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# Some Common Fixed Point Theorems in 0-$\sigma$-Complete Metric-Like Spaces

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**Abstract** In this paper, we introduce the notion of 0-$\sigma$-complete metric-like space and prove some common fixed point theorems in such spaces. Our results unify and generalize several well-known results in the literature and the recent result of Amini-Harandi [Fixed Point Theory Appl. 2012:204, 2012]. Some examples are included which show that the generalization is proper.

**Keywords** Common fixed point · Metric-like space · Partial metric space

**Mathematics Subject Classification (2000)** 47H10 · 54H25

## 1 Introduction and Preliminaries

Matthews [20] introduced the notion of partial metric space as a part of the study of denotational semantics of dataflow network. In this space, the usual metric is replaced by partial metric with an interesting property that the self-distance of any point of the space may not be zero. Further, Matthews showed that the Banach contraction principle is valid in partial metric spaces and can be applied in program verification. Later, several authors generalized the result of Metthews (see, for example, [2–19, 21–31]). O'Neill [22] generalized the concept of partial metric space a bit further by admitting negative distances. The partial

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metric defined by O'Neill is called dualistic partial metric. Heckmann [17] generalized it by omitting small self-distance axiom. The partial metric defined by Heckmann is called weak partial metric. Romaguera [24] introduced the notion of 0-Cauchy sequence, 0-complete partial metric spaces and proved some characterizations of partial metric spaces in terms of completeness and 0-completeness.

Recently, Amini-Harandi [7] generalized the partial metric spaces by introducing the metric-like spaces and proved some fixed point theorems in such spaces. Amini-Harandi defined the $\sigma$-completeness of metric-like spaces. In this paper, we introduce the notion of 0-$\sigma$-completeness which generalizes the notion of the $\sigma$-completeness of [7] as well as the notion of the 0-completeness of [24]. Also, we prove common fixed point results in such spaces which generalize the results of Amini-Harandi and several well-known results of metric, partial metric spaces in metric-like spaces.

First we recall some definitions and facts about partial metric and metric-like spaces.

**Definition 1** [20, 24] A partial metric on a nonempty set $X$ is a function $p : X \times X \to \mathbb{R}^+$ ($\mathbb{R}^+$ stands for nonnegative reals) such that for all $x, y, z \in X$:

(p1) $x = y$ if and only if $p(x, x) = p(x, y) = p(y, y)$;
(p2) $p(x, x) \leq p(x, y)$;
(p3) $p(x, y) = p(y, x)$;
(p4) $p(x, y) \leq p(x, z) + p(z, y) - p(z, z)$.

A partial metric space is a pair $(X, p)$ such that $X$ is a nonempty set and $p$ is a partial metric on $X$. A sequence $\{x_n\}$ in $(X, p)$ converges to a point $x \in X$ if and only if $p(x, x) = \lim_{n\to\infty} p(x_n, x)$. A sequence $\{x_n\}$ in $(X, p)$ is called a $p$-Cauchy sequence if $\lim_{n,m\to\infty} p(x_n, x_m)$ exists and is finite. $(X, p)$ is said to be complete if every $p$-Cauchy sequence $\{x_n\}$ in $X$ converges to a point $x \in X$ such that $p(x, x) = \lim_{n,m\to\infty} p(x_n, x_m)$. A sequence $\{x_n\}$ in $(X, p)$ is called 0-Cauchy sequence if $\lim_{n,m\to\infty} p(x_n, x_m) = 0$. The space $(X, p)$ is said to be 0-complete if every 0-Cauchy sequence in $X$ converges to a point $x \in X$ such that $p(x, x) = 0$.

**Definition 2** [7] A metric-like on a nonempty set $X$ is a function $\sigma : X \times X \to \mathbb{R}^+$ such that for all $x, y, z \in X$:

($\sigma$1) $\sigma(x, y) = 0$ implies $x = y$;
($\sigma$2) $\sigma(x, y) = \sigma(y, x)$;
($\sigma$3) $\sigma(x, y) \leq \sigma(x, z) + \sigma(z, y)$.

A metric-like space is a pair $(X, \sigma)$ such that $X$ is a nonempty set and $\sigma$ is a metric-like on $X$. Note that a metric-like satisfies all the conditions of metric except that $\sigma(x, x)$ may be positive for $x \in X$. Each metric-like $\sigma$ on $X$ generates a topology $\tau_\sigma$ on $X$ whose base is the family of open $\sigma$-balls

$$B_\sigma(x, \epsilon) = \left\{y \in X : |\sigma(x, y) - \sigma(x, x)| < \epsilon\right\} \quad \text{for all } x \in X \text{ and } \epsilon > 0.$$

A sequence $\{x_n\}$ in $X$ converges to a point $x \in X$ if and only if $\lim_{n\to\infty} \sigma(x_n, x) = \sigma(x, x)$. A sequence $\{x_n\}$ is said to be $\sigma$-Cauchy if $\lim_{n,m\to\infty} \sigma(x_n, x_m)$ exists and is finite. A metric-like space $(X, \sigma)$ is called complete if for each $\sigma$-Cauchy sequence $\{x_n\}$, there exists $x \in X$ such that

$$\lim_{n\to\infty} \sigma(x_n, x) = \sigma(x, x) = \lim_{m,n\to\infty} \sigma(x_n, x_m).$$

Every partial metric space is a metric-like space but the converse may not be true.

