



Weighted Error Estimates for Transient Transport Problems Discretized Using Continuous Finite Elements with Interior Penalty Stabilization on the Gradient Jumps

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Abstract

In this paper we consider the semi-discretization in space of a first order scalar transport equation. For the space discretization we use standard continuous finite elements with a stabilization consisting of a penalty on the jump of the gradient over element faces. We recall some global error estimates for smooth and rough solutions and then prove a new local error estimate for the transient linear transport equation. In particular we show that for the stabilized method the effect of non-smooth features in the solution decay exponentially from the space time zone where the solution is rough so that smooth features will be transported unperturbed. Locally the L^2 -norm error converges with the expected order $O(h^{k+\frac{1}{2}})$, if the exact solution is locally smooth. We then illustrate the results numerically. In particular we show the good local accuracy in the smooth zone of the stabilized method and that the standard Galerkin fails to approximate a solution that is smooth at the final time if underresolved features have been present in the solution at some time during the evolution.

Keywords Continuous Galerkin · Stability · Scalar hyperbolic transport equations · Initial-boundary value problem · Stabilized methods

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1 Introduction

The discretization of transport problems has traditionally been dominated by discontinuous Galerkin methods or finite volume methods, typically of low order, since the continuous Galerkin method is known to have robustness problems for first order partial differential

Dedicated to Professor Alfio Quarteroni on his 70th birthday.

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equations (see [23, Chapter 5]), or convection–diffusion equations in the convection dominated regime. In certain situations the use of high order continuous Galerkin methods is appealing, for instance in the case of convection–diffusion equations, in particular where the diffusion is nonlinear, or more complex situations such as large eddy simulation of turbulent flows, where the pressure-velocity coupling can be decoupled using a pressure projection method and the convective part handled explicitly. In such situations, if continuous finite element spaces are used, one must resort to a stabilized method to avoid a reduction of accuracy due to spurious oscillations. There is a very wide literature on stabilized methods and for an overview of the topic see for example [24]. In the high order case, the Spectral Vanishing Velocity method has been a popular choice [34–36], but other methods have also been designed to work for high order, see the discussion in [17]. In this work we will focus on the continuous interior penalty (CIP) stabilization, that was shown to allow for close to hp -optimal error estimates in the high Peclet regime in [10]. Recently [37] this method was applied to under resolved simulations of turbulent flows using high order polynomial approximation and shown to perform very well in this context. Therein an eigenanalysis was performed which showed that the CIP finite element method has similar advantageous dispersion properties as the discontinuous Galerkin method (see also the report [19]) and in the computations it was verified that its numerical dissipation was less important than that of the spectral vanishing viscosity.

Ideally stability of the finite element method should match that of the continuous problem. This is typically, by and large, true for elliptic pde, but much harder to achieve in the hyperbolic case. Indeed, this would mean satisfaction of a discrete maximum principle and stability and error estimates in L^1 . Both which typically remain open questions. Herein we will only consider the stability in the L^2 -norm for continuous finite element approximations and linear symmetric stabilization of gradient penalty type applied to the transient scalar, linear first order equation. The analysis will mainly focus on semi discretization in space on periodic domains, but the extension to the fully discrete case and weakly imposed boundary conditions will be sketched. The classical estimate for smooth solutions that is proven for stabilized finite element methods is on the form

$$\|(u - u_h)(\cdot, T)\|_{\Omega} \leq C(u)h^{k+\frac{1}{2}}, \quad (1.1)$$

where $C(u)$ is a constant that depends on Sobolev norms of the exact solution and on equation data, h is the mesh-size and k the polynomial order. This estimate that is suboptimal by $h^{\frac{1}{2}}$ is known to be sharp on general meshes [38] (see also [7] for the sharpness of the estimate for the CIP method). The continuous Galerkin method without stabilization, however, only admits a bound of order h^k . The lost factor $h^{\frac{1}{2}}$ is of little consequence for smooth solutions, and high polynomial order. However for low polynomial order or rough solutions it becomes significant. In Section 4 below, we prove this type of error estimate and some variations in weak norm for rough solutions. This analysis uses ideas from [9, 12]. Some remarks on the time discretization will be added in Section 4.2. In particular we will point out the situations where the stabilization actually improves the stability of time stepping methods.

The estimate (1.1) is a weak result, but it has become a proxy for stronger estimates that give convergence also of the material derivative (see [13, 27] and Theorem 2 below) and importantly, local estimates, using weighted norms, well known in the stationary case

[14, 28, 32, 33]. In the context of time dependent problems such a weighted estimate takes the form

$$\|\varpi(u - u_h)(\cdot, T)\|_{\Omega} \leq Ch^{k+\frac{1}{2}} \left(\int_0^T \|\varpi D^{k+1} u\|_{\Omega}^2 dt \right)^{\frac{1}{2}}, \quad (1.2)$$

where D^m is a multi-index differential operator of order m and the ϖ is a weight function that is aligned with the characteristics and decays exponentially away from some zone of interest. This means that if $\varpi = 1$ in some zone where the solution is smooth the influence of locally large derivatives and underresolution at some distance d from this zone will be damped with a factor $e^{-d/\sqrt{h}}$. We prove such an estimate in Section 5 for the space semi-discretized stabilized formulation. To the best of my knowledge there are no previous such estimates for continuous finite element methods using symmetric stabilization. For earlier works on Streamline Upwind Petrov–Galerkin methods (SUPG) in this direction see [20, 44]. The approach in [44] relies strongly on the space time finite element discretisation and an additional artificial viscosity term and in [20] the authors consider the SUPG method together with a first order backward differentiation in time, on a form that can not easily be extended to higher order time-discretizations. In neither case can the arguments be applied independently of the time discretization. In this paper we apply the ideas from [14] where weighted estimates were proved for the stationary convection–diffusion equation with CIP-stabilization and [16], where they were applied to an inverse boundary value problem subject to a convection–diffusion equation. The result is presented in detail for the semi-discretized case only, but the extension to standard stable time discretizations is sketched. The results can also be extended to the case of convection–diffusion equations with Neumann conditions on the outflow boundary, by straightforward addition of the diffusive terms and following the argument of [14].

In the numerical section (Section 6) we will illustrate this localization property of the error and show that it is not shared by the standard (unstabilized) Galerkin finite element method. Indeed, as we shall see, without stabilization Galerkin FEM fails to approximate even smooth solutions satisfactory in case the solution has had non-smooth features at any time during the computation. Indeed it appears that the standard Galerkin method does not propagate underresolved features of the solution with the right speed, making it impossible for the method to evacuate high frequency content from the computational domain. For the stabilized method on the other hand the weighted estimate (1.2) guarantees that smooth components of the solution are untainted by spurious high frequency content at all times, since perturbations are damped exponentially when crossing the characteristics.

2 Model Problem and Finite Element Discretization

We will discuss a first order hyperbolic problem in a periodic domain, $\Omega = [-L, L]^n$, where $n \geq 1$ is the space dimension. Let $\beta \in C^0([0, T]; [C^m(\bar{\Omega})]^n)$, $m \geq 1$, be a periodic vector field satisfying $\nabla \cdot \beta = 0$ and consider the first order hyperbolic problem

$$\mathcal{L}u := \partial_t u + \beta \cdot \nabla u = f \quad \text{in } (0, T) \times \Omega, \quad (2.1)$$

$$u(\cdot, 0) = u_0 \quad \text{in } \Omega. \quad (2.2)$$

For smooth data β , u_0 and f there exists a unique solution by the method of characteristics, but the problem admits a unique solution also for more rough data [26]. The solution

satisfies the following regularity estimate (a proof of this can be obtained after minor modifications of [5, Lemma 2]),

$$\|u(t)\|_{H^j(\Omega)} \leq C_\beta \left(\|f\|_{L^2((0,T);H^j(\Omega))} + \|u_0\|_{H^j(\Omega)} \right), \quad t > 0, \quad j \geq 0 \text{ when } m \geq j. \quad (2.3)$$

Below we will always assume that β is smooth enough for (2.3) to hold. The constant C_β grows exponentially in time, with coefficient dependent on the sup-norm of β , and its derivatives of order up to j . Below the notation $\beta_\infty = \sup_{x \in \bar{\Omega}} |\beta(x)|$ will be used. The L^2 -norm over a domain $X \subset \Omega$ will be denoted by $\|\cdot\|_X = (\cdot, \cdot)_X^{\frac{1}{2}}$, where $(\cdot, \cdot)_X^{\frac{1}{2}}$ is the L^2 -scalar product over X , also $\|\cdot\|_\infty$ will denote the norm on $C^0(\bar{\Omega})$.

Let $\{\mathcal{T}\}_h$ be a family of shape regular decompositions of Ω in simplices S , $\mathcal{T} = \{S\}$, indexed by the (uniform) mesh size h . Let \mathcal{F} denote the set of faces of \mathcal{T} . C will denote a generic constant that can have different value at each appearance, but is always independent of the mesh-parameter h . Now define the finite element space

$$V_h := \left\{ v \in H_{per}^1(\Omega) : v|_S \in \mathbb{P}_k(S), \text{ for all } S \in \mathcal{T} \right\},$$

where $\mathbb{P}_k(S)$ denotes the set of polynomials of degree less than or equal to k on S and $H_{per}^1(\Omega)$ denotes the set of periodic functions in H^1 on Ω . We may then write a semi-discretization in space, for $t > 0$ find $u_h(t) \in V_h$, with $u_h(0) = \pi_h u_0$, such that

$$(\mathcal{L}u_h(t), v_h)_\Omega = F(v_h), \quad \forall v_h \in V_h \quad (2.4)$$

where $F(v_h) := (f, v_h)_\Omega$. Above π_h denotes the L^2 -projection onto the finite element space V_h . For all $v \in L^2(\Omega)$, $\pi_h v \in V_h$ satisfies

$$(\pi_h v, w_h)_\Omega = (v, w_h)_\Omega, \quad \forall w_h \in V_h.$$

It is well known that on locally quasi-uniform meshes the L^2 -projection satisfies the approximation bound,

$$\|v - \pi_h v\|_\Omega + h \|\nabla(v - \pi_h v)\|_\Omega \leq Ch^{k+1} \|v\|_{H^{k+1}(\Omega)}, \quad \forall v \in H^{k+1}(\Omega).$$

The formulation (2.4) defines a dynamical system that admits a unique solution for $m \geq 0$ using standard techniques. Taking $v_h = u_h$ in (2.4) and integrating in time we see that (2.4) satisfies the bound (2.3) with $j = 0$

$$\|u_h(t)\|_\Omega \leq C_\beta \|f\|_{L^2((0,T);\Omega)} + \|u_0\|_\Omega, \quad t > 0. \quad (2.5)$$

Since $\nabla \cdot \beta = 0$ the bound holds with $C_\beta = T^{\frac{1}{2}}$. Actually a stronger results holds for the L^2 -norm when the norm on f is weakened. Indeed one may use that

$$\int_0^T (f, u_h)_\Omega \, dt + \|u_0\|_\Omega^2 \leq \sup_{t \in (0,T)} \|u_h(t)\|_\Omega \left(\|f\|_{L^1((0,T);L^2(\Omega))} + \|u_0\|_\Omega \right)$$

to show that

$$\sup_{t \in (0,T)} \|u_h(t)\|_\Omega \leq \|f\|_{L^1((0,T);L^2(\Omega))} + \|u_0\|_\Omega.$$

However (2.3) does not hold for u_h for $j = 1$. A natural question to ask is then if the solution to (2.4) gives any control of the derivatives. In case $f \in L^2((0,T);\Omega)$ the immediate control offered by (2.1) is $\mathcal{L}u \in L^2((0,T);\Omega)$, that is the material derivative is bounded in L^2 . For (2.4) we get the corresponding bound $\pi_h \mathcal{L}u_h \in L^2((0,T);\Omega)$. Since $\mathcal{L}u_h$ may be discontinuous over element faces (due to the presence of derivatives in space) and $V_h \subset C^0(\Omega)$, we see that $\pi_h \mathcal{L}u_h \neq \mathcal{L}u_h$. It follows that not even this weakest measure of derivatives of u is controlled by (2.4). However since we are looking for control in a discrete

space we can use norm equivalence on discrete spaces in the form of the inverse inequality [3, Lemma 4.5.3],

$$\|\nabla u_h\|_S \leq Ch^{-1}\|u_h\|_S, \quad \forall S \in \mathcal{T} \quad (2.6)$$

and observing that $\partial_t u_h \in V_h$, we see that

$$\|\mathcal{L}u_h\|_\Omega \leq \|\pi_h \mathcal{L}u_h\|_\Omega + C\beta_\infty h^{-1}\|u_h\|_\Omega. \quad (2.7)$$

Combining (2.7) with the bound (2.5)

$$\|\mathcal{L}u_h\|_{L^2((0,T);\Omega)} \leq (1 + C\beta h^{-1}) (\|f\|_{L^2((0,T);\Omega)} + \|u_0\|_\Omega).$$

So the constant in the control of the material derivative grows as $O(h^{-1})$ under mesh refinement. Hence there is no improvement compared to obtaining an H^1 estimate by combining the L^2 -stability of (2.5) with (2.6).

The rationale for the addition of stabilizing terms is to improve the control of derivatives of u_h . As an example of stabilization we here propose the gradient penalty term, introduced in [21] and shown to result in improved robustness and error estimates for convection dominated flows in [15],

$$s(w_h, v_h) = \sum_{F \in \mathcal{F}} \left(h_F^2 |\boldsymbol{\beta}| \llbracket \nabla u_h \rrbracket, \llbracket \nabla v_h \rrbracket \right)_F \quad (2.8)$$

where $\langle u, v \rangle_F = \int_F uv \, ds$, $\llbracket \nabla v_h \rrbracket|_F = \nabla v_h|_{F \cap \partial S_1} \cdot n_1 + \nabla v_h|_{F \cap \partial S_2} \cdot n_2$ for $F = \bar{S}_1 \cap \bar{S}_2$ and n_1 and n_2 denote the outward pointing normals of the simplices S_1 and S_2 respectively. To reduce the amount of crosswind diffusion the $|\boldsymbol{\beta}|$ factor may be replaced by $|\boldsymbol{\beta} \cdot n|$. Define the stabilization semi norm by

$$|w_h|_s := s(w_h, w_h)^{\frac{1}{2}}.$$

Also recall the following inverse inequality

$$|w_h|_s \leq Ch^{-\frac{1}{2}} \beta_\infty^{\frac{1}{2}} \|w_h\|_\Omega, \quad \forall w_h \in V_h \quad (2.9)$$

which is a consequence of the scaled trace inequality, [3, Theorem 1.6.6],

$$\|v\|_{\partial S} \leq C_S \left(h^{-\frac{1}{2}} \|v\|_S + h^{\frac{1}{2}} \|\nabla v\|_S \right), \quad \forall v \in H^1(S) \quad (2.10)$$

and (2.6).

The enhanced control of derivatives offered by this stabilization term can be expressed as

$$\inf_{v_h \in V_h} \|h^{\frac{1}{2}} (\boldsymbol{\beta} \cdot \nabla u_h - v_h)\|_\Omega^2 \leq C_s \left(\beta_\infty |u_h|_s^2 + h \|\nabla \boldsymbol{\beta}\|_\infty^2 \|u_h\|_\Omega^2 \right). \quad (2.11)$$

This is an immediate consequence of the local estimate of [10, Lemma 5.3] and local approximation of $\boldsymbol{\beta}$ using lowest order Raviart–Thomas functions (for details see the discussion [13, Page 4]). In particular this implies (since $\partial_t u_h \in V_h$) that

$$\|\mathcal{L}u_h\|_\Omega \leq C \|\pi_h \mathcal{L}u_h\|_\Omega + C_s^{\frac{1}{2}} \left(h^{-\frac{1}{2}} \beta_\infty^{\frac{1}{2}} |u_h|_s + \|\nabla \boldsymbol{\beta}\|_\infty \|u_h\|_\Omega \right). \quad (2.12)$$

It follows that when the finite element method has the additional stability offered by the operator s , the constant in the bound for $\mathcal{L}u_h$ will grow at the rate $O(h^{-\frac{1}{2}})$ under mesh refinement. Therefore we propose the stabilized method, find $u_h(t) \in V_h$, with $u_h(0) = \pi_h u_0$, such that

$$(\mathcal{L}u_h(t), v_h)_\Omega + \gamma s(u_h, v_h) = F(v_h), \quad \forall v_h \in V_h \quad (2.13)$$

for $\gamma > 0$. Clearly for $\gamma = 0$ we recover the standard Galerkin method.

