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ORIGINAL ARTICLE

Boundary Controllability and Asymptotic Stabilization of a Nonlocal Traffic Flow Model

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Abstract
We study the exact boundary controllability of a class of nonlocal conservation laws modeling traffic flow. The velocity of the macroscopic dynamics depends on a weighted average of the traffic density ahead and the averaging kernel is of exponential type. Under specific assumptions, we show that the boundary controls can be used to steer the system towards a target final state or out-flux. The regularizing effect of the nonlocal term, which leads to the uniqueness of weak solutions, enables us to prove that the exact controllability is equivalent to the existence of weak solutions to the backwards-in-time problem. We also study steady states and the long-time behavior of the solution under specific boundary conditions.

Keywords Conservation laws · Nonlocal flux · Traffic flow · Exact controllability · Boundary controllability · Stabilization · Characteristics

Mathematics Subject Classification (2010) 35L65 · 35L02 · 35L04 · 35L60 · 93C20 · 93B05

1 Introduction

Conservation laws with nonlocal fluxes arise in many application areas and have thus attracted much attention in recent years: e.g., traffic flow [23–26, 50, 61], crowd dynamics [2, 34–36], sedimentation phenomena [13], slow erosion of granular matter [4, 31, 32], materials with fading memory effects [19], and biological and industrial models [37]. However, few papers have dealt with the controllability and long-time behaviour of solutions of nonlocal conservation laws. In [38, 44], some results on state and out-flux controllability and asymptotic exponential stabilization have been obtained for a very specific nonlocal model describing manufacturing systems, where the velocity is strictly positive and the nonlocal term is independent of the spatial domain. Moreover, for nonlocal dynamics with a

Dedicated to Enrique Zuazua on the occasion of his 60th birthday.

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backwards-looking nonlocal term, the authors of [61] studied the well-posedness and stability of classical solutions on a ring road, i.e., with periodic boundary datum, using a finite difference approximation of solutions. They also succeeded in showing the exponential stability of the solution to the steady state (constant) solution and provided a counterexample that the claimed stability cannot be expected for constant nonlocal kernels. This is due to the existence of traveling wave solutions and heavily depends on the kernel used (compare also [58]). Indeed, for periodic solutions, the authors of [58] were able to demonstrate exponential stability to the steady state solution for every monotone kernel and a linear velocity function.

In this contribution, we investigate the controllability of a class of nonlocal conservation laws with an explicitly space-dependent nonlocal term (on a one-dimensional bounded domain) modeling traffic flow: the velocity of the density at a given space-time point depends on a weighted average of the traffic density ahead (see Fig. 1). This work is motivated by the question of whether it is possible to steer the traffic state on a road towards a target end-state or to reach a given out-flux.

The model is similar to the problems considered in [58, 61]. However, rather than focusing on periodic solutions, we consider a control at the entrance point of the road in terms of density and at the exit point in terms of velocity (realized by an appropriate definition of the nonlocal term). This precise model was introduced in [64] and has already been studied for its analytical properties, such as existence and uniqueness of weak solutions, and maximum principle. In [22, 48], the authors considered a similar nonlocal conservation law with boundary values and showed existence and uniqueness of solutions, however the right-hand side boundary datum is different from the datum in this work, which is realized in the nonlocal term. In comparison, the considered model has the advantage that the velocity of the dynamics remains Lipschitz-continuous as a function in space. Thus, it can be considered the "natural way" of prescribing boundary conditions downstream for nonlocal conservation laws with a sign restricted velocity.

It is described by the following initial-boundary value problem.

Definition 1.1 (The nonlocal dynamics) We consider the following initial-boundary value problem:

$$\partial_t \rho(t,x) + \partial_x(V(\mathcal{W}[\rho](t,x))\rho(t,x)) = 0, \qquad (t,x) \in \Omega_T, \quad (1.1)$$

$$\rho(0,x) = \rho_0(x), \qquad x \in (0,1), \quad (1.2)$$

$$V(\mathcal{W}[\rho](t,0))\rho(t,0) = V(\mathcal{W}[\rho](t,0))u_\ell(t), \quad t \in (0,T) \quad (1.3)$$

with $\Omega_T := (0,T) \times (0,1)$, supplemented by the nonlocal operator

$$\mathcal{W}[\rho](t,x) := \frac{1}{\eta}\int_x^\infty \mathrm{e}^{-\frac{y-x}{\eta}} \begin{pmatrix} \rho(t,y) & \text{if } y < 1 \\ u_r(t) & \text{if } y \geq 1 \end{pmatrix} \mathrm{d}y, \quad (1.4)$$

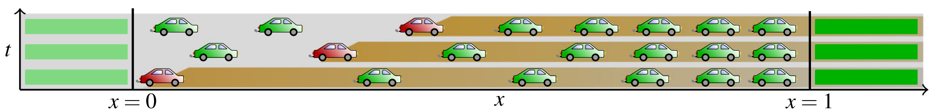

Fig. 1 The nonlocal impact (in gold) in traffic flow modeling. The red car looks ahead within the golden region and adjusts its velocity in response to a high car density far away. Inflow and the speed of the cars leaving the road segment are located at $x = 0$ and $x = 1$, respectively. The green areas to the left and right of the road segment indicate the respective boundary data, u_ℓ and u_r

for $(t,x) \in \Omega_T$ and $\rho : \bar{\Omega}_T \to [0,1]$. Here, $\rho_0 : (0,1) \to [0,1]$ is the initial datum, $u_\ell : (0,T) \to [0,1]$ is the (entering) boundary datum at $x = 0$; $u_r : (0,T) \to [0,1]$ is the nonlocal impact on the right-hand side; $V : [0,1] \to \mathbb{R}_{\geq 0}$ is the velocity; $\eta \in \mathbb{R}_{>0}$ is the nonlocal average parameter; and the exponential function in the nonlocal operator is the weight of the nonlocal term.

At the left-hand side of the road, the entry point, we prescribe an in-flux boundary condition that can be interpreted as an on-ramp of a road and can be used as on-ramp metering to control the dynamics. The function u_r in the nonlocal operator $\mathcal{W}$ in (1.4) can be interpreted as a parameter influencing the velocity with which the right-hand side boundary datum leaves at $x = 1$. As a result of the nonlocal term, it also influences the velocity on the entire link. The need to define u_r outside of the considered domain Ω_T stems from the fact that the nonlocal term, which is dependent on the nonlocal weight, requires input from $(1,\infty)$. This nonlocal term can be used to model traffic lights: if $u_r = 1$, no density leaves (red light); if $u_r = 0$, the adjacent road is fully evacuated and density can leave as fast as possible (green light). Thus, for any control purpose, both the left-hand boundary datum u_ℓ and the right-hand term u_r can be used (compare [9] for the corresponding local dynamics on bounded domains).

The choice of an averaging kernel of exponential type is motivated by the analysis in [17], where the authors showed that, under specific additional assumptions on input datum and velocity V, the solution of the Cauchy problem converges to the entropy solution of the corresponding conservation law as the nonlocal impact η approaches zero. This was refined in [16] where the authors showed that, as long as the nonlocal solutions converges strongly in L^1_{loc}, the limit is entropy admissible. Finally, [29] recently proved the convergence nonlocal to local for exponential kernel and general velocity function and initial datum. In traffic models the exponential kernel is not really standard. A kernel with compact support is usually selected as cars look ahead within a finite space horizon. However, this kind of behavior is nicely approximated by an exponentially decaying kernel. The results shown here should also hold true for more general kernels (compare [63]), though we will not detail this in the present contribution (see also Item 1 in Section 7).

The choice of an exponential kernel also enables the boundary condition prescribed on the flux in (1.3) to be be given directly in terms of density, i.e. as

$$\rho(t,0) = u_\ell(t), \quad t \in [0,T], \tag{1.5}$$

provided $\min\{\|\rho_0\|_{L^1((0,1))}, u_r(t)\} < 1$ for all $t \in [0,T]$. Indeed, in this case, the velocity V is never zero at the boundary (or anywhere else), so that the boundary datum always enters the domain and is thus always attained.

1.1 Literature on the Control of Conservation Laws

Previous boundary control results for nonlocal conservation laws have predominantly focused on a simpler version of the equation presented in Definition 1.1, which was introduced in [8] to model semiconductor manufacturing systems:

$$\begin{aligned} \partial_t \rho(t,x) + \partial_x(\rho(t,x)V(W[\rho](t))) &= 0, && (t,x) \in \Omega_T, \\ \rho(0,x) &= \rho_0(x), && x \in (0,1), \\ V(W(t))\rho(t,0) &= u(t), && t \in (0,T), \end{aligned} \tag{1.6}$$

with a strictly positive velocity function $V \in C^1(\mathbb{R}; \mathbb{R}_{>0})$, namely

$$V(s) = \frac{1}{1+s}, \quad s \geq 0, \qquad \text{and} \qquad W[\rho](t) = \int_0^1 \rho(t,x)\mathrm{d}x.$$

Difficulties to the proposed model in Definition 1.1 arise in comparison to the latter model due to the fact that the nonlocal term also depends on the spatial component, i.e. the spatial dependency is not integrated out as in (1.6). In [21, 38, 43–45, 76], for the open-loop system, the authors studied an optimal control problem, state controllability and out-flux controllability; and for the closed-loop system, they use a Lyapunov function approach to prove some exponential stabilization results. These results were generalized in [27], where (local) state controllability and out-flux controllability results were established for a space-dependent velocity (but still space-independent nonlocal term) $V(x, W[\rho](t))$. In [58, 61] the authors did consider the dynamics in Definition 1.1, though they focused on a ring road which renders the boundary datum in (1.3) and u_r as part of the nonlocal operator in (1.4) unnecessary. They showed that stability cannot be expected for specific kernels and that exponential stability to the constant steady state solution holds for monotonically decreasing kernel and linear velocity function (see also Section 1).

The optimal control problem for an analogous system of scalar nonlocal conservation laws on networks that model a highly re-entrant multi-commodity manufacturing system was analyzed in [54]. Moreover, a more general conservation law with explicitly space dependent nonlocal flow that describes a supply chain model and the description of pedestrian flows was considered in [33].

Questions related to the control of local scalar conservation laws and systems have received considerably more attention in the past few decades, resulting in a very large body of literature on this topic. For an overview of controllability results for hyperbolic conservation laws in the case of solutions without shocks, we refer the reader to [40, 55, 65] and references therein; and to [1, 5–7, 69] in the case of solutions developing shocks. For asymptotic stabilization of hyperbolic systems, see [39, 41, 42, 68, 73, 78, 79]. Nodal profile controllability for quasi-linear hyperbolic systems has also been considered in [52, 53, 66, 67].

1.2 Outline of the Paper

This paper is organized as follows. In Section 2, we recall some preliminary results on well-posedness for the class of nonlocal conservation laws introduced in Definition 1.1.

In Section 3, we prove that any end state can be reached from appropriately defined initial and boundary datum on a sufficiently small time horizon. This phenomenon is exclusively "nonlocal": for local conservation laws, the reachable targets are characterized by the Oleinik entropy condition [3, 71]. We also provide some numerical examples to illustrate this result.

In Section 4, we discuss exact controllability to a given end-state or out-flux of the nonlocal model with boundary controls on the left (in-flux) and on the right (out-flux) of the domain. We prove that this is equivalent to the existence of a solution of the corresponding backwards-in-time nonlocal conservation law. This is related to the fact that weak solutions of the nonlocal conservation law are unique so that we have no loss of information over time.

Section 5 centers on the long-time behavior of solutions when constant boundary conditions are prescribed and the initial condition is suitably chosen. We show that the solution converges to the corresponding constant steady state. Some numerical simulations verify our results and suggest that they should hold for every initial datum.

In Section 6, we state existence and uniqueness of steady state solutions for constant u_ℓ and u_r.

Finally, in Section 7, we conclude this contribution and present some open problems.

2 Preliminaries and Basic Results

We first recall some well-known results on the existence and uniqueness of solutions to the initial-boundary value problem described in Definition 1.1. To this end, we introduce the following (regularity) assumptions.

Assumption 1 (Assumption on the data of Definition 1.1) *For $T \in \mathbb{R}_{>0}$ we assume*

Nonlocal parameter: $\eta \in \mathbb{R}_{>0}$;
Initial datum: $\rho_0 \in L^\infty((0,1);[0,1])$;
Velocity: $V \in W^{1,\infty}((0,1);\mathbb{R}_{\geq 0}) : \ V' \leqq 0,\ V' \not\equiv 0,\ (V(s)=0 \iff s=1)$;
Boundary: $(u_\ell, u_r) \in L^\infty((0,T);[0,1])^2$.

