□

If B is a nilpotent block with canonical character $\theta \neq \overline{\theta}$, the common F-S indicator of the real characters in $\mathrm{Irr}(B)$ is not always $\epsilon(\theta^T)$ as in Theorem 3 . A counterexample is given by a certain 3-block of $G = \texttt{SmallGroup}(288, 924)$ with defect group $D \cong C_3 \times C_3$.

We now restrict ourselves to 2-blocks. Héthelyi–Horváth–Szabó [12] introduced four conjectures, which are real versions of Brauer's conjecture, Olsson's conjecture and Eaton's conjecture. We only state the strongest of them, which implies the remaining three. Let $D^{(0)} := D$ and $D^{(k+1)} := [D^{(k)}, D^{(k)}]$ for $k \geq 0$ be the members of the derived series of D.

Conjecture 4 (Héthelyi–Horváth–Szabó) *Let B be a 2-block with defect group D. For every $h \geq 0$, the number of real characters in* $\mathrm{Irr}(B)$ *of height $\leq h$ is bounded by the number of elements of $D/D^{(h+1)}$ which are real in* $\mathrm{N}_G(D)/D^{(h+1)}$.

A conjugacy class K of G is called *real* if $K = K^{-1} := \{x^{-1} : x \in K\}$. A conjugacy class K of a normal subgroup $N \trianglelefteq G$ is called *real under G* if there exists $g \in G$ such that $K^g = K^{-1}$.

Proposition 5 *Let B be a nilpotent 2-block with defect group D and Brauer correspondent b_D* in $D\mathrm{C}_G(D)$. *Then the number of real characters in* $\mathrm{Irr}(B)$ *of height $\leq h$ is bounded by the number of conjugacy classes of $D/D^{(h+1)}$ which are real under* $\mathrm{N}_G(D, b_D)^*/D^{(h+1)}$. *In particular, Conjecture 4 holds for B.*

Proof We may assume that B is real. As in the proof of Theorem A, we fix some 2-rational real character $\chi_0 \in \mathrm{Irr}(B)$ of height 0. Now $\lambda * \chi_0$ has height $\leq h$ if and only if $\lambda(1) \leq p^h$ for $\lambda \in \mathrm{Irr}(B)$. By [14, Theorem 5.12], the characters of degree $\leq p^h$ in $\mathrm{Irr}(D)$ lie in $\mathrm{Irr}(D/D^{(h+1)})$. By Theorem A, $\lambda * \chi_0$ is real if and only if $\lambda^t = \overline{\lambda}$. By Brauer's permutation lemma (see [23, Theorem 2.3]), the number of those characters λ coincides with the number of conjugacy classes K of $D/D^{(h+1)}$ such that $K^t = K^{-1}$. Now Conjecture 4 follows from $\mathrm{N}_G(D, b_D)^* \leq \mathrm{N}_G(D)$. □

3 Extended Defect Groups

We continue to assume that $p = 2$. As usual we choose a complete discrete valuation ring $\mathcal{O}$ such that $F := \mathcal{O}/J(\mathcal{O})$ is an algebraically closed field of characteristic 2. Let $\mathrm{Cl}(G)$ be the set of conjugacy classes of G. For $K \in \mathrm{Cl}(G)$ let $K^+ := \sum_{x \in K} x \in \mathrm{Z}(FG)$ be the class sum of K. We fix a 2-block B of FG with block idempotent $1_B = \sum_{K \in \mathrm{Cl}(G)} a_K K^+$, where $a_K \in F$. The central character of B is defined by

$$\lambda_B : \mathrm{Z}(FG) \to F, \quad K^+ \mapsto \left(\frac{|K|\chi(g)}{\chi(1)}\right)^*,$$

where $g \in K$, $\chi \in \mathrm{Irr}(B)$ and * denotes the canonical reduction $\mathcal{O} \to F$ (see [22, Chapter 2]).

Since $\lambda_B(1_B) = 1$, there exists $K \in \mathrm{Cl}(G)$ such that $a_K \neq 0 \neq \lambda_B(K^+)$. We call K a *defect class* of B. By [22, Corollary 3.8], K consists of elements of odd order. According to [22, Corollary 4.5], a Sylow 2-subgroup D of $\mathrm{C}_G(x)$, where $x \in K$, is a defect group of B. For $x \in K$ let

$$\mathrm{C}_G(x)^* := \{g \in G : gxg^{-1} = x^{\pm 1}\} \leq G$$

be the *extended centralizer* of x.

