Surprise at Adjoining an Identity to an Algebra

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Abstract

This note contains an example of two non-isomorphic algebras A and B over an arbitrary field K such that the algebras $\tilde{A}$ and $\tilde{B}$ obtained from A and B , respectively, by the standard process of adjoining an identity, are isomorphic. In addition, the dimension of A may be arbitrary ≥ 2 .

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1 Introduction

Throughout this paper, all considered algebras are algebras over an arbitrary field K . An element e of an algebra A is called a left (right) identity of A if $e \cdot a = a$ ($a \cdot e = a$) for each $a \in A$.

Remark 1 If e is a left identity of A and f is a right identity of A , then $e = f$. Indeed, $e \cdot a = a$ and $b \cdot f = b$ for all $a, b \in A$. Substituting $a = f$ and $b = e$ yields $e \cdot f = f = e$ and $e = f$.

Therefore, if e and f are different left identities (right identities) of an algebra A , then A has no right identity (left identity). In particular, A has no identity.

A straightforward computation shows that any isomorphism of algebras preserves both the left and right identities.

Example 1 In the algebra $M_2(K)$ of square 2×2 matrices over the field K , let $A_0 = \left\{ \begin{bmatrix} a & b \\ 0 & 0 \end{bmatrix} : a, b \in K \right\}$ and $B_0 = \left\{ \begin{bmatrix} 0 & x \\ 0 & y \end{bmatrix} : x, y \in K \right\}$. It is easily seen that A_0 and B_0 are subalgebras of $M_2(K)$. Moreover, for any $a, b, c \in K$ we have $\begin{bmatrix} 1 & c \\ 0 & 0 \end{bmatrix} \cdot \begin{bmatrix} a & b \\ 0 & 0 \end{bmatrix} =$

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$\begin{bmatrix} a & b \\ 0 & 0 \end{bmatrix}$, so $\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}$ and $\begin{bmatrix} 1 & 1 \\ 0 & 0 \end{bmatrix}$ are different left identities of A_0 . By Remark 1, the algebra A_0 has neither a right identity nor an identity. Similarly, for any $x, y, z \in K$ we obtain $\begin{bmatrix} 0 & x \\ 0 & y \end{bmatrix} \cdot \begin{bmatrix} 0 & z \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 0 & x \\ 0 & y \end{bmatrix}$, so $\begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}$ and $\begin{bmatrix} 0 & 1 \\ 0 & 1 \end{bmatrix}$ are different right identities of the algebra B_0 . Applying Remark 1 again, we infer that the algebra B_0 has neither a left identity nor an identity.

In the theory of algebras, there is a well-known method of adjoining an identity. Namely, on the set $\tilde{A} = \{(a, \alpha) : a \in A, \alpha \in K\}$ we define the multiplication, addition and scalar multiplication by the rules:

$$(a_1, \alpha_1) \cdot (a_2, \alpha_2) = (a_1 \cdot a_2 + \alpha_2 a_1 + \alpha_1 a_2, \alpha_1 \alpha_2), \quad (1)$$

$$(a_1, \alpha_1) + (a_2, \alpha_2) = (a_1 + a_2, \alpha_1 + \alpha_2), \quad (2)$$

$$\beta \cdot (a, \alpha) = (\beta a, \beta \alpha) \quad (3)$$

(see, [1]). A trivial verification shows that $\tilde{A}$ is an algebra with identity $(0, 1)$ and $A \cong \{(a, 0) : a \in A\} = \bar{A}$. Moreover, the function $\pi : \tilde{A} \rightarrow K$ given by $\pi((a, \alpha)) = \alpha$ is a homomorphism of the algebra $\tilde{A}$ onto K and $\text{Ker}(\pi) = \bar{A}$. Hence $\bar{A}$ is an ideal of $\tilde{A}$ and $\tilde{A}/\bar{A} \cong K$. Note that the algebra $\tilde{A}$ seemingly not much different from the algebra A since $\tilde{A} = \bar{A} + K \cdot (0, 1)$. It turns out that this is misleading!

An easy computation shows that, if $f : A \rightarrow B$ is an algebra isomorphism, then the function $F : \tilde{A} \rightarrow \tilde{B}$ given by $F((a, \alpha)) = (f(a), \alpha)$ is also an isomorphism. Therefore, the following natural question arises: does the fact that algebras $\tilde{A}$ and $\tilde{B}$ are isomorphic imply that algebras A and B are isomorphic? A positive answer to this question is suggested by [1, p. 12, Exercise 12]. We will show that this is not true!