*Example 1* [7] Let $X = \{0, 1\}$ and $\sigma : X \times X \to \mathbb{R}^+$ be defined by

$$\sigma(x, y) = \begin{cases} 2 & \text{if } x = y = 0, \\ 1 & \text{otherwise.} \end{cases}$$

Then $(X, \sigma)$ is a metric-like space, but it is not a partial metric space, as $\sigma(0, 0) \nleq \sigma(0, 1)$.

*Example 2* Let $X = \mathbb{R}$, $k \geq 0$ and $\sigma : X \times X \to \mathbb{R}^+$ be defined by

$$\sigma(x, y) = \begin{cases} 2k & \text{if } x = y = 0, \\ k & \text{otherwise.} \end{cases}$$

Then $(X, \sigma)$ is a metric-like space, but for $k > 0$, it is not a partial metric space, as $\sigma(0, 0) \nleq \sigma(0, 1)$.

**Definition 3** Let $(X, \sigma)$ be a metric-like space. A sequence $\{x_n\}$ in $X$ is called a 0-$\sigma$-Cauchy sequence if $\lim_{n,m\to\infty} \sigma(x_n, x_m) = 0$. The space $(X, \sigma)$ is said to be 0-$\sigma$-complete if every 0-$\sigma$-Cauchy sequence in $X$ converges with respect to $\tau_\sigma$ to a point $x \in X$ such that $\sigma(x, x) = 0$.

It is obvious that every 0-$\sigma$-Cauchy sequence is a $\sigma$-Cauchy sequence in $(X, \sigma)$ and every $\sigma$-complete metric-like space is 0-$\sigma$-complete. Also, every 0-complete partial metric space is a 0-$\sigma$-complete metric-like space. The following example shows that the converse assertions of these facts do not hold.

*Example 3* Let $X = [0, \infty) \cap \mathbb{Q}$ and $\sigma : X \times X \to \mathbb{R}^+$ be defined by

$$\sigma(x, y) = \begin{cases} 2x & \text{if } x = y, \\ \max\{x, y\} & \text{otherwise} \end{cases}$$

for all $x, y \in X$. Then $(X, \sigma)$ is a metric-like space. Note that $(X, \sigma)$ is not a partial metric space, as $\sigma(1, 1) = 2 \nleq \sigma(1, 0) = 1$. Now, it is easy to see that $(X, \sigma)$ is a 0-$\sigma$-complete metric-like space, while it is not a $\sigma$-complete metric-like space.

*Remark 1* It is not hard to see that, if $\sigma(x_n, x) \to \sigma(x, x) = 0$, then $\sigma(x_n, y) \to \sigma(x, y)$ for all $y \in X$.

For the following definition and proposition we refer to [1].

**Definition 4** Let $f$ and $g$ be self maps of a set $X$. If $w = fx = gx$ for some $x \in X$, then $x$ is called a coincidence point of $f$ and $g$, and $w$ is called a point of coincidence of $f$ and $g$. The pair $f, g$ of self maps is weakly compatible if they commute at their coincidence points.

**Proposition 1** *Let $f$ and $g$ be weakly compatible self maps of a set $X$. If $f$ and $g$ have a unique point of coincidence $w = fx = gx$, then $w$ is the unique common fixed point of $f$ and $g$.*

The following lemmas will be useful in the sequel.

**Lemma 1** *Let $(X,\sigma)$ be a metric-like space and let $\{x_n\}$ be a sequence in $X$ such that*

$$\lim_{n\to\infty}\sigma(x_{n+1},x_n)=0. \tag{1}$$

*If $\{x_n\}$ is not a 0-$\sigma$-Cauchy sequence in $(X,\sigma)$, then there exist $\varepsilon>0$ and two sequences $\{m_k\}$ and $\{n_k\}$ of positive integers such that $n_k>m_k>k$ and the following four sequences tend to $\varepsilon^+$ when $k\to\infty$:*

$$\big\{\sigma(x_{m_k},x_{n_k})\big\},\ \big\{\sigma(x_{m_k},x_{n_k+1})\big\},\ \big\{\sigma(x_{m_k-1},x_{n_k})\big\},\ \big\{\sigma(x_{m_k-1},x_{n_k+1})\big\}. \tag{2}$$