Remark 1 Although we only consider continuous FEM below all the results holds true for dG methods if the standard Galerkin method (without stabilization) is replaced by the standard dG method with central flux and the stabilized finite element method is replaced by the standard dG method with upwind flux. There is indeed a common misconception that the enhanced stability of the dG methods (space discretization) is due to the discontinuity of the element. The discontinuity only allows for the improved control of the material derivative if there is sufficient control on the solution jump. This can be introduced through upwind fluxes, or otherwise. Indeed it is easy to see that the upwind flux formulation is obtained from the central flux formulation by adding the following stabilization term [4]

$$s_{up}(v_h, w_h) := \frac{1}{2} \sum_{F \in \mathcal{F}} \langle |\beta \cdot n_F| [v_h], [w_h] \rangle_F,$$

where $[\cdot]$ simply denotes the jump of the function over the element face F . In general the full jump needs to be penalized, but the minimal stabilization needed to make the dG method satisfy the bound (2.12) depends on the mesh geometry and the polynomial order [6, 39].

3 Stability Estimate of the Finite Element Method

Here we will formalize the discussion of the previous section to obtain a stability estimate that will be useful for the subsequent error analysis. First define the operator norms

$$\|F\|_0 := \sup_{v_h \in V_h} \frac{|F(v_h)|}{\|v_h\|_\Omega} \quad \text{and} \quad \|F\|_h := \sup_{v_h \in V_h} \frac{|F(v_h)|}{\|v_h\|_\Omega + |v_h|_s}. \quad (3.1)$$

With these definitions the arguments discussed in the previous section may be written as follows.

Theorem 1 *Let u_h solve (2.13) with $\gamma > 0$ then for all $\tau \in [0, T]$*

$$\|u_h(\tau)\|_\Omega^2 + \gamma \int_0^\tau |u_h|_s^2 \, dt \leq C_\beta \left(\int_0^\tau \|F\|_h^2 \, dt + \|u_h(0)\|_\Omega^2 \right)$$

where $C_\beta = O(\gamma^{-1} + T)$.

Proof First take $v_h = u_h$ in (2.13) to obtain using the skew symmetry of the convective operator

$$(\mathcal{L}u_h, u_h)_\Omega = \frac{1}{2} \frac{d}{dt} \|u_h(t)\|_\Omega^2$$

and therefore after integration in time over $(0, \tau)$

$$\begin{aligned} \frac{1}{2} \|u_h(\tau)\|_\Omega^2 + \gamma \int_0^\tau |u_h(t)|_s^2 \, dt &\leq \frac{1}{2} \|u_h(0)\|_\Omega^2 + \int_0^\tau F(u_h) \, dt \\ &\leq \frac{1}{2} \|u_h(0)\|_\Omega^2 + \int_0^\tau \|F\|_h (\|u_h(t)\|_\Omega + |u_h(t)|_s) \, dt. \end{aligned}$$

Using the arithmetic-geometric inequality $ab \leq \frac{1}{2}a^2 + \frac{1}{2}b^2$ it follows that $\|F\|_h (\|u_h(t)\|_\Omega + |u_h(t)|_s) \leq (\gamma^{-1} + T) \|F\|_h^2 + \frac{1}{2} T^{-1} \|u_h(t)\|_\Omega^2 + \gamma \frac{1}{2} |u_h(t)|_s^2$ leading to

$$\|u_h(\tau)\|_\Omega^2 + \gamma \int_0^\tau |u_h(t)|_s^2 \, dt \leq \|u_h(0)\|_\Omega^2 + (\gamma^{-1} + T) \int_0^\tau \|F\|_h^2 \, dt + \int_0^\tau T^{-1} \|u_h(t)\|_\Omega^2 \, dt.$$

By Gronwall's inequality we have

$$\begin{aligned}\|u_h(\tau)\|_{\Omega}^2 &\leq \left(\exp \int_0^\tau T^{-1} dt\right) \left(\|u_h(0)\|_{\Omega}^2 + (\gamma^{-1} + T) \int_0^\tau \|F\|_h^2 dt\right) \\ &\leq C \left(\|u_h(0)\|_{\Omega}^2 + (\gamma^{-1} + T) \int_0^\tau \|F\|_h^2 dt\right).\end{aligned}$$

We may then bound

$$\begin{aligned}\gamma \int_0^\tau |u_h(t)|_s^2 dt &\leq \|u_h(0)\|_{\Omega}^2 + (\gamma^{-1} + T) \int_0^\tau \|F\|_h^2 dt + \int_0^\tau T^{-1} \|u_h(t)\|_{\Omega}^2 dt \\ &\leq C \left(\|u_h(0)\|_{\Omega}^2 + (\gamma^{-1} + T) \int_0^\tau \|F\|_h^2 dt\right)\end{aligned}$$

which concludes the proof. \square

For the material derivative we can prove the similar bound

Corollary 1 *Let u_h solve (2.13) with $\gamma > 0$ then there holds*

$$\int_0^T \|h^{\frac{1}{2}} \mathcal{L}u_h\|_{\Omega}^2 dt \leq C_{\beta} \zeta(\gamma)^2 \left(\|u_h(0)\|_{\Omega}^2 + \int_0^T (h\|F\|_0^2 + (\beta_{\infty} + h\|\nabla \beta\|_{\infty}^2 T) \|F\|_h^2) dt\right),$$

where $\zeta(\gamma) = \gamma^{\frac{1}{2}} + \gamma^{-\frac{1}{2}}$.

Proof

$$\int_0^T \|h^{\frac{1}{2}} \mathcal{L}u_h\|_{\Omega}^2 dt = \int_0^T (\mathcal{L}u_h, h\pi_h \mathcal{L}u_h)_{\Omega} dt + \int_0^T \|h^{\frac{1}{2}} (I - \pi_h) \mathcal{L}u_h\|_{\Omega}^2 dt = T_1 + T_2.$$

To bound the term T_1 we use the formulation (2.13) to obtain

$$(\mathcal{L}u_h, h\pi_h \mathcal{L}u_h)_{\Omega} = F(h\pi_h \mathcal{L}u_h) - \gamma s(u_h, h\pi_h \mathcal{L}u_h).$$

For the first term on the right hand side we see that using the first definition of (3.1) and the stability of the L^2 -projection there holds

$$F(h\pi_h \mathcal{L}u_h) \leq \|F\|_0 \|h\pi_h \mathcal{L}u_h\|_{\Omega} \leq h^{\frac{1}{2}} \|F\|_0 \|h^{\frac{1}{2}} \mathcal{L}u_h\|_{\Omega}.$$

For the second term we use (2.9) and the L^2 -stability of the projection to get

$$\gamma s(u_h, h\pi_h \mathcal{L}u_h) \leq \gamma s(u_h, u_h)^{\frac{1}{2}} s(h\pi_h \mathcal{L}u_h, h\pi_h \mathcal{L}u_h)^{\frac{1}{2}} \leq C \gamma \beta_{\infty}^{\frac{1}{2}} |u_h|_s \|h^{\frac{1}{2}} \mathcal{L}u_h\|_{\Omega}.$$

Observe that in the last inequality a factor $h^{\frac{1}{2}}$ is lost due to the application of (2.9). Collecting these bounds we see that

$$T_1 \leq \int_0^T \left(h\|F\|_0^2 + C^2 \gamma^2 \beta_{\infty} |u_h|_s^2 + \frac{1}{2} \|h^{\frac{1}{2}} \mathcal{L}u_h\|_{\Omega}^2\right) dt.$$

To bound T_2 we note that by the definition of the L^2 -projection $\|h^{\frac{1}{2}} (I - \pi_h) \mathcal{L}u_h\|_{\Omega} \leq \|h^{\frac{1}{2}} (\mathcal{L}u_h - v_h)\|_{\Omega}$ for all $v_h \in V_h$ and apply (2.11) and the fact that $\partial_t u_h \in V_h$, leading to

$$T_2 = \int_0^T \inf_{v_h \in V_h} \|h^{\frac{1}{2}} (\beta \cdot \nabla u_h - v_h)\|_{\Omega}^2 dt \leq C_s \int_0^T \left(\beta_{\infty} |u_h|_s^2 + h\|\nabla \beta\|_{\infty}^2 \|u_h\|_{\Omega}^2\right) dt.$$

The claim follows by the bounds on T_1 and T_2 and the result of Theorem 1. \square

Remark 2 Observe that the presence of both positive and negative powers of γ in ζ , shows that the estimate degenerates both for vanishing stabilization and for too strong stabilization. If γ goes to infinity the solution has to become C^1 and the solution will in this case coincide with the standard Galerkin approximation in the C^1 -subspace, which is unstable, see discussion in [18].

4 Error Estimates for the Stabilized Formulation (2.13)

Using the stability estimates of Theorem 1 it is straightforward to derive the error estimate (1.1) for smooth solutions. Below we will also use Corollary 1 to obtain an optimal order $O(h^k)$ error estimate for the material derivative.

Then we will assume that $f \in L^2(0, T; \Omega)$ in (2.3) so that we only have $u \in L^2(0, T; \Omega)$. In this case we will show that the stabilized finite element method still converges in a weaker norm.

Theorem 2 Let $u_0 \in H^{k+1}(\Omega)$, $f \in L^2(0, T; H^{k+1}(\Omega))$, let u be the solution of (2.1) and u_h the solution of (2.13). Then there holds, for all $T > 0$

$$\|(u - u_h)(\cdot, T)\|_{\Omega} + \gamma \left(\int_0^T |u_h|_s^2 dt \right)^{\frac{1}{2}} \leq C_{\beta} \zeta(\gamma) h^{k+\frac{1}{2}} (\|f\|_{L^2(0, T; H^{k+1}(\Omega))} + \|u_0\|_{H^{k+1}(\Omega)})$$

and

$$\left(\int_0^T \|\mathcal{L}(u - u_h)\|_{\Omega}^2 dt \right)^{\frac{1}{2}} \leq C_{\beta} \zeta(\gamma)^2 h^k \|u\|_{H^1(0, T; H^{k+1}(\Omega))},$$

where $\zeta(\gamma) := \gamma^{\frac{1}{2}} + \gamma^{-\frac{1}{2}}$ and C_{β} depends on β_{∞} and $\|\nabla \beta\|_{\infty}$ and T .

Proof This result is a consequence of the stability of Theorem 1, the consistency and (2.11). It is standard material (see [24, Section 76.4]) however for completeness we include the short proof.

Using standard approximation estimates there holds [10, Lemma 5.6]

$$\|\beta_{\infty}^{\frac{1}{2}} h^{-\frac{1}{2}} (u - \pi_h u)\|_{\Omega} + |u - \pi_h u|_s \leq C \beta_{\infty}^{\frac{1}{2}} h^{k+\frac{1}{2}} |u|_{H^{k+1}(\Omega)}. \quad (4.1)$$

Hence by applying a triangle inequality we only need to consider the discrete error $e_h = \pi_h u - u_h$. Injecting it in (2.1) and using (2.13) we see that

$$(\mathcal{L}e_h, v_h) + \gamma s(e_h, v_h) = F_{\pi}(v_h)$$

with $F_{\pi}(v_h) = (\partial_t(\pi_h u - u), v_h)_{\Omega} + (\beta \cdot \nabla(\pi_h u - u), v_h)_{\Omega} + \gamma s(\pi_h u, v_h)$. Applying Theorem 1 we see that

$$\|e_h(T)\|_{\Omega}^2 + \gamma \int_0^T |e_h|_s^2 dt \leq C_{\beta} \int_0^T \|F_{\pi}\|_h^2 dt + \|e_h(0)\|_{\Omega}^2.$$

By the definition of $u_h(0)$, $e_h(0) = 0$. Since $\partial_t \pi_h u = \pi_h \partial_t u$ we have using L^2 -orthogonality and integration by parts

$$F_{\pi}(v_h) = (u - \pi_h u, \beta \cdot \nabla v_h - w_h)_{\Omega} + \gamma s(\pi_h u, v_h), \quad \forall w_h \in V_h.$$

It now follows using the Cauchy–Schwarz inequality, (2.11) and (4.1) and recalling that under the regularity assumptions on data $u(t) \in H^2(\Omega)$, that

$$\|F_\pi\|_h \leq C_\beta \zeta(\gamma) h^{k+\frac{1}{2}} |u|_{H^{k+1}(\Omega)}. \quad (4.2)$$

The first claim then follows after an application of (2.3).

For the second inequality we apply Corollary 1 to see that, since $e_h(0) = 0$,

$$\int_0^T \|h^{\frac{1}{2}} \mathcal{L}e_h\|_\Omega^2 dt \leq C\zeta(\gamma)^2 \int_0^T (h\|F_\pi\|_0^2 + (\beta_\infty + h\|\nabla \beta\|_\infty^2 T)\|F_\pi\|_h^2) dt. \quad (4.3)$$

It follows that we only need to bound F in the stronger topology $\|\cdot\|_0$ to conclude. Using the Cauchy–Schwarz inequality and the inverse inequalities (2.6) and (2.9)

$$\begin{aligned} F_\pi(v_h) &= (u - \pi_h u, \beta \cdot \nabla v_h)_\Omega + \gamma s(\pi_h u, v_h) \\ &\leq C\beta_\infty \|h^{-1}(u - \pi_h u)\|_\Omega \|v_h\|_\Omega + C\gamma h^{-\frac{1}{2}} \beta_\infty^{\frac{1}{2}} |\pi_h u|_s \|v_h\|_\Omega. \end{aligned}$$

It follows from (4.1) that

$$\|F\|_0 \leq C_\beta (1 + \gamma) h^k |u|_{H^{k+1}(\Omega)}.$$

Combining this bound for $\|F\|_0$ with the bound (4.2) in (4.3) we see that

$$\int_0^T \|h^{\frac{1}{2}} \mathcal{L}e_h\|_\Omega^2 dt \leq C_\beta \zeta(\gamma)^4 h^{2k+1} \int_0^T |u|_{H^{k+1}(\Omega)}^2 dt$$

and we conclude using the approximation bound

$$\|\mathcal{L}(u - \pi_h u)\|_\Omega \leq C \left(h^{k+1} \|\partial_t u\|_{H^{k+1}(\Omega)} + \beta_\infty h^k \|u\|_{H^{k+1}(\Omega)} \right)$$

and the triangle inequality. \square

Remark 3 Note that the error estimate on the material derivative is optimal compared with the approximation properties of the finite element space. In the corresponding analysis for (2.4) only $\|F\|_0$ may be used for the upper bound in Theorem 1, resulting in a bound that is suboptimal by $O(h^{\frac{1}{2}})$.

