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LIBOR and swap market models and measures*

Farshid Jamshidian

New Products and Strategic Trading, Sakura Global Capital, 42 New Broad Street,
London EC2M 1JX, UK (e-mail: farshid@sgc.com)

Abstract. A self-contained theory is presented for pricing and hedging LIBOR and swap derivatives by arbitrage. Appropriate payoff homogeneity and measurability conditions are identified which guarantee that a given payoff can be attained by a self-financing trading strategy. LIBOR and swap derivatives satisfy this condition, implying they can be priced and hedged with a finite number of zero-coupon bonds, even when there is no instantaneous saving bond. Notion of locally arbitrage-free price system is introduced and equivalent criteria established. Stochastic differential equations are derived for term structures of forward libor and swap rates, and shown to have a unique positive solution when the percentage volatility function is bounded, implying existence of an arbitrage-free model with such volatility specification. The construction is explicit for the lognormal LIBOR and swap "market models", the former following Musiela and Rutkowski (1995). Primary examples of LIBOR and swap derivatives are discussed and appropriate practical models suggested for each.

Key words: LIBOR and swap derivatives, self-financing trading strategies, homogenous payoffs, stochastic differential equations

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1. Introduction

Traditionally, models of interest rate have dealt with continuously compounded, instantaneous rates. The instantaneous, continuously compounded spot rate and

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forward rates and a continuum of discount factors are constructed and options are evaluated, using the risk-neutral measure. The latter is arrived at by taking as numeraire the continuously compounded saving account (bond). Yet, in the actual market place, the rates applicable to interest-rate derivatives, foremost among them LIBOR and swap derivatives, are quoted for accrual periods of at least a month, commonly three or six months, and their calculation is simple rather than continuously compounded. Moreover, the market quotes liquid caps and (European) swaptions in terms of implied Black-Scholes volatilities, implicitly assuming forward LIBOR and swap rates follow lognormal processes with the quoted volatilities.

The traditional models take a continuum of initial instantaneous forward rates or discount factors as given, and construct a continuum of processes, making assumptions either on the dynamics of the instantaneous spot rate (possibly dependent on several state variables) or volatilities of instantaneous forward rates. (A comprehensive treatment can be found in Musiela and Rutkowski (1997).) In order to match market quoted prices of caps or European swaptions, they need to suitably parametrize degrees of freedom in their specified dynamics of instantaneous spot rate or volatilities of instantaneous forward rates, and then "calibrate" these parameters to quoted prices by, in general, a multidimensional and often highly computationally intensive, numerical root search algorithm. The resultant processes for forward LIBOR or swap rates are analytically intractable, and generally bear no resemblance to lognormality.

Only recently, arbitrage-free models have appeared that model LIBOR or swap rates directly as the primary process rather than a secondary process derived from instantaneous rates. By a direct hedging argument, Neuberger (1990) derived the industry standard Black-Scholes formula for European swaptions. However, a term structure of swap rates (or LIBOR rates), which is necessary for modelling of more complex derivatives, such as Bermudan swaptions, was not developed.

Sandmann and Sondermann (1993, 1994) proposed a lognormal model for the effective rate and showed that it circumvents certain instabilities (particularly with Eurodollars) present in lognormal, continuously compounded rate models. This was further developed within the framework of Heath et al. (1992) by Goldys et al. (1994) and Musiela (1994). Continuing in this framework, the emphasis was shifted to LIBOR rates in Miltersen et al. (1995, 1997) and Brace et al. (1997), where by different techniques a term structure model of lognormal LIBOR rates was constructed which priced caplets by the industry standard Black-Scholes formula – and for this reason termed the "market model" by the latter. Such a model is now automatically calibrated to caplet prices and can be used to evaluate more complex products like captions and callable capped floating rate notes.

Brace, Gatarek and Musiela's approach was complicated by the fact that the LIBOR market model dynamics was specified in the risk-neutral measure, and as such still relied on the continuously compounded spot rate, which they (implicitly) assumed to have finite variation. A satisfactory and transparent construction was subsequently carried out by Musiela and Rutkowski (1995) in the forward-risk-adjusted measure (of the final payment). Their "backward induction" was an explicit recursive equation for the term structure of forward LIBOR rates with a lognormal (i.e., deterministic percentage) volatility. In particular, it followed that prices of all LIBOR derivatives depended only on the finite number of discount factors that define the LIBOR rates.

Our approach to LIBOR and swap derivatives is largely motivated by Musiela and Rutkowski (1995). We do not use continuously compounded instantaneous interest rates, or the risk-neutral measure: LIBOR and swap derivatives are evaluated and hedged without them. Key to this is the fact that payoffs of LIBOR and swap derivatives are homogeneous of degree one in the discount factors that define these rates[1]. As such, relative to a zero-coupon bond numeraire (or a linear combination of them), the payoff is a function only of (the path of) LIBOR and swap rates. As we will show, under suitably general conditions, such payoffs can be attained by self-financing trading strategies involving only the finite number of zero-coupon bonds that define LIBOR rates, even in some situations where there is no instantaneous saving bond, or the market is incomplete.

The no-arbitrage framework adopted here assumes the existence of a *state price deflator* which makes deflated zero-coupon bond prices martingales (in the actual measure), as in Duffie (1992, Chapter 6). This framework does not require the instantaneous saving bond. Moreover, the condition is evidently only on the ratios of zero-coupon bond prices. But, forward LIBOR (and swap) rates are defined in terms of these ratios, and vice versa. As such, the no-arbitrage condition naturally translates into a constraint on forward LIBOR (or swap) rates, an equation which relates the drifts and covariance matrix of the rates. This leads to the existence of a unique arbitrage-free term structure of forward LIBOR (or swap) rates from an arbitrary specification of forward LIBOR (or or swap) volatility function, with an explicit construction in the lognormal case. LIBOR and swap derivative prices and hedges are then determined, because, as already mentioned, their payoffs (relative to appropriate numeraires) are specified directly in terms of LIBOR and swap rates.

The content is as follows. The next section will establish notation, recalls various mathematical facts and records some preliminary results. We work in a continuous semimartingale framework. Not depending on the choice of a Brownian motion as a basis, it makes the derivations and formulae more conceptual and arguably simpler. We will specialize to Ito processes when constructing models as solutions of stochastic differential equations (SDE).

Section 3 discusses self-financing trading strategies (SFTS). A useful criteria is established which facilitates several examples. The European swaption (including stepup and amortizing varieties) is treated as an instance of the option to exchange two assets, leading to its market model Black-Scholes formula. A more general example derives the SFTS (and price) for a “trigger swap.” In this section we also point out a fact which becomes a theme for the rest of the paper. Suppose we are given a finite number of assets, whose covariance matrix of instantaneous returns is nonsingular. Then a payoff which is a function of these asset prices at expiration cannot be attained by a SFTS, unless the payoff function is homogeneous of degree one in the prices. This is a situation where the market is incomplete, e.g., a European call option on an individual asset cannot be replicated, unless another asset is a zero-coupon bond maturing at the option expiration. The reason is that an instantaneous saving bond cannot be replicated in the non-singular covariance case. Yet homogeneous payoffs, such as an option to exchange two assets, can be replicated, because a long position in one asset can be financed by a short position in the other. Thus, as long as the homogeneity property is satisfied, the existence of an instantaneous saving bond is an unnecessary restriction.

Section 4 introduces "locally arbitrage-free" (LAF) price systems. This enables to bypass technical integrability conditions, and instead concentrate on the linear constraint between the drift and covariance matrix that underlines the essence of the no-arbitrage condition. These constraints are also formulated in terms of forward LIBOR rates, leading to an SDE, and in the case of the LIBOR market model to its explicit solution by the "backward induction" technique of Musiela and Rutkowski (1995). The LAF condition already implies to some extent that there are no "free lunches." It also enables deriving prices and hedging strategies for path-independent payoffs, as solution to a "fundamental differential equation" expressed in terms of forward LIBOR.

Section 5 strengthens the LAF condition by assuming that the relevant local martingales are actually martingales. This enables stronger no-free lunch results, representation of appropriately measurable payoffs by self-financing trading strategies, and their valuation by taking expectation. We also prove the existence of an arbitrage-free term structure of positive forward LIBOR rates, given an absolute forward LIBOR volatility function of linear growth. We use the technique of change of numeraire as applied for the special case of zero-coupon bonds and forward risk adjustment in Jamshidian (1987) and El Karoui and Rochet (1989) (and in connection with exchange rates in Jamshidian (1993)) and described for general numeraires in, for instance, Babbs and Selby (1993), and more fully in this connection, by Geman et al. (1995).

In Sect. 6, we introduce a tenor structure and with it the notion of a tenor adapted self-financing trading strategy. A "spot LIBOR measure" is constructed which shares many characteristics of the risk neutral measure (e.g., prices are "discounted along the path before averaging"), yet is well-adapted to LIBOR and swap derivatives. The SDE for forward LIBOR here resembles the Heath et al. (1992) "forward rate restriction" for instantaneous forward rates. They had to assume a bounded absolute volatility function for otherwise the solution may explode. But here, a unique positive solution exists when the percentage volatility is bounded (absolute volatility having linear growth).

Assuming a tenor structure, Sect. 7 finally imposes the unity constraint ($B_i(T_i) = 1$) for zero-coupon bonds at maturity. The existence results are extended to enforce this constraint. Forward LIBOR rates are uniquely determined as before as the solution of an SDE. But the numeraire is uniquely determined only at the tenor dates. Different continuous interpolation of these values give rise to different arbitrage-free models. However, relative prices of LIBOR and swap derivatives are independent of the choice of interpolation, as are the prices themselves at all tenor dates. Moreover, implementation algorithms need only to construct the LIBOR process, not any numeraire.

Section 8 extends the results for LIBOR to the term structure of forward swap rates with a fixed end date. The dynamics of this term structure in the last-maturity forward-risk-adjusted measure is derived by noting that each forward swap rate is a martingale in an associated "forward swap measure" whose numeraire is an annuity. As in the LIBOR case, the SDE has a unique positive solution, and the swap market model can be constructed explicitly by backward induction. However, we point out that the lognormal LIBOR and swap market models are inconsistent with each other.

The next four sections are devoted to applications. We discuss in some detail some primary examples of LIBOR and swap derivatives, including knockout, Asian, periodic, and flexible caps in Sect. 9, Bermudan swaptions, captions, and callable capped floating rate swaps in Sect. 10, LIBOR in arrears and constant maturity swaps in Sect. 11, and spread options, triggered swaps, index amortizing swaps, and callable reverse floaters in Sect. 12.

The path-dependent derivatives of Sect. 9 can be accurately evaluated by constructing random paths of the LIBOR process using either the forward-risk-adjusted or spot LIBOR dynamics, followed by averaging. By "forward transporting" contingent payoffs, we can actually cast the Bermudan derivatives of Sect. 10 as ordinary path-dependent options. But since their definition is recursive, a multitude of conditional expectations have to be computed, for which conventional Monte Carlo simulation is inadequate, unless there are only two (possibly three) exercise dates. "Bushy trees" are more appropriate, but they have their own limitations. A simple "non-arbitrage free approximation" is suggested as a numerically robust alternative. Section 11 deals with "convexity adjustment." Interestingly, a related problem arises in statistical genetics, for which a closed-form solution is now provided. The options of Sect. 12 depend on both caplet and swaption volatilities. We indicate how to construct a swap market model which is root-search calibrated to caplet prices. But, since root-search calibration is not in the spirit of market models, we suggest alternative "improvised models."

A concluding section elaborates on the policy of adopting different models for different products.