*Proof* Suppose that $\{x_n\}$ is a sequence in $(X,\sigma)$ satisfying (1) which is not 0-$\sigma$-Cauchy. Then there exist $\varepsilon>0$ and sequences $\{m_k\}$ and $\{n_k\}$ of positive integers such that

$$n_k>m_k>k,\qquad \sigma(x_{m_k},x_{n_k-1})<\varepsilon,\qquad \sigma(x_{m_k},x_{n_k})\ge\varepsilon$$

for all positive integers $k$. Then

$$\varepsilon\le\sigma(x_{m_k},x_{n_k})\le\sigma(x_{n_k},x_{n_k-1})+\sigma(x_{n_k-1},x_{m_k})<\varepsilon+\sigma(x_{n_k},x_{n_k-1}).$$

Using (1) we conclude that

$$\lim_{k\to\infty}\sigma(x_{m_k},x_{n_k})=\varepsilon^+. \tag{3}$$

Further,

$$\sigma(x_{m_k},x_{n_k})\le\sigma(x_{m_k},x_{n_k+1})+\sigma(x_{n_k+1},x_{n_k})$$

and

$$\sigma(x_{m_k},x_{n_k+1})\le\sigma(x_{m_k},x_{n_k})+\sigma(x_{n_k},x_{n_k+1}).$$

Passing to the limit when $k\to\infty$ and using (1) and (3), we obtain

$$\lim_{k\to\infty}\sigma(x_{m_k},x_{n_k+1})=\varepsilon^+.$$

The remaining two sequences in (2) tend to $\varepsilon^+$ can be proved in a similar way. □

**Lemma 2** *Let $(X,\sigma)$ be a metric-like space, $f,g\colon X\to X$ be mappings such that the following condition is satisfied:*

$$\sigma(fx,fy)\le\psi\big(M(x,y)\big)\quad \textit{for all } x,y\in X, \tag{4}$$

*where $\psi\colon[0,\infty)\to[0,\infty)$ is a function satisfying $\psi(t)<t$ for all $t>0$ and*

$$M(x,y)=\max\big\{\sigma(gx,gy),\sigma(gx,fx),\sigma(gy,fy),\sigma(gx,fy),\\ \sigma(gy,fx),\sigma(gx,gx),\sigma(gy,gy)\big\}.$$

*If $f$ and $g$ have a point of coincidence $z\in X$, then $\sigma(z,z)=0$.*

*Proof* Let $z\in X$ be the point of coincidence of $f$ and $g$ and $u$ be the corresponding coincidence point, that is, $gu=fu=z$. Suppose to the contrary that $\sigma(z,z)>0$.

Using (4) we obtain

$$\begin{aligned}\sigma(z,z) &= \sigma(fu,fu) \le \psi\big(M(u,u)\big)\\ &\le \psi\big(\max\{\sigma(gu,gu),\sigma(gu,fu),\sigma(gu,fu),\sigma(gu,fu),\sigma(gu,fu),\sigma(gu,gu),\\ &\qquad \sigma(gu,gu)\}\big)\\ &\le \psi\big(\max\{\sigma(z,z),\sigma(z,z),\sigma(z,z),\sigma(z,z),\sigma(z,z),\sigma(z,z),\sigma(z,z)\}\big)\\ &= \psi\big(\sigma(z,z)\big).\end{aligned}$$

Thus, using the definition of $\psi$ it follows from the above inequality that $\sigma(z,z) < \sigma(z,z)$, a contradiction. Therefore, $\sigma(z,z) = 0$. □

Now we can state our main results.

## 2 Main Results

The following theorem is a generalization and improvement of Theorem 2.4 of Amini-Harandi [7].

**Theorem 1** *Let $(X,\sigma)$ be a metric-like space. Suppose mappings $f,g\colon X \to X$ satisfy*

$$\sigma(fx,fy) \le \psi\big(M(x,y)\big) \quad \textit{for all } x,y \in X, \tag{5}$$

*where $\psi\colon [0,\infty) \to [0,\infty)$ is a nondecreasing function satisfying $\psi(t) < t$ for all $t > 0$, $\lim_{s\to t^+}\psi(s) < t$ for all $t > 0$, $\lim_{t\to\infty}(t-\psi(t)) = \infty$ and*

$$\begin{aligned}M(x,y) = \max\{&\sigma(gx,gy),\sigma(gx,fx),\sigma(gy,fy),\sigma(gx,fy),\sigma(gy,fx),\\ &\sigma(gx,gx),\sigma(gy,gy)\}.\end{aligned}$$

*If the range of $g$ contains the range of $f$ and $f(X)$ or $g(X)$ is a closed subset of $X$, then $f$ and $g$ have a unique point of coincidence in $X$. Moreover, if $f$ and $g$ are weakly compatible, then $f$ and $g$ have a unique common fixed point $v$ and $\sigma(v,v) = 0$.*

*Proof* We construct a sequence $\{y_n\}$ in $X$ as follows: let $x_0$ be an arbitrary point in $X$. Choose a point $x_1 \in X$ such that $fx_0 = gx_1 = y_1$ (say). This can be done, since the range of $g$ contains the range of $f$. Continuing this process, having chosen $x_n \in X$, we obtain $x_{n+1} \in X$ such that $fx_n = gx_{n+1} = y_n$ (say). Thus, we obtain the sequence $\{y_n\} = \{gx_{n+1}\}$ such that $fx_n = gx_{n+1} = y_n$ for all $n \in \mathbb{N}$. Consider the two possible cases.