4.1 Rough Solutions: Convergence in Weak Norms

Assume now that we have $f \in L^2((0, T); \Omega)$ in (2.13) and $u_0 \in L^2(\Omega)$. Then $u \in L^2((0, T); \Omega)$ is the best we can hope for, making the error estimates of Theorem 2 invalid. However if we estimate the error in a weaker norm, we can still obtain an error bound with convergence order, provided a stabilized method is used. For $\psi \in H_{per}^1(\Omega)$ consider the adjoint problem

$$\begin{aligned} -\mathcal{L}\varphi &= 0, \\ \varphi(\cdot, T) &= \psi. \end{aligned}$$

This problem admits a unique solution and by (2.3)

$$\sup_{t \in (0, T)} \|\varphi(t)\|_{H^1(\Omega)} \leq C_\beta \|\psi\|_{H^1(\Omega)}. \quad (4.4)$$

Let $V := H_{per}^1(\Omega)$ and introduce the dual norm

$$\|v\|_{V'} := \sup_{w \in V \setminus 0} \frac{\langle v, w \rangle_{V', V}}{\|w\|_V},$$

where $\langle v, w \rangle_{V', V}$ is a space duality pairing that we can identify with the L^2 -scalar product for $v \in L^2(\Omega)$. We now proceed using duality to prove an a posteriori bound

Proposition 1 (A posteriori error bound) *Let u be the solution of (2.1) with $f \in L^2(0, T; \Omega)$ and $u_0 \in L^2(\Omega)$ and u_h the solution of (2.13), with $\gamma \geq 0$. Then there holds, for all $T > 0$ and for all $\psi \in V$,*

$$\begin{aligned} \frac{((u - u_h)(\cdot, T), \psi)_\Omega}{\|\psi\|_V} &\leq C_\beta h \|u_0 - \pi_h u_0\|_\Omega \\ &\quad + C_\beta \int_0^T \left(\inf_{v_h \in V_h} h \|f - \beta \cdot \nabla u_h - v_h\|_\Omega + \gamma h^{\frac{1}{2}} |u_h|_s \right) dt. \end{aligned}$$

Proof Using the adjoint equation and integration by parts we see that for any $\psi \in H_{per}^1(\Omega)$,

$$\begin{aligned} ((u - u_h)(\cdot, T), \psi)_\Omega &= ((u - u_h)(\cdot, T), \psi)_\Omega + \int_0^T (u - u_h, -\mathcal{L}\varphi)_\Omega dt \\ &= (u_0 - \pi_h u_0, \varphi(\cdot, 0))_\Omega + \int_0^T (\mathcal{L}(u - u_h), \varphi)_\Omega dt \\ &= (u_0 - \pi_h u_0, (I - \pi_h)\varphi(\cdot, 0))_\Omega \\ &\quad + \int_0^T ((\mathcal{L}(u - u_h), \varphi - \pi_h \varphi)_\Omega + \gamma s(u_h, \pi_h \varphi)) dt. \end{aligned}$$

Considering the terms of the right hand side we see that

$$\begin{aligned} (u_0 - \pi_h u_0, (I - \pi_h)\varphi(\cdot, 0))_\Omega &\leq Ch \|u_0 - \pi_h u_0\|_\Omega \|\nabla \varphi(\cdot, 0)\|_\Omega, \\ ((\mathcal{L}(u - u_h), \varphi - \pi_h \varphi)_\Omega &\leq Ch \inf_{v_h \in V_h} \|f - \mathcal{L}u_h - v_h\|_\Omega \|\nabla \varphi\|_\Omega \\ &= Ch \inf_{v_h \in V_h} \|f - \beta \cdot \nabla u_h - v_h\|_\Omega \|\nabla \varphi\|_\Omega \end{aligned}$$

and

$$s(u_h, \pi_h \varphi) \leq |u_h|_s h^{\frac{1}{2}} \beta_\infty^{\frac{1}{2}} \|\nabla \varphi\|_\Omega.$$

It follows that

$$\begin{aligned} &(u_0 - \pi_h u_0, (I - \pi_h)\varphi(\cdot, 0))_\Omega + \int_0^T ((\mathcal{L}(u - u_h), \varphi - \pi_h \varphi)_\Omega + \gamma s(u_h, \pi_h \varphi)) dt \\ &\leq C \left(h \|u_0 - \pi_h u_0\| + \int_0^T \left(\inf_{v_h \in V_h} h \|f - \beta \cdot \nabla u_h - v_h\|_\Omega + \gamma \beta_\infty^{\frac{1}{2}} h^{\frac{1}{2}} |u_h|_s \right) dt \right) \\ &\quad \times \sup_{t \in (0, T)} \|\varphi(t)\|_{H^1(\Omega)}. \end{aligned}$$

We end the proof by applying the stability (4.4). \square

Remark 4 A posteriori error estimates in negative norms for stationary first order pde was introduced in [30] and the case of transient problems using stabilized FEM in [9]. Observe that this a posteriori error estimate can not in general be sharp, indeed for a smooth solution, by Theorem 2 we get $O(h^{k+1})$ convergence in the dual norm. This follows by observing that since we may take $v_h = \partial_t u_h$ and $f = \mathcal{L}u$,

$$\inf_{v_h \in V_h} h \|f - \beta \cdot \nabla u_h - v_h\|_\Omega \leq h \|\mathcal{L}(u - u_h)\|_\Omega$$

and then applying the second bound of Theorem 2. We see that compared to the L^2 -estimate we have lost another power $h^{\frac{1}{2}}$. Sharp residual type a posteriori error estimates in the L^2 -norm for transport equations in dimension > 1 , so far to the best of my knowledge, have only been obtained under a saturation assumption and using a stabilized finite element method, or a dG method with upwind flux [8].

Theorem 3 (A priori error estimate for rough solutions) *Let u be the solution of (2.1) with $f \in L^2(0, T; L^2(\Omega))$ and $u_0 \in L^2(\Omega)$ and u_h that of (2.13) with $\gamma > 1$. Then there holds*

$$\sup_{t \in [0, T]} \|(u - u_h)(\cdot, t)\|_{V'} \leq C_\beta (\zeta(\gamma) + 1) h^{\frac{1}{2}} (\|f\|_{L^2(0, T; L^2(\Omega))} + \|u_0\|_\Omega),$$

with $\zeta(\gamma) = \gamma^{\frac{1}{2}} + \gamma^{-\frac{1}{2}}$.

Proof By definition

$$\|u - u_h\|_{V'} = \sup_{w \in V \setminus 0} \frac{(u - u_h, w)_\Omega}{\|w\|_V}.$$

Applying Proposition 1 we see that, after a Cauchy–Schwarz inequality in time, for any $T > 0$,

$$\begin{aligned} \|(u - u_h)(\cdot, T)\|_{V'} &\leq C_\beta h \|u_0 - \pi_h u_0\|_\Omega \\ &\quad + C_\beta h^{\frac{1}{2}} T^{\frac{1}{2}} \left(\int_0^T \left(\inf_{v_h \in V_h} h \|f - \beta \cdot \nabla u_h - v_h\|_\Omega^2 + \gamma^2 |u_h|_s^2 \right) dt \right)^{\frac{1}{2}}. \end{aligned}$$

Then noting that by (2.11) there holds

$$\inf_{v_h \in V_h} h \|f - \beta \cdot \nabla u_h - v_h\|_\Omega^2 \leq h \|f\|_\Omega^2 + C_s (|u_h|_s^2 + h \|\nabla \beta\|_\infty^2 \|u_h\|_\Omega^2)$$

we see that all the a posteriori terms depending on u_h are either on the form $|u_h|_s$ or on the form $\|u_h\|_\Omega^2$ and we conclude by applying Theorem 1. \square

4.2 Time Discretization and Stabilized Methods

As a rule of thumb any time integrator with non-trivial imaginary stability boundary extending into the complex plane will be stable and accurate in the sense (1.1), possibly under a CFL condition depending on β and γ . In particular any time discretization method allowing for a time discrete version of an energy estimate of the type in Theorem 1 may be applied and will lead to optimal error estimates similar to those above. This includes all A-stable schemes, backward differentiation methods of first and second order, the Crank–Nicolson method. Explicit methods with good stability properties such as explicit strongly stable Runge–Kutta (RK) methods of order higher than, or equal to, 3 are stable [12, 40–43]. Similar stability results are expected to hold for Adams–Bashforth (AB) methods of order 3, 4, 7, 8 under standard hyperbolic CFL, $\delta t \leq C_0 h$, where δt denotes the timestep and C_0 the Courant number. See for instance [31] for a discussion of time-discretization of advection–diffusion equation, [25] for a discussion of the stability boundaries of AB methods and [13] for numerical experiments using AB3. All these methods are energy stable regardless of whether or not stabilization is added. The second order RK method is energy stable under hyperbolic CFL only for piecewise affine approximation and with added stabilization of the form (2.8) [12] (for dG FEM and affine approximation

upwind stabilization must be added [42]). In the general case (no stabilization, higher polynomial approximation) the RK2 method is stable only under a slightly more strict CFL condition, indeed one needs to assume $dt \leq Co h^{\frac{4}{3}}$, with Co fixed, but small enough. This condition is the same for both cG and dG methods (see [12, 42]). Recently an analysis of the second order backward differentiation formula and the Crank–Nicolson method (AB2) with convection extrapolated to second order from previous time steps was proposed for the discretization of (2.13) [13]. It was shown that these schemes are stable under similar conditions as the RK2 scheme. Such multi step schemes are particularly appealing in the context of IMEX methods for convection–diffusion and hence provide a one-stage alternative to the RK2 IMEX method analysed in [11].

5 Weighted Error Estimates

In this section we will consider the slightly more technically advanced case of weighted estimates. The idea is to show that stabilization makes information follow the characteristics similarly as in the physics. This means that for solutions with a localized sharp layer, the dependence of a local error in the smooth zone on the regularity of the exact solution decreases exponentially with the distance to the singularity. Hence locally large gradients in the solution can not destroy the solution globally. This is not the case for approximations produced using cG FEM without stabilization. These results touch at the very essence of stabilized FEM, unfortunately their proofs are quite technical and therefore these results in my opinion have received less attention than they deserve. Here we try to give the simplest possible exposition of these ideas, without striving for optimality of exponential decay or generality of meshes. We let the domain be infinite ($L = \infty$) and let u_0 have compact support. To simplify the discussion assume that $\beta \equiv e_x$, where e_x is the Cartesian unit vector in the x -direction, so that $\beta \cdot \nabla u = \partial_x u$. Since here $\beta_\infty = 1$, below the dependence on the speed will not be tracked. First the case of a globally smooth solution will be considered (Theorem 4). The objective is to obtain an estimate for the error in some subdomain $\Omega_0(t) \subset \Omega$ defined as

$$\Omega_0(t) := \{x \in \Omega : |x_0 + \beta t - x| < r_0\}$$

for some $x_0 \in \Omega$ and some $r_0 > 0$. The derivatives of u are assumed to be moderate in a neighbourhood of Ω_0 and we will prove that the accuracy in this subdomain is independent of large derivatives in other parts of the domain, provided they are sufficiently far away, relative to the mesh size. This is achieved using weights so that the effect of portions of the domain where locally the Sobolev norm is large decays exponentially with the distance to Ω_0 . Then we will show how the arguments of the smooth case can be used to prove accuracy in Ω_0 in the case where the solution is locally only L^2 in the far field (Corollary 2). The key message is that the local accuracy of the approximation depends only on the local smoothness of the exact solution and that perturbations due to roughness in the solution is exponentially damped, except along characteristics. Finally we will discuss how the arguments can be extended to bounded domains with weakly imposed boundary condition and time discretization.

Let $\varphi \in C^{k+1}(\Omega)$ be a smooth positive function defined using polar/spherical coordinates, depending only on $r(x) = |x_0 - x|$, with $\varphi'(r) \leq 0$, $\varphi(r) = 1$, $r \leq r_0$, $\varphi(r) \sim \exp(-(r - r_0)/\sigma)$, $r > r_0$, with $\sigma = K\sqrt{h}$, $K > 1$, and for some $C > 0$,

$$|\partial_r^l \varphi(r)| \leq C \sigma^{-l} \varphi(r), \quad l \geq 1.$$

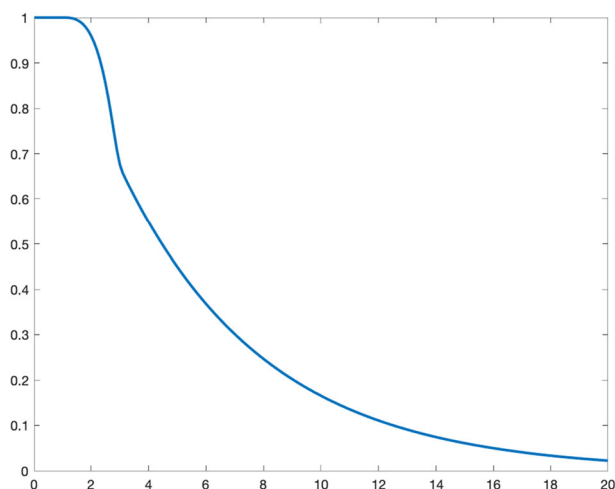


Fig. 1 Example of the radial cross section of $\varphi \in C^1(\Omega)$ with $r_0 = 1$ and $\sigma = 5$

Remark 5 For the case $k = 1$ we only require $\varphi \in C^1(\Omega)$. An example of such a function with $r_0 = 1$ and $\sigma = 5$ is given in Fig. 1 for illustration.

Define $\varpi(\mathbf{x}, t) = \varphi(r(\mathbf{x} - \boldsymbol{\beta}t))$ then, since ϖ follows the characteristics $\mathcal{L}\varpi = 0$, and

$$|D^l \varpi| \leq C\sigma^{-l}\varpi, \quad l \geq 1 \quad (5.1)$$

where the derivatives are taken with respect to space or time. The objective is to prove stability and error estimates in the weighted norm

$$\|v\|_{\varpi} := \|\varpi v\|_{\Omega}.$$

The same notation will be used occasionally below with different weight functions. The rationale for the design of the weight function is that for all $v \in L^\infty(0, T; L^2(\Omega))$ with $\mathcal{L}v \in L^2(0, T; \Omega)$, by partial integration in space and time,

$$\int_0^T (\partial_t v, \varpi^2 v)_{\Omega} dt = \|v(\cdot, T)\|_{\varpi}^2 - \|v(\cdot, 0)\|_{\varpi}^2 - \int_0^T (v, \partial_t \varpi^2 v + \varpi^2 \partial_t v)_{\Omega} dt$$

and

$$(\boldsymbol{\beta} \cdot \nabla v, \varpi^2 v)_{\Omega} = -(v, (\boldsymbol{\beta} \cdot \nabla \varpi^2) v + \varpi^2 \boldsymbol{\beta} \cdot \nabla v)_{\Omega},$$

there holds

$$\begin{aligned} \int_0^T (\mathcal{L}v, \varpi^2 v)_{\Omega} dt &= \|v(\cdot, T)\|_{\varpi}^2 - \|v(\cdot, 0)\|_{\varpi}^2 \\ &\quad - \int_0^T (v, \underbrace{(\mathcal{L}\varpi^2)}_{=0} v)_{\Omega} + (v, \varpi^2 \mathcal{L}v)_{\Omega} dt. \end{aligned}$$

Hence

$$\int_0^T (\mathcal{L}v, \varpi^2 v)_{\Omega} dt = \frac{1}{2} \|v(\cdot, T)\|_{\varpi}^2 - \frac{1}{2} \|v(\cdot, 0)\|_{\varpi}^2 \quad (5.2)$$

and therefore the following stability is satisfied by the continuous equation, (2.1), $\forall \sigma > 0$,

$$\frac{1}{2} \|u(\cdot, T)\|_{\varpi}^2 \leq \frac{1}{2} \|u(\cdot, 0)\|_{\varpi}^2 + \int_0^T \|f\|_{\varpi} \|u\|_{\varpi} \, dt$$

from which we conclude

$$\sup_{t \in (0, T)} \|u(\cdot, t)\|_{\varpi} \leq \|u(\cdot, 0)\|_{\varpi} + 2 \int_0^T \|f\|_{\varpi} \, dt.$$

This relation expresses that the solution is transported along the characteristics. The influence across characteristics will be damped exponentially as $\exp(-d/\sigma)$. However in the continuous case, since the bound holds for all $\sigma > 0$ the cut-off is sharp.