2 Main Results

Lemma 1 *Let A be an algebra without identity. If A is a subalgebra of an algebra S with the identity 1, then $\tilde{A} \cong A + K \cdot 1$.*

Proof Since $1 \notin A$, we get $A \cap (K \cdot 1) = \{0\}$. Therefore, the function $f : \tilde{A} \rightarrow S$ given by $f((a, \alpha)) = a + \alpha \cdot 1$ is injective. From (2) and (3) we conclude that f is K -linear. Moreover, (1) implies that for arbitrary $a, b \in A$ and $\alpha, \beta \in K$, we obtain $f((a, \alpha) \cdot (b, \beta)) = f((a \cdot b + \beta a + \alpha b, \alpha \beta)) = a \cdot b + \beta a + \alpha b + (\alpha \beta) \cdot 1$. But $f((a, \alpha)) \cdot f((b, \beta)) = (a + \alpha \cdot 1) \cdot (b + \beta \cdot 1) = a \cdot b + \beta a + \alpha b + (\alpha \beta) \cdot 1$, so $f((a, \alpha) \cdot (b, \beta)) = f((a, \alpha)) \cdot f((b, \beta))$. Thus f is an embedding of algebras. Moreover, $f(\tilde{A}) = A + K \cdot 1$, so $\tilde{A} \cong A + K \cdot 1$. $\square$

Theorem 1 *There exist two non-isomorphic algebras A and B of arbitrary dimension ≥ 2 and without identities for which $\tilde{A} \cong \tilde{B}$.*

Proof Let C be an arbitrary algebra with identity 1. Then $S = M_2(K) \times C$ is an algebra with the identity $(I, 1)$, where $I = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$. Let $A = A_0 \times C$ and $B = B_0 \times C$, where the algebras A_0 and B_0 are as in Example 1. Then A and B are subalgebras of the algebra S . Moreover, by Example 1, $\left(\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}, 1\right)$ and $\left(\begin{bmatrix} 1 & 1 \\ 0 & 0 \end{bmatrix}, 1\right)$ are different left identities of

the algebra A . It follows from Remark 1 that the algebra A has neither a right identity nor an identity. Similarly, the algebra B has at least two right identities and has neither a left identity nor an identity. Hence, by Remark 1, the algebras A and B are not isomorphic.

By Lemma 1, $\tilde{A} \cong A + K \cdot (I, 1)$ and $\tilde{B} \cong B + K \cdot (I, 1)$. But one can easily check that $A + K \cdot (I, 1) = B + K \cdot (I, 1) = \left\{ \begin{bmatrix} p & q \\ 0 & r \end{bmatrix} : p, q, r \in K \right\} \times C$, so $\tilde{A} \cong \tilde{B}$.

Since the algebra C is arbitrary and $\dim(A_0) = 2$ we have that $\dim(A)$ can take arbitrary values ≥ 2 . $\square$

Theorem 2 *Let A be an algebra without identity, which cannot be homomorphically mapped onto the field K . If B is an algebra without identity such that $\tilde{A} \cong \tilde{B}$, then $A \cong B$.*

Proof Let $f: \tilde{A} \rightarrow \tilde{B}$ be any isomorphism of algebras. Then $f((0, 1)) = (0, 1)$ and, consequently, $f((0, \alpha)) = \alpha f((0, 1)) = \alpha(0, 1) = (0, \alpha)$ for each $\alpha \in K$. For any $a \in A$ and $\alpha \in K$ we have $(a, \alpha) = (a, 0) + (0, \alpha)$, so $f((a, \alpha)) = f((a, 0)) + (0, \alpha)$. Hence $\tilde{B} = f(\tilde{A}) + K \cdot (0, 1)$.

Suppose, contrary to our claim, that $f(\tilde{A}) \not\subseteq \tilde{B}$. Since $\tilde{B}$ is the kernel of the natural epimorphism $(b, \alpha) \mapsto \alpha$ of algebra $\tilde{B}$ onto the field K , $\pi(f(\tilde{A}))$ is a non-zero ideal of K . Thus $\pi(f(\tilde{A})) = K$. But $A \cong \tilde{A}$, so K is a homomorphic image of the algebra A . This contradicts our assumption.

Hence $f(\tilde{A}) \subseteq \tilde{B}$. But $\tilde{B} = f(\tilde{A}) + K \cdot (0, 1)$, so by the modularity of the lattice of K -subspaces of the algebra $\tilde{B}$, we get $\tilde{B} = f(\tilde{A}) + [\tilde{B} \cap K \cdot (0, 1)] = f(\tilde{A})$. It follows that $f(\tilde{A}) = \tilde{B}$ and thus $\tilde{A} \cong \tilde{B}$. But $A \cong \tilde{A}$ and $B \cong \tilde{B}$, so $A \cong B$. $\square$

Note that the assumptions of Theorem 2 are satisfied by a large class of algebras; for instance, every nil-algebra satisfies these assumptions. Moreover, if an algebra A of dimension 1 has no identity, then $A^2 = \{0\}$ and A satisfies the assumptions of Theorem 2.

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