Suppose that $y_n = y_{n+1}$ for some $n \in \mathbb{N}$. Then $gx_n = fx_n = y_n$ is a point of coincidence and the proof is finished.

Suppose that $y_n \ne y_{n+1}$ for all $n \ge 0$. We shall show that $\{y_n\}$ is a 0-$\sigma$-Cauchy sequence in $X$.

We denote by $O(y_k;n)$ the set of points $\{y_k, y_{k+1},\dots,y_{k+n}\}$ and by $O(y_k)$ the set of points $\{y_k,y_{k+1},\dots,y_{k+n},\dots\}$. First we show that $O(y_0)$ is a bounded set and for each $n \in \mathbb{N}$ there exists $k = k(n) \in \{0,1,2,\dots,n\}$ such that

$$\delta_n(y_0) = \operatorname{diam}\big(O(y_0;n)\big) = \sigma(y_0,y_k). \tag{6}$$

Suppose, to the contrary that there are positive integers $i$, $j$ such that $1 \le i \le j \le n$ and

$$\delta_n(y_0) = \sigma(y_i, y_j) > 0.$$

By (5) we obtain

$$\begin{aligned}
\delta_n(y_0) &= \sigma(y_i, y_j) = \sigma(fx_i, fx_j) \le \psi\big(M(x_i, x_j)\big) \\
&\le \psi\big(\max\{\sigma(gx_i, gx_j), \sigma(gx_i, fx_i), \sigma(gx_j, fx_j), \sigma(gx_i, fx_j), \sigma(gx_j, fx_i), \\
&\quad \sigma(gx_i, gx_i), \sigma(gx_j, gx_j)\}\big) \\
&= \psi\big(\max\{\sigma(y_{i-1}, y_{j-1}), \sigma(y_{i-1}, y_i), \sigma(y_{j-1}, y_j), \sigma(y_{i-1}, y_j), \sigma(y_{j-1}, y_i), \\
&\quad \sigma(y_{i-1}, y_{i-1}), \sigma(y_{j-1}, y_{j-1})\}\big) \\
&\le \psi\big(\delta_n(y_0)\big) < \delta_n(y_0),
\end{aligned}$$

a contradiction and the proof of (6) is complete.

Again, using the triangle inequality, (5) and (6) we obtain

$$\begin{aligned}
\delta_n(y_0) &\le \sigma(y_0, y_1) + \sigma(y_1, y_k) \\
&= \sigma(y_0, y_1) + \sigma(fx_1, fx_k) \\
&\le \sigma(y_0, y_1) + \psi\big(M(x_1, x_k)\big) \\
&\le \sigma(y_0, y_1) + \psi\big(\max\{\sigma(gx_1, gx_k), \sigma(gx_1, fx_1), \sigma(gx_k, fx_k), \sigma(gx_1, fx_k), \\
&\quad \sigma(gx_k, fx_1), \sigma(gx_1, gx_1), \sigma(gx_k, gx_k)\}\big) \\
&= \sigma(y_0, y_1) + \psi\big(\max\{\sigma(y_0, y_{k-1}), \sigma(y_0, y_1), \sigma(y_{k-1}, y_k), \sigma(y_0, y_k), \\
&\quad \sigma(y_{k-1}, y_1), \sigma(y_0, y_0), \sigma(y_{k-1}, y_{k-1})\}\big) \\
&\le \sigma(y_0, y_1) + \psi\big(\delta_n(y_0)\big),
\end{aligned}$$

that is

$$\delta_n(y_0) - \psi\big(\delta_n(y_0)\big) \le \sigma(y_0, y_1). \tag{7}$$

By definition, $\{\delta_n(y_0)\}$ is a nonincreasing sequence of nonnegative numbers, therefore there exists $\lim_{n\to\infty} \delta_n(y_0)$. Suppose $\lim_{n\to\infty} \delta_n(y_0) = \infty$, then by (7) and the choice of $\psi$ we obtain

$$\infty = \lim_{n\to\infty}\big[\delta_n(y_0) - \psi\big(\delta_n(y_0)\big)\big] \le \sigma(y_0, y_1) < \infty,$$

a contradiction. Therefore, $\lim_{n\to\infty} \delta_n(y_0) < \infty$. Let $\lim_{n\to\infty} \delta_n(y_0) = \delta(y_0)$ then

$$\delta(y_0) = \operatorname{diam}\big(\{y_0, y_1, \ldots, y_n, \ldots\}\big) = \operatorname{diam}\big(O(y_0)\big) < \infty.$$

Thus, $O(y_0)$ is bounded. Now we shall show that $\{y_n\}$ is a 0-$\sigma$-Cauchy sequence.