Following [64, Definition 2.4], we provide the definition of solutions:

Definition 2.1 (Weak solutions) We call $\rho \in C\big([0,T];L^1((0,1))\big) \cap L^\infty((0,T);L^\infty((0,1)))$ a *weak solution* to the initial-boundary value problem introduced in Definition 1.1 if for every $\varphi \in W^{1,\infty}((0,T)\times(0,1))$ with $\varphi(T,\cdot)=0$ and $\varphi(\cdot,1)=0$, we have

$$
\begin{aligned}
0 = &\iint_{\Omega_T} \left(\partial_t \varphi(t,x) + V(\mathcal{W}[\rho])(t,x)\partial_x\varphi(t,x)\right)\rho(t,x)\mathrm{d}x\mathrm{d}t \\
&+\int_0^1 \rho_0(x)\varphi(0,x)\mathrm{d}x + \int_0^T \varphi(t,0)V(\mathcal{W}[\rho])(t,0)u_\ell(t)\mathrm{d}t.
\end{aligned}
\tag{2.1}
$$

Existence and uniqueness of weak solutions were investigated in [64]. We recall the principal well-posedness result in the following theorem.

Theorem 2.1 (Existence, uniqueness, maximum principle) *Given Assumption 1, the nonlocal initial-boundary value problem introduced in Definition 1.1 admits a unique weak solution $\rho \in C\big([0,T];L^1((0,1))\big) \cap L^\infty((0,T);L^\infty((0,1)))$ in the sense of Definition 2.1. Moreover, the solution can be stated in terms of characteristics for $(t,x)\in\Omega_T$ as*

$$
\rho(t,x) = \begin{cases} \rho_0(\xi_{w^*}(t,x;0))\partial_2\xi_{w^*}(t,x;0), & x \geq \xi_{w^*}(0,0;t), \\ u(\xi_{w^*}[t,x]^{-1}_{\max}(0))\partial_2\xi_{w^*}(t,x;\xi_{w^*}[t,x]^{-1}_{\max}(0)), & x \leq \xi_{w^*}(0,0;t), \end{cases}
\tag{2.2}
$$

where $\xi : [0,T]\times[0,1]\times[0,T] \to \mathbb{R}_{\geq 0}$ is the characteristic curve that satisfies

$$
\xi_{w^*}(t,x;\tau) = x + \int_t^\tau V(w^*(s,\xi_{w^*}(t,x;s)))\mathrm{d}s, \quad (t,x,\tau)\in\overline{\Omega_T}\times[0,T],
\tag{2.3}
$$

$\xi^{-1}_{\max}[t,x]$ denotes the time-inverted characteristics tracing back the points $(t,x)\in\{(t,x)\in\Omega_T : x \leq \xi_{\tilde{w}}(0,0;t)\}$ to the boundary ([64, Definition 2.5, Equation (2.3)]*) and $w^* \in L^\infty\big((0,T);W^{1,\infty}((0,1))\big)$ is the unique solution of a fixed-point equation on $(t,x)\in\Omega_T$ given in* [64, Theorem 3.1, Eq. (3.2)]. *In addition, the following maximum principle holds for a.e. $(t,x)\in\Omega_T$*

$$
0 \leq \rho(t,x) \leq \max\{\|\rho_0\|_{L^\infty((0,1))}, \|u_\ell\|_{L^\infty((0,t))}, \|u_r\|_{L^\infty((0,t))}\}.
\tag{2.4}
$$

Proof For a compactly supported nonlocal kernel that is monotonically decreasing, the proof can be found in [64, Theorem 3.1, Theorem 4.2, Corollary 5.9]. The exponential kernel considered in Definition 1.1 actually simplifies the analysis and the proof can be obtained analogously. We omit the details. □

It should be noted that, in contrast to local conservation laws (see [15, 47, 57]), nonlocal models of this kind do not require an entropy condition to select a unique solution. Having stated these fundamental results, we now study the controllability properties of the nonlocal weak solutions.

3 Reachability for Sufficiently Small Times

In this section, we show that, for any given function in $L^\infty((0,1);[0,1])$, we can select suitable boundary and initial datum so that the solution of the corresponding nonlocal conservation law reaches the target at a (sufficiently small) time $T > 0$. The key idea behind the proof is to consider the backward-in-time problem, whose solvability is equivalent to the controllability of the given forward problem. Owing to the results presented in [62], the backward problem is solvable for any terminal data for a sufficiently small time horizon. This is because the nonlocal velocity function is Lipschitz-continuous for small time (independent of the specific nonlocal weight and area of integration provided the initial datum is essentially bounded).

This result differs from that of local conservation laws, where the attainable set necessarily needs to satisfy an Oleinik inequality [3, 71], also for arbitrary small time. This prevents downward jumps in the dynamics (in the case where the flux function is strictly concave) and thus postulates a rarefaction wave that becomes less steep over time.

Theorem 3.1 (Exact controllability on small time horizon) *For every $\rho_{des} \in L^\infty((0,1);[0,1))$ with $\|\rho_{des}\|_{L^\infty((0,1))} < 1$, there exists a time $T \in \mathbb{R}_{>0}$, controls $u_\ell, u_r \in L^\infty((0,T);[0,1))$, and initial datum $\rho_0 \in L^\infty((0,1);[0,1))$ such that the corresponding weak solution*

$$\rho \in C\Big([0,T];L^1((0,1))\Big) \cap L^\infty\big((0,T);L^\infty((0,1))\big)$$

to the conservation law in Definition 1.1 satisfies

$$\rho(T,\cdot) \equiv \rho_{des}.$$

Here, ρ_{des} stands for the ***des****ired state one wants to achieve.*

Proof For $u_r \equiv c$ with $c \in [0,1)$, as shown in [62, Theorem 2.20], there exists a sufficiently small time-horizon $T \in \mathbb{R}_{>0}$ such that the auxiliary end-value problem

$$\begin{aligned} \partial_t p(t,x) + \partial_x\left(V(\mathcal{W}[p](t,x))p(t,x)\right) &= 0, && (t,x) \in (0,T)\times\mathbb{R},\\ p(T,x) &= \rho_{\text{des}}(x), && x \in (0,1),\\ p(T,x) &= c, && x \in \mathbb{R}\setminus(0,1), \end{aligned} \tag{3.1}$$

with

$$\mathcal{W}[p](t,x) := \frac{1}{\eta}\int_x^\infty \exp\left(\frac{x-y}{\eta}\right) p(t,y)\,\mathrm{d}y, \tag{3.2}$$

admits a unique solution $p \in C\left([0, T]; L^1_{\text{loc}}(\mathbb{R})\right)$. Moreover, by [62, Lemma 2.6, Item 2], there exists $d \in \mathbb{R}_{\geq 0}$ (depending on η, ρ_{des}, c and V) such that

$$\|p(t, \cdot)\|_{L^\infty(\mathbb{R})} \leq \max\{\|\rho_{\text{des}}\|_{L^\infty((0,1))}, c\}\mathrm{e}^{d(T-t)}.$$

The key idea of interpreting the control problem as a Cauchy problem on $\mathbb{R}$ backwards in time is illustrated in Fig. 2. Thus, for

$$T \leq \frac{1}{d} \log\left(\max\{\|\rho_{\text{des}}\|_{L^\infty((0,1))}, c\}^{-1}\right),$$

we obtain $\|p(t, \cdot)\|_{L^\infty(\mathbb{R})} \leq 1$ for all $t \in [0, T]$. Consequently, by choosing

$$\begin{aligned} u_\ell(t) &= p(t, 0), & t &\in (0, T), \\ u_r(t) &= c, & t &\in (0, T), \\ \rho_0(x) &= p(0, x), & x &\in (0, 1), \end{aligned}$$

the boundary and initial data are admissible and the solution to the corresponding IVP (1.1) satisfies $\rho(T, \cdot) \equiv \rho_{\text{des}}$ on $(0, 1)$. Note that u_ℓ is given by $p(\cdot, 0)$, which can be evaluated as an L^1 function at $x = 0$ as the backwards "velocity" is not zero. □

Remark 3.1 (Surjectivity of state to control map over small times) We remark that the statement in Theorem 3.1 amounts to

$$\bigcup_{t \in (0,T]} \bigcup_{\substack{u_\ell \in L^\infty((0,T);[0,1)) \\ u_r \in L^\infty((0,T);[0,1)) \\ \rho_0 \in L^\infty((0,1);[0,1))}} \rho[\rho_0, u_\ell, u_r](t, \cdot) = L^\infty((0, 1); [0, 1)),$$

where $\rho[\rho_0, u_\ell, u_r] \in C\left([0, T]; L^1((0, 1))\right) \cap L^\infty((0, T); L^\infty((0, 1)))$ denotes the weak solution of the nonlocal conservation law in Definition 1.1, with initial datum ρ_0, left-hand side boundary datum u_ℓ, and nonlocal right-hand side u_r. It thus states the surjectivity of the control to state map when uniting over sufficiently small times. Note that this differs from the local dynamics, where for strictly concave flux—due to Oleinik's entropy condition [3, 71]— downward jumps are "smoothed" over time.

Example 3.1 (Num. examp. for exact controllability on a sufficiently small time horizon) We consider a target function

$$\rho_{\text{des}} :\equiv \frac{1}{2} + \frac{1}{4}\chi_{\left(\frac{1}{4}, \frac{1}{2}\right)} - \frac{1}{4}\chi_{\left(\frac{1}{2}, \frac{3}{4}\right)}.$$

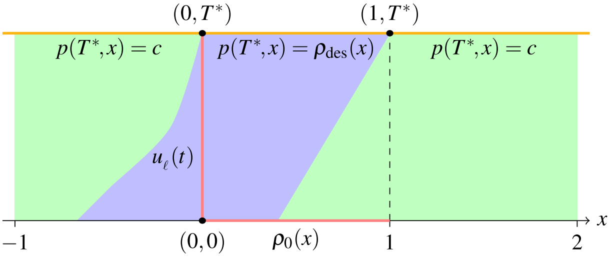

Fig. 2 Transformation of the end boundary value problem into a backward in time Cauchy problem on $\mathbb{R}$. The given desired state and the "boundary" data are in gold, the corresponding datum ρ_0 and u_ℓ are in red, yielding—forward in time—the desired state ρ_{des}

We verify numerically that we can find suitable initial ρ_0 and boundary data u_ℓ, u_r such that $\rho(T,\cdot) \equiv \rho_{\text{des}}$ for the sufficiently small time horizon $T = 0.6$ (see Fig. 3). We note that, for local conservation laws, Oleinik's entropy condition would prevent the reachability of this state. The important role of the nonlocal parameter $\eta \in \mathbb{R}_{>0}$ can also be observed. The smaller the η (here $\eta \in \{1, 0.9, 0.8\}$) in the given example, the more the solution increases backwards over time. This is illustrated in the first three rows of Fig. 3 and in particular in the boundary datum, so that for $\eta = 0.8$ the backward solution has already exceeded 1