2. Notation and mathematical preliminaries

Let $T > 0$, and $(\Omega, \mathscr{F}, P, \mathscr{F}_t)$, $t \in [0, T]$, be a complete filtered probability space satisfying the usual hypotheses. The value at time t of a stochastic process X will be denoted by X_t or $X(t)$, as convenient. The $\mathscr{F}_t$ conditional expectation operator is denoted E_t, or E_t^P. Let

$$\mathscr{E} \equiv \{\text{continuous semimartingales on } [0, T]\}\ , \quad \mathscr{E}_+ \equiv \{X \in \mathscr{E} : X_t > 0,\ \forall t\}\ .$$

The quadratic covariation process of $X, Y \in \mathscr{E}$ is denoted $\langle X, Y \rangle$. In our convention, $\langle X, Y \rangle_0 = 0$ rather than $X_0 Y_0$. The quadratic variation $\langle X \rangle \equiv \langle X, X \rangle$, being a continuous increasing process, induces pathwise a measure $d\langle X\rangle$ on $[0,T]$, and $d\langle X,Y\rangle$ is likewise a signed measure by polarization. Other notations for $d\langle X,Y\rangle$ often seen are $dX\cdot dY$, $\mathrm{cov}(dX,dY)$ and $\langle dX,dY\rangle$. For $X\in\mathscr{E}$, and an appropriate integrand σ, we denote by $\int\sigma dX$ the process in $\mathscr{E}$ whose value at time t is the stochastic integral $\int_0^t\sigma_s dX_s$. In our convention $\int_0^0\sigma_s dX_s$ is 0 rather than σ_0X_0. With these conventions, Ito's product rule (or integration by parts) states

$$XY=X_0Y_0+\int XdY+\int YdX+\langle X,Y\rangle.$$

If $X,Y\in\mathscr{E}_+$, Ito's lemma applied to $\log(x)$ gives

$$X=X_0\,e^{\int\frac{dX}{X}-\frac{\langle\log X\rangle}{2}}\;,\quad\langle\log X,\log Y\rangle=\langle\int\frac{dX}{X},\int\frac{dY}{Y}\rangle=\int\frac{d\langle X,Y\rangle}{XY}\;.$$

The compensator of $X\in\mathscr{E}$ is here denoted u_X or u_X^P; it is the unique process of finite variation $u_X\in\mathscr{E}$ such that $u_X(0)=0$ and $X-u_X$ is a P-local martingale. If $X>0$, there is an also unique process of finite variation $U_X\equiv U_X^P\in\mathscr{E}_+$ such that $U_X(0)=1$ and X/U_X is a P-local martingale. For $X>0$, the compensator u_X and the multiplicative compensator U_X are related by

$$u_X=\int Xd(\log U_X),\quad U_X=e^{\int\frac{du_X}{X}}\;.$$

For $X,Y\in\mathscr{E}$, we get from Ito's product rule

$$u_{XY}=\int(Xdu_Y+Ydu_X)+\langle X,Y\rangle.\tag{1}$$

If $X,Y\in\mathscr{E}_+$, we have in addition

$$U_{XY}=U_XU_Y\,e^{\langle\log X,\log Y\rangle}\;.\tag{2}$$

We recall some well-known facts about the Girsanov change of measure needed for the change of numeraire. Let Q be a measure equivalent to P. Denote its $\mathscr{F}_t$ conditional expectation operator by E_t^Q. Set $M_t\equiv E_t[dQ/dP]$. Then, $M_t>0$ is a P-martingale, $M_0=1$, and if X_s is a $\mathscr{F}_s$ measurable, Q-integrable random variable, then $E_t^Q[X_s]=E_t^P[X_sM_s]/M_t$, $t<s$. This implies that X is a Q-martingale iff XM is a P-martingale, and by localization, X is a Q-local martingale iff XM is a P-local martingale. Now, assume M is continuous. For $X\in\mathscr{E}$, by Ito's product rule,

$$(X-u_X-\langle X,\log M\rangle)M=X_0+\int(X-u_X-\langle X,\log M\rangle)dM+\int Md(X-u_X)\;.$$

The first integral is a P-local martingale because M is, and so is the second integral because $X-u_X$ is a P-local martingale. So, $(X-u_X-\langle X,\log M\rangle)M$ is a P-local martingale, implying that $X-u_X-\langle X,\log M\rangle$ is a Q-local martingale. Therefore, the Q-compensator u_X^Q of X is

$$u_X^Q = u_X^P + \langle X, \log M \rangle, \quad \left(M_t \equiv E_t^P \left[\frac{dQ}{dP} \right] \right). \tag{3}$$

In particular, if X is a P-Brownian motion, then $Y \equiv X - \langle X, \log M \rangle$ is a continuous Q-local martingale satisfying $\langle Y \rangle_t = \langle X \rangle_t = t$; hence Y is a Q-Brownian motion by Levy's characterization. (It is known that stochastic integration and quadratic variation with respect to P and Q coincide.) If $X \in \mathscr{E}_+$, the multiplicative Q-compensator U_X^Q of X is similarly given by

$$U_X^Q = U_X^P e^{\langle \log X, \log M \rangle}. \tag{4}$$

For $X \in \mathscr{E}$, we consider the following space $L(X)$ of stochastic integrands:

$$L(X) \equiv \left\{ \text{predictable processes } \sigma : \int_0^T \sigma_s^2 d\langle X \rangle_s + \int_0^T |\sigma_s| \, |du_X(s)| < \infty \text{ a.s.} \right\}.$$

For $\sigma \in L(X)$, the stochastic integral $\int \sigma dX$ is well defined, and is a continuous semimartingale. It is known that $L(X)$ and $\int \sigma dX$ depend only on the equivalence class of the measure P. Clearly, for any $X \in \mathscr{E}$, $L(X)$ contains all predictable processes with a.s. bounded paths on $[0, T]$. In particular, for any $X \in \mathscr{E}$, $L(X)$ contains all adapted processes whose paths are left continuous and have right limits a.s. For our purposes, the latter would have sufficed and been more convenient, except for one result (Theorem 5.1) where we need the larger space $L(X)$.

Lemma 1. *Let $X, Y \in \mathscr{E}$ and $\sigma \in L(X) \cap L(Y)$. Then $\sigma \in L(XY)$, and if $Y > 0$, $\sigma \in L(X/Y)$.*

Proof. Since X and Y have bounded paths on $[0, T]$, to show $\int \sigma^2 d\langle XY \rangle < \infty$, it suffices by the product rule and the assumption $\sigma \in L(X) \cap L(Y)$, to show that $\int \sigma^2 d\langle X, Y \rangle < \infty$. But, using again $\sigma \in L(X) \cap L(Y)$, this is a direct consequence of the Kunita-Watanabe inequality. That $\int |\sigma| \, |du_{XY}| < \infty$ now follows from this, Eq. (1), and σ being in $L(X) \cap L(Y)$. The second statement is similar. $\square$

We will deal with vector processes. Set

$$\mathscr{E}^n \equiv \{(B_1, \dots, B_n) : B_i \in \mathscr{E}\}; \quad \mathscr{E}_+^n \equiv \{(B_1, \dots, B_n) : B_i \in \mathscr{E}_+\}.$$

We regard elements of $\mathscr{E}^n$ as column vectors. For $B \in \mathscr{E}^n$, B_0 and B_t will denote $B(0)$ and $B(t)$, whereas B_i will denote the i-th component of B. This should not cause confusion. For $B \in \mathscr{E}^n$, set

$$L(B) \equiv \{\theta = (\theta_1, \dots, \theta_n) : \theta_i \in L(B_i)\}, \quad B \in \mathscr{E}^n.$$

We regard elements of $L(B)$ as row vectors. For $\theta \in L(B)$, $\int \theta dB$ denotes $\sum \int \theta_i dB_i$, or in differential notation, $\theta dB \equiv \sum \theta_i dB_i$. The following simple result will be important. Interestingly, Eq. (5) and (6) below have the same form in the stochastic case as in the deterministic case.

Lemma 2. *(a) If $\xi \in \mathscr{E}$, $B \in \mathscr{E}^n$, and $\theta \in L(B)$ with $\theta_i \in L(\xi)$ for all i, then*

$$d\left(\xi \int \theta dB\right) = \theta d(\xi B) + \left(\int \theta dB - \theta B\right) d\xi \ . \tag{5}$$

(b) If $B \in \mathscr{E}^n$ with $B_j > 0$ for some j, and $\theta \in L(B)$ with $\theta_i \in L(B_j)$ for all i, then

$$d\left(B_j \int \theta d\frac{B}{B_j}\right) = \theta dB + \left(\int \theta d\frac{B}{B_j} - \frac{\theta B}{B_j}\right) dB_j \ . \tag{6}$$

Proof. Using the product rule, then Lemma 1 and associativity, then again the product rule, gives

$$\begin{aligned} d\left(\xi \int \theta dB\right) - \left(\int \theta dB\right) d\xi &= \xi(\theta dB) + d\left\langle \xi, \int \theta dB \right\rangle \\ &= \theta(\xi dB + d\langle \xi, dB\rangle) = \theta(d(\xi B) - Bd\xi) \ . \end{aligned}$$

(b) Follows by applying (a) with ξ replaced by B_j and B replaced by B/B_j. □

The next result will be used (in different measures) for construction of forward LIBOR or swap rates processes, given a covariance function and prescribed initial forward rates.

Lemma 3. *Let w_t, be a d-dimensional Brownian motion on $(\Omega, \mathscr{F}, P, \mathscr{F}_t)$. Let scalar functions $\mu_i(t,x)$ and d-dimensional row-vector valued functions $\sigma_i(t,x)$, $t \in [0,T]$, $x \in \mathbb{R}^m_+$, $1 \le i < m$, be measurable, bounded and locally Lipschitz in x. Then, there exist unique w_t-Ito processes $X_i > 0$, with a given initial condition $X(0) \in \mathbb{R}^m_+$, satisfying the SDE $dX_i = X_i \mu_i(t,X)dt + X_i \sigma_i(t,X)dw$. Moreover, the solution is square-integrable.*

Proof. It suffices to show that the log-transformed SDE $dY_i = \upsilon_i(t,Y)dt + \gamma_i(t,Y)dw$ has a unique solution, where $\gamma_i(t,y) = \sigma_i(t,e^y)$, $\upsilon_i(t,y) = \mu_i(t,e^y) - |\gamma_i(t,y)|^2/2$. But it is easy to see that $\gamma_i(t,y)$ and $\nu_i(t,y)$ are bounded and locally Lipschitz. The desired result now follows from a standard existence and uniqueness theorem for solutions of SDEs.[2] Moreover, a standard argument shows that the SDE for X implies $E[|X_t|^2] < K(1+|X_0|^2)\exp(Kt)$ for some $K > 0$. □

Corollary 1. *With w_t and $\sigma_i(t,X)$ as in Lemma 2, there exists a unique (square-integrable) w_t-Ito process $X_i > 0$, with a given initial condition $X(0) \in \mathbb{R}^m_+$, satisfying the SDE*

$$dX_i = -\sum_{j=i+1}^{m} \frac{X_i X_j \sigma_i(t,X)\sigma_j(t,X)^t}{1+X_j} dt + X_i \sigma_i(t,X)dw \ .$$

Moreover, the processes $Y_i \equiv (1+X_i)\ldots(1+X_m)$ are square-integrable martingales, for all i.