Set $\delta(y_n) = \operatorname{diam}(O(y_n))$, then $\delta(y_n) \le \delta(y_0) < \infty$ for all $n = 0, 1, 2, \ldots$, therefore $\{\delta(y_n)\}$ is a nonincreasing sequence of nonnegative numbers and so it converges to some $\delta \ge 0$. We shall show that $\delta = 0$. Suppose $\delta > 0$, then for any $n \ge 1$ and $r, s \ge n + 1$ we have $y_{r-1}, y_{s-1} \in O(y_n)$ and so

$$\begin{aligned}
\delta(y_{n+1}) &= \sup\bigl\{\sigma(y_r, y_s) \colon r, s \geq n+1\bigr\} \\
&= \sup\bigl\{\sigma(fx_r, fx_s) \colon r, s \geq n+1\bigr\} \\
&\leq \sup\bigl\{\psi\bigl(M(x_r, x_s)\bigr) \colon r, s \geq n+1\bigr\} \\
&\leq \sup\bigl\{\psi\bigl(\sigma(gx_r, gx_s), \sigma(gx_r, fx_r), \sigma(gx_s, fx_s), \sigma(gx_r, fx_s), \sigma(gx_s, fx_r)\bigr), \\
&\quad \sigma(gx_r, gx_r), \sigma(gx_s, gx_s) \colon r, s \geq n+1\bigr\} \\
&= \sup\bigl\{\psi\bigl(\sigma(y_{r-1}, y_{s-1}), \sigma(y_{r-1}, y_r), \sigma(y_{s-1}, y_s), \sigma(y_{r-1}, y_s), \sigma(y_{s-1}, y_r)\bigr) \\
&\quad \sigma(y_{r-1}, y_{r-1}), \sigma(y_{s-1}, y_{s-1}) \colon r, s \geq n+1\bigr\} \\
&\leq \psi\bigl(\delta(y_n)\bigr).
\end{aligned}$$

As $\delta \leq \delta(y_n)$ for all $n = 0, 1, 2, \ldots$, from the above inequality we have $\delta < \psi(\delta(y_n))$ and by the choice of $\psi$ we obtain

$$0 < \delta \leq \lim_{n\to\infty} \psi\bigl(\delta(y_n)\bigr) = \lim_{s\to\delta^+} \psi(s) < \delta,$$

a contradiction. Therefore, we have $\lim_{n\to\infty} \operatorname{diam}(\{y_n, y_{n+1}, \ldots\}) = \lim_{n\to\infty} \delta(y_n) = \delta = 0$, and so, $\lim_{n,m\to\infty} \sigma(y_n, y_m) = 0$, that is, $\{y_n\} = \{gx_{n+1}\}$ is a 0-$\sigma$-Cauchy sequence. If $g(X)$ is a closed set in $(X, \sigma)$, there exist $u, v \in X$ such that $v = gu$ and

$$\lim_{n\to\infty} \sigma(y_n, v) = \lim_{n,m\to\infty} \sigma(y_n, y_m) = \sigma(v, v) = 0.$$

We shall show that $\sigma(v, fu) = 0$. Suppose $\sigma(v, fu) > 0$, then using (5) we obtain

$$\begin{aligned}
\sigma(v, fu) &\leq \sigma(v, y_{n+1}) + \sigma(y_{n+1}, fu) = \sigma(v, y_{n+1}) + \sigma(fx_{n+1}, fu) \\
&\leq \sigma(v, y_{n+1}) + \psi\bigl(M(x_{n+1}, u)\bigr) \\
&\leq \sigma(v, y_{n+1}) + \psi\bigl(\max\bigl\{\sigma(gx_{n+1}, gu), \sigma(gx_{n+1}, fx_{n+1}), \sigma(gu, fu), \\
&\quad \sigma(gx_{n+1}, fu), \sigma(gu, fx_{n+1}), \sigma(gx_{n+1}, gx_{n+1}), \sigma(gu, gu)\bigr\}\bigr) \\
&\leq \sigma(v, y_{n+1}) + \psi\bigl(\max\bigl\{\sigma(y_n, v), \sigma(y_n, y_{n+1}), \sigma(v, fu), \sigma(y_n, fu), \\
&\quad \sigma(v, y_{n+1}), \sigma(y_n, y_n), \sigma(v, v)\bigr\}\bigr).
\end{aligned} \tag{8}$$

By Remark 1, $\lim_{n\to\infty} \sigma(y_n, fu) = \sigma(v, fu)$. Thus for large enough $n$, from (8) we obtain

$$\sigma(v, fu) \leq \sigma(v, y_{n+1}) + \psi\bigl(\max\bigl\{\sigma(v, fu), \sigma(y_n, fu)\bigr\}\bigr). \tag{9}$$