The aim is to make the error analysis for the solution of (2.13) reproduce this type of localization. For the purposes of analysis we introduce the weighted stabilization operator

$$s_{\varpi}(v_h, w_h) = \sum_{F \in \mathcal{F}} \int_F h_F^2 \varpi^2 \llbracket \nabla v_h \rrbracket \llbracket \nabla w_h \rrbracket \, ds, \text{ with semi-norm } |w|_{s, \varpi} := s_{\varpi}(w, w)^{\frac{1}{2}}$$

and note that $s(v_h, \varpi^2 w_h) = s_{\varpi}(v_h, w_h)$. Also recall the following weighted versions of (2.11) from [14, Lemma 3.1, equation (3.1) and (3.2)], here $\beta_0|_S \in \mathbb{R}^n$ is some piecewise constant per element,

$$\|h^{\frac{1}{2}}(\beta_0 \cdot \nabla v_h - \pi_h \beta_0 \cdot \nabla v_h)\|_{\varpi}^2 \leq C_{ws} |\beta_0|_{v_h|_{s, \varpi}}^2 \quad (5.3)$$

and

$$\|h^{\frac{1}{2}}(\beta \cdot \nabla(\varpi^2 v_h) - \pi_h(\beta \cdot \nabla(\varpi^2 v_h)))\|_{\varpi^{-1}}^2 \leq C_{ws} |v_h|_{s, \varpi}^2 + C_{\beta} K^{-2} \|v_h\|_{\varpi}^2. \quad (5.4)$$

The second bound differs from the bound in [14], since there the derivative of v_h appears in the second term of the right hand side. The proof however is similar. For completeness we detail it in [Appendix](#). We will need to use approximation in the weighted norm and therefore collect some results on the L^2 -projection in the following lemmas. The first one is taken from [2] and we refer to this reference for the proof. The following two are variations on results from [14] and for completeness we give the proofs in [Appendix](#). We note that all the above inequalities hold both for the weight ϖ and ϖ^{-1} , since by the construction of the weight,

$$|\nabla \varpi^{-1}| = |\varpi^{-2} \nabla \varpi| \leq C \varpi^{-2} \sigma^{-1} \varpi = C \sigma^{-1} \varpi^{-1}.$$

It follows that (5.1) is satisfied also for ϖ^{-1} .

Lemma 1 (Stability L^2 -projection) *Let π_h denote the L^2 -projection onto V_h . Then, if ϕ is a function satisfying*

$$|\nabla \phi(x)| \leq v h^{-1} |\phi(x)|,$$

for some $v > 0$, sufficiently small then there holds

$$\|\pi_h v\|_{\phi} \leq C \|v\|_{\phi}, \quad (5.5)$$

$$\|\nabla \pi_h v\|_{\phi} \leq C \|\nabla v\|_{\phi} \quad (5.6)$$

and

$$\|\nabla \pi_h v\|_{\phi} \leq C h^{-1} \|v\|_{\phi}, \quad \forall v \in H^1(\Omega). \quad (5.7)$$

Proof The estimates (5.5)–(5.7) are taken verbatim from [2, bounds (1.7)–(1.9)] (see also [22, Appendix]). \square

The above stability estimates allows us to prove bounds on the L^2 -error in the weighted norm.

Lemma 2 (Weighted approximation) *Let π_h denote the L^2 -projection onto V_h . Then for $h^{\frac{1}{2}}/K$ sufficiently small and $I_\delta = [t - \delta t, t + \delta t] \cap [0, T]$ with $\delta t \in \mathbb{R}^+$, $\delta t \sim h$, there holds*

$$\max_{(x,t) \in S \times I_\delta} \varpi(x, t) \|v\|_S \leq 2 \min_{t \in I_\delta} \|v \varpi(\cdot, t)\|_S, \quad \forall v \in L^2(S), \quad (5.8)$$

$$\|(v - \pi_h v)_{\varpi} + h \|\nabla(v - \pi_h v)\|_{\varpi} \leq Ch^{k+1} \|D^{k+1} v\|_{\varpi}, \quad \forall v \in H^{k+1}(\Omega) \quad (5.9)$$

and

$$\|v - \pi_h v\|_{s, \varpi} \leq Ch^{k+\frac{1}{2}} \|D^{k+1} v\|_{\varpi}, \quad \forall v \in H^{k+1}(\Omega). \quad (5.10)$$

For the analysis we also need the following interpolation estimates on weighted discrete functions.

Lemma 3 (Super approximation) *Let $v_h \in V_h$. Assume that $h^{\frac{1}{2}}/K$ is sufficiently small. Then there holds*

$$\|\varpi^2 v_h - \pi_h(\varpi^2 v_h)\|_{\varpi^{-1}} + h \|\nabla(\varpi^2 v_h - \pi_h(\varpi^2 v_h))\|_{\varpi^{-1}} \leq Ch^{\frac{1}{2}} K^{-1} \|v_h\|_{\varpi} \quad (5.11)$$

and

$$\left(\sum_{S \in \mathcal{T}} \|\varpi^{-1} \nabla(\varpi^2 v_h - \pi_h(\varpi^2 v_h))\|_{\partial S}^2 \right)^{\frac{1}{2}} \leq Ch^{-1} K^{-1} \|v_h\|_{\varpi}. \quad (5.12)$$

We will now derive a weighted stability estimate for the finite element formulation (2.13). First use similar arguments as for (5.2) to obtain for any $v_h \in C^1(0, T; V_h)$,

$$\int_0^T (\mathcal{L} v_h, \varpi^2 v_h)_{\Omega} \, dt = \frac{1}{2} \|v_h(\cdot, T)\|_{\varpi}^2 - \frac{1}{2} \|v_h(\cdot, 0)\|_{\varpi}^2$$

and, since $\varpi \in C^1(\Omega)$ we see that

$$s(v_h, \varpi^2 v_h) = |v_h|_{s, \varpi}^2.$$

Therefore,

$$\|v_h(\cdot, T)\|_{\varpi}^2 + 2\gamma \int_0^T |v_h|_{s, \varpi}^2 \, dt = 2 \int_0^T ((\mathcal{L} v_h, \varpi^2 v_h)_{\Omega} + \gamma s(v_h, \varpi^2 v_h)) \, dt + \|v_h(\cdot, 0)\|_{\varpi}^2. \quad (5.13)$$

However, since $\varpi^2 v_h \notin V_h$ the equality can not be used directly for the finite element formulation. We need to show that stability similar to (5.13) can be obtained by testing by some interpolant of $\varpi^2 v_h$.

Proposition 2 (Weighted stability) *Let $\gamma > 0$, $K > 1$. Assume that $h^{\frac{1}{2}}/K$ is sufficiently small. For all $v_h \in C^1(0, T; V_h)$ there holds*

$$\begin{aligned} \|v_h(\cdot, T)\|_{\varpi}^2 + \gamma \int_0^T |v_h|_{s, \varpi}^2 \, dt &\leq C/K^2 \int_0^T \|v_h\|_{\varpi}^2 \, dt \\ &\quad + 2 \int_0^T ((\mathcal{L} v_h, w_h)_{\Omega} + \gamma s(v_h, w_h)) \, dt + \|v_h(\cdot, 0)\|_{\varpi}^2, \end{aligned}$$

where $w_h = \pi_h \varpi^2 v_h$ and the constant $C \sim \gamma + \gamma^{-1}$.

Proof Starting from the equality (5.13) we add and subtract the finite element formulation tested with some function w_h ,

$$\begin{aligned} \|v_h(\cdot, T)\|_{\varpi}^2 + 2\gamma \int_0^T |v_h|_{s,\varpi}^2 dt &= 2 \int_0^T ((\mathcal{L}v_h, \varpi^2 v_h - w_h)_{\Omega} + \gamma s(v_h, \varpi^2 v_h - w_h)) dt \\ &\quad + 2 \int_0^T ((\mathcal{L}v_h, w_h)_{\Omega} + \gamma s(v_h, w_h)) dt + \|v_h(\cdot, 0)\|_{\varpi}^2. \end{aligned}$$

We choose $w_h = \pi_h(\varpi^2 v_h)$ to obtain, for an arbitrary $y_h \in V_h$

$$\begin{aligned} (\mathcal{L}v_h, \varpi^2 v_h - \pi_h(\varpi^2 v_h))_{\Omega} &= (\beta \cdot \nabla v_h - y_h, \varpi^2 v_h - \pi_h(\varpi^2 v_h))_{\Omega} \\ &\leq \inf_{y_h \in V_h} \|h^{\frac{1}{2}}(\beta \cdot \nabla v_h - y_h)\|_{\varpi} h^{-\frac{1}{2}} \|(\varpi^2 v_h - \pi_h(\varpi^2 v_h))\|_{\varpi^{-1}}. \end{aligned}$$

Considering the stabilization term we see that

$$s(v_h, \varpi^2 v_h - \pi_h(\varpi^2 v_h)) \leq |v_h|_{s,\varpi} h \beta_{\infty}^{\frac{1}{2}} \left(\sum_{F \in \mathcal{F}} \|\varpi^{-1} \llbracket \nabla(\varpi^2 v_h - \pi_h(\varpi^2 v_h)) \rrbracket \rrbracket_F^2 \right)^{\frac{1}{2}}. \quad (5.14)$$

Using the arithmetic-geometric inequality $ab \leq (2\epsilon)^{-1}a^2 + (\epsilon 2^{-1})b^2$, to split the terms in the right hand side, with $\epsilon = 2$ in (5.14), we obtain

$$\begin{aligned} \|v_h(\cdot, T)\|_{\varpi}^2 + \frac{7}{4}\gamma \int_0^T |v_h|_{s,\varpi}^2 dt &\leq \epsilon^{-1}\gamma^{-1}h^{-1} \underbrace{\int_0^T \|(\varpi^2 v_h - \pi_h(\varpi^2 v_h))\|_{\varpi^{-1}}^2 dt}_{T_1} \\ &\quad + \underbrace{\gamma h^2 \beta_{\infty} \int_0^T \sum_{F \in \mathcal{F}} \|\varpi^{-1} \llbracket \nabla(\varpi^2 v_h - \pi_h(\varpi^2 v_h)) \rrbracket \rrbracket_F^2 dt}_{T_2} \\ &\quad + \underbrace{\epsilon \gamma \int_0^T \inf_{y_h \in V_h} \|h^{\frac{1}{2}}(\beta \cdot \nabla v_h - y_h)\|_{\varpi}^2 dt}_{T_3} \\ &\quad + 2 \int_0^T ((\mathcal{L}v_h, w_h)_{\Omega} + \gamma s(v_h, w_h)) dt + \|v_h(\cdot, 0)\|_{\varpi}^2. \end{aligned}$$

We need to bound the contributions T_1 , T_2 and T_3 in terms of the quantities of the left hand side and $\|v_h\|_{\varpi}$. Using (5.11) immediately yields

$$T_1 = \|(\varpi^2 v_h - \pi_h(\varpi^2 v_h))\|_{\varpi^{-1}}^2 \leq CK^{-2}h\|v_h\|_{\varpi}^2.$$

By distribution of the integrals over the faces on simplices, splitting the jumps on the contributions from the two sides and applying (5.12) there holds

$$T_2 \leq C \sum_{S \in \mathcal{T}} \|\varpi^{-1} \nabla(\varpi^2 v_h - \pi_h(\varpi^2 v_h))\|_{\partial S}^2 \leq C/K^2 h^{-2} \|v_h\|_{\varpi}^2.$$

Finally for the term T_3 apply the weighted stabilization bound (5.3), with $\beta_0 \equiv e_x$, where e_x is the Cartesian unit vector in the x -direction

$$T_3 = \inf_{y_h \in V_h} \|h^{\frac{1}{2}}(\beta \cdot \nabla v_h - y_h)\|_{\varpi}^2 \leq C_{ws} |v_h|_{s,\varpi}^2.$$

Collecting the bounds for T_1 - T_3 and choosing $\epsilon = (2C_{ws})^{-1}$ we see that

$$\begin{aligned} \|v_h(\cdot, T)\|_{\varpi}^2 + \gamma \int_0^T |v_h|_{s,\varpi}^2 dt &\leq (\gamma^{-1} + \gamma)C/K^2 \int_0^T \|v_h\|_{\varpi}^2 dt \\ &\quad + 2 \int_0^T ((\mathcal{L}v_h, w_h)_{\Omega} + \gamma s(v_h, w_h)) dt + \|v_h(\cdot, 0)\|_{\varpi}^2. \end{aligned}$$

□

Theorem 4 Assume that the hypothesis of Proposition 2 are satisfied. Let $u \in L^\infty(0, T; H^{k+1}(\Omega))$ be the solution of (2.1) and u_h the solution of (2.13). Then for all $T > 0$ there holds

$$\|(u - u_h)(\cdot, T)\|_{\varpi} \leq C_K h^{k+\frac{1}{2}} \left(h \|D^{k+1}u(\cdot, T)\|_{\varpi}^2 + (\gamma + \gamma^{-1}) \int_0^T \|D^{k+1}u\|_{\varpi}^2 dt \right)^{\frac{1}{2}}.$$

The constant C_K grows exponentially in time with coefficient proportional to $(\gamma + \gamma^{-1})K^{-2}$.

First note that we may split the error as $u - u_h = \underbrace{u - \pi_h u}_{=-\eta} + \underbrace{\pi_h u - u_h}_{=e_h}$ and by (5.9),

$$\|(u - \pi_h u)(\cdot, T)\|_{\varpi} \leq C h^{k+1} \|D^{k+1}u(\cdot, T)\|_{\varpi}.$$

By the triangle inequality we only need to prove the bound on $\|e_h(\cdot, T)\|_{\varpi}$.