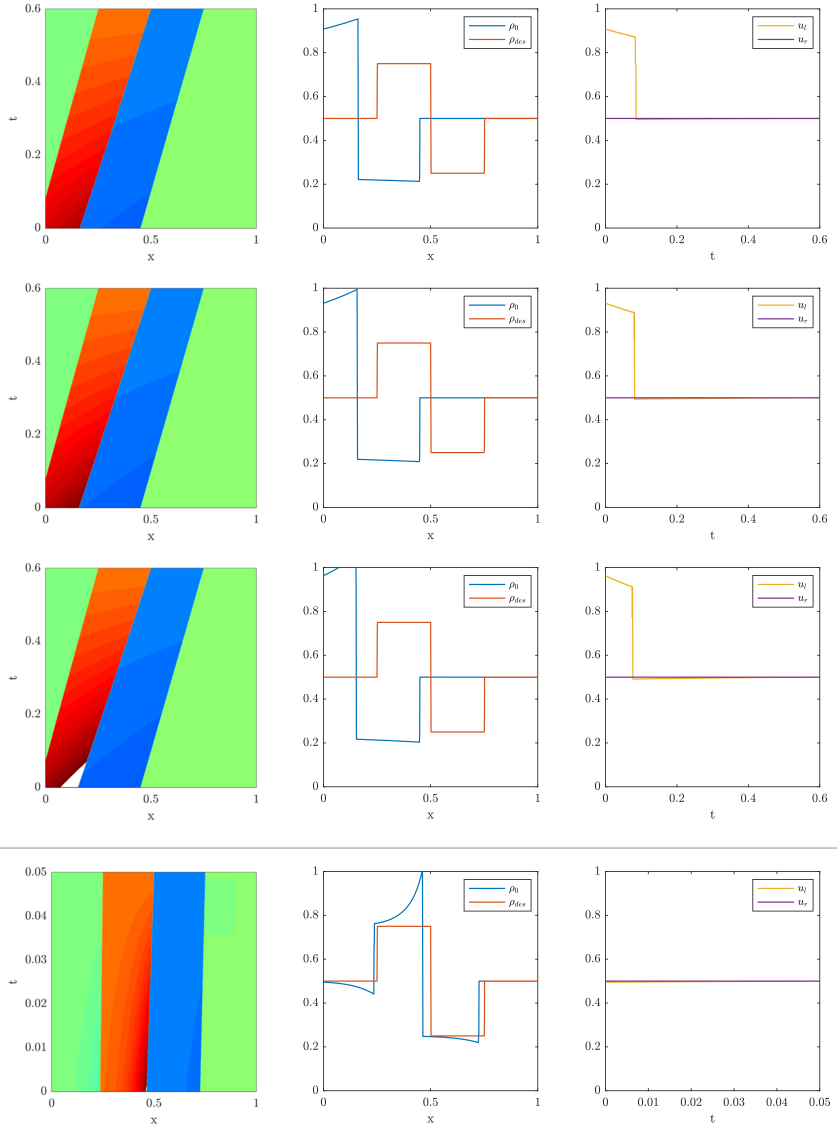

Fig. 3 Illustration of Example 3.1 for different $\eta \in \{0.8, 0.9, 1, 0.1\}$. **Left:** The solution with the proper boundary and initial datum to reach the desired state $\rho_{\text{des}} \equiv \frac{1}{2} + \frac{1}{4}\chi_{(0.25,0.5)} - \frac{1}{4}\chi_{(0.5,0.75)}$ for $T = 0.6$. **Middle:** Desired state ρ_{des} and the corresponding initial state ρ_0 to steer the system to ρ_{des}. **Right:** Boundary data, i.e. u_ℓ, u_r, to steer the system to ρ_{des}. **Color bar:** 0 1

and is thus not admissible for $T = 0.6$. The fourth row in Fig. 3 represents the solution and control for a sufficiently smaller $\eta = 0.1$. Here, the final time needs to be chosen to be much smaller, $T = 0.05$, and even then the backwards solution reaches the bound 1 and would cease to exist if we were to consider it on a sufficiently larger time horizon. Due to the short time horizon in the fourth row, the significant changes in the desired datum ρ_{des} are tackled mainly through the initial datum, and the boundary datum is almost constant. As a result of the short time, the relation between the desired state and the initial state can also be seen in the middle pictures. Here the initial state is shifted slightly in front of the desired state but also needs to compensate for the nonlocal term, necessitating the peaks at the jump discontinuities.

4 Exact Boundary Controllability and Time-inverted Dynamics

In this section, we consider two control problems:

- steering a given initial state towards a prescribed target state;
- achieving a prescribed out-flux on the right-hand side of the road.

In both cases, we show that exact controllability holds if and only if the corresponding backwards-in-time dynamics admit a weak solution and satisfy some bounds. This result is essentially due to the fact that, for nonlocal conservation laws, there is no loss of information (with regard to initial and boundary data), i.e. initial and left side boundary data can be uniquely identified from a given final state, right boundary datum and right side nonlocal impact. This is also related to the key result that weak solutions of nonlocal conservation laws are per se unique and no entropy condition is required (see [62, 64]).

Our approach is reminiscent of the one used to obtain an exact controllability result for the linear transport equation (see [41, Section 2.1])

$$\begin{aligned} \partial_t \rho(t,x) + \partial_x \rho(t,x) &= 0, && (t,x) \in (0,T) \times (0,1),\\ \rho(0,x) &= \rho_0(x), && x \in (0,1),\\ \rho(t,0) &= u(t), && t \in (0,T). \end{aligned}$$

Namely, given $\rho_0 \in L^p((0,1))$ with $p \in \mathbb{R}_{\geq 1} \cup \{\infty\}$ and a target profile $\rho_{\text{des}} \in L^p((0,1))$, a control $u \in L^p((0,1))$ exists so that $\rho(T,\cdot) = \rho_{\text{des}}$ if and only if $T \geq 1$. The key to the proof is observing that the solution of the initial-boundary value problem is given explicitly by

$$\rho(t,x) := \begin{cases} \rho_0(x-t), & (t,x) \in (0,T) \times (0,1),\ t \leq x,\\ u(t-x), & (t,x) \in (0,T) \times (0,1),\ t > x; \end{cases}$$

therefore, if $T \geq 1$, we can choose

$$u(t) := \begin{cases} \rho_{\text{des}}(T-t), & t \in (T-1,T),\\ 0, & t \in (0,T-1), \end{cases}$$

and the solution then satisfies $\rho(t,x) = u(T-x) = \rho_{\text{des}}(x)$ for $(t,x) \in (0,T) \times (0,1)$. In other words, after the initial data (which moves along characteristics) has left the domain, we can inject the solution of the backward-in-time problem having ρ_{des} as initial data in the left-hand boundary.

Since the waves of hyperbolic equations have a finite speed of propagation and the control is applied at the boundary, an exact controllability result requires that the time horizon T must be sufficiently large. Similarly, in our nonlocal model, the first crucial step is to know that the initial state leaves the domain in finite time. This seems very natural when prescribing a density $u_r \in [0, 1)$ as right-hand side boundary term, which then necessarily "pulls out" the initial data for non-zero velocities. However, in contrast to the linear case, for the nonlocal conservation law considered here the initial datum—even after leaving the domain—still has an impact on the solution. This is because it changes the shape of the solution stemming from the boundary datum through the nonlocal term.

The result regarding initial datum leaving the domain is detailed in the following Lemma and is illustrated in Fig. 4.

Lemma 4.1 (Initial datum leaving domain in finite time) *Given Assumption 1 and a large enough $T \in \mathbb{R}_{>0}$, assume that $\|u_r\|_{L^\infty((0,T))} < 1$. Then, the initial datum—evolving with the dynamics in Definition 1.1—leaves the domain in finite time, i.e. the corresponding characteristics ξ as in (2.3) emanating from $(0, 0)$ satisfy*

$$\exists! T^* \in (0, T] : \ \xi_{\tilde{w}}(0, 0; T^*) = 1 \ \textit{with} \ T^* \leq V\left(1 - \frac{1 - \|u_r\|_{L^\infty((0,T))}}{\mathrm{e}}\right)^{-1}. \quad (4.1)$$

Proof We show that the zero characteristics move with positive speed bounded away from zero. To this end, we use the maximum principle in Theorem 2.1 and estimate the nonlocal operator in (1.4) as follows for $(t, x) \in \Omega_T$:

$$\begin{aligned}
\mathcal{W}[q](t,x) &= \frac{1}{\eta}\int_x^\infty \mathrm{e}^{-\frac{y-x}{\eta}} \left(\begin{cases}\rho(t,y) & \text{if } y < 1\\ u_r(t) & \text{if } y \geq 1\end{cases}\right)\mathrm{d}y\\
&\leq \frac{1}{\eta}\int_0^\infty \mathrm{e}^{-\frac{y}{\eta}} \left(\begin{cases}1 & \text{if } y < 1\\ u_r(t) & \text{if } y \geq 1\end{cases}\right)\mathrm{d}y\\
&= \frac{1}{\eta}\int_0^1 \mathrm{e}^{-\frac{y}{\eta}}\mathrm{d}y + \frac{u_\ell(t)}{\eta}\int_1^\infty \mathrm{e}^{-\frac{y}{\eta}}\mathrm{d}y\\
&= 1 - \mathrm{e}^{-1} + u_r(t)\mathrm{e}^{-1} = 1 - \frac{1-u_r(t)}{\mathrm{e}} \leq 1 - \frac{1 - \|u_r\|_{L^\infty((0,T))}}{\mathrm{e}}.
\end{aligned}$$

Using this estimate, which is uniform in $(t, x) \in \Omega_T$, and the monotonicity of V in Assumption 1, we can bound the zero characteristics in (2.3) from below:

$$\begin{aligned}
\xi_{w^*}(0,0;t) &= \int_0^t V(\mathcal{W}[q](s, \xi[0,0](s)))\mathrm{d}s\\
&\geq \int_0^t V\left(1 - \frac{1 - \|u_r\|_{L^\infty((0,T))}}{\mathrm{e}}\right)\mathrm{d}s = tV\left(1 - \frac{1 - \|u_r\|_{L^\infty((0,T))}}{\mathrm{e}}\right). \quad (4.2)
\end{aligned}$$

As V is non-zero at $1 - \frac{1-\|u_r\|_{L^\infty((0,T))}}{\mathrm{e}}$ (again by Assumption 1), we have the upper bound

$$\tilde{T} = V\left(1 - \frac{1 - \|u_r\|_{L^\infty((0,T))}}{\mathrm{e}}\right)^{-1} \quad (4.3)$$

when the initial datum has necessarily left the domain Ω_T. This also explains the assumption of T being sufficiently large, as we require $T \geq \tilde{T}$. As $\xi_{w^*}(0, 0; \cdot) \in C([0, T])$, i.e. it is continuous, a $T^* \in (0, T]$ satisfying $\xi_{\tilde{w}}(0, 0; T^*) = 1$ indeed exists. As $t \mapsto \xi_{w^*}(0, 0; t)$ is also strictly monotone, such a T^* is unique. □

Remark 4.1 (Improved upper bounds on T^* for affine linear velocities) In particular, in the case of an affine linear Greenshields velocity function (see [51]), i.e. $V(s) \equiv 1 - s$, we obtain an improved estimate on the bound in (4.3). We make the same ansatz as in (4.2) and write for the time derivative of the zero characteristics

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}t}\xi_{w^*}(0,0;t) &= V\big(\mathcal{W}[\rho,u_r](t,\xi_{w^*}(0,0;t))\big) = 1-\mathcal{W}[\rho,u_r](t,\xi_{w^*}(0,0;t))\\ &= 1-\frac{1}{\eta}\int_{\xi_{w^*}(0,0;t)}^{\infty}\mathrm{e}^{\frac{\xi_{w^*}(0,0;t)-y}{\eta}}\left(\begin{cases}\rho(t,y) & \text{if } y\le 1\\ u_r(t) & \text{if } y\ge 1\end{cases}\right)\mathrm{d}y,\end{aligned}$$

taking advantage of the maximum principle (2.4) in Theorem 2.1

$$\begin{aligned}&\ge 1-\frac{1}{\eta}\int_{\xi_{w^*}(0,0;t)}^{\infty}\mathrm{e}^{\frac{\xi_{w^*}(0,0;t)-y}{\eta}}\left(\begin{cases}\max\{\|\rho_0\|_{L^\infty((0,1))},u_r(t)\} & \text{if } y\le 1\\ u_r(t) & \text{if } y\ge 1\end{cases}\right)\mathrm{d}y\\ &= 1-\frac{\max\{\|\rho_0\|_{L^\infty((0,1))},u_r(t)\}}{\eta}\int_{\xi_{w^*}(0,0;t)}^{1}\mathrm{e}^{\frac{\xi_{w^*}(0,0;t)-y}{\eta}}\mathrm{d}y-\frac{u_r(t)}{\eta}\int_1^\infty \mathrm{e}^{\frac{\xi_{w^*}(0,0;t)-y}{\eta}}\mathrm{d}y\\ &= 1+\max\{\|\rho_0\|_{L^\infty((0,1))},u_r(t)\}\left(\mathrm{e}^{\frac{\xi_{w^*}(0,0;t)-1}{\eta}}-1\right)-u_r(t)\mathrm{e}^{\frac{\xi_{w^*}(0,0;t)-1}{\eta}}\\ &= \mathrm{e}^{\frac{\xi_{w^*}(0,0;t)-1}{\eta}}\big(\max\{\|\rho_0\|_{L^\infty((0,1))},u_r(t)\}-u_r(t)\big)\\ &\quad +1-\max\{\|\rho_0\|_{L^\infty((0,1))},u_r(t)\}\\ &\ge \mathrm{e}^{\frac{\xi_{w^*}(0,0;t)-1}{\eta}}\max\{\|\rho_0\|_{L^\infty((0,1))}-\|u_r\|_{L^\infty((0,T))},0\}\\ &\quad +1-\max\{\|\rho_0\|_{L^\infty((0,1))},\|u_r\|_{L^\infty((0,T))}\}.\end{aligned}$$