Proposition 6 (Gow, Murray) *Every real 2-block B has a real defect class K.* Let $x \in K$. *Choose a Sylow 2-subgroup E of* $\mathrm{C}_G(x)^*$ *and put $D := E \cap \mathrm{C}_G(x)$. Then the G-conjugacy class of the pair (D, E) does not depend on the choice of K* or x.

Proof For the principal block (which is always real since it contains the trivial character), $K = \{1\}$ is a real defect class and $E = D$ is a Sylow 2-subgroup of G. Hence, the uniqueness follows from Sylow's theorem. Now suppose that B is non-principal. The existence of K was first shown in [8, Theorem 5.5]. Let L be another real defect class of B and choose $y \in L$. By [9, Corollary 2.2], we may assume after conjugation that E is also a Sylow 2-subgroup of $\mathrm{C}_G(y)^*$. Let $D_x := E \cap \mathrm{C}_G(x)$ and $D_y := E \cap \mathrm{C}_G(y)$. We may assume that $|E : D_x| = 2 = |E : D_y|$ (cf. the remark after the proof).

We now introduce some notation in order to apply [17, Proposition 14]. Let $\Sigma = \langle\sigma\rangle \cong C_2$. We consider FG as an $F[G \times \Sigma]$-module, where G acts by conjugation and $g^\sigma = g^{-1}$ for $g \in G$ (observe that these actions indeed commute). For $H \le G \times \Sigma$ let

$$\mathrm{Tr}_H^{G\times\Sigma} : (FG)^H \to (FG)^{G\times\Sigma}, \quad \alpha \mapsto \sum_{x\in\mathcal{R}} \alpha^x$$

be the *relative trace* with respect to H, where $\mathcal{R}$ denotes a set of representatives of the right cosets of H in $G \times \Sigma$. By [17, Proposition 14], we have $1_B \in \mathrm{Tr}_{E_x}^{G\times\Sigma}(FG)$, where $E_x := D_x\langle e_x\sigma\rangle$ for some $e_x \in E \setminus D_x$. By the same result we also obtain that $D_y\langle e_y\sigma\rangle$ with $e_y \in E \setminus D_y$ is G-conjugate to E_x. This implies that D_y is conjugate to D_x inside $\mathrm{N}_G(E)$. In particular, (D_x, E) and (D_y, E) are G-conjugate as desired. □

Definition 7 In the situation of Proposition 6 we call E an *extended defect group* and (D, E) a *defect pair* of B.

We stress that real 2-blocks can have non-real defect classes and non-real blocks can have real defect classes (see [10, Theorem 3.5]).

It is easy to show that non-principal real 2-blocks cannot have maximal defect (see [22, Problem 3.8]). In particular, the trivial class cannot be a defect class and consequently, $|E : D| = 2$ in those cases. For non-real blocks we define the extended defect group by $E := D$ for convenience. Every given pair of 2-groups $D \le E$ with $|E : D| = 2$ occurs as a defect pair of a real (nilpotent) block. To see this, let $Q \cong C_3$ and $G = Q \rtimes E$ with $\mathrm{C}_E(Q) = D$. Then G has a unique non-principal block with defect pair (D, E).

We recall from [14, p. 49] that

$$\sum_{\chi\in\mathrm{Irr}(G)} \epsilon(\chi)\chi(g) = |\{x \in G : x^2 = g\}| \tag{3}$$

for all $g \in G$. The following proposition provides some interesting properties of defect pairs.

Proposition 8 (Gow, Murray) *Let B be a real 2-block with defect pair (D, E). Let b_D be a Brauer correspondent of B in $D\mathrm{C}_G(D)$. Then the following holds:*

(i) $\mathrm{N}_G(D, b_D)^* = \mathrm{N}_G(D, b_D)E$. *In particular, b_D is real if and only if $E = D\mathrm{C}_E(D)$.*
(ii) *For $u \in D$, we have $\sum_{\chi\in\mathrm{Irr}(B)} \epsilon(\chi)\chi(u) \ge 0$ with strict inequality if and only if u is G-conjugate to e^2 for some $e \in E \setminus D$. In particular, E splits over D if and only if $\sum_{\chi\in\mathrm{Irr}(B)} \epsilon(\chi)\chi(1) > 0$.*
(iii) *E/D' splits over D/D' if and only if all height zero characters in* $\mathrm{Irr}(B)$ *have non-negative F-S indicator.*

Proof (i) See [19, Lemma 1.8] and [18, Theorem 1.4].
(ii) See [19, Lemma 1.3].
(iii) See [8, Theorem 5.6]. □

The next proposition extends [18, Lemma 1.3].