Proof. Since $(X_j/(1+X_j))$ and sum and product of bounded, locally Lipschitz functions are bounded and locally Lipschitz, by Lemma 2 a unique solution $X_i > 0$ exists. We claim that

$$dY_i = Y_i \gamma_i dw, \quad \gamma_i(t) \equiv \sum_{j=i}^{m} \frac{X_j(t)\sigma_j(t, X_t)}{1 + X_j(t)} .$$

This is obvious for $i = m$ because $Y_m = 1 + X_m$, and $dX_m = X_m \sigma_m dw$. By (2) and backward induction

$$\begin{aligned} U_{Y_i} &= U_{(1+X_i)Y_{i+1}} = U_{(1+X_i)} U_{Y_{i+1}} \exp(\langle \log(1+X_i), \log Y_{i+1} \rangle >) \\ &= \exp\left(\int \frac{du_{X_i}}{1+X_i} \right) \times 1 \times \exp\left(\int \frac{X_i \sigma_i}{1+X_i} \gamma_{i+1}^t dt \right) = 1 , \end{aligned}$$

where the last equality is a direct consequence of the drift of dX_i given by the SDE and the definition of γ_{i+1}. So, Y_i is a local martingale, and has zero drift. That the dispersion coefficient of Y_i is $Y_i \gamma_i$ follows directly from the product rule. Finally, since γ_i is a bounded process, it now follows that Y_i is a square-integrable martingale. In fact, by a standard argument, for some $K > 0$,

$$E[Y_i(t)^2] \leq K(1 + Y_i^2(0))\, e^{Kt} < \infty, \quad \forall t, i . \quad \square$$

3. Self-financing trading strategies (SFTS)

Definition. *A pair* (θ, B), $B \in \mathscr{E}^n$, $\theta \in L(B)$, *is called a* **self-financing trading strategy** *(SFTS) if* $d(\theta B) = \theta dB$ *(i.e.,* $\theta B = \theta_0 B_0 + \int \theta dB$*). We also say* θ *is a SFTS if* B *is understood.*

In this definition, B_i is thought of as the price of the i-th asset, and θ_i denotes the number of shares held in asset i. These assets are considered to be cash securities without any continuous cash flows, such as stocks, bonds and options. Here, the B_i do not represent forward or futures prices or exchange rates.[3] The quantity $\theta dX - d(\theta X)$ can be thought of as the continual financing needed to maintain the trading strategy. So, if it vanishes, it means that, except initially at time 0, no money is put into or taken out of the strategy until maturity T. The price $C = \theta B$ of an SFTS is always continuous, even when θ is discontinuous (because it equals $\theta_0 B_0 + \int \theta dB$). Sometimes, θ may only be defined on a subinterval $[0, T^*]$ satisfying $d(\theta B) = \theta dB$. Then, provided $B_n > 0$, we can extend θ to an SFTS by setting on $(T^*, T]$, $\theta_i = 0$ for $i < n$, and $\theta_n(t) = \theta(T^*)B(T^*)/B_n(T^*)$. This can be done also in the cases where B_i are defined only on $[0, T^*]$ for $i < n$.

We will only study securities which can be replicated by an SFTS θ with respect to a given price system $B \in \mathscr{E}^n$. As such, in this paper, the term “security price” is synonymous to θB for some SFTS θ. There may exist two distinct SFTS θ and θ' with the same payoffs, i.e., $\theta(T)B(T) = \theta'(T)B(T)$. In that case, unless they have the same prices, i.e., $\theta B = \theta' B$, there is obviously an arbitrage opportunity. Later we will impose a condition which ensures that such a "law of one price" will hold, and in fact provides the price as an appropriate expectation of the payoff.

Theorem 1. *If (θ, B) is an SFTS, then so is $(\theta, \xi B)$ for any $\xi \in \mathscr{E}$ such that $\theta_i \in L(\xi)$ for all i.*

Proof. This follows immediately by substituting $\theta B - \theta_0 B_0 = \int \theta dB$ in both sides of Eq. (2.5). $\square$

Corollary 1. *Let $B \in \mathscr{E}^n$ with $B_j > 0$ for some j. Let $\theta \in L(B)$ with $\theta_i \in L(B_j)$ for all i. Then (θ, B) is an SFTS if and only if $(\theta, B/B_j)$ is an SFTS, and this holds if and only if*

$$\theta_j = \theta_j(0) + \sum_{i \neq j} \left(\theta_i(0) \frac{B_i(0)}{B_j(0)} - \theta_i \frac{B_i}{B_j} + \int \theta_i \, d\frac{B_i}{B_j} \right) . \tag{1}$$

Proof. This follows by setting in Theorem 1, $\xi = B_j$ ($\xi = 1/B_j$) for the "if" ("only if") part. $\square$

The significance of this result is that we can choose all but a single θ_j arbitrarily, i.e., we can trade in all except the j-th security arbitrarily, while using the j-th security to appropriately finance the strategy. Also, when $n = 1$, Corollary 1 implies $(\theta, 1)$ is an SFTS, so θ is a constant.

We call a price system $B \in \mathscr{E}^n$ *non-degenerate* if $\theta \in L(B)$ and $\langle \int \theta dB \rangle = 0$ imply $\theta = 0$. An example is when $\langle B_i, B_j \rangle$ is absolutely continuous, and the n by n matrix $d\langle B_i, B_j \rangle / dt$ is nonsingular for all (t, ω). By an *instantaneous saving bond* (with respect to a given $B \in \mathscr{E}^n$), we mean a nonzero SFTS θ such that $\langle \theta B \rangle = 0$. Thus, an instantaneous saving bond is a replicable security whose price θB has finite variation. Note, an instantaneous saving bond exists only when B is degenerate. The following result provides a useful criteria for a "path-independent" SFTS.

Theorem 2. *Let $B \in \mathscr{E}^n$ and $\theta(t, B)$ be a C^1 function of $n+1$ real variables. Set $\theta_t = \theta(t, B_t)$, $\partial\theta/\partial t = \partial\theta/\partial t(t, B_t)$, etc. Then (θ, B) is an SFTS if the following two equations are satisfied*

$$\sum_j B_j \frac{\partial \theta_j}{\partial B_i} = 0, \quad \forall i \quad (\text{equivalently } \theta_i = \frac{\partial \theta B}{\partial B_i}) , \tag{2}$$

$$\frac{\partial \theta}{\partial t} \cdot B \; dt + \frac{1}{2} \sum_{i,j} \frac{\partial \theta_i}{\partial B_j} d\langle B_i, B_j \rangle = 0 . \tag{3}$$

Further, if $B \in \mathscr{E}_+^n$ and Eq. (2) is satisfied, then for any $1 \le k \le n$, Eq. (3) is equivalent to

$$\frac{\partial \theta}{\partial t} \cdot B \; dt + \frac{1}{2} \sum_{i \neq k, j \neq k} \frac{\partial \theta_i}{\partial B_j} B_i B_j \, d\langle \log \frac{B_i}{B_k}, \; \log \frac{B_j}{B_k} \rangle = 0 . \tag{4}$$

Conversely, if (θ, B) is an SFTS and $B \in \mathscr{E}^n$ is nondegenerate, then (2) and (3) are satisfied.

Proof. Set $C(t, B) = \theta B$, $C_t = C(t, B_t)$, $\partial C/\partial t = \partial C/\partial t(t, B_t)$, etc. Since $\partial C/\partial B_i = \theta_i + \sum B_j \partial\theta_j/\partial B_i$, if (2) holds, then $\partial C/\partial B_i = \theta_i$, and $\partial\theta_i/\partial B_j = \partial^2 C/\partial B_i \partial B_j = \partial\theta_j/\partial B_i$. Hence by Ito's lemma, $dC - \partial C/\partial B \cdot dB$ equals the left hand side of (3), which, if zero implies $dC = \theta dB$, i.e., θ is an SFTS. The equivalence of (3) and (4) follows easily from Eq. (2) and the symmetry of $\partial\theta_i/\partial B_j$. As for the converse, since now $\theta \in \mathscr{E}^n$, and θ is an SFTS, we have $\int B d\theta = -\langle B, \theta\rangle$. Thus $0 = \langle \int B d\theta\rangle = \int \sum (B_j \partial\theta_j/\partial B) \cdot dB$, which by nondegeneracy implies (2). Equation (3) follows as before. □

Note from the proof that (2) implies $\partial\theta_i/\partial B_j = \partial\theta_j/\partial B_i$, and plugging this back into (2), it follows $\theta(t, B)$ $(C(t, B))$ is homogeneous of degree zero (one) in B. In particular, when B is non-degenerate, all path-independent SFTS θ are homogeneous of degree 0 in B. By a similar argument one sees that if B is non-degenerate and an Ito diffusion, and the payoff $C(B(T))$ is a function of $B(T)$, then there is no SFTS θ such that $\theta_T B_T = C(B((T))$, unless the payoff is homogeneous of degree 1 in B, in which case the price will be a function $C(t, B)$ of $B(t)$ satisfying a PDE provided by (3) or (4).

Example 1: *Option to exchange two assets* (Margrabe 1978; Geman et al. 1995). Let $B = (B_1, B_2) \in \mathscr{E}_+^2$. This option has the payoff at time $T > 0$ of $\max(B_1(T) - B_2(T), 0)$. Assuming $\langle \log B_1/B_2\rangle$ is positive and deterministic, for $t < T$, set

$$\theta_1(t) = N(h_+(t)), \quad \theta_2(t) = -N(h_-(t)) \,,$$

where

$$\begin{aligned} V^2(t,T) &= \langle \log \frac{B_1}{B_2}\rangle_T - \langle \log \frac{B_1}{B_2}\rangle_t, \quad h_\pm(t) = \frac{\log(B_1(t)/B_2(t))}{V(t,T)} \pm \frac{V(t,T)}{2}, \\ n(x) &= \frac{e^{-x^2/2}}{\sqrt{2\pi}}, \quad N(x) = \int_{-\infty}^{x} n(y)dy \ . \end{aligned}$$

It is standard that θB approaches $\max(B_1(T) - B_2(T), 0)$ at time T. Also, (θ, B) is an SFTS. Indeed, from the relation $B_1 n(h_+) = B_2 n(h_-)$, it follows that the function $C(t, B) = \theta B$ satisfies $\partial C/\partial B_i = \theta_i$. Therefore, Eq. (2) in Theorem 2 holds. It remains to show Eq. (4). This follows from

$$\begin{aligned} \frac{\partial C}{\partial t} dt &= \left(B_1(n(h_+) \frac{\partial h_+}{\partial t} - B_2 n(h_-) \frac{\partial h_-}{\partial t} \right) dt \\ &= B_1 n(h_+) dV = -\frac{1}{2} \frac{\partial \theta_1}{\partial B_1} B_1^2 d\langle \log \frac{B_1}{B_2}\rangle \ . \end{aligned}$$

If B_2 is a zero-coupon bond price in this example, i.e., $B_2(T) = 1$, then the option is a European call option on the first asset, as in Merton (1973). But, otherwise, the converse part of Theorem 2 and remarks following it imply that it is not possible to replicate a call option on the first asset (by trading in B_1 and B_2) because a call option payoff is not homogeneous of degree 1 in (B_1, B_2).

Example 2: *European swaption* (Neuberger 1990). Let $B \in \mathscr{E}_+^n$, $\delta \in \mathbb{R}_+^{n-1}$, $K > 0$. Assume $B_1 > B_n$. Set $X = B_1 - B_n$, $Y = \delta_1 B_2 + \ldots + \delta_{n-1} B_n$. A European payer's swaption with coupon K, (fixed) day-count fractions δ and expiration T is a security whose payoff at time T is $\max(0, X_T - KY_T)$. So, it is an option to exchange "fixed cash flows" KY_T for "floating cash flows" X. The pricing and hedging formulae of Example 1 therefore apply, provided the "forward swap rate" $S \equiv X/Y$ has deterministic volatility, i.e., $\langle \log(S) \rangle$ is deterministic. This is the industry-standard approach. A *caplet* is a European swaption with $n = 2$. A European swaption can also be considered as an option with strike price 1 on a forward bond with coupon K.

A variation often seen in steep yield curve environments is a "stepup swaption", where the coupon K is nonconstant, depending on i. This can be reduced to the standard case by replacing δ_i by $\delta_i K_i$ and setting $K = 1$. Sometimes one has a non-constant notional, e.g., an option to enter (or cancel) an amortizing swap with notional $N \in \mathbb{R}_+^{n-1}$. The payoff is again $\max(0, X_T - KY_T)$, where now $X = N_1(B_1 - B_2) + \ldots + N_{n-1}(B_{n-1} - B_n)$, and $Y = N_1\delta_1 B_2 + \ldots + N_{n-1}\delta_{n-} B_n$. We can again apply Example 1 if we assume that the "break-even forward swap rate" $S \equiv X/Y$ has deterministic volatility. These assumptions of deterministic volatility for different swap rates S may not be consistent with each other. Yet, taken individually, they appear to be quite reasonable assumptions – certainly robust and convenient.