Recalling that $\xi(0,0;0)=0$ and solving the previous differential inequality in the case of equality, we obtain the following expression for the corresponding solution called y_η

$$\begin{aligned}y_\eta(t) &= 1+bt-\eta\ln\left(a\left(1-\mathrm{e}^{\frac{bt}{\eta}}\right)+b\mathrm{e}^{\frac{1}{\eta}}\right)+\eta\ln(b),\\ a &:= \max\{\|\rho_0\|_{L^\infty((0,1))}-\|u_r\|_{L^\infty((0,T))},0\},\\ b &:= 1-\max\{\|\rho_0\|_{L^\infty((0,1))},\|u_r\|_{L^\infty((0,T))}\}.\end{aligned}\tag{4.4}$$

Solving for $T^*\in\mathbb{R}_{>0}$ such that $y(T^*)=1$ gives the upper bound on T^*

$$T^*_{\text{improved}}(\eta)=\frac{\eta}{b}\ln\left(\frac{a+b\exp(\frac{1}{\eta})}{a+b}\right).\tag{4.5}$$

Let us compare the results in Lemma 4.1 with the improved estimate in this remark. The velocity function is required to satisfy $V(x)=1-x$, $x\in[0,1]$, and we assume that $\rho_0\equiv\frac{1}{2}$ and $u_r\equiv 0$. Then, for the estimate in (4.1) we obtain the upper bound on T^*, which we call T_1^* and is given by

$$T_1^*=\frac{1}{1-\left(1-\frac{1}{\mathrm{e}}\right)}=\mathrm{e}.$$

From (4.5), we obtain, for $\eta\in\mathbb{R}_{>0}$,

$$T^*_{\text{improved}}(\eta)\le 2\eta\ln\left(\frac{1}{2}\left(1+\mathrm{e}^{\frac{1}{\eta}}\right)\right),$$

which is illustrated for $\eta\in(0,2)$ in Fig. 4. Clearly, the improved estimate on T^*, i.e. $T^*_{\text{improved}}(\eta)$, $\eta\in\mathbb{R}_{>0}$, is sharper. It also depends on the nonlocal reach $\eta\in\mathbb{R}_{>0}$. As the right-hand side datum is minimal here, it is expected that the nonlocal term $W(t,\xi[0,0](t))$ becomes smaller with rising η as the nonlocal right-hand side u_r has an increasingly powerful influence on the nonlocal term. Thus, the velocity is higher and the upper bound on the time when the initial datum has left becomes smaller.

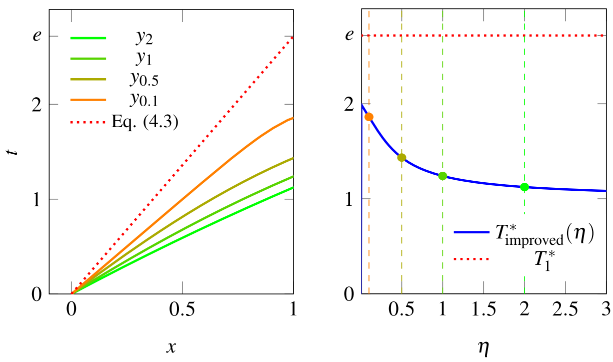

Fig. 4 Illustration of the different upper bounds for the initial datum leaving the domain as defined in (4.4). We chose $u_r = 0$ and $\rho_0 \equiv \frac{1}{2}$. **Left:** The different upper bounds for the zero characteristics $t \mapsto \xi(0, 0; t)$. The dashed red line is the—rather coarse—estimate uniform in η given in Lemma 4.1. The solid lines, which represent the improved upper bounds on T^* for affine linear velocity functions, exhibit higher accuracy. **Right:** The improved bounds on T^* for different values of the nonlocal reach η. As η becomes larger, the upper bound becomes smaller. This is because we have initialized the right-hand side u_r as zero so that for larger η this zero becomes more and more dominant, leading to an increased velocity approaching 1 in the limit. Consequently $\lim_{\eta\to\infty} T^*(\eta) = 1$

Remark 4.2 (Equivalent to Remark 4.1 for $\eta \to 0$, i.e. local conservation laws) The upper bound $T^*_{\text{improved}}(\eta)$ (see (4.5)) on T^* (see (4.1)) (the time, when the initial datum has left the domain) is a function of $\eta \in \mathbb{R}_{>0}$. For $\eta \to 0$, we formally obtain the local conservation law. For specific cases, i.e. for the Cauchy problem on $\mathbb{R}$, it has been proved that the solution of the Cauchy problem associated to the nonlocal conservation law converges to the entropy solution of the corresponding local conservation law [16, 17, 29] (see also [30] for the nonlocal-to-local limit of the same equation with additional (vanishing) viscosity effects). Although this cannot easily be extended to the initial boundary value problem considered in this work, it is still interesting to compute the limit for $\eta \to 0$ of the upper bound $T^*_{\text{improved}}(\eta)$, $\eta \in \mathbb{R}_{>0}$. We obtain

$$\lim_{\eta\to 0} T^*_{\text{improved}}(\eta) = \lim_{\eta\to 0} \frac{\eta}{b} \ln\left(\frac{a + b\exp(\frac{1}{\eta})}{a+b}\right) = \frac{1}{b},$$

(compare also Fig. 4 for $b = \frac{1}{2}$). This is then an upper bound for the time the local conservation law needs to transport the mass of the initial datum, i.e., $\int_0^1 \rho_0(x)\mathrm{d}x$, out of the domain. For constant initial datum and constant right-hand side this estimate is actually sharp.

Having shown that for a reasonable nonlocal right-hand side u_r the initial datum leaves the domain in finite time, we can state our main result in this section.

Theorem 4.1 (Equivalence controllability/time-inverted dynamics) *Let Assumption 1 and the following hold:*

- $\rho_0 \in L^\infty((0, 1); [0, 1])$
- $\rho_{des} \in L^\infty((0, 1); [0, 1))$
- $u_r \in L^\infty(\mathbb{R}_{>0}; [0, c]),\ c \in [0, 1)$
- $\rho_r \in L^\infty((0, \infty); [0, 1])$

Define

$$T^* := T^*_{\rho_0,u_r} := \operatorname*{arg\text{-}min}_{t\in\mathbb{R}_{\geq 0}} \left\{\xi[\rho_0, u_r](0,0;t) = 1\right\}, \tag{4.6}$$

$$\Xi_{\rho_0,u_r} := \{(t,x) \in \Omega_{T^*} : \xi[\rho_0, u_r](0,0;t) < x < 1\}, \tag{4.7}$$

$$v[\rho, u_r](t,x) := \begin{cases} \rho(t,x) & \text{if } (t,x) \in \Xi_{\rho_0,u_r}, \\ u_r(t) & \text{if } x > 1, \\ 0 & \text{otherwise}, \end{cases} \quad (t,x) \in \Xi \cup [0,T^*] \times \mathbb{R}_{>1}, \tag{4.8}$$

$$\tilde{\mathcal{W}}[p, v](t,x) := \frac{1}{\eta}\int_x^\infty \mathrm{e}^{\frac{x-y}{\eta}} \left(\begin{cases} p(t,y) & (t,y) \in \Omega_{T^*} \setminus \Xi_{\rho_0,u_r} \\ v[\rho,u_r](t,y) & \text{otherwise} \end{cases}\right) \mathrm{d}y, \quad (t,x) \in \Omega_{T^*}. \tag{4.9}$$

With these definitions, the following two results hold:

1. *There exists* $u_\ell \in L^\infty((0,T^*);[0,1])$ *such that* $\rho(T^*,\cdot) \equiv \rho_{des}$ *if and only if the backward nonlocal balance law*

$$\partial_t p(t,x) = -\partial_x \Big(V(\tilde{\mathcal{W}}[p, v[\rho,u_r]](t,x) p(t,x)\Big), \quad (t,x) \in \Omega_{T^*} \setminus \Xi_{\rho_0,u_r}, \tag{4.10}$$

$$p(T^*,x) = \rho_{des}(x), \quad x \in (0,1) \tag{4.11}$$

with $v[\rho,u_r]$ *as in* (4.8) *and* $\tilde{\mathcal{W}}$ *as in* (4.9)*, admits a solution satisfying*

$$\|p\|_{L^\infty((0,T^*);L^\infty((0,1)))} \leq 1.$$

2. *There exists* $T \in [T^*,\infty)$ *and* $u_\ell \in L^\infty((0,T);[0,1])$ *such that* $\rho(t,1) \equiv \rho_r(t)$ *for a.e.* $t \in (T^*,T)$ *if and only if the backward nonlocal balance law*

$$\partial_t p(t,x) = -\partial_x \Big(V(\tilde{\mathcal{W}}[p, v[\rho,u_r]](t,x) p(t,x)\Big), \tag{4.12}$$

$$p(t,1) = \rho_r(t), \quad t \in (T^*,T), \tag{4.13}$$

$$p(T,x) = 0, \quad x \in [0,1], \tag{4.14}$$

admits a solution, satisfying $\|p\|_{L^\infty((0,T);L^\infty((0,1)))} \leq 1$.

Proof First, we mention that T^* as in (4.6) exists and is unique according to Lemma 4.1. We prove only the result in Item 1 as the second result can be obtained analogously.

Let us assume that we can control the system to the desired end state/boundary state. Then, we can time-invert the dynamics; the solution to the corresponding backwards-in-time system exists and satisfies (4.10) to (4.11).

Conversely, let us assume that the backward system admits a weak solution. Then, we can evaluate the solution at $x = 0$ to obtain the proper boundary data, which indeed serves as a control to steer the system towards the desired state. The regularity required for this to hold is $C\left([0,1]; L^1((0,T))\right)$. Although such regularity generally does not hold (compare also [64, Remark 5.6]), it does hold provided the corresponding velocity is bounded away from zero. This is true in the underlying case, as also illustrated in Lemma 4.1, as long as $\|u_r\|_{L^\infty((0,T))} < 1$. □

Remark 4.3 (Explanation of Theorem 4.1) The backwards in time nonlocal conservation laws and the suitable domain on which they need to be solved are illustrated in Fig. 5.

The red lines indicate the data that needs to be fitted, the blue areas illustrate datum that is given by a prescribed initial datum and right-hand side nonlocal impact. The backwards problem is—for both cases—considered on the grey area/domain.

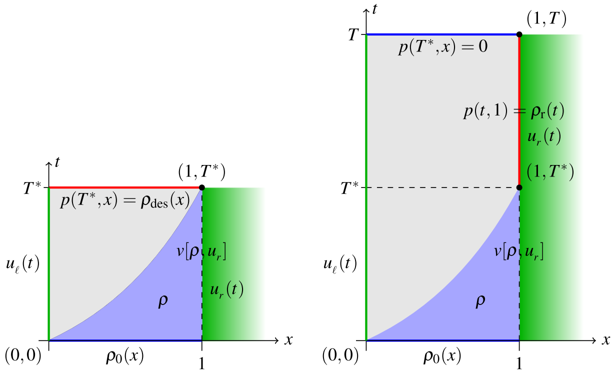

Fig. 5 **Left**: Illustration of the statement in Theorem 4.1, Item 1. Red) shows the desired end value we wish to control the system to and blue shows the known quantities. The green colors indicate functions that we wish to control to reach the desired state $\rho(T^*, \cdot)$. The backward in time equation is considered on the grey area. **Right:** Illustration of the statement in Theorem 4.1, Item 2. Red again indicates (here) the boundary value we would like to control the system to and in blue we have the quantities that are given (in particular, the end value can be chosen arbitrarily). Green indicates the quantity we can use to control the system, i.e. left-hand side boundary datum and right-hand side nonlocal impact. The backward system is considered on the grey area with explicit boundary conditions from $(1, T^*)$ to $(1, T)$

Remark 4.4 (Controlling to target state and out-flux simultaneously) It is straightforward to generalize the previous result to the case where we seek a left-hand side boundary datum u_ℓ and nonlocal right-hand side u_r so that for significantly large time $T \in \mathbb{R}_{>T^*}$ the end state satisfies

$$\rho(T, \cdot) \equiv \rho_{\text{des}}$$

and the boundary state

$$\rho(\cdot, 1) \equiv \rho_{\text{r}}.$$

We do not go into details.