Corollary 9 *Suppose that B is a 2-block with defect pair (D, E) where D is abelian. Then E splits over D if and only if all characters in* $\mathrm{Irr}(B)$ *have non-negative F-S indicator.*

Proof If B is non-real, then $E = D$ splits over D and all characters in $\mathrm{Irr}(B)$ have F-S indicator 0. Hence, let $\overline{B} = B$. By Kessar–Malle [15], all characters in $\mathrm{Irr}(B)$ have height 0. Hence, the claim follows from Proposition 8 (iii). □

Theorem 10 *Let B be a real, nilpotent 2-block with defect pair (D, E), where D is abelian. If E splits over D, then all real characters in* $\mathrm{Irr}(B)$ *have F-S indicator* 1. *Otherwise exactly half of the real characters have F-S indicator* 1. *In either case, Conjecture B holds for B.*

Proof If E splits over D, then all real characters in $\mathrm{Irr}(B)$ have F-S indicator 1 by Corollary 9. Otherwise we have $\sum_{\chi\in\mathrm{Irr}(B)}\epsilon(\chi) = 0$ by Proposition 8(ii), because all characters in $\mathrm{Irr}(B)$ have the same degree. Hence, exactly half of the real characters have F-S indicator 1. Using Theorem A we can determine the number of characters for each F-S indicator. For the last claim, we may therefore replace B by the unique non-principal block of $G = Q \rtimes E$, where $Q \cong C_3$ and $\mathrm{C}_E(Q) = D$ (mentioned above). In this case Conjecture B follows from Gow [8, Lemma 2.2] or Theorem E. □

Example 11 Let B be a real block with defect group $D \cong C_4 \times C_2$. Then B is nilpotent since $\mathrm{Aut}(D)$ is a 2-group and D is abelian. Moreover $|\mathrm{Irr}(B)| = 8$. The F-S indicators depend not only on E, but also on the way D embeds into E. The following cases can occur (here M_{16} denotes the modular group and [16, 3] refers to the small group library):

F-S indicators	E
$+ + + + + + + +$	$D_8 \times C_2$
$+ + + + - - - -$	$Q_8 \times C_2,\ C_4 \rtimes C_4$ with $\Phi(D) = E'$
$+ + + + 0\ 0\ 0\ 0$	$D,\ D \times C_2,\ D_8 * C_4,\ [16, 3]$
$+ + - - 0\ 0\ 0\ 0$	$C_4^2,\ C_8 \times C_2,\ M_{16},\ C_4 \rtimes C_4$ with $\Phi(D) \neq E'$

The F-S indicator $\epsilon(\Phi)$ appearing in Conjecture C has an interesting interpretation as follows. Let $\Omega := \{g \in G : g^2 = 1\}$. The conjugation action of G on Ω turns $F\Omega$ into an FG-module, called the *involution module*.

Lemma 12 (Murray) *Let B be a real* 2*-block and $\varphi \in \mathrm{IBr}(B)$. Then $\epsilon(\Phi_\varphi)$ is the multiplicity of φ as a constituent of the Brauer character of $F\Omega$.*

Proof See [18, Lemma 2.6]. □

Next we develop a local version of Conjecture C. Let B be a real 2-block with defect pair (D, E) and B-subsection (u, b). If $E = D\mathrm{C}_E(u)$, then b is real and $(\mathrm{C}_D(u), \mathrm{C}_E(u))$ is a defect pair of b by [19, Lemma 2.6] applied to the subpair $(\langle u\rangle, b)$. Conversely, if b is real, we may assume that $(\mathrm{C}_D(u), \mathrm{C}_E(u))$ is a defect pair of b by [19, Theorem 2.7]. If b is non-real, we may assume that $(\mathrm{C}_D(u), \mathrm{C}_D(u)) = (\mathrm{C}_D(u), \mathrm{C}_E(u))$ is a defect pair of b.