If $\sigma(v, fu) \leq \sigma(y_n, fu)$ then from (9) we obtain

$$\sigma(v, fu) \leq \sigma(v, y_{n+1}) + \psi\bigl(\sigma(y_n, fu)\bigr)$$

and as $n \to \infty$, by the definition of $\psi$ we obtain $\sigma(v, fu) < \sigma(v, fu)$, a contradiction. If $\sigma(y_n, fu) \leq \sigma(v, fu)$ then again from (9) we obtain

$$\sigma(v, fu) \leq \sigma(v, y_{n+1}) + \psi\bigl(\sigma(v, fu)\bigr)$$

and as $n \to \infty$ we obtain $\sigma(v, fu) < \psi(\sigma(v, fu)) < \sigma(v, fu)$, a contradiction. Therefore $\sigma(v, fu) = 0$, that is, $fu = gu = v$. Thus $u$ is a coincidence point and $v$ is a point of coincidence of $f$ and $g$. Suppose there exists another point of coincidence $v_1$ of $f$ and $g$ and $u_1$ is the corresponding coincidence point, that is, $fu_1 = gu_1 = v_1$. Then by Lemma 2, we obtain $\sigma(v_1, v_1) = 0$.

If $\sigma(v, v_1) > 0$ then it follows from (5) that

$$\begin{aligned}\sigma(v, v_1) = \sigma(fu, fu_1) &\le \psi\big(M(u, u_1)\big)\\ &\le \psi\big(\max\big\{\sigma(gu, gu_1), \sigma(gu, fu), \sigma(gu_1, fu_1), \sigma(gu, fu_1), \sigma(gu_1, fu),\\ &\quad \sigma(gu, gu), \sigma(gu_1, gu_1)\big\}\big)\\ &\le \psi\big(\max\big\{\sigma(v, v_1), \sigma(v, v), \sigma(v_1, v_1), \sigma(v, v_1), \sigma(v_1, v), \sigma(v, v), \sigma(v_1, v_1)\big\}\big)\\ &\le \psi\big(\sigma(v, v_1)\big) < \sigma(v, v_1),\end{aligned}$$

a contradiction. Therefore we obtain $\sigma(v, v_1) = 0$, that is, $v = v_1$. Thus, the point of coincidence of $f$ and $g$ is unique.

Suppose that $f$ and $g$ are weakly compatible, then by Remark 1, $f$ and $g$ have a unique common fixed point $v$ which is also a unique point of coincidence of $f$ and $g$ and $\sigma(v, v) = 0$.

In the case when $f(X)$ is a closed set in $(X, \sigma)$ the proof is similar. □

Taking $g = I_X$ (the identity mapping of $X$) in the above theorem we obtain the following improvement of Theorem 2.4 of Amini-Harandi [7] (in the sense that $\sigma$-completeness of space is replaced by 0-$\sigma$-completeness as well as the uniqueness of fixed point is established).

**Corollary 1** *Let $(X, \sigma)$ be a 0-$\sigma$-complete metric-like space. Suppose a mapping $f : X \to X$ satisfies*

$$\sigma(fx, fy) \le \psi\big(M(x, y)\big) \quad \textit{for all } x, y \in X,$$

*where $\psi : [0, \infty) \to [0, \infty)$ is a nondecreasing function satisfying $\psi(t) < t$ for all $t > 0$, $\lim_{s\to t^+} \psi(s) < t$ for all $t > 0$, $\lim_{t\to\infty}(t - \psi(t)) = \infty$ and*

$$M(x, y) = \max\big\{\sigma(x, y), \sigma(x, fx), \sigma(y, fy), \sigma(x, fy), \sigma(y, fx), \sigma(x, x), \sigma(y, y)\big\}.$$

*Then $f$ has a unique fixed point $v \in X$ and $\sigma(v, v) = 0$.*

Now we give some examples which illustrate our results.

*Example 4* Let $X = \{0, 1, 2, 3\}$ and $\sigma : X \times X \to \mathbb{R}^+$ be defined by

$$\sigma(x, y) = \begin{cases} 2x & \text{if } x = y,\\ \max\{x, y\} & \text{otherwise.}\end{cases}$$

Then $(X, \sigma)$ is a $\sigma$-complete metric-like space. Note that $(X, \sigma)$ is not a partial metric space as $\sigma(2, 2) = 4 \not\le \sigma(1, 2) = 2$. Define $f, g : X \to X$ by

$$f0 = 0, f1 = 0, f2 = 2, f3 = 1 \quad \text{and} \quad g0 = 0, g1 = 1, g2 = 3, g3 = 3.$$

Note that Theorem 2.4 of [7] is not applicable on $f$ because $\sigma(f2, f2) = 4$, $\sigma(2, 2) = 4$, $\sigma(2, f2) = 4$, therefore there is no function $\psi$ which satisfies the conditions of Theorem 2.4

of [7] in such a way that the contractive condition of Theorem 2.4 of [7] is satisfied. On the other hand, if we take $\psi(t) = \frac{2}{3}t$ then all the conditions of Theorem 1 are satisfied and $f$ and $g$ have a unique common fixed point, namely 0.