As the previous result is not explicit in the sense that we cannot "a priori" determine which final states we can control the system to, we show in the following that a constant state can always be reached in a sufficiently large time when also controlling u_r.

Lemma 4.2 (Controllability to constant state) *Let* $\rho_{des} \equiv c \in [0, 1)$ *and* $\rho_0 \in L^\infty((0, 1); [0, 1])$ *be given. Then,*

$$\exists T \in \mathbb{R}_{>0}\ \exists (u_\ell, u_r) \in L^\infty((0, T); [0, 1])^2 :\ \rho(T, \cdot) \equiv \rho_{des},$$

where ρ *denotes the solution of the initial boundary value problem in Definition 1.1 for boundary datum* u_ℓ*, nonlocal impact on the right-hand side* u_r*, and initial datum* ρ_0*.*

Proof We prove this result by introducing different steps in which we control the solution to a specific datum.

First, following Lemma 4.1, there exists $T^* \in \mathbb{R}_{>0}$ so that for $u_\ell(t) = u_r(t) = 0\ \forall t \in [0, T^*]$ the initial datum has left the domain. Thus, we have

$$\rho(T^*, \cdot) \equiv 0,$$

i.e. the road is fully evacuated. Second, we show that the initial state zero can be controlled in finite time to the steady state $\varepsilon \in \mathbb{R}_{>0}$ for ε sufficiently small and continue this process until we have reached the constant state. We take advantage of Theorem 4.1 and consider the following sequences of surrogate problems for $n \in \mathbb{N}_{\geq 1}$ backwards in time and $(t, x) \in \Omega_{T_n^*} \setminus \Xi_{\varepsilon(n-1),\varepsilon(n-1)}$:

$$\begin{aligned} \partial_t p_n(t,x) &= -\partial_x \Big(V(\tilde{\mathcal{W}}[p_n, v[\varepsilon(n-1), \varepsilon(n-1)]](t,x) p_n(t,x) \Big), \\ p_n(T_n^*, x) &= n\varepsilon, \end{aligned} \tag{4.15}$$

and $T_n^* := \sum_{k=1}^n T^*_{(k-1)\varepsilon,(k-1)\varepsilon}$ as in (4.6). As we will stop when we have found $n^* \in \mathbb{N}_{\geq 1}$: $n^*\varepsilon = c$ (we can always choose ε so that this holds), we can immediately provide a uniform upper bound on T_n^* by invoking Lemma 4.1:

$$T^*_{(n-1)\varepsilon,(n-1)\varepsilon} \leq \frac{1}{V\left(1 - \frac{1-c}{\mathrm{e}}\right)} \quad \text{and} \quad T_n^* \leq \frac{n}{V\left(1 - \frac{1-c}{\mathrm{e}}\right)} \qquad \forall n \in \mathbb{N}_{\geq 1} \tag{4.16}$$

thanks to the monotonicity of V in Assumption 1. Now, we show that for sufficiently small ε the system in (4.15) admits a solution on the entire time horizon $\frac{1}{V(1-\frac{1-c}{\mathrm{e}})}$. To this end, we examine how at a given space time point $(t, x) \in (T^*_{n-1}, T^*_n) \setminus \Xi_{\varepsilon(n-1),\varepsilon(n-1)}$ a maximum evolves backwards in time. Assuming that at such a (t, x) the solution is maximal, parametrized on the characteristics, i.e., $p_n(t, \xi(T^*, x; t)) = \|p_n(t, \cdot)\|_{L^\infty(\mathbb{R})}$ (and thus also $\partial_2 p_n(t, \xi(T^*, x; t)) = 0$), we estimate (recall the required definition of the operator v in (4.8))

$$\begin{aligned} &-\frac{\mathrm{d}}{\mathrm{d}t} p_n(t, \xi(T_n^*, x; t)) \\ &= V'(\tilde{\mathcal{W}}[p_n, v[\varepsilon(n-1), \varepsilon(n-1)]](t, \xi(T_n^*, x; t)) \\ &\quad \cdot \partial_2 \tilde{\mathcal{W}}[p, v[\varepsilon(n-1), \varepsilon(n-1)]](t, \xi(T_n^*, x; t)) \\ &= V'(\tilde{\mathcal{W}}[p_n, v[\varepsilon(n-1), \varepsilon(n-1)]](t, \xi(T_n^*, x; t)) \\ &\quad \cdot \frac{1}{\eta} (\tilde{\mathcal{W}}[p_n, v[\varepsilon(n-1), \varepsilon(n-1)]](t, \xi(T_n^*, x; t)) - p_n(t, \xi(T_n^*, x; t))) \\ &\leq \frac{1}{\eta} \|V'\|_{L^\infty((0,1))} p_n(t, \xi(T_n', x; t))^2. \end{aligned}$$

Integrating the previous differential inequality backwards in time from T_n^* to t yields as upper bound

$$\|p_n(t, \cdot)\|_{L^\infty((0,1))} \leq \frac{1}{\frac{n}{\varepsilon} - \frac{1}{\eta}\|V'\|_{L^\infty((0,1))}(T_n^* - t)} \overset{!}{\leq} 1.$$

For admissibility we need to ensure that the previous $\|p_n(t, \cdot)\|_{L^\infty((0,1))}$ is less than or equal to one, which is satisfied if

$$t \geq T_n^* - \eta \frac{1 - n\varepsilon}{\varepsilon \|V'\|_{L^\infty((0,1))}} \geq T_n^* - \eta \frac{1 - \kappa}{\varepsilon \|V'\|_{L^\infty((0,1))}}.$$

However, for $\varepsilon \in \mathbb{R}_{>0}$ sufficiently small, we obtain the well-posedness of the backwards system in (4.15) on every time horizon and, thus, particularly on the time horizon required for the initial datum to leave, i.e. (4.16). As the estimates are uniform in $n \in \mathbb{N}_{\geq 1}$, we can then pick as many sequences as needed to iteratively control the zero initial datum to $\varepsilon, 2\varepsilon, \ldots$ until we have reached the constant state c. This concludes the proof. □

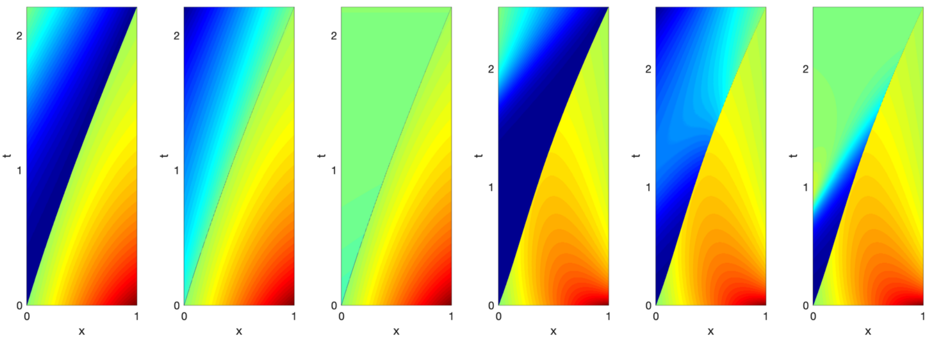

Fig. 6 The three images on the left correspond to $\eta = 1$, the three on the right to $\eta = 0.1$. In the left images of both triples, $\rho_{\text{des}}(x) = \frac{1}{2}(1-x)$, in the middle $\rho_{\text{des}}(x) = \frac{1}{2}x$ and in the right images $\rho_{\text{des}} \equiv \frac{1}{2}$. In all images, the initial datum is given by $\rho_0(x) = \frac{1}{2}(1+x)$ and the right boundary $u_r = \frac{1}{2}$. **Color bar:** 0 1

Remark 4.5 (Extensions of Lemma 4.2) The previous Lemma 4.2 can be generalized. For instance, the solution can also be steered to monotonically increasing ρ_{des} by first controlling it to the sufficiently large constant state $\mathbb{R} \ni c \geq \|\rho_{\text{des}}\|_{L^\infty((0,1))}$ and then showing that the backward in time system does not blow up (this cannot happen due to the assumed monotonicity). Another extension might consist of slightly disturbing the constant ρ_{des} with regard to the L^∞ norm and still achieving controllability (compare also Remark 3.1). We do not go into details.

Example 4.1 (Controllability and lack of controllability in minimal time) We present some examples related to the state controllability result in Theorem 4.1. In Fig. 6, we consider three cases: $\rho_{\text{des}}(x) = \frac{1}{2}(1-x)$, $\rho_{\text{des}}(x) = \frac{1}{2}x$, and $\rho_{\text{des}} \equiv \frac{1}{2}$, with initial and right

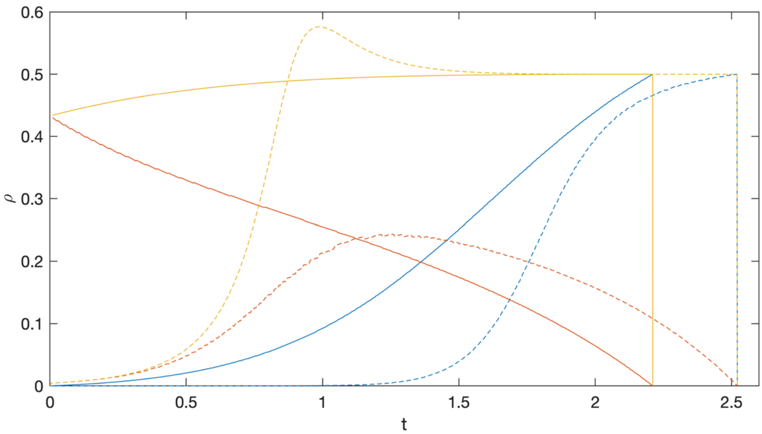

Fig. 7 Corresponding to Fig. 6 in Example 4.1, the left-hand side boundary datum u_ℓ to achieve the desired final state ρ_{des} in minimal time. Solid lines represent the boundary datum for $\eta = 1$, dashed lines for $\eta = 0.1$. The colors represent the related desired state ρ_{des} that we wish to achieve: For $x \in [0, 1]$ we have in red $\rho_{\text{des}}(x) = \frac{1}{2}x$, in blue $\rho_{\text{des}}(x) = \frac{1}{2}(1-x)$ and in yellow $\rho_{\text{des}}(x) = \frac{1}{2}$

boundary data given by $\rho_0(x) = \frac{1}{2}(1+x)$ and $u_r = \frac{1}{2}$. Figure 7 also shows the left-hand side boundary datum u_ℓ to achieve the desired final state ρ_{des} in minimal time. As can be observed in the left three pictures in Fig. 6, the initial datum leaves faster. This is due to the fact that η is larger, meaning that the nonlocal right-hand side $u_r = \frac{1}{2}$ has a higher impact on the velocity of the entire road. Another noteworthy point is that, for smaller η and end-datum, $\rho_{\text{des}} = \frac{1}{2}x,\ x \in [0,1]$, (see the fifth pictures in Fig. 6 or the maximum of the yellow dotted curve in Fig. 7) actually becomes larger than the desired state and then becomes smaller again to compensate for the later velocity of the system. This clearly indicates that in general not every end state can be tracked, as the corresponding control could exceed 1 and would thus not be admissible.

Finally, all the images indicate that the solution below the characteristics emanating from $(0,0)$, i.e. the solution which only depends on initial datum and the right-hand side nonlocal impact u_r, stays the same independent of the boundary datum. Thus, clearly the time when the initial datum has left is the same.

5 Long-time Behavior

In this section, we are concerned with the long-time behavior of the solution to Definition 1.1 when prescribing constant $(u_\ell, u_r) \in [0,1)^2$. Under the assumption that the initial datum is uniformly less than or equal to (or greater than or equal to) $u_\ell = u_r$, we can show that the solution converges to a given constant. We detail this in the following Theorem.