Theorem 13 *Let B be* 2*-block of a finite group G with defect pair (D, E). Suppose that Conjecture C holds for all Brauer correspondents of B in sections of G. Let (u, b) be a B-subsection with defect pair $(\mathrm{C}_D(u), \mathrm{C}_E(u))$ such that $\mathrm{IBr}(b) = \{\varphi\}$. Then*

$$\sum_{\chi\in\mathrm{Irr}(B)} \epsilon(\chi) d^u_{\chi\varphi} = \begin{cases} |\{x \in D : x^2 = u\}| & \textit{if } B \textit{ is the principal block},\\ |\{x \in E \setminus D : x^2 = u\}| & \textit{otherwise}. \end{cases}$$

Proof If B is not real, then B is non-principal and $E = D$. It follows that $\epsilon(\chi) = 0$ for all $\chi \in \mathrm{Irr}(B)$ and

$$|\{x \in E \setminus D : x^2 = u\}| = 0.$$

Hence, we may assume that B is real. By Lemma 2, we have

$$\sum_{\chi\in\mathrm{Irr}(B)} \epsilon(\chi) d^u_{\chi\varphi} = \sum_{\psi\in\mathrm{Irr}(b)} \epsilon(\psi) d^u_{\psi\varphi} = \frac{1}{\varphi(1)} \sum_{\psi\in\mathrm{Irr}(b)} \epsilon(\psi)\psi(u). \tag{4}$$

Suppose that B is the principal block. Then b is the principal block of $\mathrm{C}_G(u)$ by Brauer's third main theorem (see [22, Theorem 6.7]). The hypothesis $l(b) = 1$ implies that $\varphi = 1_{\mathrm{C}_G(u)}$ and $\mathrm{C}_G(u)$ has a normal 2-complement N (see [22, Corollary 6.13]). It follows that $\mathrm{Irr}(b) = \mathrm{Irr}(\mathrm{C}_G(u)/N) = \mathrm{Irr}(\mathrm{C}_D(u))$ and

$$\sum_{\psi\in\mathrm{Irr}(b)} \epsilon(\psi)d^u_{\psi\varphi} = \sum_{\lambda\in\mathrm{Irr}(\mathrm{C}_D(u))} \epsilon(\lambda)\lambda(u) = |\{x \in \mathrm{C}_D(u) : x^2 = u\}|$$

by (3). Since every $x \in D$ with $x^2 = u$ lies in $\mathrm{C}_D(u)$, we are done in this case.

Now let B be a non-principal real 2-block. If b is not real, then (4) shows that $\sum_{\chi\in\mathrm{Irr}(B)} \epsilon(\chi)d^u_{\chi\varphi} = 0$. On the other hand, we have $\mathrm{C}_E(u) = \mathrm{C}_D(u) \leq D$ and $|\{x \in E \setminus D : x^2 = u\}| = 0$. Hence, we may assume that b is real. Since every $x \in E$ with $x^2 = u$ lies in $\mathrm{C}_E(u)$, we may assume that $u \in \mathrm{Z}(G)$ by (4).

Then $\chi(u) = d^u_{\chi\varphi}\varphi(1)$ for all $\chi \in \mathrm{Irr}(B)$. If $u^2 \notin \mathrm{Ker}(\chi)$, then $\chi(u) \notin \mathbb{R}$ and $\epsilon(\chi) = 0$. Thus, it suffices to sum over χ with $d^u_{\chi\varphi} = \pm d_{\chi\varphi}$. Let $Z := \langle u\rangle \leq \mathrm{Z}(G)$ and $\overline{G} := G/Z$. Let $\hat{B}$ be the unique (real) block of $\overline{G}$ dominated by B. By [19, Lemma 1.7], $(\overline{D}, \overline{E})$ is a defect pair for $\hat{B}$. Then, using [14, Lemma 4.7] and Conjecture C for B and $\hat{B}$, we obtain

