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ORIGINAL ARTICLE

# Surprise at Adjoining an Identity to an Algebra

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**Abstract**

This note contains an example of two non-isomorphic algebras $A$ and $B$ over an arbitrary field $K$ such that the algebras $\widetilde{A}$ and $\widetilde{B}$ obtained form $A$ and $B$, respectively, by the standard process of adjoining an identity, are isomorphic. In addition, the dimension of $A$ may be arbitrary $\geq 2$.

**Keywords** Algebra over a field · Matrix

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## 1 Introduction

Throughout this paper, all considered algebras are algebras over an arbitrary field $K$. An element $e$ of an algebra $A$ is called a left (right) identity of $A$ if $e \cdot a = a$ ($a \cdot e = a$) for each $a \in A$.

*Remark 1* If $e$ is a left identity of $A$ and $f$ is a right identity of $A$, then $e = f$. Indeed, $e \cdot a = a$ and $b \cdot f = b$ for all $a, b \in A$. Substituting $a = f$ and $b = e$ yields $e \cdot f = f = e$ and $e = f$.

Therefore, if $e$ and $f$ are different left identities (right identities) of an algebra $A$, then $A$ has no right identity (left identity). In particular, $A$ has no identity.

A straightforward computation shows that any isomorphism of algebras preserves both the left and right identities.

*Example 1* In the algebra $M_2(K)$ of square $2 \times 2$ matrices over the field $K$, let $A_0 = \left\{ \begin{bmatrix} a & b \\ 0 & 0 \end{bmatrix} : a, b \in K \right\}$ and $B_0 = \left\{ \begin{bmatrix} 0 & x \\ 0 & y \end{bmatrix} : x, y \in K \right\}$. It is easily seen that $A_0$ and $B_0$ are subalgebras of $M_2(K)$. Moreover, for any $a, b, c \in K$ we have $\begin{bmatrix} 1 & c \\ 0 & 0 \end{bmatrix} \cdot \begin{bmatrix} a & b \\ 0 & 0 \end{bmatrix} =$

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$\begin{bmatrix} a & b \\ 0 & 0 \end{bmatrix}$, so $\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}$ and $\begin{bmatrix} 1 & 1 \\ 0 & 0 \end{bmatrix}$ are different left identities of $A_0$. By Remark 1, the algebra $A_0$ has neither a right identity nor an identity. Similarly, for any $x, y, z \in K$ we obtain $\begin{bmatrix} 0 & x \\ 0 & y \end{bmatrix} \cdot \begin{bmatrix} 0 & z \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 0 & x \\ 0 & y \end{bmatrix}$, so $\begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}$ and $\begin{bmatrix} 0 & 1 \\ 0 & 1 \end{bmatrix}$ are different right identities of the algebra $B_0$. Applying Remark 1 again, we infer that the algebra $B_0$ has neither a left identity nor an identity.

In the theory of algebras, there is a well-known method of adjoining an identity. Namely, on the set $\widetilde{A} = \{(a, \alpha) \colon a \in A, \alpha \in K\}$ we define the multiplication, addition and scalar multiplication by the rules:

$$(a_1, \alpha_1) \cdot (a_2, \alpha_2) = (a_1 \cdot a_2 + \alpha_2 a_1 + \alpha_1 a_2, \alpha_1 \alpha_2), \tag{1}$$

$$(a_1, \alpha_1) + (a_2, \alpha_2) = (a_1 + a_2, \alpha_1 + \alpha_2), \tag{2}$$

$$\beta \cdot (a, \alpha) = (\beta a, \beta \alpha) \tag{3}$$

(see, [1]). A trivial verification shows that $\widetilde{A}$ is an algebra with identity $(0, 1)$ and $A \cong \{(a, 0) \colon a \in A\} = \overline{A}$. Moreover, the function $\pi \colon \widetilde{A} \to K$ given by $\pi((a, \alpha)) = \alpha$ is a homomorphism of the algebra $\widetilde{A}$ onto $K$ and $\mathrm{Ker}(\pi) = \overline{A}$. Hence $\overline{A}$ is an ideal of $\widetilde{A}$ and $\widetilde{A}/\overline{A} \cong K$. Note that the algebra $\widetilde{A}$ seemingly not much different from the algebra $A$ since $\widetilde{A} = \overline{A} + K \cdot (0, 1)$. It turns out that this is misleading!

An easy computation shows that, if $f \colon A \to B$ is an algebra isomorphism, then the function $F \colon \widetilde{A} \to \widetilde{B}$ given by $F((a, \alpha)) = (f(a), \alpha)$ is also an isomorphism. Therefore, the following natural question arises: does the fact that algebras $\widetilde{A}$ and $\widetilde{B}$ are isomorphic imply that algebras $A$ and $B$ are isomorphic? A positive answer to this question is suggested by [1, p. 12, Exercise 12]. We will show that this is not true!