Using the stability of Proposition 2 we see that, since $e_h(\cdot, 0) = 0$,

$$\begin{aligned} \|e_h(\cdot, T)\|_{\varpi}^2 + \gamma \int_0^T |e_h|_{s,\varpi}^2 dt &\leq C/K^2 \int_0^T \|e_h\|_{\varpi}^2 dt \\ &\quad + 2 \int_0^T ((\mathcal{L}e_h, w_h)_{\Omega} + \gamma s(e_h, w_h)) dt \end{aligned}$$

with $w_h = \pi_h(\varpi^2 e_h)$. Now observe that the following consistency property holds

$$\int_0^T (\mathcal{L}(e_h - \eta), v_h)_{\Omega} - \gamma s(u_h, v_h) dt = 0, \quad \forall v_h \in V_h$$

and hence

$$\int_0^T ((\mathcal{L}e_h, w_h)_{\Omega} + \gamma s(e_h, w_h)) dt = \int_0^T ((\mathcal{L}\eta, w_h)_{\Omega} + \gamma s(\pi_h u_h, w_h)) dt.$$

This leads to a perturbation equation on the form

$$\begin{aligned} \|e_h(\cdot, T)\|_{\varpi}^2 + \gamma \int_0^T |e_h|_{s,\varpi}^2 dt &\leq C K^{-2} \int_0^T \|e_h\|_{\varpi}^2 dt \\ &\quad + 2 \int_0^T ((\mathcal{L}\eta, w_h)_{\Omega} + \gamma s(\pi_h u_h, w_h)) dt. \end{aligned} \quad (5.15)$$

Considering the first term of the second integral in the right hand side we have using that time derivation and the L^2 -projection commute and the L^2 -orthogonality of η

$$\begin{aligned} (\mathcal{L}\eta, w_h)_{\Omega} &= -(\eta, \beta \cdot \nabla w_h - y_h)_{\Omega} \leq h^{-\frac{1}{2}} \|\eta\|_{\varpi} h^{\frac{1}{2}} \inf_{y_h \in V_h} \|\beta \cdot \nabla w_h - y_h\|_{\varpi^{-1}} \\ &\leq h^{-1} \gamma^{-1} C \|\eta\|_{\varpi}^2 + \frac{1}{4} \gamma |e_h|_{s,\varpi}^2 + C \gamma / K^2 \|e_h\|_{\varpi}^2. \end{aligned}$$

Here we used the inequality $ab \leq 4^{-1}a^2 + b^2$ and that by the triangle inequality followed by the bounds (5.11), and (5.4) there holds

$$\begin{aligned} h^{\frac{1}{2}} \inf_{y_h \in V_h} \|\beta \cdot \nabla w_h - y_h\|_{\varpi^{-1}} &\leq h^{\frac{1}{2}} \|\beta \cdot \nabla \pi_h(\varpi^2 e_h) - \beta \cdot \nabla(\varpi^2 e_h)\|_{\varpi^{-1}} \\ &\quad + h^{\frac{1}{2}} \inf_{y_h \in V_h} \|\beta \cdot \nabla(\varpi^2 e_h) - y_h\|_{\varpi^{-1}} \\ &\leq h^{\frac{1}{2}} \beta_\infty \|\nabla(\pi_h(\varpi^2 e_h) - \varpi^2 e_h)\|_{\varpi^{-1}} \\ &\quad + (C_{ws} |e_h|_{s,\varpi}^2 + C_\beta K^{-2} \|e_h\|_{\varpi}^2)^{\frac{1}{2}} \\ &\leq CK^{-1} \|e_h\|_{\varpi} + C_{ws} |e_h|_{s,\varpi}. \end{aligned}$$

For the last term in the right hand side of (5.15) we have

$$\begin{aligned} s(\pi_h u_h, w_h) &= s(\pi_h u_h, \pi_h(\varpi^2 e_h) - \varpi^2 e_h) + s(\pi_h u_h, \varpi^2 e_h) \\ &\leq C |\pi_h u_h|_{s,\varpi}^2 + \frac{1}{4} |e_h|_{s,\varpi}^2 \\ &\quad + h^2 \beta_\infty^2 \sum_{F \in \mathcal{F}} \|\varpi^{-1} \llbracket \nabla(\varpi^2 e_h - \pi_h(\varpi^2 e_h)) \rrbracket \|_F^2. \end{aligned}$$

Applying the bound (5.12) to each term of the jump separately in the last term in the right hand side and collecting the estimates it follows that

$$(\mathcal{L}\eta, w_h)_\Omega + \gamma s(\pi_h u_h, w_h) \leq C(\gamma |\pi_h u_h|_{s,\varpi}^2 + h^{-1} \gamma^{-1} \|\eta\|_{\varpi}^2) + \frac{1}{2} \gamma |e_h|_{s,\varpi}^2 + \gamma C / K^2 \|e_h\|_{\varpi}^2.$$

Applying this bound in (5.15) we have

$$\begin{aligned} \|e_h(\cdot, T)\|_{\varpi}^2 + \frac{1}{2} \gamma \int_0^T |e_h|_{s,\varpi}^2 dt &\leq C(\gamma + \gamma^{-1}) / K^2 \int_0^T \|e_h\|_{\varpi}^2 dt \\ &\quad + C \int_0^T \left(\gamma |\pi_h u_h|_{s,\varpi}^2 + h^{-1} \gamma^{-1} \|\eta\|_{\varpi}^2 \right) dt. \end{aligned} \quad (5.16)$$

Since the solution is assumed regular, $u(\cdot, t) \in H^{\frac{3}{2}+\epsilon}(\Omega)$, $\epsilon > 0$ we have $|\pi_h u_h|_{s,\varpi}^2 = |\eta|_{s,\varpi}^2$. Applying Lemma 2 yields

$$\int_0^T \left(\gamma |\eta|_{s,\varpi}^2 + h^{-1} \gamma^{-1} \|\eta\|_{\varpi}^2 \right) dt \leq Ch^{2k+1} (\gamma + \gamma^{-1}) \int_0^T \|D^{k+1} u\|_{\varpi}^2 dt.$$

The claim now follows by an application of Gronwall's inequality.

5.1 Discussion of Estimates for Rough Solutions

Consider the following subsets of Ω , $\Omega_0(t) := \{x \in \Omega : \varpi(x, t) = 1\}$ and $\Omega_p(t) := \{x \in \Omega : \varpi(x, t) \leq h^p, p > 0\}$. Then denoting $d = \text{dist}(\Omega_0, \Omega_p)$ it follows by the construction of ϖ that

$$d \sim Kp\sqrt{h} |\log(h)|,$$

and the following bound holds

$$\|(u - u_h)(\cdot, T)\|_{\Omega_0} \leq Ch^{k+\frac{1}{2}} \left(\max_{t \in [0, T]} \|D^{k+1} u\|_{L^2(\Omega \setminus \Omega_p)} + h^p \max_{t \in [0, T]} \|D^{k+1} u\|_{L^2(\Omega_p)} \right).$$

It follows that $D^{k+1}u$ can be large, $O(h^{-p})$, in Ω_p without destroying the solution in Ω_0 . To apply the argument to u_0 that is only piecewise in H^{k+1} one can use the weighted L^2 -stability in the error analysis above and still obtain estimates. We present a sketch of this result in a corollary.

Corollary 2 *Assume that the hypothesis of Proposition 2 are satisfied. Let $p = k + 1$. Assume that $u \in L^\infty(0, T; L^2(\Omega))$, with $u|_{\Omega \setminus \Omega_p} \in H^{k+1}(\Omega \setminus \Omega_p)$, for all $t \in [0, T]$ is the solution of (2.1) and u_h the solution of (2.13). Then there holds (omitting for simplicity the dependence on γ).*

$$\|(u - u_h)(\cdot, T)\|_{\Omega_0} \leq C_K h^{k+\frac{1}{2}} \left(\max_{t \in [0, T]} \|u\|_{H^{k+1}(\Omega \setminus \Omega_p)} + \max_{t \in [0, T]} \|u\|_{L^2(\Omega_p)} \right).$$

Proof The proof follows that of Theorem 4 closely. We only need to substitute the L^2 -projection for an interpolant with more local properties before applying approximation. Let the domain $\Omega_{p,ih}(t)$ be defined by the union of all the elements that intersect $\Omega_p(t)$ and an integer i layers of nearest neighbours. The norm over $\Omega_{p,ih}(t)$ will be denoted $\|\cdot\|_{\Omega_{p,ih}}$. Let C_h denote the Clément interpolant defined using local projections. It is well known [23, Lemma 1.127] that if for a given $S \in \mathcal{T}$, Δ_S denotes the set of simplices sharing at least one vertex with S and for a face F , Δ_F denotes the set of simplices sharing at least one vertex with F , then

$$\begin{aligned} \|v - C_h v\|_{H^m(S)} &\leq Ch^{l-m} \|v\|_{H^l(\Delta_S)}, \quad \|v - C_h v\|_{H^m(F)} \leq Ch^{l-m-\frac{1}{2}} \|v\|_{H^l(\Delta_F)}, \\ 0 &\leq m \leq l \leq k+1. \end{aligned} \quad (5.17)$$

It is then straightforward to use the approximation properties of C_h in $\Omega \setminus \Omega_{p,1h}$ and the local stability of C_h in $\Omega_{p,1h}$ to show the estimates

$$\begin{aligned} \|(u - C_h u)(\cdot, t)\|_{\varpi} &\leq C \left(h^{k+1} \|D^{k+1}u(\cdot, t)\|_{\Omega \setminus \Omega_p} + h^p \|u(\cdot, t)\|_{\Omega_{p,2h}} \right) \\ &\leq Ch^{k+1} \left(\|u(\cdot, t)\|_{H^{k+1}(\Omega \setminus \Omega_p)} + \|u(\cdot, t)\|_{\Omega_p} \right) \end{aligned} \quad (5.18)$$

and

$$\begin{aligned} |C_h u(\cdot, t)|_{s, \varpi} &\leq C \left(h^{k+\frac{1}{2}} \|D^{k+1}u(\cdot, t)\|_{\Omega \setminus \Omega_p} + h^{-\frac{1}{2}+p} \|u(\cdot, t)\|_{\Omega_{p,2h}} \right) \\ &\leq Ch^{k+\frac{1}{2}} \left(\|u(\cdot, t)\|_{H^{k+1}(\Omega \setminus \Omega_p)} + \|u(\cdot, t)\|_{\Omega_p} \right). \end{aligned} \quad (5.19)$$

For the second inequality we divide $|C_h u(\cdot, t)|_{s, \varpi}$ into the sum over faces in $\Omega \setminus \Omega_{p,1h}$ and $\Omega_{p,1h}$. The two different sets are treated differently. For faces in $\Omega \setminus \Omega_{p,1h}$ we proceeded as usual using that $u(\cdot, t)|_{\Omega \setminus \Omega_p} \in H^{\frac{3}{2}+\epsilon}(\Omega \setminus \Omega_p)$ and apply the local approximation properties on faces of C_h (second inequality of (5.17)). For faces in $\Omega_{p,1h}$ we can not use approximation and instead apply (2.10) and (2.6). We also used that $\varpi|_{\Omega_{p,1h}} \leq Ch^p$ by construction. Observe that by the weighted L^2 -stability (5.5) we have

$$\|(u - \pi_h u)(\cdot, T)\|_{\varpi} \leq \|\pi_h(u - C_h u)(\cdot, T)\|_{\varpi} + \|(u - C_h u)(\cdot, T)\|_{\varpi} \leq C \|(u - C_h u)(\cdot, T)\|_{\varpi} \quad (5.20)$$

and hence as before we only need to prove the bound for $e_h(\cdot, T)_{\varpi}$. The inequality (5.16) still holds. To conclude we observe that using (5.20)

$$\int_0^T h^{-1} \|\eta\|_{\varpi}^2 dt \leq C \int_0^T h^{-1} \|u - C_h u\|_{\varpi}^2 dt. \quad (5.21)$$

By combining the inequality

$$|v_h|_{s,\varpi} \leq Ch^{-\frac{1}{2}} \|v_h\|_{\varpi}$$

(that is immediate by (2.10), (2.6) and (5.8)) with (5.20) we also have

$$\begin{aligned} \int_0^T |\pi_h u|_{s,\varpi}^2 dt &\leq 2 \int_0^T \left(|\pi_h u - C_h u|_{s,\varpi}^2 + |C_h u_h|_{s,\varpi}^2 \right) dt \\ &\leq C \int_0^T \left(h^{-1} \|u - C_h u\|_{\varpi}^2 + |C_h u_h|_{s,\varpi}^2 \right) dt. \end{aligned} \quad (5.22)$$

We conclude as before after applying (5.18) and (5.19) in (5.21) and (5.22). \square

5.2 Time Discretization and Weakly Imposed Boundary Conditions

In practice and in the numerical section below of course we need to include boundary conditions and time discretizations in the above arguments. Depending on the time-discretization this can be a challenging exercise, but we will here focus on the θ -scheme and the main steps of its analysis using the ideas above in the case of the backward Euler scheme ($\theta = 1$). Boundary conditions are imposed weakly using the standard upwind technique known from discontinuous Galerkin methods. We consider a polygonal domain Ω and denote its boundary by $\Gamma := \partial\Omega$ with outward pointing normal n . We decompose Γ into an inflow part

$$\Gamma_- := \{x \in \Gamma : \beta(x) \cdot n < 0\}$$

and an outflow part $\Gamma_+ := \partial\Omega \setminus \Gamma_-$. The space V_h will here denote the standard finite element space of continuous piecewise polynomial functions, without boundary conditions defined on \mathcal{T} . We are now interested in the the solution of (2.1) with the additional inflow boundary condition

$$u = g \text{ on } \Gamma_-,$$

where $g \in L^2(0, T; L^2_{\beta \cdot n}(\Gamma_-))$ with $L^2_{\beta \cdot n}(\Gamma_-) := \{v : \Gamma_- \mapsto \mathbb{R} : \| |\beta \cdot n|^{\frac{1}{2}} v \|_{L^2(\Gamma_-)} < \infty\}$. We will assume that the g , Γ_- and Γ_+ are such that the exact solution is smooth enough for our purposes. The timestep $\delta t := T/N$ for some $N \in \mathbb{N}^+$ will be assumed to satisfy $\delta t \leq Ch$ for some $C > 0$, and the discrete solution $u_h := \{u_h^n\}_{n=0}^N$ collects the finite element approximations on the discrete time levels $t^n = n\delta t$. The so-called θ -scheme takes the form: find $u_h^n \in V_h$ such that for $n = 1, 2, 3 \dots N$,

$$(\mathcal{L}_\theta^n u_h, v_h)_\Omega + \langle |\beta \cdot n| u_h^{n\theta}, v_h \rangle_{\Gamma_-} + s(u_h^{n\theta}, v_h) = (f^{n\theta}, v_h)_\Omega + \langle |\beta \cdot n| g^{n\theta}, v_h \rangle_{\Gamma_-}, \quad \forall v_h \in V_h, \quad (5.23)$$

where $u_h^{n\theta} := \theta u_h^n + (1 - \theta)u_h^{n-1}$, $g^{n\theta} := g(\cdot, t^n + \theta\delta t)$, $f^{n\theta} := f(\cdot, t^n + \theta\delta t)$,

$$\mathcal{L}_\theta^n u_h := \delta t^{-1} (u_h^n - u_h^{n-1}) + \beta \cdot \nabla u_h^{n\theta}, \quad \theta \in [1/2, 1]$$

and $u_h^0 = \pi_h u_0$. Compared to the time continuous analysis we have two additional points to study

1. the time discrete character of the equation,
2. the boundary penalty term.

We recall that the theta scheme includes the well-known backward Euler scheme ($\theta = 1$) and the Crank–Nicolson scheme ($\theta = 1/2$). A complete analysis of the θ scheme is beyond the scope of the present paper. To give some insight in the validity of the above arguments in the fully discrete case we will show the modifications necessary to prove Proposition 2 in the time discrete case with weakly imposed boundary conditions, for $\theta = 1$. Theorem 4 then follows using the arguments above and standard truncation error analysis. We will then show numerically that also the Crank–Nicolson scheme enjoys the local accuracy property. For further evidence of the local accuracy property we refer to [12, Section 5.2 and Fig. 1] for examples using explicit Runge–Kutta methods and [13, Section 6] for examples using explicit extrapolated multistep methods. For the analysis we need the following Lemma the proof of which is given in the [Appendix](#).

Lemma 4 *Let $\varpi_n(x) = \varpi(x, t_n)$, where ϖ is a weightfunction satisfying (5.1) and $v_h \in V_h$, then for δt small enough there holds*

$$\left\| v_h \int_{t_{n-1}}^{t_n} \partial_t \varpi \, dt \right\|_{\Omega} + \left\| v_h \left| \int_{t_{n-1}}^{t_n} \int_t^{t_n} \partial_t^2 \varpi^2 \, ds \, dt \right|^{\frac{1}{2}} \right\|_{\Omega} \leq C K^{-1} \delta t^{\frac{1}{2}} \|v_h\|_{\varpi_n}.$$

The following weighted L^2 -stability estimate is the key ingredient of the analysis of the fully discrete scheme.