$$\begin{aligned}
\sum_{\chi\in\mathrm{Irr}(B)} \epsilon(\chi)d^u_{\chi\varphi} &= \sum_{\chi\in\mathrm{Irr}(B)} \epsilon(\chi)(d_{\chi\varphi} + d^u_{\chi\varphi}) - \sum_{\chi\in\mathrm{Irr}(B)} \epsilon(\chi)d_{\chi\varphi}\\
&= 2\sum_{\chi\in\mathrm{Irr}(\hat{B})} \epsilon(\chi)d_{\chi\varphi} - \sum_{\chi\in\mathrm{Irr}(B)} \epsilon(\chi)d_{\chi\varphi}\\
&= 2|\{\overline{x} \in \overline{E} \setminus \overline{D} : \overline{x}^2 = 1\}| - |\{x \in E \setminus D : x^2 = 1\}|\\
&= \sum_{\lambda\in\mathrm{Irr}(E)} \epsilon(\lambda)(\lambda(1) + \lambda(u)) - \sum_{\lambda\in\mathrm{Irr}(D)} \epsilon(\lambda)(\lambda(1) + \lambda(u))\\
&\quad - \sum_{\lambda\in\mathrm{Irr}(E)} \epsilon(\lambda)\lambda(1) + \sum_{\lambda\in\mathrm{Irr}(D)} \epsilon(\lambda)\lambda(1)\\
&= \sum_{\lambda\in\mathrm{Irr}(E)} \epsilon(\lambda)\lambda(u) - \sum_{\lambda\in\mathrm{Irr}(D)} \epsilon(\lambda)\lambda(u) = |\{x \in E \setminus D : x^2 = u\}|.
\end{aligned}$$

□

4 Theorems D and E

The following result implies Theorem D.

Theorem 14 *Suppose that B is a real, nilpotent, non-principal 2-block fulfilling the statement of Theorem 13. Then Conjecture B holds for B.*

Proof Let (D, E) be defect pair of B. By Gow [8, Theorem 5.1], there exists a 2-rational character $\chi_0 \in \mathrm{Irr}(B)$ of height 0 and $\epsilon(\chi_0) = 1$. Let

$$\Gamma : \mathrm{Irr}(D) \to \mathrm{Irr}(B), \qquad \lambda \mapsto \lambda * \chi_0$$

be the Broué–Puig bijection. Let $(u_1, b_1), \ldots, (u_k, b_k)$ be representatives for the conjugacy classes of B-subsections. Since B is nilpotent, we may assume that $u_1, \ldots, u_k \in D$ represent the conjugacy classes of D. Let $\mathrm{IBr}(b_i) = \{\varphi_i\}$ for $i = 1, \ldots, k$. Since χ_0 is 2-rational, we have $\sigma_i := d^u_{\chi_0,\varphi_i} \in \{\pm 1\}$ for $i = 1, \ldots, k$. Hence, the generalized decomposition matrix of B has the form

$$Q = (\lambda(u_i)\sigma_i : \lambda \in \mathrm{Irr}(D), i = 1, \ldots, k)$$

(see [16, Section 8.10]). Let $v := (\epsilon(\Gamma(\lambda)) : \lambda \in \mathrm{Irr}(D))$ and $w := (w_1, \ldots, w_k)$ where $w_i := |\{x \in E \setminus D : x^2 = u_i\}|$. Then Theorem 13 reads as $vQ = w$.

Let $d_i := |\mathrm{C}_D(u_i)|$ and $d = (d_1, \ldots, d_k)$. Then the second orthogonality relation yields $Q^{\mathrm{t}}\overline{Q} = \mathrm{diag}(d)$, where Q^{t} denotes the transpose of Q. It follows that $Q^{-1} = \mathrm{diag}(d)^{-1}\overline{Q}^{\mathrm{t}}$ and

$$v = w\mathrm{diag}(d)^{-1}\overline{Q}^{\mathrm{t}} = w\mathrm{diag}(d)^{-1}Q^{\mathrm{t}},$$

because $\overline{v} = v$. Since $w_i = |\{x \in E \setminus D : x^2 = u_i^y\}|$ for every $y \in D$, we obtain $\sum_{i=1}^k w_i|D : \mathrm{C}_D(u_i)| = |E \setminus D| = |D|$. In particular,

$$1 = \epsilon(\chi_0) = \sum_{i=1}^k \frac{w_i\sigma_i}{|\mathrm{C}_D(u_i)|} \le \sum_{i=1}^k \frac{w_i|\sigma_i|}{|\mathrm{C}_D(u_i)|} = 1.$$