## 2 Main Results

**Lemma 1** *Let $A$ be an algebra without identity. If $A$ is a subalgebra of an algebra $S$ with the identity $1$, then $\widetilde{A} \cong A + K \cdot 1$.*

*Proof* Since $1 \notin A$, we get $A \cap (K \cdot 1) = \{0\}$. Therefore, the function $f \colon \widetilde{A} \to S$ given by $f((a, \alpha)) = a + \alpha \cdot 1$ is injective. From (2) and (3) we conclude that $f$ is $K$-linear. Moreover, (1) implies that for arbitrary $a, b \in A$ and $\alpha, \beta \in K$, we obtain $f((a, \alpha) \cdot (b, \beta)) = f((a \cdot b + \beta a + \alpha b, \alpha\beta)) = a \cdot b + \beta a + \alpha b + (\alpha\beta) \cdot 1$. But $f((a, \alpha)) \cdot f((b, \beta)) = (a + \alpha \cdot 1) \cdot (b + \beta \cdot 1) = a \cdot b + \beta a + \alpha b + (\alpha\beta) \cdot 1$, so $f((a, \alpha) \cdot (b, \beta)) = f((a, \alpha)) \cdot f((b, \beta))$. Thus $f$ is an embedding of algebras. Moreover, $f(\widetilde{A}) = A + K \cdot 1$, so $\widetilde{A} \cong A + K \cdot 1$. □

**Theorem 1** *There exist two non-isomorphic algebras $A$ and $B$ of arbitrary dimension $\geq 2$ and without identities for which $\widetilde{A} \cong \widetilde{B}$.*

*Proof* Let $C$ be an arbitrary algebra with identity $1$. Then $S = M_2(K) \times C$ is an algebra with the identity $(I, 1)$, where $I = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$. Let $A = A_0 \times C$ and $B = B_0 \times C$, where the algebras $A_0$ and $B_0$ are as in Example 1. Then $A$ and $B$ are subalgebras of the algebra $S$. Moreover, by Example 1, $\left( \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}, 1 \right)$ and $\left( \begin{bmatrix} 1 & 1 \\ 0 & 0 \end{bmatrix}, 1 \right)$ are different left identities of the algebra $A$. It follows from Remark 1 that the algebra $A$ has neither a right identity nor an identity. Similarly, the algebra $B$ has at least two right identities and has neither a left identity nor an identity. Hence, by Remark 1, the algebras $A$ and $B$ are not isomorphic.

By Lemma 1, $\widetilde{A} \cong A + K \cdot (I, 1)$ and $\widetilde{B} \cong B + K \cdot (I, 1)$. But one can easily check that $A + K \cdot (I, 1) = B + K \cdot (I, 1) = \left\{ \begin{bmatrix} p & q \\ 0 & r \end{bmatrix} : p, q, r \in K \right\} \times C$, so $\widetilde{A} \cong \widetilde{B}$.

Since the algebra $C$ is arbitrary and $\dim(A_0) = 2$ we have that $\dim(A)$ can take arbitrary values $\geq 2$. $\square$

**Theorem 2** *Let $A$ be an algebra without identity, which cannot be homomorphically mapped onto the field $K$. If $B$ is an algebra without identity such that $\widetilde{A} \cong \widetilde{B}$, then $A \cong B$.*

*Proof* Let $f : \widetilde{A} \to \widetilde{B}$ be any isomorphism of algebras. Then $f((0, 1)) = (0, 1)$ and, consequently, $f((0, \alpha)) = \alpha f((0, 1)) = \alpha(0, 1) = (0, \alpha)$ for each $\alpha \in K$. For any $a \in A$ and $\alpha \in K$ we have $(a, \alpha) = (a, 0) + (0, \alpha)$, so $f((a, \alpha)) = f((a, 0)) + (0, \alpha)$. Hence $\widetilde{B} = f(\overline{A}) + K \cdot (0, 1)$.

Suppose, contrary to our claim, that $f(\overline{A}) \not\subseteq \overline{B}$. Since $\overline{B}$ is the kernel of the natural epimorphism $(b, \alpha) \overset{\pi}{\longmapsto} \alpha$ of algebra $\widetilde{B}$ onto the field $K$, $\pi(f(\overline{A}))$ is a non-zero ideal of $K$. Thus $\pi(f(\overline{A})) = K$. But $A \cong \overline{A}$, so $K$ is a homomorphic image of the algebra $A$. This contradicts our assumption.

Hence $f(\overline{A}) \subseteq \overline{B}$. But $\widetilde{B} = f(\overline{A}) + K \cdot (0, 1)$, so by the modularity of the lattice of $K$-subspaces of the algebra $\widetilde{B}$, we get $\overline{B} = f(\overline{A}) + [\overline{B} \cap K \cdot (0, 1)] = f(\overline{A})$. It follows that $f(\overline{A}) = \overline{B}$ and thus $\overline{A} \cong \overline{B}$. But $A \cong \overline{A}$ and $B \cong \overline{B}$, so $A \cong B$. $\square$

Note that the assumptions of Theorem 2 are satisfied by a large class of algebras; for instance, every nil-algebra satisfies these assumptions. Moreover, if an algebra $A$ of dimension 1 has no identity, then $A^2 = \{0\}$ and $A$ satisfies the assumptions of Theorem 2.



## References

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