Example 3: *Trigger swap*. Let $B = (B_1, B_2, B_3, B_4) \in \mathscr{E}_+^4$. A trigger swap with maturity T is a security whose payoff at time T is $B_3(T) - B_4(T)$ if $B_1(T) > B_2(T)$ and zero otherwise. If $B_3 = B_1$ and $B_4 = B_2$, then the trigger swap is the same as the option to exchange assets 1 and 2. Assuming $\langle \log B_1/B_2 \rangle$ and $\langle \log B_1/B_2, \log B_3/B_2 \rangle$ are deterministic, for $t < T$ set

$$\theta_1(t) = \frac{n(h(t))B_3(t)}{V(t,T)B_1(t)}\ , \quad \theta_2(t) = -\frac{n(h(t))B_3(t)}{V(t,T)B_2(t)}\ , \quad \theta_3(t) = N(h(t))\ ,$$

where,

$$h(t) = h_-(t) + \left(\langle \log \frac{B_1}{B_2}, \log \frac{B_3}{B_2} \rangle_T - \langle \log \frac{B_1}{B_2}, \log \frac{B_3}{B_2} \rangle_t \right) / V(t,T)\ .$$

We claim that (θ, B) is an SFTS and

$$\theta_t \cdot B_t = B_3(t)N(h(t)) \to 1_{B_1(T) > B_2(T)} B_3(T) \quad \text{a.s. as } t \to T\ .$$

This then provides the SFTS for the receive leg of the trigger swap, and the pay leg is given by a similar SFTS with B_3 replaced by B_4. (A digital option is the special case of this receive leg with $B_2(T) = B_3(T) = 1$.) To establish the claim, note $\theta_1 B_1 + \theta_2 B_2 = 0$, hence the value of the receive leg is $\theta_t B_t = B_3(t)N(h(t))$.

Now, if $B_1(T) > B_2(T)$, then $\log(B_1/B_2)$ is bounded above zero near T, and since $V(t,T) \to 0$, we see that $h_-(t)$ and hence $h(t)$ approach infinity. Hence, $\theta B \to B_3(T)$. Similarly, if $B_1(T) < B_2(T)$, then $\theta B \to 0$. To show θ is self-financing, we use Theorem 2. Set $C(t,B) = \theta B = B_3 N(h)$. Clearly, $\partial C/\partial B_i = \theta_i$ for all i; hence it remains to show (4) holds. Since $\partial\theta_3/\partial B_3 = 0$, Eq. (4) for $k = 2$ simply reads

$$B_3 \frac{\partial N(h(t))}{\partial t} dt + \frac{1}{2}\frac{\partial\theta_1}{\partial B_1} B_1^2 dV_{1\,1} + \frac{\partial\theta_1}{\partial B_3} B_1 B_3 dV_{1\,3} = 0 \ , \quad V_{ij} \equiv \langle \log\frac{B_i}{B_2}, \log\frac{B_j}{B_2}\rangle \ .$$

This equation is easily verified by the calculations

$$\frac{\partial\theta_1}{\partial B_1} B_1^2 = -n(h)\frac{B_3}{V}\left(1 + \frac{h}{V}\right) \ , \quad dV_{1\,1} = -2VdV \ , \quad \frac{\partial\theta_1}{\partial B_3} B_1 B_3 = n(h)\frac{B_3}{V} \ ,$$

$$\frac{\partial h}{\partial t} dt = -\left(1 + \frac{h_-}{V}\right) dV + d\left(\frac{V_{1\,3}(T) - V_{1\,3}(t)}{V}\right) = -\left(1 + \frac{h}{V}\right) dV - \frac{dV_{1\,3}}{V} \ .$$

We can rederive the SFTS for an option to exchange two assets by considering a trigger swap with $B_3 = B_1$ and $B_4 = B_2$. For the receive (pay) leg, $h(t)$ becomes the same as $h_+(t)$ ($h_-(t)$). Aggregating and using the identity $B_1 n(h_+) = B_2 n(h_-)$, one gets $\theta_1 = N(h_+)$ and $\theta_2 = -N(h_-)$, as before.

Example 4: Suppose B_1, B_2, B_3 are geometric Brownian motions with respect to a one-dimensional Brownian motion $z(t)$: $dB_i/B_i = \mu_i dt + \sigma_i dz$. Set

$$\theta_1 = \Theta_{1\,2} \ , \quad \theta_2 = \Theta_{2\,1} - \Theta_{2\,3} \ , \quad \theta_3 = -\Theta_{3\,2} \ ,$$

where

$$\Theta_{ij}(t) \equiv \frac{\sigma_j C_{ij}(t)}{(\sigma_j - \sigma_i)B_i(t)} \ , \quad C_{ij}(t) \equiv e^{\mu_{ij} t} \ , \quad \mu_{ij} \equiv \frac{\sigma_i \mu_j - \sigma_j \mu_i}{\sigma_i - \sigma_j} \ .$$

It is easy to see that (θ, B) is an SFTS, $\theta B = C_{1\,2} - C_{2\,3}$, and and $\theta_0 B_0 = 0$. This is an example of an SFTS with zero initial investment, and a deterministic value $\theta_t B_t$ thereafter. Unless this value is zero, i.e, $\mu_{1\,2} = \mu_{2\,3}$, it is clear that there will be an arbitrage opportunity. Note, the condition $\mu_{1\,2} = \mu_{2\,3}$ is equivalent to the condition $(\mu_1 - \mu_2)/(\sigma_1 - \sigma_2) = (\mu_3 - \mu_2)/(\sigma_3 - \sigma_2)$.

The trigger swap example illustrates that, regardless of whether or not there is an arbitrage opportunity among the underlying asset prices $B_1, \ldots, B_4$, the payoff of a trigger swap can be replicated by dynamic trading. For example, in the situation of Example 4 with $\mu_{1\,2} \neq \mu_{2\,3}$, the SFTS in Example 3 still replicates the payoff of the trigger swap, although in this case we can do better by combining it with the SFTS in Example 4. A bookrunner engaged in structured customers' business is not normally required to take advantage of arbitrage opportunities. Normally, his primary task is to preserve the initial revenue of the trade by hedging. What he needs is a systematic way of deriving the price and the hedge. Arbitrage-free models not only guarantee absence of "free lunches", but, perhaps more importantly, provide a systematic method to arrive at the price and the replicating SFTS.

4. Locally arbitrage-free (LAF) price systems

Definition. *$B \in \mathscr{E}^n$ is said to be* **locally arbitrage-free** *(LAF) if there exists $\xi \in \mathscr{E}_+$ (called the* state price deflator*) with $\xi_0 = 1$ such that ξB_i are P-local martingales for all i.*

In the next section we will strengthen this to ξB_i being martingales. But, as we see in this section, the LAF condition already suffices for most of the basic properties. Note, if B is LAF, then so is ζB for any $\zeta \in \mathscr{E}_+$. So, for $B \in \mathscr{E}_+^n$, the LAF condition is only a condition on the ratios B_i/B_j. In particular, it does not depend on the choice of currency. Clearly, it is also invariant under change of equivalent measure.

Theorem 1. *Let $B \in \mathscr{E}^n$ be LAF. Then $\xi(\theta B)$ is a P-local martingale for all SFTS θ such that $\theta_i \in L(\xi)$ for all i.*

Proof. By Theorem 3.1, $\xi\theta B = \theta_0 B_0 + \int \theta d(\xi B)$, which is a local martingale if ξB_i are, provided we show $\theta_i \in L(\xi B_i)$. But $\theta_i \in L(\xi)$ by assumption and $\theta_i \in L(B_i)$ because θ is an SFTS. Hence, $\theta_i \in L(\xi B_i)$ by Lemma 2.1. □

The LAF condition implies there is no "free lunch" in the weak sense below.

Proposition 1. *Let $B \in \mathscr{E}^n$, $C \in \mathscr{E}$, $\xi \in \mathscr{E}_+$. If $C_0 = 0$, $C \geq 0$, and ξC is a P-local martingale then $C = 0$.*

Proof. Since ξC is a non-negative local martingale, it is a supermartingale. So $E_0[\xi_t C_t] \leq \xi_0 C_0 = 0$. But $\xi_t C_t \geq 0$. Hence, $\xi_t C_t = 0$ and $C_t = 0$ a.s.$\forall t$. Being continuous, C is indistinguishable from 0. □

Theorem 2. *Let $B \in \mathscr{E}^n$ be LAF and $C \in \mathscr{E}$ be such that ξC is a P-local martingale. Then*

$$u_C + \int C \frac{dU_\xi}{U_\xi} + \langle C, \log \xi \rangle = 0 \,. \tag{1}$$

If further $B_j > 0$ for some j, then

$$u_{C/B_j} = -\langle \frac{C}{B_j}, \log \xi B_j \rangle \,. \tag{2}$$

If further $C > 0$, then the following three equations also hold.

$$U_C e^{\langle \log C, \log \xi \rangle} U_\xi = 1 \,; \tag{3}$$

$$U_C = U_{B_j} e^{-\langle \log \frac{C}{B_j}, \log \xi \rangle} \,; \tag{4}$$

$$U_{\frac{C}{B_j}} = e^{-\langle \log \frac{C}{B_j}, \log \xi B_j \rangle} \,. \tag{5}$$

Proof. By (2.1), Eq. (1) is equivalent to $u_{\xi C} = 0$, i.e., to ξC being a local martingale. In particular, if B is LAF, then (1) holds for B_j. Combining (1) for B_j with (1) for C, and using (2.1) gives (2). By (2.2), Eq. (3) is equivalent to $U_{\xi C} = 1$, i.e., to ξC being a local martingale. Eq. (3) applied to B_j and then combined with (3) itself to eliminate U_ξ gives (4). Equation (5) follows from (2). □

Equations (1)–(5) hold in particular for $C = B_i$, and by Theorem 1, more generally for prices C of SFTSs. Either of these equations for $C = B_i$ completely characterizes the LAF condition:

Theorem 3. *Let $B \in \mathscr{E}_+^n$. Then B is LAF if there exists $\xi \in \mathscr{E}_+$ such that, for all i, either Eq. (1) or Eq. (2) or Eq. (3) or Eq. (4) or Eq. (5) holds for all $C = B_i$ and some j.*

Proof. Eq. (1) is equivalent to $u_{\xi C} = 0$ by (2.1). So, if (1) holds for $C = B_i$, then ξB_i is a local martingale and B is LAF. If Eq. (2) holds for some ξ, then by replacing ξ by $\xi / U_{\xi B_j}$, we may assume that Eq. (2) holds for some ξ such that ξB_j is a local martingale, i.e., that (1) holds with $C = B_j$. Applying (2.1) to the product $B_i = (B_i/B_j)B_j$, and using (1) for $C = B_j$ and (2) for $C = B_i$ all terms cancel, showing ξB_i is a local martingale; so B is LAF. The rest follows similarly. □

Remark. If B is LAF and C is the price of an instantaneous saving bond price (so $\langle C \rangle = 0$) then by (3) $U_C = 1/U_\xi$, hence $C = C_0/U_\xi$. Here, we do not assume that an instantaneous saving bond necessarily exists (though we allow it). In practice, B is usually degenerate and it is expected to exist.

Equations (3)–(5) suggest that the LAF condition is essentially a linear constraint between the "covariance matrix" $d\langle \log B_i, \log B_j \rangle$ and the "drifts" $d(\log U_i)$, as made explicit below.