Theorem 5.1 (Long-time behavior) *Suppose that $\kappa \in (0,1)$ is given. Let Assumption 1 hold, $\eta \in \mathbb{R}_{>0}$ and assume $u_r \equiv \kappa$, $u_\ell \equiv \kappa$ and the derivative of V may satisfy $V'(s) < 0\ \forall s \in [0,1)$. In addition, let*

$$\big(\rho_0 \geqq \kappa \ \text{ on } \ (0,1)\big) \quad \text{or} \quad \big(\rho_0 \leqq \kappa \ \text{ on } \ (0,1)\big). \tag{5.1}$$

Then, the corresponding solution ρ converges exponentially in time to κ:

$$\|\rho(t,\cdot)-\kappa\|_{L^1((0,1))} \leq \eta\left(\exp\left(\frac{\|\rho_0-\kappa\|_{L^1((0,1))}}{\eta}\right)-1\right)\mathrm{e}^{\frac{K(\eta)}{\eta}t} \quad \forall t \in \mathbb{R}_{\geq 0}$$

with

$$\begin{aligned}
\bar{\kappa} &:= (1-\kappa)\left(1-\exp\left(-\eta^{-1}\right)\right),\\
K(\eta) &:= (1-\kappa)\kappa \sup_{s\in\langle\kappa,\bar{\kappa}\rangle} V'(s)\exp\left(-\eta^{-1}\right) < 0,\\
\langle a,b\rangle &:= (\min\{a,b\},\max\{a,b\}), \quad (a,b)\in\mathbb{R}^2.
\end{aligned}$$

Proof Let us define the difference between $\rho(t,\cdot)$ and κ in the integral sense for $t \in [0,T]$

$$M(t) := \int_0^1 (\rho(t,x)-\kappa)\mathrm{d}x.$$

As we want to compute the time-derivative of $M(t)$, we first need to show that $t \mapsto M(t)$ is differentiable. This can be achieved by taking advantage of the solution formula in terms of the characteristics in (2.2). Assuming that $T^* \in \mathbb{R}_{>0}$ and $\xi(0, 0; T^*) = 1$, we can write for $t \in [0, T^*]$

$$\begin{aligned} M(t) &= \int_0^{\xi(0,0;t)} u\left(\xi_{w^*}[t,x]^{-1}_{\max}(0)\right) \partial_2 \xi_{w^*}\left(t, x; \xi_{w^*}[t,x]^{-1}_{\max}(0)\right) \mathrm{d}x \\ &\quad + \int_{\xi(0,0;t)}^1 \rho_0(\xi(t,x;0)) \partial_2 \xi(t,x;0) \mathrm{d}x - \kappa \\ &= \int_0^t u(z) V(\mathcal{W}[\rho](z,0)) \mathrm{d}z + \int_0^{\xi(t,1;0)} \rho_0(z) \mathrm{d}z - \kappa, \end{aligned}$$

which is clearly differentiable with regard to t sufficiently small. Taking the time derivative we obtain

$$\begin{aligned} M'(t) &= u(t) V(\mathcal{W}[\rho](t,0)) + \rho_0(\xi(t,1;0)) \partial_1 \xi(t,1;0) \\ &= \kappa V(\mathcal{W}[\rho](t,0)) - \rho(t,1) V(\mathcal{W}[\rho](t,1)). \end{aligned} \tag{5.2}$$

As the previous result does not depend explicitly on the initial datum (we have replaced part of the initial datum with the general expression for the solution ρ), this result holds for every time $t \in \mathbb{R}_{>0}$. Assume for now that $\rho_0 \geq \kappa$. Then, thanks to the maximum principle in (2.4) in Theorem 2.1, we know

$$\rho(t,x) \geq \kappa \quad \forall (t,x) \in [0,T] \times (0,1) \text{ a.e.} \tag{5.3}$$

It follows directly

$$1 \geq M(t) \geq 0 \quad \forall t \in [0,T]. \tag{5.4}$$

The upper bound 1 is a consequence of the maximum principle and the fact that $\kappa \in [0, 1]$ by assumption. Then, we estimate the nonlocal term as follows:

$$\begin{aligned} \mathcal{W}[\rho](t,0) &:= \frac{1}{\eta} \int_0^\infty \exp\left(-\frac{s}{\eta}\right) \left(\begin{cases} \rho(t,s), & s < 1 \\ \kappa, & s \geq 1 \end{cases}\right) \mathrm{d}s, \\ &\overset{(5.3)}{\geq} \frac{\kappa}{\eta} \int_0^{1-M(t)} \exp\left(-\frac{s}{\eta}\right) \mathrm{d}s + \frac{1}{\eta} \int_{1-M(t)}^1 \exp\left(-\frac{s}{\eta}\right) \mathrm{d}s + \frac{\kappa}{\eta} \int_1^\infty \exp\left(-\frac{s}{\eta}\right) \mathrm{d}s \\ &= \frac{\kappa}{\eta} \int_0^\infty \exp\left(-\frac{s}{\eta}\right) \mathrm{d}s + \frac{1-\kappa}{\eta} \int_{1-M(t)}^1 \exp\left(-\frac{s}{\eta}\right) \mathrm{d}s \\ &= \kappa + (1-\kappa)\left(\exp\left(-\frac{1-M(t)}{\eta}\right) - \exp\left(-\frac{1}{\eta}\right)\right) \\ &= \kappa + \exp\left(-\frac{1}{\eta}\right)(1-\kappa)\left(\exp\left(\frac{M(t)}{\eta}\right) - 1\right), \end{aligned}$$

from which we can continue the estimate on $M(t)$ in (5.2). Again recalling that $\rho(t, 1) \geq \kappa \ \forall t \in [0, T]$, we obtain

$$M'(t) \overset{V' \leqq 0}{\leq} V\left(\kappa + \exp\left(-\frac{1}{\eta}\right)(1-\kappa)\left(\exp\left(\frac{M(t)}{\eta}\right) - 1\right)\right)\kappa - \underbrace{V(\kappa)\rho(t,1)}_{\geq V(\kappa)\kappa}.$$

Using the mean value theorem and defining $\bar{\kappa} := (1-\kappa)(1-\exp(-\eta^{-1})) < 1$,

$$\begin{aligned} M'(t) &\leq \sup_{s \in (\kappa,\bar{\kappa})} V'(s) \left(\exp\left(-\frac{1}{\eta}\right)(1-\kappa)\left(\exp\left(\frac{M(t)}{\eta}\right) - 1\right)\right)\kappa \\ &\leq (1-\kappa)\kappa \sup_{s \in (\kappa,\bar{\kappa})} V'(s) \exp\left(-\frac{1}{\eta}\right)\left(\exp\left(\frac{M(t)}{\eta}\right) - 1\right). \end{aligned}$$

We solve the previous differential inequality for equality, call the solution $M_=(t)$ and obtain for it

$$\begin{aligned} M_=(t) &= -\eta \ln\left(\left(\mathrm{e}^{-\frac{M(0)}{\eta}} - 1\right)\mathrm{e}^{\frac{K(\eta)t}{\eta}} + 1\right) \\ K(\eta) &:= (1-\kappa)\kappa \sup_{s\in(\kappa,\bar{\kappa})} V'(s)\exp\left(-\tfrac{1}{\eta}\right) < 0. \end{aligned} \tag{5.5}$$

From this, we have

$$0 \le M(t) \le M_=(t) \quad \forall t \in \mathbb{R}_{\ge 0}. \tag{5.6}$$

To prove the rate of convergence, we use $\ln(x) \le x - 1\ \forall x \in \mathbb{R}_{>0}$ and have

$$\begin{aligned} M_=(t) &= \eta \ln\left(\left(1 - \left(1 - \mathrm{e}^{-\frac{M(0)}{\eta}}\right)\mathrm{e}^{\frac{K(\eta)t}{\eta}}\right)^{-1}\right) \\ &\le \eta \frac{\left(1 - \mathrm{e}^{-\frac{M(0)}{\eta}}\right)\mathrm{e}^{\frac{K(\eta)t}{\eta}}}{1 - \left(1 - \mathrm{e}^{-\frac{M(0)}{\eta}}\right)\mathrm{e}^{\frac{K(\eta)t}{\eta}}} \le \eta\left(\mathrm{e}^{\frac{M(0)}{\eta}} - 1\right)\mathrm{e}^{\frac{K(\eta)}{\eta}t}. \end{aligned}$$

For initial datum $\rho_o(x) \le \kappa$ for a.e. $x \in (0, 1)$, the results follow by performing similar estimates with the opposite sign. □

Remark 5.1 (Theorem 5.1 for $\kappa = 0$ and $\kappa = 1$) The previous result with the given constants does not provide exponential stability for $\kappa \in \{0, 1\}$ as then $K(\eta) = 0$ for $\eta \in \mathbb{R}_{>0}$.

However, for $\kappa = 0$, the boundary contribution to the solution is zero and, by Lemma 4.1, we know that the initial data leaves the domain in finite time $T^* \in \mathbb{R}_{>0}$. Afterwards, the solution remains identically zero so we have stability to the zero solution in finite time and particularly exponentially.

For $\kappa = 1$, the situation is slightly more delicate. We look at the change of the L^1-norm of the solution in time,

$$\begin{aligned} \partial_t \|\rho(t,\cdot)\|_{L^1((0,1)} &= -\int_0^1 \partial_x \left(\rho(t,y)V(\mathcal{W}[\rho,u_r](t,y))\right)\mathrm{d}y \\ &= \rho(t,0)V(\mathcal{W}[\rho,u_r](t,0)) - \rho(t,1)V(\mathcal{W}[\rho,u_r](t,1)) \\ &= V(\mathcal{W}[\rho,u_r](t,0)) \\ &\ge V\left(\frac{1}{\eta}\int_0^{\|\rho(t,\cdot)\|_{L^1((0,1))}} 1\cdot\exp\left(\frac{-y}{\eta}\right)\mathrm{d}y + \exp\left(-\frac{1}{\eta}\right)\right) \\ &= V\left(1 - \exp\left(\frac{-\|\rho(t,\cdot)\|_{L^1((0,1))}}{\eta}\right) + \exp\left(-\frac{1}{\eta}\right)\right) \end{aligned} \tag{5.7}$$

and using the mean value theorem, $\exists \zeta \in (0, 1)$ s.t.

$$\begin{aligned} &= V(1) - V'(\zeta)\cdot\left(\mathrm{e}^{\frac{-\|\rho(t,\cdot)\|_{L^1((0,1))}}{\eta}} - \mathrm{e}^{-\frac{1}{\eta}}\right) \\ &\ge -\sup_{s\in(0,1)} V'(s)\cdot\left(\mathrm{e}^{\frac{-\|\rho(t,\cdot)\|_{L^1((0,1))}}{\eta}} - \mathrm{e}^{-\frac{1}{\eta}}\right), \end{aligned}$$

and consequently

$$\|\rho(t,\cdot)\|_{L^1((0,1))} \geq 1 + \eta \log\left(\left(\mathrm{e}^{\frac{\|\rho_0\|_{L^1((0,1))}-1}{\eta}} - 1\right)\exp\left(t \sup_{s\in(0,1)} V'(s)\frac{\mathrm{e}^{-\frac{1}{\eta}}}{\eta}\right) + 1\right).$$

As $\ln(x+1) \geq x(x+1)^{-1}\ \forall x > \mathbb{R}_{>-1}$ we can continue our estimate

$$\geq 1 + \eta \frac{\left(\mathrm{e}^{\frac{\|\rho_0\|_{L^1((0,1))}-1}{\eta}} - 1\right)\exp\left(t \sup_{s\in(0,1)} V'(s)\frac{\mathrm{e}^{-\frac{1}{\eta}}}{\eta}\right)}{\left(\mathrm{e}^{\frac{\|\rho_0\|_{L^1((0,1))}-1}{\eta}} - 1\right)\exp\left(t \sup_{s\in(0,1)} V'(s)\frac{\mathrm{e}^{-\frac{1}{\eta}}}{\eta}\right) + 1}$$

$$\geq 1 + \frac{\eta}{2}\left(\mathrm{e}^{\frac{\|\rho_0\|_{L^1((0,1))}-1}{\eta}} - 1\right)\exp\left(t \sup_{s\in(0,1)} V'(s)\frac{\mathrm{e}^{-\frac{1}{\eta}}}{\eta}\right).$$

This is the exponential convergence to the steady state solution in the case that $\kappa = 1$, i.e., in the case that the road is blocked at the right-hand side and $u_\ell \equiv 1$. For the statement to hold we require

$$\sup_{s\in(0,1)} V'(s) < 0.$$

In the case that this assumption does not hold and only Assumption 1 applies, we can still show that the solution converges to the 1 solution for $t \to \infty$, though without any order of convergence. The convergence is then due to the fact that the mapping $t \mapsto \|\rho(t,\cdot)\|_{L^1((0,1))}$ is monotonically increasing in t and bounded from above by 1. Then, we know that a limit point for this sequence exists, i.e., $\exists A \in (0,1] : \lim_{t\to\infty} \|\rho(t,\cdot)\|_{L^1((0,1))} = A$. Thanks to (5.7), the time derivative of $\|\rho(t,\cdot)\|_{L^1((0,1))}$ is nonnegative and only zero for $\|\rho(t,\cdot)\|_{L^1((0,1))} = 1$, which implies $A = 1$.

