Finance Stochast. 1, 345–352 (1997)

Finance and Stochastics

A note on pricing interest rate derivatives when forward LIBOR rates are lognormal

Beniamin Goldys

School of Mathematics, The University of New South Wales, Sydney 2052, Australia
(e-mail: b.goldys@unsw.edu.au)

Abstract. We derive the closed form pricing formulae for contracts written on zero coupon bonds for the lognormal forward LIBOR rates. The method is purely probabilistic in contrast with the earlier results obtained by Miltersen et al. (1997).

Key words: Lognormal model of LIBOR rates, contracts on zero-coupon bonds, Girsanov transformation

JEL classification: E43, G13

Mathematics Subject Classification (1991): 90A09, 60H10

0. Introduction

The aim of this note is to obtain closed form pricing formulae for contracts written on zero coupon bonds when the forward LIBOR rates are lognormal. Let $T_0 = 0 < T_1 < \ldots < T_N \leq T^*$ be a given sequence of settlement dates in an economy with the time horizon T^* and let $\delta_j = T_j - T_{j-1}$ for $j = 1, 2, \ldots, N$. Let $B(t, T)$ be the time t price of the zero coupon bond with maturity T. For a given $j \leq N - 1$ the LIBOR rate prevailing at time $t \leq T_j$ over the time interval $\left(T_j, T_j + \delta_{j+1}\right)$ is defined by

$$1 + \delta_{j+1} L\left(t, T_j\right) = \frac{B\left(t, T_j\right)}{B\left(t, T_{j+1}\right)}.$$

The lognormal model of LIBOR rates given by the Itô equation

$$dL(t, T_j) = \mu\left(t, T_j\right) L\left(t, T_j\right) dt + \gamma\left(t, T_j\right) L\left(t, T_j\right) dW(t) \tag{1}$$

Manuscript received: November 1995; final version received: June 1997

has attracted much attention recently because it avoids many shortcomings of previous models: it agrees with the market practice, provides positive rates and gives finite Eurodollar futures prices. Sandmann and Sondermann (1993) noticed for the first time that the continuously compounded interest rate is not the right one to be modelled. They have shown that the lognormal model of the effective annual rates leads to finite Eurodollar futures prices (see also Sandmann and Sondermann (1997)). The idea to build a model based on a version of effective rates has been developed in Goldys et al. (1995) and in Brace et al. (1997), where equation (1) is introduced and studied in the framework of infinite dimensional diffusion processes. A thorough discussion of this model can be also found in Musiela and Rutkowski (1995). A different approach, based on the inspired guess of the equation for bond prices has been proposed in Rady and Sandmann (1994). Their approach also produced positive yields and the closed form formulae for the option prices.

In a series of papers Miltersen, Sandmann and Sondermann (see for example Miltersen et al. (1997) and references therein) demonstrated how to use the hedging argument in forward market to derive the closed form pricing formulae for caps. It turned out that the derived formulae agree with the market practice to price the options with the Black futures formula. The main step in Sandmann et al. (1995) consisted of the derivation of the option price on zero coupon bond by solving the partial differential equation corresponding to the forward price process. A different approach has been proposed by Brace et al. (1997), where the price of a cap is obtained directly by the convenient forward measure transformation.

Given the advantages of the lognormal model of forward LIBOR rates, it seems that the model given by (1) might provide a basis for pricing various related contingent claims. In this paper we propose a purely probabilistic method of price calculation for a large class of contingent claims based on the LIBOR rate. This method allows to obtain the result of Miltersen et al. (1997) in a natural way and can be also useful for other, more general, pricing problems in the lognormal model. Results close to ours have been recently obtained by Rady (1997), where the approach is also probabilistic. Rady applies the change of numeraire method to derive prices of some standard options in the model slightly more general than ours.