Proposition 3 *Consider the scheme (5.23) with $\theta = 1$, then assuming $\delta t < 1$ small enough there holds, with $w_h^n = \pi_h \varpi^2 v_h^n$,*

$$\begin{aligned} & \|v_h^N\|_{\varpi_N}^2 + \sum_{n=1}^N \|v_h^n - v_h^{n-1}\|_{\varpi_n}^2 + \delta t \sum_{n=1}^N \left(\| |\boldsymbol{\beta} \cdot \mathbf{n}|^{\frac{1}{2}} v_h^n \varpi_n \|_{\Gamma}^2 + \gamma |v_h^n|_{s, \varpi_n}^2 \right) \\ & \leq C_K \left(\|v_h^0\|_{\varpi_0}^2 + \delta t \sum_{n=1}^N \left((\mathcal{L}_\theta^n v_h, w_h^n)_{\Omega} + \langle |\boldsymbol{\beta} \cdot \mathbf{n}| v_h^n, w_h^n \rangle_{\Gamma_-} + \gamma s(v_h^n, w_h^n) \right) \right). \end{aligned}$$

The constant C_K grows exponentially in time with exponential coefficient $1/K^2$.

Proof First we observe that using standard partial integration and $\nabla \cdot \boldsymbol{\beta} = 0$ we have

$$\begin{aligned} & (\boldsymbol{\beta} \cdot \nabla v_h, \varpi^2 v_h)_{\Omega} + \langle |\boldsymbol{\beta} \cdot \mathbf{n}| v_h, \varpi^2 v_h \rangle_{\Gamma_-} \\ & = -(\boldsymbol{\beta} \cdot \nabla v_h, \varpi^2 v_h)_{\Omega} - (v_h, (\boldsymbol{\beta} \cdot \nabla \varpi^2) v_h)_{\Omega} + \langle |\boldsymbol{\beta} \cdot \mathbf{n}| v_h, \varpi^2 v_h \rangle_{\Gamma_+}. \end{aligned}$$

As a consequence

$$(\boldsymbol{\beta} \cdot \nabla v_h, \varpi^2 v_h)_{\Omega} + \langle |\boldsymbol{\beta} \cdot \mathbf{n}| v_h, \varpi^2 v_h \rangle_{\Gamma_-} = -\frac{1}{2} (v_h, (\boldsymbol{\beta} \cdot \nabla \varpi^2) v_h)_{\Omega} + \frac{1}{2} \langle |\boldsymbol{\beta} \cdot \mathbf{n}| v_h, \varpi^2 v_h \rangle_{\Gamma}.$$

We also have

$$(v_h^n - v_h^{n-1}, \varpi_n^2 v_h^n)_{\Omega} = \frac{1}{2} \|v_h^n\|_{\varpi_n}^2 + \frac{1}{2} \|v_h^n - v_h^{n-1}\|_{\varpi_n}^2 - \frac{1}{2} \|v_h^{n-1}\|_{\varpi_n}^2.$$

It follows that

$$\begin{aligned} & \delta t \sum_{n=1}^N ((\mathcal{L}_\theta^n v_h, \varpi_n^2 v_h^n)_\Omega + \langle |\boldsymbol{\beta} \cdot \mathbf{n}| v_h^n, \varpi_n^2 v_h^n \rangle_{\Gamma_-} + \gamma s(v_h^n, \varpi_n^2 v_h^n)) \\ &= \frac{1}{2} \|v_h^N\|_{\varpi_N}^2 + \frac{1}{2} \sum_{n=1}^N \left(\|v_h^n - v_h^{n-1}\|_{\varpi_n}^2 - ((v_h^{n-1})^2, \varpi_n^2 - \varpi_{n-1}^2)_\Omega \right) - \frac{1}{2} \|v_h^0\|_{\varpi_0}^2 \\ & \quad - \frac{1}{2} \delta t \sum_{n=1}^N ((v_h^n)^2, \boldsymbol{\beta} \cdot \nabla \varpi_n^2)_\Omega + \frac{1}{2} \delta t \sum_{n=1}^N \left(\| |\boldsymbol{\beta} \cdot \mathbf{n}|^{\frac{1}{2}} v_h^n \varpi_n \|_{\Gamma}^2 + 2\gamma s(v_h^n, \varpi_n^2 v_h^n) \right). \end{aligned}$$

Identifying the terms in the right hand side that do not have a sign we see that we need to control

$$\sum_{n=1}^N ((v_h^{n-1})^2, \varpi_n^2 - \varpi_{n-1}^2)_\Omega + \delta t ((v_h^n)^2, \boldsymbol{\beta} \cdot \nabla \varpi_n^2)_\Omega.$$

We rewrite the first term

$$((v_h^{n-1})^2, (\varpi_n^2 - \varpi_{n-1}^2))_\Omega = ((v_h^{n-1})^2 - (v_h^n)^2, \varpi_n^2 - \varpi_{n-1}^2)_\Omega + ((v_h^n)^2, (\varpi_n^2 - \varpi_{n-1}^2))_\Omega.$$

For the first term on the right hand side we develop $a^2 - b^2 = (a + b)(a - b)$ and apply Cauchy–Schwarz inequality and the arithmetic-geometric inequality, followed by Lemma 4 and the inequality (5.8) to obtain the bound

$$\begin{aligned} & ((v_h^{n-1})^2 - (v_h^n)^2, \varpi_n^2 - \varpi_{n-1}^2)_\Omega = ((v_h^{n-1} + v_h^n)(v_h^{n-1} - v_h^n), \varpi_n^2 - \varpi_{n-1}^2)_\Omega \\ &= \left((v_h^{n-1} + v_h^n)(v_h^{n-1} - v_h^n), (\varpi_n + \varpi_{n-1}) \int_{t_{n-1}}^{t_n} \partial_t \varpi(\cdot, t) dt \right)_\Omega \\ &\geq -\epsilon^{-1} \left\| (v_h^n + v_h^{n-1}) \int_{t_{n-1}}^{t_n} \partial_t \varpi(\cdot, t) dt \right\|_\Omega^2 - \frac{\epsilon}{2} ((v_h^n - v_h^{n-1})^2, \varpi_n^2 + \varpi_{n-1}^2)_\Omega \\ &\geq -CK^{-2}\epsilon^{-1} \delta t \left(\|v_h^n\|_{\varpi_n}^2 + \|v_h^{n-1}\|_{\varpi_{n-1}}^2 \right) - \frac{C\epsilon}{2} \|v_h^n - v_h^{n-1}\|_{\varpi_n}^2. \end{aligned}$$

Considering the remaining terms, using the relation $\mathcal{L}\varpi^2 = 0$, and applying once again Lemma 4, yields the bound

$$\begin{aligned} & ((v_h^n)^2, \varpi_n^2 - \varpi_{n-1}^2)_\Omega + \delta t ((v_h^n)^2, \boldsymbol{\beta} \cdot \nabla \varpi_n^2)_\Omega = \left((v_h^n)^2, \int_{t_{n-1}}^{t_n} \partial_t \varpi^2 dt - \delta t \partial_t \varpi_n^2 \right)_\Omega \\ &= \left((v_h^n)^2, \int_{t_{n-1}}^{t_n} \int_{t_n}^t \partial_{tt} \varpi^2 ds dt \right)_\Omega \\ &\geq -\delta t C/K^2 \|v_h^n\|_{\varpi_n}^2. \end{aligned}$$

Taking ϵ sufficiently small so that $C\epsilon/2 \leq 1/4$ it follows that

$$\begin{aligned} & \|v_h^N\|_{\varpi_N}^2 + \sum_{n=1}^N (\|v_h^n - v_h^{n-1}\|_{\varpi_n}^2 + \delta t \sum_{n=1}^N (\| |\boldsymbol{\beta} \cdot \mathbf{n}|^{\frac{1}{2}} v_h^n \varpi_n \|_{\Gamma}^2 + \gamma |v_h^n|_{s, \varpi_n}^2)) \\ &\leq C \left(\|v_h^0\|_{\varpi_0}^2 + \delta t \sum_{n=1}^N ((\mathcal{L}_\theta^n v_h, \varpi_n^2 v_h^n)_\Omega + \langle |\boldsymbol{\beta} \cdot \mathbf{n}| v_h^n, \varpi_n^2 v_h^n \rangle_{\Gamma_-} \right. \\ & \quad \left. + \gamma s(v_h^n, \varpi_n^2 v_h^n) + CK^{-2} \|v_h^n\|_{\varpi_n}^2 \right). \end{aligned}$$

Proceeding as before we add and subtract $w_h^n := \pi_h(\varpi_n^2 v_h^n)$ in the right slot of the bilinear forms of the right hand side

$$\begin{aligned} & \|v_h^N\|_{\varpi_N}^2 + \sum_{n=1}^N \|v_h^n - v_h^{n-1}\|_{\varpi_n}^2 + \delta t \sum_{n=1}^N \left(\| |\boldsymbol{\beta} \cdot n|^{\frac{1}{2}} v_h^n \varpi_n \|_{\Gamma}^2 + \gamma |v_h^n|_{s, \varpi_n}^2 \right) \\ & \leq C \left(\|v_h^0\|_{\varpi_0}^2 + \delta t \sum_{n=1}^N \left((\mathcal{L}_{\theta}^n v_h, w_h)_{\Omega} + \langle |\boldsymbol{\beta} \cdot n| v_h^n, w_h \rangle_{\Gamma_-} + s(v_h^n, w_h) + \delta t C \|v_h^n\|_{\varpi_n}^2 \right) \right. \\ & \quad \left. + \delta t \sum_{n=1}^N \left((\mathcal{L}_{\theta}^n v_h, \varpi_n^2 v_h^n - w_h)_{\Omega} + \langle |\boldsymbol{\beta} \cdot n| v_h^n, \varpi_n^2 v_h^n - w_h \rangle_{\Gamma_-} + \gamma s(v_h^n, \varpi_n^2 v_h^n - w_h) \right) \right). \end{aligned}$$

Only the term introduced for the weak imposition of boundary conditions differs from the time-continuous analysis. For this term we observe that

$$\left\langle |\boldsymbol{\beta} \cdot n| v_h^n, \varpi_n^2 v_h^n - w_h \right\rangle_{\Gamma_-} \geq -\epsilon \| |\boldsymbol{\beta} \cdot n|^{\frac{1}{2}} v_h^n \varpi_n \|_{\Gamma}^2 - \frac{\beta_{\infty}}{4\epsilon} \|\varpi_n^{-1} (\varpi_n^2 v_h^n - \pi_h \varpi_n^2 v_h^n)\|_{\Gamma_-}^2.$$

For the second term on the right hand side we have the bound

$$\|\varpi_n^{-1} (\varpi_n^2 v_h^n - \pi_h \varpi_n^2 v_h^n)\|_{\Gamma_-}^2 \leq C/K^2 \|v_h^n\|_{\varpi}^2.$$

This follows by applying the trace inequality (2.10), the properties of ϖ and the inequality (5.11). Proceeding as in the time-continuous case we then obtain the bound

$$\begin{aligned} & \|v_h^N\|_{\varpi_N}^2 + \sum_{n=1}^N \|v_h^n - v_h^{n-1}\|_{\varpi_n}^2 + \delta t \sum_{n=1}^N \left(\| |\boldsymbol{\beta} \cdot n|^{\frac{1}{2}} v_h^n \varpi_n \|_{\Gamma}^2 + \gamma |v_h^n|_{s, \varpi_n}^2 \right) \\ & \leq C \left(\|v_h^0\|_{\varpi_0}^2 + \delta t \sum_{n=1}^N \left((\mathcal{L}_{\theta}^n v_h, w_h)_{\Omega} + \langle |\boldsymbol{\beta} \cdot n| v_h^n, w_h \rangle_{\Gamma_-} + \gamma s(v_h^n, w_h) + K^{-2} \|v_h^n\|_{\varpi_n}^2 \right) \right). \end{aligned}$$

Choosing δt sufficiently small the term $\delta t C K^{-2} \|v_h^N\|_{\varpi_N}^2$ in the right hand side can be absorbed in the left hand side and we conclude by an application of the discrete Gronwall's inequality. \square

Remark 6 A consequence of the previous analysis is that the proposed method can be used in the context of problems, where the boundary or initial data is unknown or partially known. Assume for example that g is unknown and replaced by zero. Then, since the effect of the erroneous boundary condition is damped exponentially for non-characteristic directions, the solution can still be approximated with good accuracy in subsets Ω_0 whose domain of dependence is sufficiently far from the boundary. Similarly if the initial data is unknown in some parts of the domain, the solution will still remain accurate in subdomains where the initial data in the domain of dependence is known. This result is a time-dependent analogue to the analysis of [16].

6 Numerical Examples

All numerical examples were produced using the package FreeFEM++ [29]. The method (5.23) is considered with $\theta = 1/2$, corresponding to the second order Crank–Nicolson scheme. This choice was made to minimize the perturbation of the global energy estimate by the time-discretization. The consistent mass matrix is used and exact quadrature is applied to

all the forms. We first consider transport in the disc $\Omega := \{(x, y) \in \mathbb{R}^2 : x^2 + y^2 < 1\}$ under the velocity field $\beta = (y, -x)$. Approximations are computed on a series of unstructured meshes. We set $f = 0$ and consider two different functions u_0 as initial data. One is smooth

$$u_0 = e^{-30((x-0.5)^2+y^2)} \quad (6.1)$$

and one is rough

$$\tilde{u}_0 = \begin{cases} 1, & \sqrt{(x+0.5)^2+y^2} < 0.2, \\ 0 & \text{otherwise.} \end{cases}$$

The velocity field simply turns the disc with the initial data and one full turn is computed so that the final solution should be equal to the initial data. Two numerical experiments are considered where the solution is approximated for the initial data u_0 and $u_0 + \tilde{u}_0$.

We report the global error in the material derivative over the space time domain, the global L^2 -norm of the error at the final time, and in the case where both the rough and the smooth initial data are combined, the error obtained in the smooth part, i.e. the L^2 -norm over $\{(x, y) \in \Omega : x > 0\}$. The discretization parameters for piecewise affine (P_1 below) approximation have been chosen as $dt = \frac{1}{2}h = \pi/nele$, where $nele$ is the number of cell faces on the disc perimeter. For piecewise quadratic (P_2 below) approximation $h = 2\pi/nele$ and $dt = \frac{1}{2}h^{\frac{3}{2}}$, to make the error of the time and space discretization similar. In the left panel of Fig. 2 the smooth and rough initial data, interpolated on a very fine mesh, are presented. In the middle panel the solution after one turn without stabilization and in the right panel the solution after one turn with stabilization for P_1 , on the mesh resolution $nele = 80$ are reported. We see that the sharp layers are smeared on this coarse mesh when the stabilized method is used, but contrary to the unstabilized case the smooth part of the solution is accurately captured.

In Fig. 3 the convergence of stabilized and unstabilized methods with P_1 and P_2 elements are compared for the smooth initial data. We observe that when the solution is globally smooth both methods perform well in the L^2 -norm. Nevertheless, the improvement of the convergence rate for the stabilized method is clearly visible for both approximation spaces, both in the L^2 -error and in the material derivative. The results when part of the solution is rough (initial data from Fig. 2, left plot) are reported in Fig. 4. Note that both methods have similar global error in the L^2 -norm. The stabilized method on the other hand still has optimal convergence in the part where the solution is smooth, in accordance with the theory of Section 5. Its material derivative is also more stable under refinement. The unstabilized method has equally poor convergence in the smooth and in the rough part of the solution.