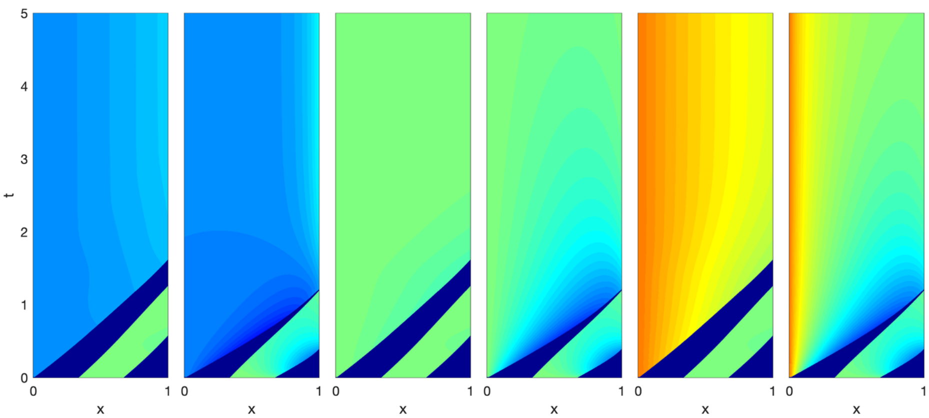

Fig. 8 The images are ordered from left to right. **First:** $u_\ell = \frac{1}{4}$, $u_r = \frac{1}{2}$, $\eta = 1$, **Second:** $u_\ell = \frac{1}{4}$, $u_r = \frac{1}{2}$, $\eta = 0.1$, **Third:** $u_\ell = \frac{1}{2}$, $u_r = \frac{1}{2}$, $\eta = 1$, **Fourth:** $u_\ell = \frac{1}{2}$, $u_r = \frac{1}{2}$, $\eta = 0.1$, **Fifth:** $u_\ell = \frac{3}{4}$, $u_r = \frac{1}{2}$, $\eta = 1$, **Sixth:** $u_\ell = \frac{3}{4}$, $u_r = \frac{1}{2}$, $\eta = 0.1$ with $\rho_0 \equiv \frac{1}{2}\chi_{(\frac{1}{3},\frac{2}{3})}$ in every case. **Color bar:** 0 1

Example 5.1 (Long-time behavior and comparison to steady-state solutions) In Fig. 8, we present some numerical simulations related to Theorem 5.1. We assume that

$$V \equiv 1 - \cdot, \quad u_r \equiv \frac{1}{2}, \quad u_\ell \in \left\{\frac{1}{4}, \frac{1}{2}, \frac{3}{4}\right\}, \quad \eta \in \{0.1, 1\}, \quad \rho_0 \equiv \frac{1}{2}\chi_{(\frac{1}{3},\frac{2}{3})}.$$

One remarkable feature which can be seen in all images is the fact that after the initial datum has left, the solution no longer changes substantially and appears to become stationary. Although we are not able to show this in the general case, it appears that all solutions converge to the corresponding steady state, anticipating the existence and uniqueness of steady state solutions in Theorem 6.1. Indeed, this is also illustrated in Fig. 9, wherein the image on the left the solutions are plotted at $t \in \{2, 4, 8\}$ and in the image on the right we see the steady state solution in comparison to the corresponding solution at time $t = 8$.

It is worth mentioning the impact of the size of the nonlocal parameter $\eta \in \mathbb{R}_{>0}$. As the initial datum's L^1-mass is smaller than $u_r = \frac{1}{2}$, in the present case the initial datum leaves more quickly when η is larger. The different η also affects how the solution will later evolve, which can be seen in particular in the two rightmost images. Larger η decreases the spatial derivative of the solution, which is clear from the fact that for larger η there is more averaging of the velocity.

6 Steady States

In the literature, steady state solutions for nonlocal conservation laws on a bounded domain have not yet been studied. On the real axis, traveling wave solutions have been discussed in [74].

However, for local conservation laws this has been a topic of much discussion over the past few decades. In [46], Dafermos (inspired by the previous analysis and numerical experiment of [75]) used the method of generalized characteristics to study the long-time behavior

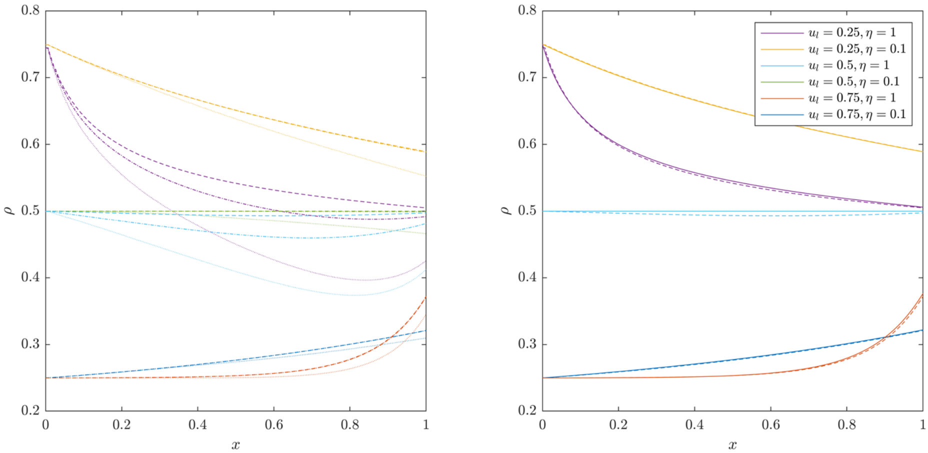

Fig. 9 **Left:** Illustrations of the evolution of solutions at different time snippets $t \in \{2, 4, 8\}$ (dotted $t = 2$, dash-dotted $t = 4$ and dashed $t = 8$). The different colors represent the six different cases in Fig. 8 for different u_ℓ, η as described in the legend of the illustration on the right and fixed $u_r = 0.5$. **Right:** Comparison of the different solutions at $t = 8$ and the corresponding steady state solutions as in Theorem 6.1

of solutions for the initial-boundary value problem for conservation law with spatial inhomogeneity. The analysis of entropic steady states of the initial-boundary value problem for a conservation law with a source term was later carried out in [70]. In [28], the existence of stationary solutions for a scalar conservation law was obtained in the case of a nonlocal source term. In [49], the authors considered stationary scalar conservation laws with a damping term and showed the existence and uniqueness of entropy solutions as time-asymptotic limits of solutions of the corresponding (evolutionary) hyperbolic conservation laws. In the case of periodic data (and boundary conditions), the time-asymptotic decay properties of entropy solutions were shown in [20, 72, 77]. For the asymptotic behavior of the closely related equations of Hamilton–Jacobi type, we refer the reader to [10, 11, 59, 60] and the references therein. For nonlinear hyperbolic systems of balance laws, there is a large body of literature dealing with the existence of global (classical) solutions around an equilibrium (see e.g., [12, 14, 56, 80]).

In the following theorem, we prove the existence and uniqueness of steady state solutions on a bounded domain when prescribing a constant left-hand side boundary datum and constant nonlocal right-hand side datum.

Theorem 6.1 (Steady state solutions on bounded domains) *In the setting of Definition 1.1, for every* $u_\ell \equiv const \in [0,1]$, $u_r \equiv const \in [0,1)$ *there exists a unique and monotone* $\bar{\rho} \in W^{1,\infty}((0,1);[\min\{u_\ell,u_r\},\max\{u_\ell,u_r\}])$ *satisfying*

$$\frac{\mathrm{d}}{\mathrm{d}x}\big(\bar{\rho}(x)V(\mathcal{W}[\bar{\rho},u_r](x))\big)=0, \qquad x\in[0,1], \tag{6.1}$$

$$\mathcal{W}[\bar{\rho},u_r](x)=\frac{1}{\eta}\int_x^\infty \mathrm{e}^{-\frac{y-x}{\eta}}\left(\begin{cases}\bar{\rho}(y) & \text{if } y<1\\ u_r & \text{if } y\geq 1\end{cases}\right)\mathrm{d}y, \qquad x\in[0,1], \tag{6.2}$$

$$\bar{\rho}(0)=u_\ell. \tag{6.3}$$

In addition, if $V\in C^\infty([0,1])$*, then* $\bar{\rho}\in C^\infty([0,1])$*. We call* $\bar{\rho}$ *the* ***steady state*** *to the corresponding boundary datum* u_ℓ, u_r.

We remark that for $u_\ell = u_r$ a solution is given by $\bar{\rho}\equiv u_r$, which can be checked by substituting it into (6.1) to (6.3). However, even in this simpler case we need to prove that this is the only solution and that one and only one solution exists in the case $u_r \neq u_\ell$. This is carried out in the following proof.

Proof As a first step, we show the existence of solutions using a Schauder fixed-point argument.

A solution of (6.1) to (6.3) can be interpreted as the solution of a fixed-point problem with the fixed-point mapping

$$\boldsymbol{F}:\begin{cases}\Omega \to \Omega\\ \bar{\rho}\mapsto \left(x\mapsto u_\ell \frac{V(\mathcal{W}[\bar{\rho},u_r](0))}{V(\mathcal{W}[\bar{\rho},u_r](x))}\right)\end{cases} \tag{6.4}$$

with a proper $\Omega\subset C([0,1])$ yet to be defined. We distinguish two different cases: $u_\ell \geq u_\ell$ and $u_\ell \leq u_r$.

If $u_\ell \geq u_r$, we define

$$\begin{aligned}\Omega &:= \left\{\bar{\rho}\in W^{1,\infty}((0,1)) : \big(u_r\leq\bar{\rho}(x)\leq u_\ell\big)\wedge\big(\mathcal{A}\leq\bar{\rho}'(x)\leq 0\big)\ \forall x\in[0,1]\right\}\\ \mathcal{A} &:= -u_\ell\frac{V(u_r)\|V'\|_{L^\infty((u_r,u_\ell))}(u_\ell-u_r)}{\eta V(u_\ell)^2}\end{aligned} \tag{6.5}$$

and show that $\boldsymbol{F}$ is a self-mapping on Ω, i.e. $\boldsymbol{F}[\Omega] \subseteq \Omega$. To this end, we take $\bar{\rho} \in \Omega$ and compute, for $x \in [0, 1]$,

$$\frac{\mathrm{d}}{\mathrm{d}x}\boldsymbol{F}[\bar{\rho}](x) = -u_{\ell}\frac{V(\mathcal{W}[\bar{\rho}, u_r](0))}{V(\mathcal{W}[\bar{\rho}, u_r](x))^2}V'(\mathcal{W}[\bar{\rho}, u_r](x))\partial_x \mathcal{W}[\bar{\rho}, u_r](x). \tag{6.6}$$

Since $V' \leqq 0$, we need to show that $\partial_x \mathcal{W}[\bar{\rho}, u_r] \leqq 0$, which we do with the following manipulations for $x \in [0, 1]$:

$$\partial_x \mathcal{W}[\bar{\rho}, u_r](x) = \partial_x\left(\frac{1}{\eta}\int_x^1 \mathrm{e}^{\frac{x-y}{\eta}}\bar{\rho}(y)\mathrm{d}y\right) + \partial_x\left(\frac{1}{\eta}u_r\int_1^\infty \mathrm{e}^{\frac{x-y}{\eta}}\mathrm{d}y\right) \tag{6.7}$$

$$= -\frac{1}{\eta}\bar{\rho}(x) + \frac{1}{\eta^2}\int_x^1 \mathrm{e}^{\frac{x-y}{\eta}}\bar{\rho}(y)\mathrm{d}y + \frac{1}{\eta^2}u_r\int_1^\infty \mathrm{e}^{\frac{x-y}{\eta}}\mathrm{d}y \tag{6.8}$$

$$= \frac{1}{\eta}(\mathcal{W}[\bar{\rho}, u_r](x) - \bar{\rho}(x)). \tag{6.9}$$

As $\bar{\rho}$ is monotonically decreasing and $\bar{\rho} \geqq u_r$, we obtain that $\mathcal{W}[\bar{\rho}, u_r] \leqq \bar{\rho}$ and consequently

$$\partial_x \mathcal{W}[\bar{\rho}, u_r] \leqq 0 \tag{6.10}$$

and thus $\partial_x \boldsymbol{F}[\bar{\rho}] \leqq 0$. From the monotonicity, it also follows that $\boldsymbol{F}[\bar{\rho}] \leqq u_{\ell}$ as by the very definition in (6.6) it holds that $\boldsymbol{F}[\bar{\rho}](0) = u_{\ell}$.