We will consider a class of contracts which pay $g(B(T,T+\delta))$ dollars at maturity T (in what follows we consider one period only and therefore for simplicity of notation we omit the period number j). Let

$$F(t,T,T+\delta) = \frac{B(t,T+\delta)}{B(t,T)}$$

be the forward price process. Then (1) implies that after the appropriate change of measure, (see Miltersen et al. (1997)) the process $F(t,T,T+\delta)$ satisfies, under the forward measure P_T, the equation

$$dF(t,T,T+\delta) = -\gamma(t,T)F(t,T,T+\delta)(1-F(t,T,T+\delta))dW_T(t), \qquad (2)$$

where (W_T) is a Wiener process under the forward measure. The time $t(\leq T)$ price of this contract is

$$V(t,T,T+\delta) = B(t,T)E_T\left(g(F(T,T,T+\delta))\middle|\,\mathscr{F}_t\right) \tag{3}$$

where E_T is the expectation with respect to the forward measure. The solution to equation (2) is a Markov process and an exponential martingale. Therefore putting $x = F(t,T,T+\delta)$ we find that (see below for details)

$$V(t,T,T+\delta) = B(t,T)E_T g\left(x\frac{F(T,T,T+\delta)}{F(t,T,T+\delta)}\right).$$

In Theorem 1 below we give an explicit formula for expressions of the type

$$E_T g\left(x\frac{F(T,T,T+\delta)}{F(t,T,T+\delta)}\right)$$

and in particular we recover the result of Miltersen et al. (1997).

1. The pricing formulae

We start with the discussion of the following abstract version of (2). Let $X(\cdot,x)$, $t \geq 0$, be a solution to the equation

$$\begin{cases} dX(t,x) = -X(t,x)(1-X(t,x))\gamma(t)dW(t), \\ X(0,x) = x,\ t \leq T, \end{cases} \tag{4}$$

where we assume that the function $\gamma : [0,T] \to \mathbf{R}$ is bounded and measurable. The process W is a Wiener process defined on a stochastic basis $\left(\Omega, \mathscr{F}, \left(\mathscr{F}_t\right), P\right)$, where the filtration $\left(\mathscr{F}_t\right)$ satisfies the usual conditions. If we fix the initial condition then we write $X(t)$ instead of $X(t,x)$. Existence of a unique global solution to (4) can be easily deduced from the general theory of stochastic differential equations. However, in Lemma 1 below we provide a direct proof by means of a simple transformation which is also crucial for the further calculations. Consider the following stochastic differential equation

$$dZ(t) = \frac{1}{2}\frac{1-e^{-Z(t)}}{1+e^{-Z(t)}}\gamma^2(t)dt - \gamma(t)dW(t). \tag{5}$$

Since the drift term in this equation is defined by a bounded and globally Lipschitz function, equation (5) has a unique strong nonexploding solution (see for example Theorem 5.2.9 in Karatzas and Shreve (1988)).

Lemma 1. *For every* $x \in (0,1)$ *the process*

$$X(t,x) = \frac{1}{1+e^{-Z(t,z)}}$$

is a unique strong and nonexploding solution to equation (4), where $Z(\cdot,z)$ *denotes the solution of (5) starting from*

$$z = \log\frac{x}{1-x}.$$

Moreover, $0 < X(t,x) < 1$ *for every* $t \geq 0$*.*

Proof. It is easy to see that equation (5) can be rewritten in the form

$$dZ(t) = \left(X(t) - \frac{1}{2}\right)\gamma^2(t)dt - \gamma(t)dW(t). \tag{6}$$

Hence, applying the Itô formula to the process X we find that

$$\begin{aligned} dX(t) &= -X(t)(1 - X(t))dZ(t) + X(t)(1 - X(t))\left(X(t) - \frac{1}{2}\right)\gamma^2(t)dt \\ &= -X(t)(1 - X(t))\gamma(t)dW(t). \end{aligned}$$

Therefore, the process X is a solution to (4). Conversely, if X is any local weak solution to (4) then it is in fact a strong solution because the diffusion coefficient in (4) is locally Lipschitz. Moreover, using the Itô formula in the same way as in the first part of the proof we can show that the process

$$Z(t) = \log\frac{X(t)}{1 - X(t)}$$

is a strong solution of equation (5), hence can be continued to a global one which is unique. The last part of the lemma follows trivially from the definition of the process X and uniqueness of solutions to (4).