Fig. 2 From left to right: rough initial data on fine mesh $u_0 + \tilde{u}_0$, unstabilized solution, stabilized solution ($nele = 80$, one turn)

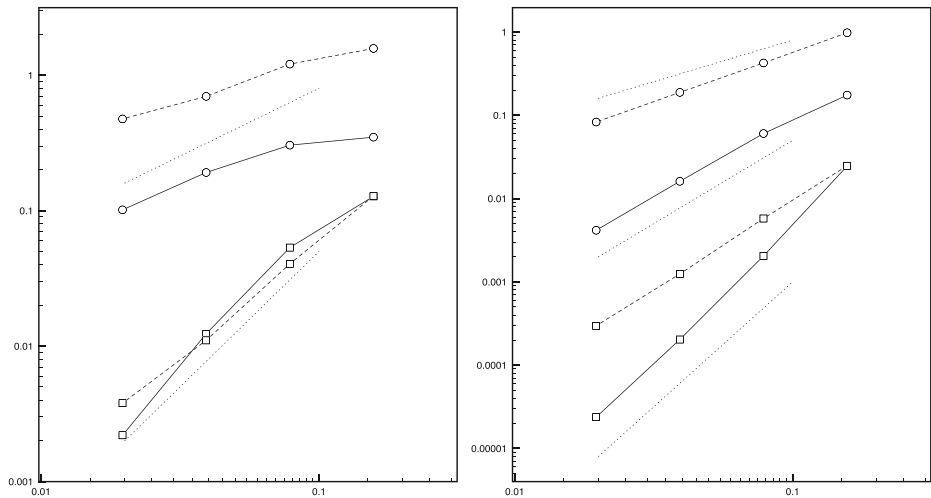


Fig. 3 Comparison of errors plotted against mesh size h for stabilized (full line) and unstabilized (dashed line) methods with P_1 (left) and P_2 (right) approximation. Globally smooth initial data (6.1). The space time error in material derivative has circle markers. The final time global L^2 -error has square markers. The dotted reference lines have slope 1,2 from top to bottom in the left graphic and 1,2,3 from top to bottom in the right graphic

6.1 An Example with Inflow and Outflow and Weakly Imposed Boundary Conditions

Here we consider transport in the unit square with $\beta = (1, 0)^T$. We use a structured mesh with $nele$ cell faces on the side of the square. The initial data consists of a cylinder of

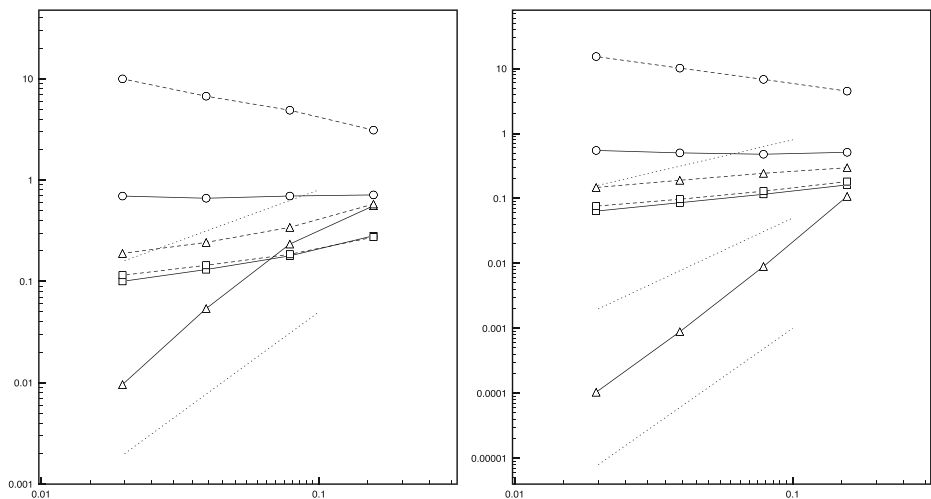


Fig. 4 Comparison of errors plotted against mesh size h for stabilized (full line) and unstabilized (dashed line) methods with P_1 (left) and P_2 (right). Initial data from Fig. 2 (left plot). The space time error in material derivative has circle markers. The final time global L^2 -error has square markers and the final time local L^2 -error has triangle markers. The dotted reference lines have slope 1,2 from top to bottom in the left graphic and 1,2,3 from top to bottom in the right graphic

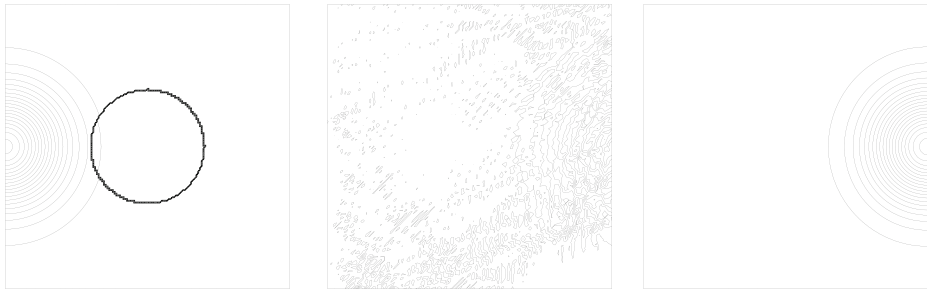


Fig. 5 From left to right: initial data on fine mesh, unstabilized solution, stabilized solution ($nele = 80$, final time $t = 1$)

radius $r = 0.2$ centered in the middle of the square and a Gaussian centered on the left boundary (see Fig. 5, left plot). The exact shapes are the same as those of the previous example. The solution is approximated over the time interval $(0, 1]$ so that the cylinder leaves the domain at $t = 0.7$ and at $t = 1$ the Gaussian is centered on the right boundary. The time dependent inflow boundary condition $u = g$ on Γ_- is imposed weakly as described in (5.23) (g is chosen as the trace of the known exact solution). In Fig. 5, the final time approximation is reported in the middle plot without stabilization and in right plot with stabilization. Observe that from $t = 0.7$ the solution is smooth. Nevertheless the unstabilized Galerkin method fails to produce an accurate approximation of the smooth final time solution. Spurious oscillations from the discontinuity have spread over the whole computational domain and remain also when the rough part of the solution has left. The convergence of the L^2 -error at final times for the stabilized and unstabilized approaches is shown in Fig. 6 ($h = 1/nele$, $nele = 40, 80, 160, 320$). We see that for the stabilized method both

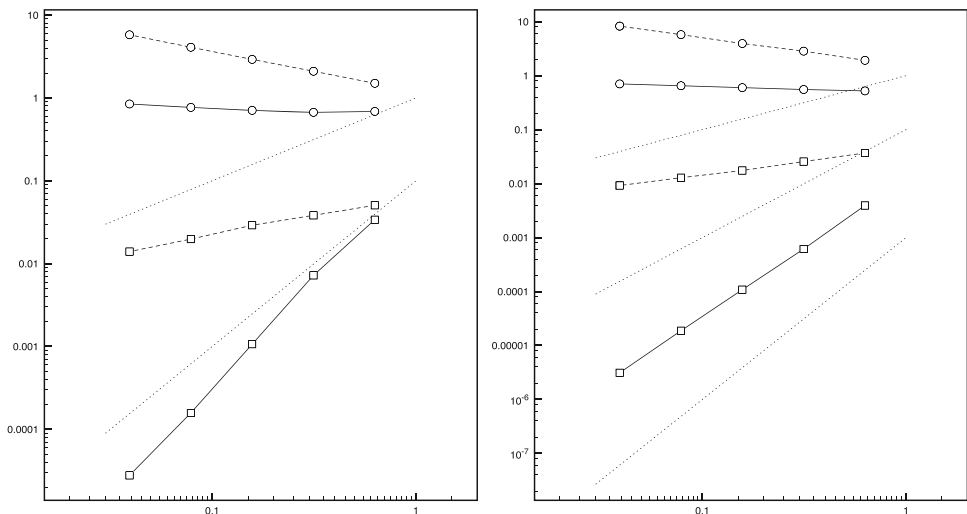


Fig. 6 Comparison of errors plotted against mesh size h for stabilized (full line) and unstabilized (dashed line) methods with P_1 (left) and P_2 (right). Initial data from Fig. 5 (left plot). The space time error in material derivative has circle markers. The final time global L^2 -error has square marker. The dotted reference lines have slope 1,2 from top to bottom in the left graphic and 1,2,3 from top to bottom in the right graphic

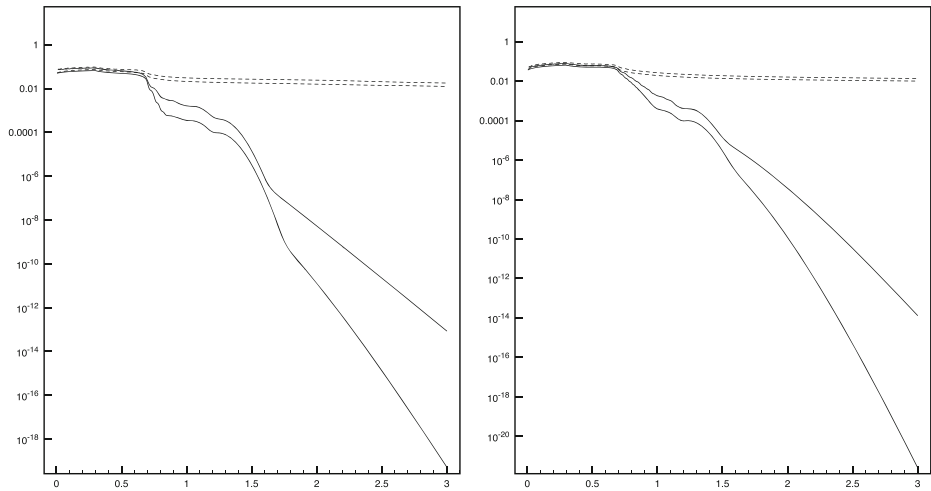


Fig. 7 Comparison of stabilized (full line) and unstabilized (dashed line) methods with P_1 (left) and P_2 (right) approximation. Evolution of the global L^2 -error in time. Initial data from Fig. 5, left graphic. In each case the upper curve has $nele = 40$ and the lower curve $nele = 80$

the P_1 and P_2 approximations have optimal convergence to the smooth solution. The unstabilized method converges approximately as $O(h^{\frac{1}{2}})$ in both cases and its material derivative diverges.

6.2 Long Term Stability

To see the effect of perturbations on the solution for long time we revisit the computational example of the previous section, but extend the time interval to $(0, 3)$. The cylinder leaves the domain at $t = 0.7$ and at the final time the solution is very small. One would then expect the error of the method to go to zero with machine precision, since the solution to approximate is very close to the trivial zero solution. In Fig. 7 the global L^2 -norm is reported, for two consecutive meshes ($nele = 40$ and $nele = 80$) and both the stabilized (full line) and the unstabilized (dashed line) methods. In the stabilized case the improvement of the approximation at $t = 0.7$, when the cylinder leaves the domain, is clearly visible and the solution also improves as the Gaussian is evacuated. We see convergence to zero at machine precision of the error and also convergence under mesh refinement. In the unstabilized case the change at time $t = 0.7$ is barely visible, the error decreases only very slowly in time and not noticeably under mesh refinement. Similarly as in the previous example, we conclude that the standard Galerkin method with weakly imposed boundary conditions in our simulations fails to evacuate the high frequency perturbations produced by the discontinuous initial data on the two meshes considered.

Appendix

Here we give the proofs of the approximation results for the L^2 -projection, Lemmas 2 and 3 and finally the weighted discrete interpolation result (5.4).

First we give a simple super approximation result for the Lagrange interpolant i_h that will be useful for the proofs of inequalities (5.11) and (5.12). For a general discussion of discrete commutator properties we refer to [1].

Lemma 5 *Let $\phi \in W^{k+1,\infty}(\Omega)$ satisfying (5.1) with $K > 1$ and $h < 1$. Then for $h^{\frac{1}{2}}/K$ sufficiently small, there holds for all $v_h \in V_h$, $S \in \mathcal{T}$,*

$$|\phi v_h - i_h(\phi v_h)|_{H^s(S)} \leq Ch^{\frac{1}{2}-s}/K \|\phi v_h\|_S, \quad 0 \leq s \leq 2.$$

Proof By the approximation properties of i_h there holds

$$|\phi v_h - i_h(\phi v_h)|_{H^s(S)} \leq Ch^{k+1-s} \|D^{k+1}(\phi v_h)\|_S. \quad (6.2)$$

Using the product rule and the fact that $D^{k+1}v_h = 0$ since $v_h|_S \in \mathbb{P}_k(S)$, we see that

$$\|D^{k+1}(\phi v_h)\|_S \leq C \sum_{l=1}^{k+1} |\phi|_{W^{l,\infty}(S)} |v_h|_{H^{k+1-l}(S)}.$$

By applying the inverse inequality (2.6) repeatedly the derivatives on v_h can be eliminated at the price of factors of the inverse of h ,

$$h^{k+1-s} \|D^{k+1}(\phi v_h)\|_S \leq Ch^{1-s} \|v_h\|_S \sum_{l=1}^{k+1} h^{l-1} |\phi|_{W^{l,\infty}(S)}. \quad (6.3)$$

Using the bound (5.1) it then follows that

$$\sum_{l=1}^{k+1} h^{l-1} |\phi|_{W^{l,\infty}(S)} \leq C \sum_{l=1}^{k+1} h^{l-1} (Kh^{\frac{1}{2}})^{-l} \|\phi\|_{L^\infty(S)} \leq C (Kh^{\frac{1}{2}})^{-1} \|\phi\|_{L^\infty(S)}. \quad (6.4)$$

Where we used the assumption that $h < 1$ and $K > 1$ in the last inequality. Combining the bounds (6.2), (6.3) and (6.4) it follows that

$$\|\phi v_h - i_h(\phi v_h)\|_{H^s(S)} \leq Ch^{1-s} (Kh^{\frac{1}{2}})^{-1} \|\phi\|_{L^\infty(S)} \|v_h\|_S.$$

The claim now follows by applying (5.8). \square

Proof (Lemma 2) First note that by the construction of ϖ there holds

$$|\nabla \varpi| \leq C(\sqrt{h}K)^{-1} \varpi \leq (C\sqrt{h}/K)h^{-1} \varpi$$

and we see that we may apply (5.5)–(5.7) with $\phi = \varpi$ for $(C\sqrt{h}/K)$ small enough.

Proof of (5.8). To prove (5.8), consider a triangle S , assume that the max value in $\max_{(x,t) \in S \times I_\delta} \varpi(x, t)$ is taken at $(x^*, t^*) \in S \times I_\delta$. Then

$$\begin{aligned} \max_{(x,t) \in S \times I_\delta} \varpi(x, t) \|v\|_S &= \|\varpi(x^*, t^*)v\|_S \leq \|(\varpi(x^*, t^*) - \varpi(\cdot, \tilde{t}))v\|_S + \|\varpi(\cdot, \tilde{t})v\|_S \\ &\leq Ch^{\frac{1}{2}}K^{-1} \varpi(x^*, t^*) \|v\|_S + \|\varpi(\cdot, \tilde{t})v\|_S, \end{aligned}$$

for any $\tilde{t} \in I_\delta$. Assuming that $Ch^{\frac{1}{2}}K^{-1} \leq \frac{1}{2}$ we see that

$$\max_{(x,t) \in S \times I_\delta} \varpi(x, t) \|v\|_S \leq 2\|\varpi(\cdot, \tilde{t})v\|_S, \quad \forall \tilde{t} \in I_\delta.$$

Proof of (5.9). For the proof of (5.9) first apply the stabilities (5.5)–(5.6). For the L^2 -norm this yields

$$\|\varpi(v - \pi_h v)\|_{\Omega} \leq \|\varpi(v - i_h v)\|_{\Omega} + \|\varpi \pi_h(i_h v - v)\|_{\Omega} \leq C \|\varpi(v - i_h v)\|_{\Omega}.$$

Then apply interpolation locally and (5.8).

$$\begin{aligned} \|\varpi(v - i_h v)\|_S &\leq \max_{x \in S} \varpi(x) \|v - i_h v\|_S \leq C \max_{x \in S} \varpi(x) h^{k+1} \|D^{k+1} v\|_S \\ &\leq 2Ch^{k+1} \|\varpi D^{k+1} v\|_S. \end{aligned}$$

The claim follows by summing over $S \in \mathcal{T}$. The bound on the H^1 -norm is identical.