Next, we show that $\boldsymbol{F}[\bar{\rho}] \geq u_r$. To this end, let us assume, for the sake of finding a contradiction, that $\exists x^* \in (0, 1) : \boldsymbol{F}[\bar{\rho}](x^*) < u_r$. As $\boldsymbol{F}[\bar{\rho}]$ is monotonically decreasing, we know that $\boldsymbol{F}[\bar{\rho}](x) < u_r \forall x \in (x^*, 1]$. For $x = 1$, it holds $\mathcal{W}[\boldsymbol{F}[\bar{\rho}], u_r](1) = u_r$ but $\mathcal{W}[\boldsymbol{F}[\bar{\rho}], u_r](x) < u_r \forall x \in [x^*, 1)$, a contradiction to the monotonicity of $\mathcal{W}[\bar{\rho}, u_r]$ as stated in (6.10). Thus, we conclude that

$$\boldsymbol{F}[\bar{\rho}](x) \geq u_r \quad \forall x \in [0, 1].$$

Next, we prove the lower bound on $\frac{\mathrm{d}}{\mathrm{d}x}\boldsymbol{F}[\bar{\rho}]$. Recalling (6.6), we estimate

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}x}\boldsymbol{F}[\bar{\rho}](x) &\geq -u_{\ell}\frac{V(\mathcal{W}[u_r, u_r](0))\|V'\|_{L^\infty((u_r,u_{\ell}))}(u_{\ell} - \mathcal{W}[u_r, u_r](0))}{\eta V(\mathcal{W}[u_{\ell}, u_r](0))^2}\\ &= -u_{\ell}\frac{V(u_r)\|V'\|_{L^\infty((u_r,u_{\ell}))}(u_{\ell} - u_r)}{\eta V(u_{\ell})^2} = \mathcal{A}.\end{aligned}$$

Thus, we have shown that $\boldsymbol{F}[\Omega] \subset \Omega$. To show the existence of solutions, we apply Schauder's fixed-point theorem, which requires the following assumptions to be satisfied:

- $\boldsymbol{F} : \Omega \to \Omega$ is continuous in a proper topology. By choosing $C([0, 1])$ with the natural maximum norm, $\boldsymbol{F}$ is indeed continuous.
- The set Ω is closed in $C([0, 1])$ and it is convex. We have closedness as we have uniform constraints on $\bar{\rho}$ in the definition of Ω, and the convexity is evident.
- Ω is compact in $C([0, 1])$. We have this as the derivatives of functions in Ω have 0 as upper bound and $\mathcal{A}$ as lower bound, which is uniform. Therefore, the functions in Ω are uniform Lipschitz-continuous with Lipschitz-constant $\mathcal{A}$. Thus, they are also equi-continuous and we can apply Ascoli-Arzelà [18, Theorem 4.25], which guarantees the claimed compactness, i.e. $\Omega \overset{\mathrm{c}}{\rightharpoonup} C([0, 1])$.

Using Schauder's fixed-point theorem in the version in [81, Corollary 2.13], we conclude that there exists a solution of (6.4) lying in Ω as defined in (6.5).

If $u_r \leq u_\ell$, the proof of existence is almost identical to the case $u_r \geq u_\ell$ when exchanging the monotonicity in Ω from decreasing to increasing. We do not go into details.

For the uniqueness, we rewrite the steady state equation in (6.1) as a system of ODEs by introducing $g(x) = \mathcal{W}[\bar{\rho}, u_r](x)$, $x \in [0, 1]$. Then, we obtain, for $x \in (0, 1)$,

$$\begin{aligned} \bar{\rho}'(x) &= -\tfrac{1}{\eta}\tfrac{\bar{\rho}(x)V'(g(x))(g(x)-\bar{\rho}(x))}{V(g(x))}, & \bar{\rho}(0) &= u_\ell, \\ g'(x) &= \tfrac{1}{\eta}(g(x) - \bar{\rho}(x)), & g(1) &= u_r. \end{aligned} \tag{6.11}$$

Consider now instead the end value problem for $s \in [0, 1]$

$$\rho'(x) = -\frac{1}{\eta}\frac{\rho(x)V'(g(x))(g(x)-\rho(x))}{V(g(x))}, \tag{6.12}$$

$$g'(x) = \frac{1}{\eta}(g(x) - \rho(x)), \tag{6.13}$$

$$\rho(1) = s, \tag{6.14}$$

$$g(1) = u_r. \tag{6.15}$$

From the Lipschitz-continuity of the right-hand side, we immediately deduce that this end value problem has a unique solution of corresponding regularity.

Assume that $u_r < 1$ and consider as a end value problem in $\bar{\rho}$

$$\bar{\rho}_s(x) = \frac{sV(u_r)}{V(W[\bar{\rho}_s, u_r](x))}, \qquad x \in [0, 1]. \tag{6.16}$$

Then, we will show that the left-hand side boundary datum (which we want to prescribe) is strictly monotone with regard to $s \in [0, 1]$. We do this by methods of calculus of variations. The differentiability of $\bar{\rho}_s$ with regard to s follows by the implicit function theorem [81, Theorem 4.B]. To obtain an expression for the derivative, we differentiate the fixed-point problem in (6.16) with regard to $s \in [0, 1]$ and have, for $x \in [0, 1]$,

$$\begin{aligned} \partial_s \bar{\rho}_s(x) &= \frac{V(u_r)}{V(W[\bar{\rho}_s, u_r](x))} - \frac{sV(u_r)}{V(W[\bar{\rho}_s, u_r](x))^2}\partial_s V(W[\bar{\rho}_s, u_r](x)) \\ &= \frac{V(u_r)}{V(W[\bar{\rho}_s, u_r](x))} - \frac{sV(u_r)}{V(W[\bar{\rho}_s, u_r](x))^2}V'(W[\bar{\rho}_s, u_r](x))W[\partial_s\bar{\rho}_s, 0](x). \end{aligned}$$

However, this is a Volterra integral equation in $\partial_s\bar{\rho}_s$ of the second kind ($\bar{\rho}_s$ is given) and admits a unique solution by classical fixed-point methods. Moreover, thanks to the specific structure of the right-hand side, we have $\partial_s\bar{\rho}_s > 0$ on $[0, 1]$ so that we can conclude that left-hand side boundary datum (which we would like to prescribe in our original problem (6.11)) is strictly monotone with regard to s. As we have shown previously by the Schauder argument that we can achieve all left-hand side boundary data, we have the existence and uniqueness of steady state solutions for $u_r < 1$.

A similar proof can be made for $u_\ell \geq u_r$ by changing the initial boundary value problem to the corresponding end value problem and again using the existence of solutions as obtained by the previous Schauder argument.

To establish the higher regularity of solutions, we recall the fixed-point problem in (6.4) which has a unique solution by the argument above. Differentiating gives

$$\bar{\rho}'(x) = -u_\ell\frac{V(\mathcal{W}[\bar{\rho}, u_r](0))}{V(\mathcal{W}[\bar{\rho}, u_r](x))^2}V'(\mathcal{W}[\bar{\rho}, u_r](x))\partial_x\mathcal{W}[\bar{\rho}, u_r](x), \quad x \in [0, 1]. \tag{6.17}$$

We know that $\bar{\rho} \in W^{1,\infty}((0,1))$ and as $\partial_x \mathcal{W}[\bar{\rho}, u_r]$ is by (6.7) to (6.9) again Lipschitz-continuous, the entire right-hand side of (6.17) is Lipschitz-continuous and thus also $\bar{\rho}'$. This can be iterated arbitrarily, and we obtain the claimed regularity. □

Remark 6.1 (The case $u_r = 1$ in Theorem 6.1) In Theorem 6.1, we were assuming that the right-hand side nonlocal impact u_r is not 1 to avoid the need of writing the left-hand side boundary term in flux and not in density (see (1.5) and the original description of the boundary values in (1.3)). However, for $u_r = 1$ this is in general not valid, in particular the boundary condition for the steady state solution in (6.3) then requires to be formulated in terms of flux, i.e.

$$V(\mathcal{W}[\bar{\rho}, 1](0))\bar{\rho}(0) = V(\mathcal{W}[\bar{\rho}, 1](0))u_\ell \tag{6.18}$$

($V(\mathcal{W}[\bar{\rho}, 1](0))$ might be zero). This is why we consider this case explicitly and obtain from the steady state formulation in (6.1), for every $x \in [0, 1]$,

$$\bar{\rho}(x)V(\mathcal{W}[\bar{\rho}, 1](x)) = \bar{\rho}(1)V(\mathcal{W}[\bar{\rho}, 1](1)) = \bar{\rho}(1)V(1) = 0.$$

For this equation to hold, the following needs to be satisfied:

$$\bar{\rho}(x) = 0 \ \vee \ \mathcal{W}[\bar{\rho}, 1](x) = 1 \quad \forall x \in [0, 1].$$

As this also has to hold at $x = 0$, we obtain

$$\bar{\rho}(0) = 0 \ \vee \ \mathcal{W}[\bar{\rho}, 1](0) = 0.$$

This can only hold if $u_\ell = 0$ (first case) or $\bar{\rho} \equiv 1 \ \forall u_\ell \in (0, 1]$ (second case, the left-hand side boundary datum is not necessarily attained, but the flux (6.18)). However, in case $u_\ell = 0$, the solution $\bar{\rho}$ is not uniquely determined on $(0, 1)$ and all solutions can be parametrized for $a \in [0, 1]$ by

$$\bar{\rho} \equiv \chi_{[a,1]}, \qquad u_\ell = 0,\ u_r = 1.$$

In traffic, this can be interpreted as a red light at the end of the road and no entering cars. A traffic jam of any length at the traffic light is a stationary solution.

For an illustration of specific steady states, we refer to Example 5.1 and particularly to Fig. 9.

7 Conclusions and Future Work

In this contribution, we have obtained the first results on the controllability of nonlocal conservation laws on bounded domains when the nonlocal term is explicitly space-dependent and a maximum principle holds. We have also studied the long-time behavior of the solutions and established their convergence towards steady states under suitable assumptions.

There remain many problems left open in this line of research, some of which include:

1. Proving the principal theorems for a general monotonically decreasing kernel, rather than only for the exponential kernel as in this work. In this case, according to [64, Corollary 5.9], the solution to the corresponding nonlocal balance law still exists, is unique and satisfies a maximum principle. However, the proofs of some of our results appear to present many more technical difficulties.
2. Studying the relationship between the controllability of nonlocal conservation laws and the corresponding local equations in detail. A first attempt in this direction is made in

Remark 4.2. Thanks to [29], for an exponential weight as used in this contribution the convergence of the nonlocal solution to the local entropy solution when the nonlocal weight approaches the Dirac distribution is known and should be extended to the initial boundary value case. This is in particular interesting as the right-hand side boundary term in this work—located in the nonlocal term—is quite different from how boundary data are prescribed for local conservation laws [9].

3. Extending the results in Theorem 5.1 to the case when initial datum does not satisfy the lower/upper bounds in (5.1). As previously pointed out, numerical simulations suggest that these results should hold for general initial datum.
4. Extending the results in Theorem 5.1 when considering constant boundary data such that $u_\ell \neq u_r$. In this case, we expect the dynamics to converge to the steady state solutions of (1.1) with the corresponding initial and boundary data (see Theorem 6.1). This is also suggested by the corresponding numerical simulations Fig. 9.

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