From now on we assume that $X(0, x) = x \in (0, 1)$.

Theorem 1. *Let* $g : \mathbf{R} \to \mathbf{R}$ *be a nonnegative Borel function. Then for every* $x \in (0, 1)$ *and* $T > 0$ *the expected value* $V(T, x) = Eg(X(T, x))$ *is given by the formula*

$$\begin{aligned} V(T, x) &= \sqrt{x(1 - x)}\exp\left(-\frac{1}{8}\langle M\rangle_T\right) \\ &\quad \times E\left(g\left(\frac{1}{1 + e^{-(z+M_T)}}\right)\left(e^{(z+M_T)/2} + e^{-(z+M_T)/2}\right)\right), \end{aligned}$$

where $z = \log\frac{x}{1-x}$ *and* $M_t = \int_0^t \gamma(s)\,dW(s)$.

Proof. The proof is based on a simple idea that for any random variable Y (say) we have

$$EY = E\left((1 - X(t))Y\right) + E\left(X(t)Y\right).$$

Then we show that both processes $X(t)$ and $1-X(t)$ are exponential martingales and therefore the Girsanov theorem can be applied to both terms of the above expression. It turns out that after the change of measure the random variable Y is an exponential of the Gaussian semimartingale which, after one more change of measure, can be transformed into a Gaussian martingale. Finally, some simple manipulations conclude the proof.

For any locally bounded and predictable process Φ defined on $(\Omega, \mathscr{F}, (\mathscr{F}_t), P)$ we will use the notation

$$\mathscr{E}_t\left(\int \Phi dW\right) = \exp\left(\int_0^t \Phi_s dW_s - \frac{1}{2}\int_0^t \Phi_s^2 ds\right).$$

It can be easily checked by the Itô formula that

$$X(t) = x\,\mathscr{E}_t\left(\int (X-1)\gamma\, dW\right). \tag{7}$$

Note that the process $X_1(t) = 1 - X(t)$ satisfies the equation

$$dX_1(t) = X_1(t)\,(1 - X_1(t))\,\gamma(t)dW(t)$$

and therefore

$$1 - X(t) = (1-x)\mathscr{E}_t\left(\int X\gamma\, dW\right)$$

Hence

$$\begin{aligned} V(T,x) &= E\left((1 - X(T,x))g\left(\frac{1}{1+e^{-Z(T,x)}}\right)\right) \\ &\quad + E\left(X(T,x)g\left(\frac{1}{1+e^{-Z(T,x)}}\right)\right) \\ &= (1-x)E\left(\mathscr{E}_T\left(\int X\gamma\, dW\right) g\left(\frac{1}{1+e^{-Z(T,x)}}\right)\right) \\ &\quad + xE\left(\mathscr{E}_T\left(\int (X-1)\gamma\, dW\right) g\left(\frac{1}{1+e^{-Z(T,x)}}\right)\right). \end{aligned}$$

Let P^X and P^{1-X} be such probability measures on $(\Omega, \mathscr{F})$ that

$$\frac{dP^X}{dP} = \mathscr{E}_T\left(\int X\gamma\, dW\right)$$

and

$$\frac{dP^{1-X}}{dP} = \mathscr{E}_T\left(\int (X-1)\gamma\, dW\right).$$

Taking into account that by (6)

$$dZ(t) = \left((X(t) - 1) + \frac{1}{2}\right)\gamma^2(t)dt - \gamma(t)dW(t)$$

we find that under the measure P^X the process Z satisfies the equation

$$dZ(t) = -\frac{1}{2}\gamma^2(t)dt - \gamma(t)dW(t)$$

and under the measure P^{1-X}

$$dZ(t) = \frac{1}{2}\gamma^2(t)dt - \gamma(t)dW(t).$$

Therefore

$$V(T,x) = (1-x)$$
$$\times E^X\left(g\left(\left(1+\exp\left(-z+\int_0^T\gamma(s)\,dW(s)+\frac{1}{2}\int_0^T\gamma^2(s)\,ds\right)\right)^{-1}\right)\right)$$
$$+xE^{1-X}\left(g\left(\left(1+\exp\left(-z+\int_0^T\gamma(s)\,dW(s)-\frac{1}{2}\int_0^T\gamma^2(s)\,ds\right)\right)^{-1}\right)\right),$$