Theorem 4. *Let $B \in \mathscr{E}_+^n$. Assume that for some pathwise bounded predictable processes υ_{ij}, μ_i,*

$$d\langle \log B_i, \log B_j \rangle = \upsilon_{ij}\, dt, \quad U_{B_i}(t) = \exp\Big(\int_0^t \mu_i(s)ds\Big), \quad i, j = 1, \ldots, n \ .$$

Then, (i) there exist pathwise bounded predictable processes α_j, r, such that

$$\mu_i = r + \sum_{j=1}^n \upsilon_{ij}\alpha_j \quad \forall i, \quad \sum_{j=1}^n \alpha_j = 0$$

if and only if (ii) for some (and hence all) k, there exist pathwise bounded predictable processes α_i, $i \neq k$, such that for all i

$$\mu_i = \mu_k + \sum_{j \neq k} \upsilon_{ij,k}\alpha_j \ ,$$

where

$$v_{ij,k} \equiv v_{ij} - v_{ik} - v_{jk} + v_{kk} = \frac{d}{dt}\langle \log \frac{B_i}{B_k}, \log \frac{B_j}{B_k} \rangle \ .$$

Moreover, if these conditions are satisfied, then B is LAF. In fact, ξB_i are P-local martingales, where ξ is given by the equivalent expressions

$$\begin{aligned} \xi &= \exp\left(\int -\sum_j \alpha_j \frac{dB_j}{B_j} + \left(\frac{1}{2}\sum_{i,j} \alpha_i v_{ij} \alpha_j - r \right) dt \right) \\ &= \frac{1}{U_k} \exp\left(\int -\sum_{j \neq k} \alpha_j \frac{B_k}{B_j} d\frac{B_j}{B_k} + \frac{1}{2}\sum_{i,j\neq k} \alpha_i v_{ij,k} \alpha_j \, dt \right) \ . \end{aligned}$$

Proof. If (ii) holds for some k, set $\alpha_k = -\alpha_i - \ldots - \alpha_{k-1} - \alpha_{k+1} - \ldots - \alpha_n$, so that $\sum \alpha_j = 0$, and set $r = \mu_k - \sum v_{kj}\alpha_j$. Then, it is easy to check that $\mu_i - \sum v_{ij}\alpha_j = r$ for all i, so that (i) holds. The converse is similar. The last statement follows from Theorem 3 by verifying that ξ, defined by the first formula above, satisfies Eq. (3) for all $C = B_i$. $\square$

Corollary 1. *Let $B \in \mathscr{E}_+^n$. Then, for some (and hence all) k, the above $n-1$ by $n-1$ matrix $v_{,k} = (v_{ij,k})$ is nonsingular if and only if for each (t,ω) there is no $0 \neq a \in \mathbb{R}^n$ such that $\sum a_j = \sum v_{ij}(t,\omega)a_j = 0$ for all i. In this case, B is LAF; in fact, there is a unique α, with $\sum \alpha_j = 0$, such that conditions (i) and (ii) in Theorem 4 are satisfied.*

Proof. The equivalence of the stated conditions is an easy argument in linear algebra, using (v_{ij}) is a symmetric, positive semidefinite matrix. These conditions clearly imply the existence of unique processes $\alpha_j(t,\omega)$ satisfying conditions (i) and (ii) of Theorem 6 for each (t,ω). But then, as a process, $\alpha_j(t,\omega)$ is necessarily predictable and locally bounded, because by (ii), it is obtained from the inverse matrix of $v_{,k}$, whose components are continuous functions of $v_{ij,k}$. $\square$

The corollary basically shows that if the covariance matrix $v = (v_{ij})$ has rank $n-1$, then B is automatically LAF. (In particular, when $n = 2$, B is LAF, provided $v_{11,2} \neq 0$.). If $v = (v_{ij})$ is non-singular, then B is LAF, but ξ is not unique ("market is incomplete") and can be chosen to be a local martingale itself. (Define ξ by the first formula in the Theorem, with $\alpha = v^{-1}(\mu)$, so that $r = 0$.) Note also, the process ξ in the first formula in Theorem 6 satisfies $U_\xi(t) = \exp(-\int_0^t r(s)ds)$. Hence, r can be interpreted as the instantaneous spot interest rate, provided $1/U_\xi$ is the price of an SFTS.

All the properties we have encountered are those of the relative prices B_i/B_j. The "forward LIBOR process" incorporates only these ratios. Given $B \in \mathscr{E}_+^n$ and "daycount fractions" $\delta = (\delta_1, \ldots, \delta_{n-1}) \in \mathbb{R}^{n-1}$, $\delta_i > 0$, define the $n-1$ dimensional process $L = (L_1, \ldots, L_{n-1}) \in \mathscr{E}^{n-1}$ by

$$L_i \equiv \delta_i^{-1}\left(\frac{B_i}{B_{i+1}} - 1 \right) \ . \qquad (6)$$

This construction is particularly important when B_i represent zero-coupon bond prices. Nevertheless, its properties below hold in general.

Theorem 5. *If $B \in \mathscr{E}_+^n$ is LAF, then $\xi B_{i+1} L_i$ are P-local martingales, and*

$$u_i \equiv u_{L_i} = -\langle L_i, \log(\xi B_{i+1})\rangle \ . \tag{7}$$

Conversely, $B \in \mathscr{E}_+^n$ is LAF if there exists $\xi \in \mathscr{E}_+$ such that (7) holds for all $i = 1, \ldots, n-1$.

Proof. Since $\xi B_{i+1} L_i = \xi(B_i - B_{i+1})/\delta_i$, we see if B is LAF then $\xi B_{i+1} L_i$ are local martingales. Equation (7) is the same as Eq. (2) with $C = B_i$, and $j = i+1$. Conversely, (7) implies that (2), and hence (5) holds for $C = B_i$ and $j = i+1$, for all i. Telescoping and using (2.2), then (5) holds for all $C = B_i$ and all j. The converse now follows from Theorem 3. $\square$

By definition (6), we have $B_{i+1} = B_n(1+\delta_{i+1}L_{i+1})\ldots(1+\delta_{n-1}L_{n-1})$. Substituting this into (7) we get

$$u_{L_i} + \langle L_i, \log(\xi B_n)\rangle = -\sum_{j=i+1}^{n-1} \langle L_i, \log(1+\delta_j L_j)\rangle = -\int \sum_{j=i+1}^{n-1} \frac{\delta_j \, d\langle L_i, L_j\rangle}{1+\delta_j L_j} \ . \tag{8}$$

Conversely, it is clear that if this equation holds for all i, then Eq. (7) also holds for all i. Theorem 5 therefore implies

Theorem 6. *$B \in \mathscr{E}_+^n$ is LAF iff there exists $\xi \in \mathscr{E}_+$ such that Eq. (8) holds for all $i = 1, \ldots, n-1$.*

If $L_i > 0$, then by Eq. (2.2), we have similar to Eq. (7) and Eq. (8)

$$U_{L_i} = e^{-\langle \log L_i, \log \xi B_{i+1}\rangle} = e^{-\langle \log L_i, \log \xi B_n\rangle - \int \sum_{j=i+1}^{n-1} \frac{\delta_j \, d\langle L_i, L_j\rangle}{L_i(1+\delta_j L_j)}} \ . \tag{9}$$

The following result provides the price and SFTS of a path-independent option.

Theorem 7. *Let $B \in \mathscr{E}^n$ be LAF, and $C \in \mathscr{E}$ be such that ξC is a P-local martingale and $C_t = C(t, B_t)$ for some C^2 function $C(t, B)$, homogenous of degree 1 in B. Then $(\partial C/\partial B, B)$ is an SFTS, $C = (\partial C/\partial B)\cdot B$, and $dC = (\partial C/\partial B)\cdot dB$. Moreover, if $B_i > 0$ for all i, then $C = B_n c(t, L_t)$ for some C^2 function $c(t, L)$ such that a.s.,*

$$\frac{\partial c}{\partial t} dt - \sum_{i=1}^{n-1} \sum_{j=i+1}^{n-1} \frac{\partial c}{\partial L_i} \frac{\delta_j \, d\langle L_i, L_j\rangle}{1+\delta_j L_j} + \frac{1}{2} \sum_{i,j=1}^{n-1} \frac{\partial^2 c}{\partial L_i \partial L_j} d\langle L_i, L_j\rangle = 0 \ . \tag{10}$$

Proof. Note $C = \partial C/\partial B \cdot B$, because $C(t, B)$ is homogenous of degree 1. Let us temporarily write $X \approx Y$ if $X - Y$ has finite variation. Then by thrice application of Ito's formula we have

$$\begin{aligned} d(\xi C) &\approx \xi dC + Cd\xi \approx \xi \frac{\partial C}{\partial B} \cdot dB + Cd\xi \\ &\approx \frac{\partial C}{\partial B} \cdot d(\xi B) + \left(C - \frac{\partial C}{\partial B} \cdot B \right) d\xi = \frac{\partial C}{\partial B} \cdot d(\xi B) \,. \end{aligned}$$

This shows $(\partial C/\partial B, \xi B)$ is an SFTS; hence by Theorem 2.1 (applied to $1/\xi$), so is $(\partial C/\partial B, B)$. As for Eq. (10), $C(t, B)/B_n$, is homogenous of degree zero, hence equals $c(t, L)$ for some C^2 function c (with L as in (6)). Applying Ito's lemma to $c(t, L_t)$, and using Eq. (2) and (8) gives (10). □

The significance of Eq. (10) is that, unlike Eq. (1)–(9), "ξ" no longer appears. If the "LIBOR covariance matrix" $d\langle L_i, L_j\rangle$ is a function of L, then (10) furnishes a "fundamental differential equation" for the price. As we pointed out in the previous section, the homogeneity condition above cannot be relaxed when B is non-degenerate. This issue will be expanded on in the next section, where a measurability condition is identified for a payoff to be representable by an SFTS.

A *LIBOR market model* (with a given daycount fraction $\delta \in \mathbb{R}_+^{n-1}$) is a LAF price system $B \in \mathscr{E}_+^n$ such that $L_i > 0$ and $\langle \log L_i, \log L_j\rangle$ is deterministic for all $1 \leq i, j \leq n-1$. The following explicit construction by "backward induction" is essentially due to Musiela and Rutkowski (1995).

Example 1: *LIBOR market model*. We are given $\delta \in \mathbb{R}_+^{n-1}$ and bounded, measurable, deterministic functions $\Lambda_{ij}(t)$, $1 \leq i, j \leq n-1$, such that the matrix $\Lambda(t) = (\Lambda_{ij}(t))$ is symmetric and positive-semidefinite. We wish to construct a LAF $B \in \mathscr{E}_+^n$, such that $d\langle \log L_i, \log L_j\rangle = \Lambda_{ij}\, dt$, with a given initial condition $L(0) \in \mathbb{R}_+^{n-1}$. We assume that there is an integer d and $n-1$ bounded, measurable $\mathbb{R}^d$ valued functions $\lambda_1(t), \ldots, \lambda_{n-1}(t)$ (viewed as row vectors) such that $\Lambda_{ij} = \lambda_i \cdot \lambda_j$ (so $\mathrm{rank}(\Lambda(t)) \leq d$) and that $\mathscr{F}_t$ supports a d-dimensional P-Brownian motion $z(t)$. Let ξ, $B_n \in \mathscr{E}_+$ be such that $\xi(0) = 1$ and ξB_n is a local martingale. For $i = n-1$, define

$$L_i = L_i(0) \exp\left(\int -\frac{\Lambda_{ii}}{2} dt + \lambda_i (d(z - \langle z, \log \xi B_{i+1}\rangle) \right), \quad B_i = B_{i+1}(1 + \delta_i L_i) \,.$$

Now, B_{n-1} is available, and we can use the above equations for $i = n-2$ to define B_{n-2}, and so on, until all B_i are defined. Evidently, $d\langle \log L_i, \log L_j\rangle = \Lambda_{ij}\, dt$. Also Eq. (7) is clearly satisfied, hence B is LAF by Theorem 5. This solution B is not unique. Aside from the freedom in the choice of ξ, B_n, a different decomposition $\Lambda_{ij} = \lambda'_i \cdot \lambda'_j$ leads to a different solution. For example, if $\lambda'_i(t) = \lambda_i(t) A(t)$ for some d by d orthogonal matrix $A(t, \omega)$, then a different solution obtains by replacing z in the above recursive formula by $z' = \int A(s)\, dz(s)$. But, by Levy's characterization, $z'(t)$ is another d-dimensional Brownian motion; so these two solutions are essentially the same. We will later amend this construction when B_i represent zero coupon bonds (to ensure $B_i(T_i) = 1$).