Therefore, $\sigma_i = 1$ or $w_i = 0$ for each i. This means that the signs σ_i have no impact on the solution of the linear system $xQ = w$. Hence, we may assume that $Q = (\lambda(u_i))$ is just the character table of D. Since Q has full rank, v is the only solution of $xQ = w$. Setting $\mu(\lambda) := \frac{1}{|D|}\sum_{e\in E\setminus D}\lambda(e^2)$, it suffices to show that $(\mu(\lambda) : \lambda \in \mathrm{Irr}(D))$ is another solution of $xQ = w$. Indeed,

$$\begin{aligned}\sum_{\lambda\in\mathrm{Irr}(D)} \frac{\lambda(u_i)}{|D|} \sum_{e\in E\setminus D} \lambda(e^2) &= \frac{1}{|D|}\sum_{e\in E\setminus D}\sum_{\lambda\in\mathrm{Irr}(D)} \lambda(u_i)\lambda(e^2)\\ &= \frac{1}{|D|}\sum_{\substack{e\in E\setminus D\\ e^2=u_i^{-1}}} |D : \mathrm{C}_D(u_i)||\mathrm{C}_D(u_i)| = w_i\end{aligned}$$

for $i = 1, \ldots, k$. □

Theorem E *Conjectures B and C hold for all nilpotent* 2*-blocks of solvable groups.*

Proof Let B be a real, nilpotent, non-principal 2-block of a solvable group G with defect pair (D, E). We first prove Conjecture C for B. Since all sections of G are solvable and all blocks dominated by B-subsections are nilpotent, Conjecture C holds for those blocks as well. Hence, the hypothesis of Theorem 13 is fulfilled for B. Now by Theorem 14, Conjecture B holds for B.

Let $N := \mathrm{O}_{2'}(G)$ and let $\theta \in \mathrm{Irr}(N)$ such that the block $\{\theta\}$ is covered by B. Since B is non-principal, $\theta \ne 1_N$ and therefore $\overline{\theta} \ne \theta$ as N has odd order. Since B also lies over $\overline{\theta}$, it follow that $G_\theta < G$. Let b be the Fong–Reynolds correspondent of B in the extended stabilizer G^*_θ. By [22, Theorem 9.14] and [20, p. 94], the Clifford correspondence $\mathrm{Irr}(b) \to \mathrm{Irr}(B)$, $\psi \mapsto \psi^G$ preserves decomposition numbers and F-S indicators. Thus, we need to show that b has defect pair (D, E). Let β be the Fong–Reynolds correspondent of B in G_θ. By [22, Theorem 10.20], β is the unique block over θ. In particular, the block idempotents $1_\beta = 1_\theta$ are the same (we identify θ with the block $\{\theta\}$). Since b is also the unique block of G^*_θ over θ, we have $1_b = 1_\theta + 1_{\overline{\theta}} = \sum_{x\in N}\alpha_x x$ for some $\alpha_x \in F$. Let S be a set of representatives for the cosets G/G^*_θ. Then

$$1_B = \sum_{s\in S}(1_\theta + 1_{\overline{\theta}})^s = \sum_{s\in S} 1_b^s = \sum_{g\in N}\left(\sum_{s\in S}\alpha_{g^{s^{-1}}}\right) g.$$

Hence, there exists a real defect class K of B such that $\alpha_{g^{s^{-1}}} \neq 0$ for some $g \in K$ and $s \in S$. Of course we can assume that $g = g^{s^{-1}}$. Then 1_b does not vanish on g. By [22, Theorem 9.1], the central characters λ_B, λ_b and λ_θ agree on N. It follows that K is also a real defect class of b. Hence, we may assume that (D, E) is a defect pair of b.

It remains to consider $G = G_\theta^*$ and $B = b$. Then D is a Sylow 2-subgroup of G_θ by [22, Theorem 10.20] and E is a Sylow 2-subgroup of G. Since $|G : G_\theta| = 2$, it follows that $G_\theta \trianglelefteq G$ and $N = \mathrm{O}_{2'}(G_\theta)$. By [21, Lemmas 1 and 2], β is nilpotent and G_θ is 2-nilpotent, i.e. $G_\theta = N \rtimes D$ and $G = N \rtimes E$. Let $\widetilde{\Phi} := \sum_{\chi\in\mathrm{Irr}(B)} \chi(1)\chi = \varphi(1)\Phi$, where $\mathrm{IBr}(B) = \{\varphi\}$. We need to show that