Proof of (5.10). The stabilization operator is defined by the sum of the jumps of the gradient over the faces of the element. The first step is to split that jump using the triangle inequality over each face. Given a face $F = \partial S_1 \cap \partial S_2$ for elements S_1 and S_2 this takes the form.

$$\|\llbracket \nabla(v - \pi_h v) \rrbracket\|_F^2 \leq 2 \left(\|\nabla(v - \pi_h v)\|_{\partial S_1 \cap F}^2 + \|\nabla(v - \pi_h v)\|_{\partial S_2 \cap F}^2 \right).$$

By breaking up the jumps on the contributions from respective element faces in this was we have

$$s_{\varpi}(v - \pi_h v, v - \pi_h v) \leq C \sum_{S \in \mathcal{T}} \left(\max_{x \in S} \varpi(x) \right)^2 h^2 \beta_{\infty} \|\nabla(v - \pi_h v)\|_S^2.$$

Now apply the trace inequality (2.10) on each element to see that

$$\|\nabla(v - \pi_h v)\|_{\partial S} \leq C \left(h^{\frac{1}{2}} |\nabla(v - \pi_h v)|_{H^1(S)} + h^{-\frac{1}{2}} \|\nabla(v - \pi_h v)\|_S \right).$$

For the first term in the right hand side add and subtract $i_h u$, split it using a triangle inequality and use an inverse inequality in one of the terms and interpolation in the other to see that

$$\begin{aligned} |\nabla(v - \pi_h v)|_{H^1(S)} &\leq C \left(|\nabla(v - i_h v)|_{H^1(S)} + |\nabla(i_h v - \pi_h v)|_{H^1(S)} \right) \\ &\leq Ch^{k-1} \|D^{k+1} v\|_S + Ch^{-1} \|\nabla(v - \pi_h v)\|_S. \end{aligned}$$

It follows using (5.8) that

$$\sum_{S \in \mathcal{T}} \varpi(x)^2 h^2 \beta_{\infty} \|\nabla(v - \pi_h v)\|_{\partial S}^2 \leq C \beta_{\infty} h^{2k+1} \|D^{k+1} v\|_{\varpi}^2 + C \beta_{\infty} h \|\nabla(v - \pi_h v)\|_{\varpi}^2.$$

The claim now follows by applying (5.9) to the second term of the right hand side. \square

Proof (Lemma 3)

Proof of (5.11). To prove (5.11) recall that

$$|\nabla \varpi^{-1}| = |\varpi^{-2} \nabla \varpi| \leq C(\sqrt{h}K)^{-1} \varpi^{-1}$$

and we may apply (5.5) with $\phi = \varpi^{-1}$ to get

$$\|\varpi^{-1}(\varpi^2 v_h - \pi_h(\varpi^2 v_h))\|_{\Omega} \leq C \|\varpi^{-1}(\varpi^2 v_h - i_h(\varpi^2 v_h))\|_{\Omega}.$$

Consider one simplex S , take out the weight and then apply Lemma 5 followed by (5.8)

$$\|\varpi^{-1}(\varpi^2 v_h - i_h(\varpi^2 v_h))\|_S \leq \left(\max_{x \in S} \varpi^{-1} \right) \|\varpi^2 v_h - i_h(\varpi^2 v_h)\|_S \leq Ch^{\frac{1}{2}}/K \|\varpi v_h\|_S.$$

Finally take the square of both sides and sum over the simplices. The H^1 -norm estimate follows using similar arguments.

Proof of (5.12). For the inequality (5.12) we consider one element of the sum and apply the trace inequality (2.10),

$$\begin{aligned} \|\varpi^{-1} \nabla(\varpi^2 v_h - \pi_h(\varpi^2 v_h))\|_{\partial S} &\leq C \left(\max_{x \in S} \varpi^{-1} h^{\frac{1}{2}} |\nabla(\varpi^2 v_h - \pi_h(\varpi^2 v_h))|_{H^1(S)} \right. \\ &\quad \left. + \max_{x \in S} \varpi^{-1} h^{-\frac{1}{2}} \|\nabla(\varpi^2 v_h - \pi_h(\varpi^2 v_h))\|_S \right). \end{aligned} \quad (6.5)$$

In the first term, add and subtract $\nabla i_h(\varpi^2 v_h)$ and use the triangle inequality followed by an inverse inequality to obtain

$$\begin{aligned} \max_{x \in S} \varpi^{-1} h^{\frac{1}{2}} |\nabla(\varpi^2 v_h - \pi_h(\varpi^2 v_h))|_{H^1(S)} \\ \leq C \max_{x \in S} \varpi^{-1} h^{\frac{1}{2}} \left(|\nabla(\varpi^2 v_h - i_h(\varpi^2 v_h))|_{H^1(S)} + h^{-1} \|\nabla(i_h \varpi^2 v_h - \pi_h(\varpi^2 v_h))\|_S \right). \end{aligned}$$

For the first term in the right hand side we use Lemma 5, with $s = 2$,

$$\begin{aligned} h^{\frac{1}{2}} \max_{x \in S} \varpi^{-1} |\nabla(\varpi^2 v_h - i_h(\varpi^2 v_h))|_{H^1(S)} &\leq C \max_{x \in S} \varpi^{-1} K^{-1} h^{-1} \|\varpi^2 v_h\|_S \\ &\leq C K^{-1} h^{-1} \|\varpi v_h\|_S. \end{aligned} \quad (6.6)$$

To bound the second term we use (5.8), sum over $S \in \mathcal{T}$ and use the stability of the L^2 -projection (5.6) to get

$$\sum_{S \in \mathcal{T}} \left(\max_{x \in S} \varpi^{-2} \right) h^{-1} \|\nabla(i_h \varpi^2 v_h - \pi_h(\varpi^2 v_h))\|_S^2 \leq C h^{-1} \|\varpi^{-1} \nabla(i_h \varpi^2 v_h - \varpi^2 v_h)\|_\Omega^2.$$

We see that after summation over S the second term in the right hand side of (6.5) also is on this form.

On every S take out the factor $\max_{x \in S} \varpi^{-1}$ and apply Lemma 5 followed by (5.8) to arrive at

$$h^{-\frac{1}{2}} \|\varpi^{-1} \nabla(i_h \varpi^2 v_h - \varpi^2 v_h)\|_\Omega \leq C K^{-1} h^{-1} \|\varpi v_h\|_\Omega$$

which together with (6.6), summed over S , concludes the proof of (5.12). \square

Proof (Inequality (5.4)). For simplicity consider the form $\beta \cdot \nabla u_h = \partial_x u_h$. Using the product rule $\partial_x(\varpi^2 v_h) = (\partial_x \varpi^2) v_h + \varpi^2 \partial_x v_h$ and the triangle inequality it follows that

$$\begin{aligned} \|h^{\frac{1}{2}} (\partial_x(\varpi^2 v_h) - \pi_h(\partial_x(\varpi^2 v_h)))\|_{\varpi^{-1}}^2 &\leq 2h \|(\partial_x \varpi^2) v_h - \pi_h(\partial_x \varpi^2 v_h)\|_{\varpi^{-1}}^2 \\ &\quad + 2h \|(\varpi^2 \partial_x v_h) - \pi_h(\varpi^2 \partial_x v_h)\|_{\varpi^{-1}}^2. \end{aligned} \quad (6.7)$$

Noting that by the L^2 -stability of π_h , the bound of ϖ , Lemma 5, (5.1) and (5.8)

$$h \|(\partial_x \varpi^2) v_h - \pi_h(\partial_x \varpi^2 v_h)\|_{\varpi^{-1}}^2 \leq C h \|(\partial_x \varpi^2) v_h - i_h(\partial_x \varpi^2 v_h)\|_{\varpi^{-1}}^2 \leq C K^{-2} \|v_h\|_\Omega^2.$$

It only remains to bound the second term of (6.7). We add and subtract $\pi_0 \varpi^2$ defined by

$$\pi_0 \varpi^2|_S = |S|^{-1} \int_S \varpi^2$$

and use the triangle inequality to obtain

$$\begin{aligned} h \|(\varpi^2 \partial_x v_h) - \pi_h(\varpi^2 \partial_x v_h)\|_{\varpi^{-1}}^2 &\leq Ch \|(\varpi^2 \partial_x v_h - (\pi_0 \varpi^2) \partial_x v_h)\|_{\varpi^{-1}}^2 \\ &\quad + Ch \|((\pi_0 \varpi^2) \partial_x v_h - \pi_h((\pi_0 \varpi^2) \partial_x v_h))\|_{\varpi^{-1}}^2 \\ &\quad + Ch \|(\pi_h((\pi_0 \varpi^2) \partial_x v_h) - \pi_h(\varpi^2 \partial_x v_h))\|_{\varpi^{-1}}^2 \\ &= T_1 + T_2 + T_3. \end{aligned}$$

First, for T_3 , observe that by the stability of the L^2 -projection (5.5) we have

$$h \|(\pi_h((\pi_0 \varpi^2) \partial_x v_h) - \pi_h(\varpi^2 \partial_x v_h))\|_{\varpi^{-1}}^2 \leq Ch \|(\varpi^2 \partial_x v_h - (\pi_0 \varpi^2) \partial_x v_h)\|_{\varpi^{-1}}^2 \leq CT_1, \quad (6.8)$$

so only T_1 and T_2 need to be bounded. For T_1 , by the approximation $\|\varpi^2 - \pi_0 \varpi^2\|_{L^\infty(S)} \leq Ch^{\frac{1}{2}}/K \|\varpi\|_{L^\infty(S)}^2$ and applying (5.8) repeatedly, we have for one simplex S ,

$$\begin{aligned} \|\varpi^{-1}(\varpi^2 \partial_x v_h - (\pi_0 \varpi^2) \partial_x v_h)\|_S &\leq h^{\frac{1}{2}}/K \max_{x \in S} \varpi^2 \max_{x \in S} \varpi^{-1} \|\partial_x v_h\|_S \\ &\leq Ch^{-\frac{1}{2}} K^{-1} \|\varpi v_h\|_S. \end{aligned}$$

Taking the square of both sides and summing over all simplices yields the bound for T_1 ,

$$h \|(\varpi^2 \partial_x v_h - (\pi_0 \varpi^2) \partial_x v_h)\|_{\varpi^{-1}}^2 \leq CK^{-2} \|v_h\|_{\varpi}^2.$$

Finally for the term T_2 we use (5.3) with $\beta_0 = (\pi_0 \varpi^2) e_x$. This leads to

$$h \|((\pi_0 \varpi^2) \partial_x v_h - \pi_h((\pi_0 \varpi^2) \partial_x v_h))\|_{\varpi^{-1}}^2 \leq C_{ws} s_{\varpi^{-1}} \left((\pi_0 \varpi^2) v_h, (\pi_0 \varpi^2) v_h \right).$$

Adding and subtracting ϖ^2 and using the triangle inequality and the fact that ϖ^2 is smooth leads to

$$s_{\varpi^{-1}} \left((\pi_0 \varpi^2) v_h, (\pi_0 \varpi^2) v_h \right) \leq 2s_{\varpi} (v_h, v_h) + 2s_{\varpi^{-1}} \left((\varpi^2 - \pi_0 \varpi^2) v_h, (\varpi^2 - \pi_0 \varpi^2) v_h \right).$$

For the second term of the right hand side consider the boundary of one triangle and apply the trace inequality (2.10), followed by the approximation of π_0 to get

$$\begin{aligned} \|h(\varpi^2 - \pi_0 \varpi^2) \nabla v_h\|_{\partial S} &\leq C \max_{x \in S} \varpi^2 K^{-1} h^{\frac{3}{2}} \left(h^{\frac{1}{2}} |\nabla v_h|_{H^1(S)} + h^{-\frac{1}{2}} \|\nabla v_h\|_S \right) \\ &\leq CK^{-1} \|\varpi^2 v_h\|_S. \end{aligned}$$

The last step followed using the inverse inequality (2.6) and (5.8). Proceeding by applying the previous bound to all triangle faces, it follows that

$$\begin{aligned} s_{\varpi^{-1}} ((\varpi^2 - \pi_0 \varpi^2) v_h, (\varpi^2 - \pi_0 \varpi^2) v_h) &\leq C \sum_{S \in \mathcal{T}} \max_{x \in S} \varpi^{-2} \|h(\varpi^2 - \pi_0 \varpi^2) \nabla v_h\|_{\partial S}^2 \quad (6.9) \\ &\leq C \sum_{S \in \mathcal{T}} \max_{x \in S} \varpi^{-2} K^{-2} \|\varpi^2 v_h\|_S^2 \leq CK^{-2} \|v_h\|_{\varpi}^2, \end{aligned}$$

where the last step follows using (5.8). The proof is now finished by collecting the bounds (6.8)–(6.9). \square

Proof (Lemma 4). Using $\delta t \leq Ch$ and (5.1) there holds

$$\begin{aligned} \left\| v_h \int_{t_{n-1}}^{t_n} \partial_t \varpi \, dt \right\|_{\Omega} &\leq C \delta t / (Kh^{\frac{1}{2}}) \left(\sum_{S \in \mathcal{T}} \max_{(x,t) \in S \times [t_{n-1}, t_n]} \varpi(x, t)^2 \|v_h\|_S^2 \right)^{\frac{1}{2}} \\ &\leq \delta t^{\frac{1}{2}} C / K \|v_h\|_{\varpi_n}. \end{aligned}$$

For the second inequality we applied (5.8) elementwise and then upper bounded $\min_{t \in [t_{n-1}, t_n]} \|v_h \varpi(\cdot, t)\|_S$ by $\|v_h\|_{\varpi_n}$. For the bound of the second term observe that, estimating

$$\left| \int_{t_{n-1}}^{t_n} \int_t^{t_n} \partial_{tt} \varpi^2 \, ds \, dt \right| \leq \delta t^2 \max_{t \in [t_{n-1}, t_n]} |\partial_{tt} \varpi^2|$$

and then applying (5.1) repeatedly with $l = 1$ and 2, to show

$$\max_{t \in [t_{n-1}, t_n]} |\partial_{tt} \varpi^2| \leq C^2 h^{-1} K^{-2} \max_{t \in [t_{n-1}, t_n]} \varpi^2.$$

It follows that for all $S \in \mathcal{T}$,

$$\left\| v_h \left| \int_{t_{n-1}}^{t_n} \int_t^{t_n} \partial_{tt} \varpi^2 \, ds \, dt \right|^{\frac{1}{2}} \right\|_S \leq \delta t^{\frac{1}{2}} C K^{-1} \max_{(x,t) \in S \times [t_{n-1}, t_n]} \varpi \|v_h\|_S.$$

Applying (5.8) we conclude that

$$\begin{aligned} \sum_{S \in \mathcal{T}} \left\| v_h \left| \int_{t_{n-1}}^{t_n} \int_t^{t_n} \partial_{tt} \varpi^2 \, ds \, dt \right|^{\frac{1}{2}} \right\|_S^2 &\leq \delta t C^2 K^{-2} \sum_{S \in \mathcal{T}} \min_{t \in [t_{n-1}, t_n]} \|v_h \varpi(\cdot, t)\|_S^2 \\ &\leq \delta t C^2 K^{-2} \|v_h\|_{\varpi_n}^2. \end{aligned}$$

□

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Availability of data and material The data used to produce figures can be made available upon reasonable request.

Code Availability Codes used to produce approximate solutions can be made available upon reasonable request.

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