Assume now that B_j (hence L_j) and ξ are Ito processes with respect to a d-dimensional $(\mathscr{F}_t, P)$ Brownian motion $z(t)$, following

$$\frac{dB_i}{B_i} = \mu_i dt + \sigma_i dz, \quad \frac{d\xi}{\xi} = -r\ dt - \varphi dz, \quad (r, \mu_i \in L(t), \sigma_i, \varphi \in L(z)) . \quad (11)$$

The d-dimensional process $\varphi = -d\langle z, \log\xi\rangle/dt$ is called the *market price of risk*, and $r(t) = -d(\log U_\xi)/dt$ represents the instantaneous interest rate, provided $1/U_\xi = \theta B$ for some SFTS θ. When B is LAF, Eq. (3) and (4) for $C = B_i$ translate to

$$\mu_i = r + \sigma_i \varphi, \quad \mu_i = \mu_j + (\sigma_i - \sigma_j)\varphi . \quad (12)$$

Conversely, an Ito process $B \in \mathscr{E}_+^n$ is LAF, if there is a $\varphi \in L(z)$ (i.e. $\int |\varphi|^2 dt < \infty$) such that the second equation in (12) holds. Then, $r(t)$ is defined by the first equation in (10) for any i, and ξ is defined by $\xi = \exp(\int -(r + |\varphi|^2/2)dt - \varphi dz)$. If $C \in \mathscr{E}_+$ with $C = \theta B$ for some SFTS θ, then

$$\frac{dC}{C} = (r + \sigma_C \varphi)dt + \sigma_C dz, \quad \sigma_C = \frac{1}{C}\sum_i \theta_i B_i \sigma_i .$$

With $\beta_i = (\sigma_i - \sigma_{i-1})(1+\delta_i L_i)/\delta_i$ denoting the absolute LIBOR volatility, Eq. (8) is rewritten as

$$dL_i = -\sum_{j=i+1}^{n-1} \frac{\delta_j \beta_i \beta_j^t}{1 + \delta_j L_j} dt + \beta_i dz_n, \quad z_n \equiv z + \int (\varphi - \sigma_n)dt . \quad (13)$$

5. Arbitrage-free price systems

The strengthening below of the LAF condition enables evaluation of contingent claims by taking expectation of their payoffs, and determination of the replicating SFTS. It also leads to existence and uniqueness of the forward LIBOR process, given a LIBOR volatility function of linear growth.

Definition. *$B \in \mathscr{E}^n$ is* **arbitrage-free** *if there is $\xi \in \mathscr{E}_+$ with $\xi_0 = 1$ such that ξB_i are P-martingales for all i.*[4]

Note, if B is arbitrage-free, then so is B/X for any $X \in \mathscr{E}_+$.[5] So, for $B \in \mathscr{E}_+^n$, the arbitrage condition is a condition only on the ratios B_i/B_j.

Convention: From now on (for convenience to avoid stating integrability conditions at $t = 0$) we will always assume $\mathscr{F}_0$ consists of null sets and their complements. Hence, $\mathscr{F}_0$ measurable random variables are deterministic constants (members of $\mathbb{R}$) and $E_0[X] = E[X]$.

A LAF B is arbitrage-free if ξB_i satisfy certain integrability conditions. We recall, (i) a local martingale X is a martingale if $E[sup_{t\le T}|X_t|] < \infty$, (ii) a local martingale X with $E[\langle X\rangle_T] < \infty$ is a square integrable martingale and $E[\langle X\rangle_t] = E[X_t^2] - X_0^2 = \mathrm{var}[X_t]$, (iii) by Novikov Theorem, a continuous local martingale $X > 0$ is a martingale (and so is $\int dX/X$) if $E[\exp(\langle \log X\rangle_T/2)] < \infty$.

Theorem 1. *Let $B \in \mathscr{E}^n$ be arbitrage-free and θ be an SFTS. Then $\xi\theta B$ is a square-integrable martingale if $\forall i$ either (i) $E[\int \theta_i^2 d\langle \xi B_i\rangle] < \infty$, or (ii) ξB_i is square-integrable and θ is bounded.*

Proof. If $E[\int \theta_i^2 d\langle \xi B_i\rangle] < \infty$, then $\theta_i \in L(\xi B_i)$. Setting $C = \theta B$, it follows as in Theorem 3.1 that $\xi C = C_0 + \int \theta d(\xi B)$. But then $E[\langle \xi C\rangle] = \sum E[\int \theta_i \theta_j d\langle \xi B_i, \xi B_j\rangle] < \infty$, implying ξC is a square integrable martingale. (ii) follows from (i). $\square$

The above result provides an intuitive interpretation for the state price deflator $\xi_t(\omega)$ as the price at time zero of a security which at time t pays the infinitesimal amount $dP(\omega)$ if state ω occurs and pays zero otherwise. More precisely, let $A \in \mathscr{F}_t$, and suppose there is a bounded SFTS θ such that $\theta_t B_t = 1_A$. By the theorem, $\xi\theta B$ is a martingale, hence $\theta_0 B_0 = E[\xi_t \theta_t B_t] = \int_A \xi_t(\omega) dP(\omega)$.

The next proposition strengthens the no-free lunch result of the LAF case. In particular, it implies "the law of one price": two SFTSs with the same payoff have the same price.

Proposition 1. *Let $B \in \mathscr{E}^n$, $C \in \mathscr{E}$, $\xi \in \mathscr{E}_+$. Assume ξC is a martingale and $C_T \geq 0$ a.s. Then (i) $C_t \geq 0$ a.s. for all $0 \leq t \leq T$, and (ii) if $C_0 \leq 0$, then $C_t = 0$ a.s. for all $0 \leq t \leq T$.*

Proof. Since ξC is a martingale, $C_t = E_t[\xi_T C_T]/\xi_t$ for all t. Hence, $C_T \geq 0$ implies $C_t \geq 0$, because $\xi > 0$. If further $C_0 \leq 0$, then $E[\xi_T C_T] \leq 0$. But $\xi_T C_T \geq 0$, hence $\xi_T C_T = 0$ and $C_T = 0$ a.s. $\square$

Let $B \in \mathscr{E}^n$ be arbitrage-free and assume $B_i > 0$ for some i. Since $M_i \equiv \xi B_i / B_i(0)$ is a positive P-martingale with $M_i(0) = 1$, an equivalent measure P_i, called the B_i *numeraire measure*, is defined by $dP_i/dP = M_i(T)$. We denote its $\mathscr{F}_t$ conditional expectation operator by E_t^i. If $B_j > 0$ too, then $dP_i/dP_j = (B_i(T)/B_j(T))(B_j(0)/B_i(0))$. More generally, if $C \in \mathscr{E}_+$ is such that ξC is a P-martingale, then the C numeraire measure P_C is defined by $dP_C/dP = \xi_T C_T / C_0$. As in Sect. 2, for a semimartingale X, ξX is a P (local) martingale iff X/C is a P_C (local) martingale. So, B_j/B_i is a P_i martingale. If ξC and ξA are P-martingales with $A > 0$, then C/A is a P_A martingale, and hence

$$C_t = \frac{1}{\xi_t} E_t^P[\xi_T C_T] = B_i(t) E_t^i \left[\frac{C(T)}{B_i(T)}\right] = A_t E_t^A \left[\frac{C_T}{A_T}\right] . \tag{1}$$

Remark 1. Let $B \in \mathscr{E}_+^n$ be arbitrage-free. If the local martingale ξ/U_ξ is actually a martingale, then the measure Q defined by $dQ/dP = \xi(T)/U_\xi(T)$ is called the *risk-neutral measure*. In this case, $U_\xi B_i$ are Q martingales (because $U_\xi B_i(\xi/U_\xi) = \xi B_i$ are P-martingales). Whether or not a continuous saving bond exists, we do not assume that the risk-neutral measure exists, because in our framework, valuation of LIBOR and swap derivatives is not facilitated in this measure.

Examples 3.1 and 3.3 revisited. We exhibited SFTS and prices for the option to exchange two assets and trigger swaps. We can now see where these formulae came from. The price C of the exchange option is given by $C_t = B_2(t)E_t^2[\max(0, B_1(T)/B_2(T) - 1]$. The pricing formula would follow by a standard calculation if we show that $\log(B_1/B_2)$ is a Gaussian process in the P_2 measure. We need to show that a continuous local martingale $X > 0$ is log-Gaussian if $\langle \log X \rangle$ is deterministic. Let $Y = \int dX/X$. Then, Y is a local martingale, hence $Y_t = B_{s(t)}$, for some Brownian B with respect to another filtration, where $s(t) = \langle Y \rangle_t$. Since a Brownian motion is Gaussian and $s(t) = \langle \log X \rangle_t$ is deterministic, Y and hence $\log(X)$ is Gaussian. Having thus found the price C, the SFTS is obtained by taking partial derivatives as in Theorem 4.7. For trigger swaps, a similar expectation is calculated to derive the price, and partial derivatives are then taken to get the hedge.

We now turn to the problem of determining an SFTS θ that replicates a given payoff C_T. When B is non-degenerate, a replicating SFTS does *not* exist for a "generic" $\mathscr{F}_T$ measurable C_T. We saw this in Sect. 3 for the path-independent case. More generally, let $B \in \mathscr{E}_+^n$ be non-degenerate and satisfy the absolute continuity assumptions of Theorem 4.4. As remarked there, this implies that B is LAF and ξ can be chosen to be a local martingale. Assume more strongly that B is arbitrage-free and ξ is a martingale. Then B_i are Q-martingales, where Q is defined by $dQ/dP = \xi(T)$. Suppose $\eta \in L(B)$ is *not* an SFTS, but $E^Q[\int \eta_i^2 d\langle B_i \rangle] < \infty$. We claim there is no SFTS θ, such that $E^Q[\int \theta_i^2 d\langle B_i \rangle] < \infty$ and $\theta_T B_T = \int_0^T \eta dB$. Indeed, otherwise $\int(\eta - \theta)dB$ would vanish at T. But, $\int(\eta - \theta)dB$ is a Q-martingale, hence it vanishes identically. Since B is non-degenerate, this implies $\eta = \theta$, a contradiction. As such, when B is non-degenerate, the market is incomplete. However, this does not prevent finding an SFTS when C_T is appropriately measurable.