*Example 5* Let $X = [0, \infty) \cap \mathbb{Q}$ and $\sigma\colon X \times X \to \mathbb{R}^+$ be defined by

$$\sigma(x, y) = \begin{cases} 2x & \text{if } x = y, \\ \max\{x, y\} & \text{otherwise.} \end{cases}$$

Then $(X, \sigma)$ is a 0-$\sigma$-complete metric-like space. Define $f, g\colon X \to X$ by

$$fx = \begin{cases} 0 & \text{if } x = 1, \\ \frac{3x}{2} & \text{otherwise,} \end{cases} \quad \text{and} \quad gx = \begin{cases} 1 & \text{if } x = 1, \\ 2x & \text{otherwise.} \end{cases}$$

Then all the conditions of Theorem 1 are satisfied with $\psi(t) = \frac{3}{4}t$ and $f$ and $g$ have a unique common fixed point, namely 0. Note that $(X, \sigma)$ is not a $\sigma$-complete metric-like space, therefore Theorem 2.4 of [7] is not applicable. Also, for $g = I_X$, contractive conditions of Theorem 2.4 of [7] are not satisfied by $f$.

In the next theorem we give an improvement to Theorem 2.7 of [7].

Let $\Psi = \{\psi \mid \psi : [0, \infty) \to [0, \infty)$ is continuous, nondecreasing and $\psi^{-1}(\{0\}) = \{0\}\}$,
$\Phi = \{\varphi \mid \varphi : [0, \infty) \to [0, \infty)$ is lower semi-continuous and $\varphi^{-1}(\{0\}) = \{0\}\}$.

**Theorem 2** *Let $(X, \sigma)$ be a 0-$\sigma$-complete metric-like space. Suppose mappings $f, g : X \to X$ satisfy*

$$\psi\bigl(\sigma(fx, fy)\bigr) \le \psi\bigl(\sigma(gx, gy)\bigr) - \varphi\bigl(\sigma(gx, gy)\bigr) \tag{10}$$

*for all $x, y \in X$, where $\psi \in \Psi$, $\varphi \in \Phi$. If the range of $g$ contains the range of $f$ and $f(X)$ or $g(X)$ is a closed subset of $X$, then $f$ and $g$ have a unique point of coincidence in $X$. Moreover, if $f$ and $g$ are weakly compatible, then $f$ and $g$ have a unique common fixed point $v$ and $\sigma(v, v) = 0$.*

*Proof* Let us prove first that the point of coincidence of $f$ and $g$ is unique (if it exists). Suppose that $w_1$ and $w_2$ are two distinct points of coincidence of $f$ and $g$. Therefore there exist two point $u_1$ and $u_2$ such that $fu_1 = gu_1 = w_1$ and $fu_2 = gu_2 = w_2$. If $\sigma(w_1, w_1) > 0$, then (10) implies that

$$\begin{aligned} \psi\bigl(\sigma(w_1, w_1)\bigr) &= \psi\bigl(\sigma(fu_1, fu_1)\bigr) \\ &\le \psi\bigl(\sigma(gu_1, gu_1)\bigr) - \varphi\bigl(\sigma(gu_1, gu_1)\bigr) \\ &= \psi\bigl(\sigma(w_1, w_1)\bigr) - \varphi\bigl(\sigma(w_1, w_1)\bigr) \\ &< \psi\bigl(\sigma(w_1, w_1)\bigr), \end{aligned}$$

a contradiction. Therefore $\sigma(w_1, w_1) = 0$. Similarly, $\sigma(w_2, w_2) = 0$. Further, if $\sigma(w_1, w_2) > 0$, we have

$$\begin{aligned} \psi\bigl(\sigma(w_1, w_2)\bigr) = \psi\bigl(\sigma(fu_1, fu_2)\bigr) &\le \psi\bigl(\sigma(gu_1, gu_2)\bigr) - \varphi\bigl(\sigma(gu_1, gu_2)\bigr) \\ &= \psi\bigl(\sigma(w_1, w_2)\bigr) - \varphi\bigl(\sigma(w_1, w_2)\bigr) < \psi\bigl(\sigma(w_1, w_2)\bigr), \end{aligned}$$

a contradiction. Therefore $\sigma(w_1, w_2) = 0$, that is, $w_1 = w_2$. Thus, the point of coincidence of $f$ and $g$ is unique (if it exists).