where E^X and E^{1-X} denote expectations with respect to the measures P^X and P^{1-X} respectively and in each term W is a Wiener process under the corresponding measure. Note also that by the Girsanov theorem the process

$$M_t^- = M_t - \frac{1}{2}\langle M\rangle_t$$

under the probability measure P^- such that

$$\frac{dP^-}{dP^{1-X}} = \mathscr{E}_T\left(\frac{1}{2}M\right)$$

is a martingale with the quadratic variation equal to $\langle M\rangle_t$ and so is the process

$$M_t^+ = M_t + \frac{1}{2}\langle M\rangle_t$$

under the measure P^+ such that

$$\frac{dP^+}{dP^X} = \mathscr{E}_T\left(-\frac{1}{2}M\right).$$

Hence, denoting by E^+ and E^- expectations with respect to the measures P^+ and P^- respectively we find that

$$V(T,x) = (1-x)E^+\left(\mathscr{E}_T\left(\frac{1}{2}M^+\right)g\left(\frac{1}{1+e^{-z+M_T^+}}\right)\right)$$
$$+xE^-\left(\mathscr{E}_T\left(-\frac{1}{2}M^-\right)g\left(\frac{1}{1+e^{-z+M_T^-}}\right)\right).$$

Since the laws of M^+ and M^- under P^+ and P^- respectively are the same as the law of M under P which is symmetric we find that

$$V(T,x) = \exp\left(-\frac{1}{8}\langle M\rangle_T\right)E^+\left(g\left(\frac{1}{1+e^{-z-M_T^+}}\right)\left((1-x)e^{-\frac{1}{2}M_T^+}+xe^{\frac{1}{2}M_T^+}\right)\right)$$
$$= \sqrt{x(1-x)}\exp\left(-\frac{1}{8}\langle M\rangle_T\right)E\left(g\left(\frac{1}{1+e^{-(z+M_T)}}\right)\left(e^{-\frac{1}{2}(z+M_T)}+e^{\frac{1}{2}(z+M_T)}\right)\right)$$

which concludes the proof of the theorem.

As a consequence of Theorem 1 we find the formula derived earlier in Miltersen et al. (1997).

Corollary 1. *The time $t(\leq T)$ price of a European call option with expiry T and strike price K on a zero coupon bond maturing at $T+\delta$ is*

$$\begin{aligned} V(t,T,T+\delta) &= (1-K)B(t,T+\delta)\,N\left(-l+\frac{1}{2}\sigma(t,T)\right) \\ &\quad -K\left(B(t,T)-B(t,T+\delta)\right)N\left(-l-\frac{1}{2}\sigma(t,T)\right), \end{aligned}$$

where

$$\sigma^2(t,T)=\int_t^T \gamma^2(s,T)\,ds$$

and

$$l(t,T)=\frac{1}{\sigma(t,T)}\log\frac{K\left(B(t,T)-B(t,T+\delta)\right)}{(1-K)B(t,T+\delta)}$$

Proof. Using (3) and putting $x=F(t,T,T+\delta)$ we find that

$$V(t,T,T+\delta)=B(t,T)E_T\left(x\frac{F(T,T,T+\delta)}{F(t,T,T+\delta)}-K\right)^+.$$