Theorem 2. *Let $B \in \mathscr{E}^n$ be LAF, $B_n > 0$, and ξB_n be a martingale. Assume that dU_ξ/dt and all $d\langle B_i, B_j \rangle/dt$ exist a.s. and have bounded paths on $[0, T]$, the matrix $d\langle B_i/B_n, B_j/B_n \rangle/dt$ has a.s. constant rank d, $d \leq n - 1$, and there exists a d-dimensional P_n Brownian motion w, such that B_i/B_n are adapted to the completed filtration $\mathfrak{J}_t$ generated by w. Let C_T be a random variable such that $C_T/B_n(T)$ is $\mathfrak{J}_T$ measurable and P_n integrable. Then there exists an SFTS θ such that $C_T = \theta_T B_T$ and $\theta B/B_n$ is a $(\mathfrak{J}_t, P_n)$ martingale.*

Proof. By the local martingale representation Theorem, $d(B/B_n) = \gamma dw$, for some $n - 1$ by d, $\mathfrak{J}_t$ (adapted and) predictable matrix $\gamma = (\gamma_{ik})$.[6] The assumption implies γ has full rank d and is pathwise bounded a.s. Let ψ be the pseudo-inverse of γ. Since γ has full rank, ψ is a continuous function of γ_{ik}, as it is given by orthogonal projection onto the image of γ followed by the inverse of γ on its image. Therefore, ψ too is $\mathfrak{J}_t$ predictable and has bounded paths. Set $c_t \equiv E^n[C_T/B_n(T)|\mathfrak{J}_t]$. c is a $(\mathfrak{J}_t, P_n)$ martingale, so there exists a $\mathfrak{J}_t$-predictable process $\eta = (\eta_1, \ldots, \eta_d)$ such that $\int |\eta|^2 dt < \infty$ and $c = c_0 + \int \eta dw$. Set $\theta \equiv \eta\psi$. Then $\int |\theta|^2 dt \leq (\sup_t |\psi|^2) \int |\eta|^2 dt < \infty$ a.s. Since $\psi\sigma = Id$, $\int \theta d(B/B_n) = \int \eta dw = c - c_0$. So, $c = c_0 + \int \theta d(B/B_n)$. Extend θ to an n-vector by setting

$\theta_n \equiv c - (\theta_1 B_1 + \ldots + \theta_{n-1}B_{n-1})/B_n$, so that $\theta B = B_n c = \theta_0 B_0 + B_n \int \theta d(B/B_n)$. Using this on both sides of Eq. (2.6) (or using Corollary 3.1) gives $d(\theta B) = \theta dB$, i.e. θ is an SFTS. But to use (2.6), we still need to show $\theta_i \in L(B_i) \cap L(B_n)$. In fact, we show $\theta_i \in L(B_j)\ \forall i,j$, i.e., $\int \theta_i^2 d\langle B_j\rangle < \infty$, and $\int |\theta_i|\ |du_j| < \infty$ a.s., where u_i is the compensator of B_i. The first folows since $\int \theta_i^2 dt < \infty$ and $d\langle B_j\rangle/dt$ have bounded paths. a.s. By Kunita-Watanabe inequality,

$$\int |\theta_i|\ |d\langle B_j, \log\xi\rangle| \le \left(\int \theta_i^2 d\langle B_j\rangle\right)^{1/2} \langle \log\xi\rangle^{1/2} < \infty \text{ a.s.}$$

The second now follows from this, Eq. (4.1), and the assumption on dU_ξ. Since $\theta B = B_n c$, $\theta_T B_T = B_n(T)c_T = C_T$, and $(\theta B)/B_n$ is a $(\mathfrak{I}_t, P_n)$ martingale. $\square$

The measurability condition on C_T essentially amounts to $C_T/B_n(T)$ being measurable with respect to the sigma algebra generated by $B_i(s)/B_j(s)$, or equivalently by the $L_i(s)$, $s \le T$. So, the appropriate payoffs C_T are those such that $C_T/B_n(T)$ (or, equivalently $C_T/B_j(T)$ for any j) is measurable with respect to the sigma algebra generated by $L(s)$, $s \le T$. This restriction on C_T is substantial when B is non-degenerate. But, otherwise, $\mathfrak{I}_T$ usually coincides with $\mathscr{F}_T$, in which case no restriction is imposed on C_T.

From the proof of Theorem 1 and the uniqueness of martingale representation, one sees that if $d = n-1$, then the replicating SFTS is unique. But, if $d < n-1$, then for any κ such that $\kappa\gamma = 0$, $\theta + (\kappa, -(\kappa_1 B_1 + \ldots + \kappa_{n-1}B_{n-1})/B_n)$ is another replicating SFTS. Nevertheless, their prices θB will be the same by the law of one price.

Let $B \in \mathscr{E}_+^n$ be arbitrage-free. The forward LIBOR L_i is by its definition an affine transformation of B_i/B_{i+1}. Therefore, L_i is a P_{i+1} martingale, as pointed out by Brace et al. (1997). Eq. (4.8) and the change of measure formula (2.3) combine to give the P_n compensator u_i^n of L_i:

$$u_i^n \equiv u_{L_i}^{P_n} = -\int \sum_{j=i+1}^{n-1} \frac{\delta_j d\langle L_i, L_j\rangle}{1+\delta_j L_j}\ . \tag{2}$$

Similarly, when $L_i > 0$, Eq. (4.9) and (2.4) imply that the P_n multiplicative compensator U_i^n of L_i is

$$U_i^n \equiv U_{L_i}^{P_n} = \exp\left(-\int \sum_{j=i+1}^{n-1} \frac{\delta_j d\langle L_i, L_j\rangle}{L_i(1+\delta_j L_j)}\right)\ . \tag{3}$$

We pose the following problem: given an $n-1$ by $n-1$ symmetric positive semidefinite matrix function $\Lambda(t,L) = (\Lambda_{ij}(t,L))$ of $t \in [0,T]$, $L \in \mathbb{R}_+^{n-1}$, is there an arbitrage-free price system B such that $L_i > 0$ and $d\langle L_i, L_j\rangle_t = \Lambda_{ij}(t, L(t))L_i(t)L_j(t)dt$? The answer is affirmative provided $\Lambda(t,L)$ is bounded and locally Lipschitz in L. For example, when $\Lambda(t,L)$ is independent of L, we saw in Sect. 4, Example 5, that a solution – the LIBOR market model – can be explicitly constructed. As in that example, we can expect uniqueness only if we pose the problem in terms of the LIBOR "percentage volatility matrix" $\lambda = \lambda(t, L)$, an $n-1$ by d matrix satisfying $\Lambda = \lambda\lambda^t$.

Theorem 3. *Let ξ, $B_n \in \mathscr{E}_+$ be such that $\xi(0) = 1$ and ξB_n is a P-martingale. Let $w(t)$ be a d-dimensional $(\mathscr{F}_t, P_n)$ Brownian motion. Let n be an integer and $\lambda_i(t, L)$ be $n-1$ d-dimensional vector valued functions on $[0, T] \times \mathbb{R}_+^{n-1}$, which are measurable, bounded, and locally Lipschitz in L. Then there exist unique w_t-Ito processes $B_1, \ldots, B_{n-1}$, such that ξB_i are P-martingales for all i, the associated LIBOR process $L(t)$ is positive, starts from a given initial condition $L(0) \in \mathbb{R}_+^{n-1}$, and satisfies $d\langle L_i, w\rangle = \lambda_i(t, L)L_i dt$ a.s. (So $d\langle \log L_i, \log L_j\rangle = \lambda_i(t, L)\lambda_j(t, L)^t dt$.)*

Proof. If a solution exists, then by Eq. (2) L satisfies the SDE

$$\frac{dL_i}{L_i} = -\sum_{j=i+1}^{n-1} \frac{\delta_j L_j \lambda_i(t, L)\lambda_j(t, L)^t}{1 + \delta_j L_j} dt + \lambda_i(t, L) dw \ . \tag{4}$$

By Corollary (2.1) (with $m = n-1$, measure P replaced by P_n, and $X_i = \delta_i L_i$), this SDE has a unique solution, and moreover, if we set $B_i = B_n(1 + \delta_i L_i) \ldots (1 + \delta_{n-1}L_{n-1})$, then B_i/B_n is a P_n martingale. $\square$

When the forward LIBOR covariance matrix is of the form $d\langle L_i, L_j\rangle = \beta_{ij}(t, L)(1 + \delta_i L_i)(1 + \delta_i L_j)dt$, for some bounded, locally Lipschitz functions $\beta_{ij}(t, L)$, we can show that a solution exits by the same method. But now, L_t may no longer be positive. An example is the Gaussian model, where $\langle \log B_i, \log B_j\rangle$ are deterministic. Then $d\langle L_i, L_j\rangle$ is of the above form with a deterministic β_{ij}. An example where no solution exists is when $\langle L_i\rangle$ is specified to be deterministic, implying L_i is Gaussian in P_{i+1}. But, this cannot happen, because we must always have $L_i > -1/\delta_i$.

In Sect. 7 we impose the zero-coupon bond constraint of unity at maturity, and modify Theorem 3 such that B_n and U_ξ are part of the solution rather than given.

6. Spot LIBOR measure

We now introduce additional structure. A *tenor structure* is a sequence of times

$$0 < T_1 < T_2 < \ldots < T_n = T \ .$$

The tenor structure is usually quarterly or semiannually, and related to the day-count fractions δ_i by $\delta_i = T_{i+1} - T_i$ (or 360/365 times that). For $t \leq T_n$, define the left continuous function $i(t)$ to be the unique integer such that

$$T_{i(t)-1} < t \leq T_{i(t)} \ .$$

Let $B \in \mathscr{E}^n$. We say an SFTS θ is *tenor adapted* if $\theta_i(t) = 0$ for $t > T_i$. Note, if $B' \in \mathscr{E}^n$ and $B_i' = B_i$ on $[0, T_i]$, then (θ, B') will also be a tenor adapted

SFTS, and $\theta B = \theta B'$. For this reason, the behaviour of B_i on $(T_i, T]$ does not matter when considering tenor adapted SFTSs. This can be used to extend the notion of a tenor adapted SFTS (θ, B) to situations where B_i is only defined on $[0, T_i]$: simply extend B_i to all of $[0, T]$ in any continuous semimartingale manner whatsoever.

For $B \in \mathscr{E}_+^n$, a particular tenor adapted SFTS θ^* suggests itself: start with 1 dollar at time 0 and buy with it asset 1; at time T_1 sell asset 1 and buy asset 2; at time T_2 sell asset 2 and buy asset 3, and so on. Formally,

$$\theta_i^*(t) \equiv \frac{1_{\{T_{i-1}<t\leq T_i\}}}{B_1(0)} \prod_{j=1}^{i(t)-1} \frac{B_j(T_j)}{B_{j+1}(T_j)}, \quad 0 < t \leq T_n, \quad \theta_i^*(0) = \frac{\delta_{i1}}{B_i(0)} .$$

Note, θ^* is a left-continuous step function. Clearly, the price B^* of this SFTS is given by

$$B_t^* \equiv \theta_t^* B_t = \frac{B_{i(t)}(t)}{B_1(0)} \prod_{j=1}^{i(t)-1} \frac{B_j(T_j)}{B_{j+1}(T_j)} = \frac{B_{i(t)}(t)}{B_1(0)} \prod_{j=1}^{i(t)-1} (1 + \delta_j L_j(T_j)) . \tag{1}$$

If B is LAF, then ξB^* is a local martingale by Theorem 4.1, and the same argument as in Theorem 4 gives that Eq. (4.2), (4.4) and (4.5) hold with B_j replaced by B^*. In particular, for all i

$$U_{B_i} = U_{B^*} \exp\left(-\langle \log \frac{B_i}{B^*}, \log \xi\rangle\right) . \tag{2}$$

Conversely, if $B \in \mathscr{E}_+^n$ and for some $\xi \in \mathscr{E}_+$ the above equation holds for all i, then, as in Theorem 4.3, one shows B is LAF. Since $B_{i(t)} = B_{i+1}(1 + \delta_{i(t)} L_{i(t)}) \ldots (1 + \delta_i L_i)$, we get from (1) and (4.7)

$$u_{L_i} + \langle L_i, \log(\xi B^*)\rangle = \int \sum_{j=i(t)}^{i} \frac{\delta_i d\langle L_i, L_j\rangle}{1 + \delta_j L_j} . \tag{3}$$

When B_i and ξ are Ito processes with respect to a d-dimensional $(\mathscr{F}_t, P)$ Brownian motion $z(t)$ as in Eq. (4.11), then $dB^*/B^* = \mu_{i(t)} dt + \sigma_{i(t)} dz$, and Eq. (2) is equivalent to

$$\mu_i(t) = \mu_{i(t)}(t) + (\sigma_i(t) - \sigma_{i(t)}(t))\varphi(t) . \tag{4}$$

Conversely, Ito process $B \in \mathscr{E}_+^n$ is LAF if for some $\varphi \in L(z)$ the above equation holds for all i. With $\beta_i(t)$ denoting the absolute volatility of L_i, Eq. (3) translates to

$$dL_i = \sum_{j=i(t)}^{i} \frac{\delta_j \beta_i \beta_j^t}{1 + \delta_j L_j} dt + \beta_i dz^*, \quad z_t^* \equiv z_t + \int_0^t (\varphi(s) - \sigma_{i(s)}(s))dt . \tag{5}$$

If B is arbitrage-free (i.e., ξB_i are martingales) then ξB^* is a martingale, as follows easily from Eq. (1). Taking B^* to be the numeraire as in Sect. 5, we call the corresponding measure P^* the *spot LIBOR measure* and denote its expectation operator E^*. As in Sect. 5, ξC is a P-(local) martingale iff C/B^* is a P^*-(local) martingale. The compensator of L_i in the P^* measure is given by (3).