$$\epsilon(\widetilde{\Phi}) = \varphi(1)|\{x \in E \setminus D : x^2 = 1\}|.$$

Note that $\chi_N = \frac{\chi(1)}{2\theta(1)}(\theta + \overline{\theta})$. By Frobenius reciprocity, it follows that $\widetilde{\Phi} = 2\theta(1)\theta^G$ and

$$\widetilde{\Phi}_N = |G : N|\theta(1)(\theta + \overline{\theta}).$$

Since Φ vanishes on elements of even order, $\widetilde{\Phi}$ vanishes outside N. Since $\widetilde{\Phi}_{G_\theta}$ is a sum of non-real characters in β, we have

$$\epsilon(\widetilde{\Phi}) = \frac{1}{|G|}\sum_{g\in G_\theta} \widetilde{\Phi}(g^2) + \frac{1}{|G|}\sum_{g\in G\setminus G_\theta} \widetilde{\Phi}(g^2) = \frac{1}{|G|}\sum_{g\in G\setminus G_\theta} \widetilde{\Phi}(g^2).$$

Every $g \in G \setminus G_\theta = NE \setminus ND$ with $g^2 \in N$ is N-conjugate to a unique element of the form xy, where $x \in E \setminus D$ is an involution and $y \in \mathrm{C}_N(x)$ (Sylow's theorem). Setting $\Delta := \{x \in E \setminus D : x^2 = 1\}$, we obtain

$$\epsilon(\widetilde{\Phi}) = \frac{\theta(1)}{|N|}\sum_{x\in\Delta} |N : \mathrm{C}_N(x)| \sum_{y\in\mathrm{C}_N(x)} (\theta(y) + \overline{\theta(y)}) = 2\theta(1)\sum_{x\in\Delta} \frac{1}{|\mathrm{C}_N(x)|}\sum_{y\in\mathrm{C}_N(x)} \theta(y). \tag{5}$$

For $x \in \Delta$ let $H_x := N\langle x\rangle$. Again by Sylow's theorem, the N-orbit of x is the set of involutions in H_x. From $\theta^x = \overline{\theta}$ we see that θ^{H_x} is an irreducible character of 2-defect 0. By [8, Theorem 5.1], we have $\epsilon(\theta^{H_x}) = 1$. Now applying the same argument as before, it follows that

$$1 = \epsilon(\theta^{H_x}) = \frac{1}{|N|}\sum_{g\in H_x\setminus N} \theta^{H_x}(g^2) = \frac{2}{|\mathrm{C}_N(x)|}\sum_{y\in\mathrm{C}_N(x)} \theta(y).$$

Combined with (5), this yields $\epsilon(\widetilde{\Phi}) = 2\theta(1)|\Delta|$. By Green's theorem (see [22, Theorem 8.11]), $\varphi_N = \theta + \overline{\theta}$ and $\epsilon(\widetilde{\Phi}) = \varphi(1)|\Delta|$ as desired. □

For non-principal blocks B of solvable groups with $l(B) = 1$ it is not true in general that G_θ is 2-nilpotent in the situation of Theorem E. For example, a (non-real) 2-block of a triple cover of $A_4 \times A_4$ has a unique simple module. Extending this group by an automorphism of order 2, we obtain the group $G = \mathtt{SmallGroup}(864, 3988)$, which fulfills the assumptions with $D \cong C_2^4$, $N \cong C_3$ and $|G : NE| = 9$.

In order to prove Conjecture C for arbitrary 2-blocks of solvable groups, we may follow the steps in the proof above until E is a Sylow 2-subgroup of G and $|G : G_\theta| = 2$. By [24, Theorem 2.1], one gets

$$\varphi(1)/\theta(1) = 2\sqrt{|G_\theta/N|_{2'}} = \sqrt{|G : EN|}.$$

With some more effort, the claim then boils down to a purely group-theoretical statement:

Let B be a real, non-principal 2-block of a solvable group G with defect pair (D, E) and $l(B) = 1$. Let $N := \mathrm{O}_{2'}(G)$ and $\overline{G} := G/N$. Let $\theta \in \mathrm{Irr}(N)$ such $\{\theta\}$ is covered by B. Then

$$|\{\overline{x} \in \overline{G} \setminus \overline{G_\theta} : \overline{x}^2 = 1\}| = |\{x \in E \setminus D : x^2 = 1\}|\sqrt{|G : EN|}.$$

Unfortunately, I am unable to prove this.

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