Let $g(y)=(y-K)^+$ with $K>0$. Using the notation $\sigma^2=\sigma^2(t,T)$ and

$$L=\frac{1}{\sigma}\left(-z+\log\frac{K}{1-K}\right)$$

we obtain from Theorem 1

$$\begin{aligned} V(T,x) &= \sqrt{x(1-x)}\exp\left(-\frac{1}{8}\sigma^2\right)\int_L^\infty\left(\frac{1}{1+e^{-(z+\sigma y)}}-K\right) \\ &\quad\times\left(e^{-\frac{1}{2}(z+\sigma y)}+e^{\frac{1}{2}(z+\sigma y)}\right)n(y)\,dy, \end{aligned}$$

where n stands for the standard normal density. Therefore

$$\begin{aligned} V(T,x) &= \sqrt{x(1-x)}\exp\left(-\frac{1}{8}\sigma^2\right) \\ &\quad\cdot\left(\int_L^\infty\frac{1}{1+e^{-(z+\sigma y)}}\left(e^{-\frac{1}{2}(z+\sigma y)}+e^{\frac{1}{2}(z+\sigma y)}\right)n(y)\,dy\right. \\ &\quad\left.-K\int_L^\infty\left(e^{-\frac{1}{2}(z+\sigma y)}+e^{\frac{1}{2}(z+\sigma y)}\right)n(y)\,dy\right) \\ &= \sqrt{x(1-x)}\exp\left(-\frac{1}{8}\sigma^2\right)(I_1-KI_2)\,. \end{aligned} \tag{8}$$

We have

$$I_1=\int_L^\infty e^{\frac{1}{2}(z+\sigma y)}n(y)\,dy=e^{z/2}e^{\sigma^2/8}\left(1-N\left(L-\frac{1}{2}\sigma\right)\right)$$

and

$$I_2 = I_1 + e^{-z/2} e^{\sigma^2/8} \left(1 - N\left(L + \frac{1}{2}\sigma\right)\right).$$

Therefore taking (8) into account we obtain

$$\begin{aligned} V(T,x) &= \sqrt{x(1-x)}(1-K)e^{z/2}\left(1 - N\left(L - \frac{1}{2}\sigma\right)\right) \\ &\quad -K\sqrt{x(1-x)}e^{-z/2}\left(1 - N\left(L + \frac{1}{2}\sigma\right)\right) \\ &= x(1-K)N\left(-L + \frac{1}{2}\sigma\right) - K(1-x)N\left(-L - \frac{1}{2}\sigma\right). \end{aligned}$$

Finally, identifying $\gamma(t)$ with $\gamma(t,T)$ we obtain the corollary.

Acknowledgement. The author is indebted to A. Brace and M. Musiela for suggesting the problem and to D. Sondermann and the anonymous referee for the information about some references unknown to the author.

References

1. Brace, A., Gatarek, D., Musiela, M.: The market model of interest rate dynamics. Mathematical Finance **7**, 127–154 (1997)
2. Goldys, B., Musiela, M., Sondermann, D.: Lognormality of rates and term structure models. Report S95-16 of the School of Mathematics, The University of New South Wales, 1995
3. Karatzas, I., Shreve, S.E.: Brownian motion and stochastic calculus. Berlin, Heidelberg, New York: Springer 1988
4. Miltersen, K.R., Sandmann, K., Sondermann, D.: Closed form solutions for term structure derivatives with log-normal interest rates. Journal of Finance **52**, 409–430 (1997)
5. Musiela, M., Rutkowski, M.: Continuous-time term structure models: Forward measure approach. Finance Stochast. **1**, 259–289 (1997)
6. Rady, S.: Option pricing with a quadratic diffusion term. Discussion Paper 226, LSE Financial Markets Group, London, 1997
7. Rady, S., Sandmann, K.: The direct approach to debt option pricing. Rev. Futures Markets **13** (2), 461–514 (1994)
8. Sandmann, K., Sondermann, D.: On the stability of lognormal interest rate models. Working paper B-263, SFB 303, University of Bonn, 1993
9. Sandmann, K., Sondermann, D.: A note on the stability of lognormal interest rate models and the pricing of Eurodollar futures. Math. Finance **7**(2), 119–125 (1997)
10. Sandmann, K., Sondermann, D., Miltersen, R.K.: Closed form term structure derivatives in a Heath-Jarrow-Morton model with log-normal annually compounded interest rates. Proceedings of the Seventh Annual European Futures Research Symposium, Bonn, September 1994, Chicago Board of Trade, 145–164, 1995