We construct a sequence $\{y_n\}$ in $X$ as follows: let $x_0$ be an arbitrary point in $X$. Choose a point $x_1 \in X$ such that $fx_0 = gx_1 = y_1$ (say). This can be done, since the range of $g$ contains the range of $f$. Continuing this process, having chosen $x_n \in X$, we obtain $x_{n+1} \in X$ such that $fx_n = gx_{n+1} = y_n$ (say). Thus, we obtain the sequence $\{y_n\} = \{gx_{n+1}\}$ such that $fx_n = gx_{n+1} = y_n$ for all $n \in \mathbb{N}$. Consider the two possible cases.

Suppose that $y_n = y_{n+1}$ for some $n \in \mathbb{N}$. Hence $y_n = gx_n = fx_n$ is a point of coincidence and then the proof is finished.

Suppose that $y_n \neq y_{n+1}$ for all $n \geq 0$. In this case, we have

$$\begin{aligned}\psi\big(\sigma(y_{n+1}, y_n)\big) &= \psi\big(\sigma(fx_{n+1}, fx_n)\big)\\ &\leq \psi\big(\sigma(gx_{n+1}, gx_n)\big) - \varphi\big(\sigma(gx_{n+1}, gx_n)\big)\\ &= \psi\big(\sigma(y_n, y_{n-1})\big) - \varphi\big(\sigma(y_n, y_{n-1})\big)\\ &< \psi\big(\sigma(y_n, y_{n-1})\big).\end{aligned} \tag{11}$$

Now, according to the properties of function $\psi$ it follows that the sequence $\{\sigma(gx_{n+1}, gx_n)\}$ is nonincreasing. Therefore, $\sigma(y_{n+1}, y_n) \to \sigma^* \geq 0$ when $n \to \infty$.

We prove now that $\sigma^* = 0$. Indeed, passing to the limit in (11) when $n \to \infty$, we obtain $\psi(\sigma^*) \leq \psi(\sigma^*) - \varphi(\sigma^*)$ and $\sigma^* = 0$, by the properties of functions $\psi \in \Psi, \varphi \in \Phi$. Hence, $\lim_{n\to\infty} \sigma(y_{n+1}, y_n) = 0$.

We next prove that $\{y_n\} = \{gx_{n+1}\}$ is a 0-$\sigma$-Cauchy sequence in the metric-like space $(X, \sigma)$. Suppose that is not the case. Then using Lemma 1, we see that there exist $\varepsilon > 0$ and two sequences $\{m_k\}$ and $\{n_k\}$ of positive integers and sequences

$$\sigma(x_{m_k}, x_{n_k}), \sigma(x_{m_k}, x_{n_k+1}), \sigma(x_{m_k-1}, x_{n_k}), \sigma(x_{m_k-1}, x_{n_k+1}),$$

all tend to $\varepsilon^+$, when $k \to \infty$. Applying condition (10) to elements $x = x_{m_k}$ and $y = x_{n_k+1}$ and putting $y_n = fx_n = gx_{n+1}$ for each $n \geq 0$, we get

$$\psi\big(\sigma(y_{m_k}, y_{n_k+1})\big) \leq \psi\big(\sigma(y_{m_k-1}, y_{n_k})\big) - \varphi\big(\sigma\big((y_{m_k-1}, y_{n_k})\big)\big). \tag{12}$$

Letting $k \to \infty$ in (12), we obtain

$$\psi(\varepsilon) \leq \psi(\varepsilon) - \varphi(\varepsilon),$$

which is a contradiction if $\varepsilon > 0$.

This shows that $\{y_n\} = \{gx_{n+1}\}$ is a 0-$\sigma$-Cauchy sequence in $(X, \sigma)$.

If $g(X)$ is closed in $(X, \sigma)$ then there exist $u, v \in X$ such that $v = gu$ and

$$\lim_{n\to\infty} \sigma(y_n, v) = \lim_{n,m\to\infty} \sigma(y_n, y_m) = \sigma(v, v) = 0.$$

Now, putting $x = x_n$, $y = u$, $gu = v$ and $y_n = fx_n = gx_{n+1}$ in (10) we obtain

$$\psi\big(\sigma(y_n, fu)\big) \leq \psi\big(\sigma(y_{n-1}, v)\big) - \varphi\big(\sigma(y_{n-1}, v)\big). \tag{13}$$

Letting $n \to \infty$ in (13) and applying Remark 1, we get

$$\psi\big(\sigma(v, fu)\big) \leq \psi\big(\sigma(v, v)\big) - \varphi\big(\sigma(v, v)\big) = \psi(0) - \varphi(0) = 0,$$

that is, $fu = gu$. Thus, $f$ and $g$ have a unique point of coincidence. By Proposition 1, $f$ and $g$ have a unique common fixed point.

In the case when $f(X)$ is a closed set in $(X, \sigma)$ the proof is similar. □

*Remark 2* For $\psi(t) = t$ in the above theorem, we obtain Theorem 2.7 of [7]. Also, in the above theorem $\varphi$ is only lower semi-continuous, while in Theorem 2.7 of [7] $\varphi$ is nondecreasing and continuous.



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