Theorem 1. *Let $B \in \mathscr{E}_+^n$ be arbitrage-free, then the following equations hold.*

$$E_t^* \left[\prod_{j=i(t)}^{i} \frac{1}{1+\delta_j L_j(T_j)} \right] = \prod_{j=i(t)}^{i} \frac{1}{1+\delta_j L_j(t)}, \quad t \le T_i \; ; \tag{6}$$

$$E_t^* \left[L_i(T_i) \prod_{j=i(t)}^{i} \frac{1}{1+\delta_j L_j(T_j)} \right] = L_i(t) \prod_{j=i(t)}^{i} \frac{1}{1+\delta_j L_j(t)}, \quad t \le T_i \; . \tag{7}$$

Proof. From the definition of L_i in (4.6) we have

$$B_{i+1}(t) = B_{i(t)}(t) \prod_{j=i(t)}^{i} \frac{1}{1+\delta_j L_j(t)} \; . \tag{8}$$

Combining this with Eq. (1) we get

$$\frac{B_{i+1}(t)}{B^*(t)} = \frac{B_1(0)}{\prod_{j=1}^{i(t)-1}(1+\delta_j L_j(T_j)) \prod_{j=i(t)}^{i}(1+\delta_j L_j(t))} \; . \tag{9}$$

Substituting (9) in both sides of the equation $E_t[B_{i+1}(T_i)/B^*(T_i)] = B_{i+1}(t)/B^*(t)$, the first product cancels, and we get (6). Equation (7) follows from (6) by subtracting (6) with $i-1$ from (6). $\square$

In the Ito case, $z^*(t)$ defined in Eq. (5) is a P^* Brownian motion. When $\beta_i(t) = \lambda_i(t, L(t))L_i(t)$, for some bounded and locally Lipschitz function $\lambda_i(t, L)$, then Eq. (5) is an SDE for L. It has a similar form to the SDE (5.4), and by Lemma 2.3 it has a unique solution L.[7] As in Corollary 2.1 one can show by induction that the process Y_i defined by the right hand side of Eq. (9) is a P^* martingale. So, the solution L of (5) satisfies Eq. (6).

Equations (6) and (7) resemble familiar formulae for the risk-neutral measure and instantaneous rates. Another resembling formula is that the price C of SFTS (for which ξC are P-martingales) satisfies

$$C(T_i) = B_i(T_i) E_{T_i}^* \left[\frac{C(T_k)}{B_k(T_k)} \prod_{j=i}^{k-1} \frac{1}{1+\delta_j L_j(T_j)} \right], \quad i < k \; . \tag{10}$$

This follows directly from Eq. (1) and C/B^* being a P^* martingale. The resemblance with the risk-neutral formula and instantaneous rates is complete, only when $B_i(T_i) = B_k(T_k) = 1$.

7. The zero-coupon bond constraint

The primary application of the forward LIBOR process L is when B represents zero-coupon bonds, i.e., $B_i(T_i) = 1$. Obviously, when B is arbitrage-free and satisfies this constraint, then B still satisfies all properties of arbitrage-free price systems established so far. What we need to address is the *existence* of an arbitrage-free B with a given LIBOR volatility function $\beta_i(t) = \lambda_i(t, L(t))L_i(t)$, safisfying this constraint.

Theorem 1. *Let Q be an equivalent measure to P and $w(t)$ be a d-dimensional $(\mathscr{F}_t, Q)$ Brownian motion. Let n be an integer, $B(0) \in \mathbb{R}_+^n$ be such $L(0) \in \mathbb{R}_+^{n-1}$, and $\lambda_i(t, L)$ be $n-1$ d-dim vector valued functions on $[0, T] \times \mathbb{R}_+^{n-1}$, which are measurable, bounded, and locally Lipschitz in L. Then there exists an arbitrage-free $B \in \mathscr{E}_+^n$ starting from $B(0)$ such that $B_i(T_i) = 1$ for all i, $P_n = Q$, and associated forward LIBOR process L is positive and satisfies $d\langle L_i, w\rangle_t = \lambda_i(t, L_t)L_i(t)dt$.*

Proof. By Corollary 2.1 the SDE (5.4) has a unique solution L starting from $L(0)$, and the processes $Y_i \equiv (1 + \delta_i L_i)\ldots(1 + \delta_{n-1}L_{n-1})$ are Q-martingales. Let $B_n \in \mathscr{E}_+$ be any process such that $B_n(0)$ is as given, $B_n(T_n) = 1$, and $B_n(T_i) = 1/Y_i(T_i)$ for $i = 1, \ldots, n-1$.[8] Define $B_i = B_n Y_i$. Then $B_i(T_i) = 1$. Define $\xi_t = B_n(0)E_t[dQ/dP]/B_n(t)$. Clearly, ξB_n is a P-martingale. In fact, ξB_i is a P-martingale for all i because B_i/B_n are Q-martingales. It is also clear that $P_n = Q$. □

Note from the proof that, to ensure $B_i(T_i) = 1$, all we have to do is to make sure that B_n passes through given random variables at $0, T_1, \ldots, T_n$. (B_n can also be chosen to be less than 1.) We do not have uniqueness of B, because any such interpolation of B_n works. However, L itself, being the solution of (5.4), is independent of the choice of interpolation. This implies that $B_i(T_j)$ are also independent of the choice of interpolation, because when $B_i(T_i) = 1$, we have

$$B_i(T_j) = \prod_{k=j}^{i-1} \frac{1}{1 + \delta_k L_k(T_j)}, \quad j < i \ .$$

More generally, let C_T be a random variable such that $C_T/B_n(T)$ (or equivalently, $C_T/B_j(T)$ for any j) is measurable with respect to the sigma algebra generated by $L(t)$, $t \leq T$. Consider an option which will pay C_T at time T. The price of this option is $C_T = B_n(t)E_t^Q[C_T/B_n(T)]$. Since L does not depend on the particular interpolation of B_n, it follows that $C_t/B_n(t)$ is independent of this interpolation for all t. It now also follows that C_t at $t = 0$ and at all $t = T_i$ are independent of the choice of interpolation, because $B_n(T_i)$ are so.

Prices of LIBOR and swap derivatives are generally of the above form. Hence, the non-uniqueness of B_n does not affect their prices. What is more, pricing algorithms (e.g., simulation, trees, or PDE) need not construct B at all – all that needs to be constructed is the LIBOR process L.

Another source of non-uniqueness is the arbitrariness in the equivalent measure Q. (In the Ito case, this is the arbitrariness in the choice of the market price of risk φ.) Another choice of measure would result in different processes for not only B, but also for L. However, the price at time $t = 0$ of an option is independent of the choice of Q and w. Indeed, by the uniqueness of weak solutions, the multidimensional distribution function (i.e., the law) of a weak solution L to the SDE (5.4) is independent of Q and w. Now, if the payoff C_T is such that $C_T/B_n(T)$ is a function of $L(t_1), \ldots, L(t_m), t_k \le T$, then $C(0) = B_n(0)E^Q[C_T/B_n(T)]$ is independent of Q and w.

The construction of Theorem 1 can also be done in the spot LIBOR measure B^*. Again we are given a Brownian motion z^* in an equivalent measure Q, and this time we solve the SDE (6.5). Then we let B^* be any process in $\mathscr{E}_+$ such that $B^*(0) = 1$ and

$$B^*(T_i) = \frac{1}{B_1(0)} \prod_{j=1}^{i-1} (1 + \delta_j L_j(T_j)), \quad i \le n-1 .$$

We then define B_{i+1} by the right hand side of (6.9) times B^*. We also define $\xi_t = E_t[dQ/dP]/B^*(t)$. Then $P^* = Q$, and ξB_i are P-martingales. As before, prices of LIBOR and swap derivatives at $t = 0$ and $t = T_i$ do not depend on the choice of interpolation B^*. A particularly simple choice is the linear interpolation (which implies B^* has finite variation, i.e., $\sigma_i(t) = 0$ for $T_{i-1} < t \le T_i$). This works because, by the above equation, $B^*(T_{i+1})$ is $\mathscr{F}_{T_i}$ measurable, implying the linearly interpolated B^* is adapted.[9] As before, the question of interpolation does not arise in actual evaluation algorithms anyway, as they only construct ratios, not any numeraires like B^* or B_n.

8. Forward swap measure

Let $B \in \mathscr{E}_+^n$, and $\delta \in \mathbb{R}_+^{n-1}$. For each $i \le n-1$, consider the SFTS ("the annuity") consisting of buying and holding δ_{j-1} shares of the j-th asset for each $j > i$. Its price is given by

$$B_{i,n} \equiv \sum_{j=i+1}^{n} \delta_{j-1} B_j , \quad i \le n-1 . \tag{1}$$

The i-th *forward swap rate* process S_i is defined by

$$S_i = S_{i,n} \equiv \frac{B_i - B_n}{B_{i,n}} . \tag{2}$$

With empty sums denoting zero and empty products denoting 1, let us set

$$s_{ij} \equiv s_{ij,n} \equiv \sum_{k=j}^{n-1} \delta_k \prod_{l=i+1}^{k} (1 + \delta_{l-1} S_l), \quad s_i \equiv s_{ii} , \quad 1 \le i \le j \le n-1 . \tag{3}$$

It is not difficult to show by induction that $B_{i,n} = B_n s_i$. This in turn easily implies[10]

$$u^n_{s_i} \equiv \langle S_i, \log \frac{B_n}{B_{i,n}} \rangle = -\int \sum_{j=i+1}^{n-1} \frac{\delta_{j-1} s_{ij}\, d\langle S_i, S_j \rangle}{(1+\delta_{j-1} S_j) s_i} \,. \tag{4}$$

One can also easily show $B_i = B_n(1 + S_i s_i)$, and this implies

$$u^*_{s_i} \equiv \langle S_i, \log \frac{B^*}{B_{i,n}} \rangle = u^n_{S_i} + \int \frac{s_{i(t)}(t)(d\langle S_i, S_{i(t)} \rangle_t - S_{i(t)}(t)\,du^n_{i(t)}(t))}{1 + s_{i(t)}(t) S_{i(t)}(t)} \,. \tag{5}$$

Assume now B is arbitrage-free. As in Sect. 5, $B_{i,n}$ induces a measure $P^{i,n}$ such that $B_j/B_{i,n}$ are $P^{i,n}$ martingales. (Note $P^{n-1,n} = P_n$.) It follows, S_i is a $P^{i,n}$ martingale. Hence, by Eq. (2.3) the compensators of S_i in the P_n and P^* measures are given respectively by formulae (4) and (5). So, if S_i are Ito processes with respect to a P_n Brownian z^n, having absolute volatility ϕ_i, then

$$dS_i = -\sum_{j=i+1}^{n-1} \frac{\delta_{j-1} s_{ij}\, \phi_i \phi_j^t}{(1+\delta_{j-1} S_j) s_i} dt + \phi_i\, dz^n \,. \tag{6}$$

If ϕ_i are functions of t and $S = (S_1, \ldots, S_{n-1})$, then this is an SDE for S. Consider the case where $\phi_i(t) = S_i(t)\psi_i(t, S_t)$, for some functions $\psi_i(t, S)$ which are measurable, bounded and locally Lipschitz in L. The drift of the SDE (6) then has these properties too, so by Lemma 2.3 the SDE has unique positive solution S. Moreover, as in Corollary 2.1, one can show that the process s_i is a square-integrable P_n martingale. We can now construct an arbitrage-free B consistent with this forward-swap rate as in Theorem 5.3, or, if a tenor structure is given, as in the previous section to enforce $B_i(T_i) = 1$. Equation (5) leads to a similar SDE with respect to a P^* Brownian motion, which can alternatively be used for this construction, as in the LIBOR case. For path-independent options, Eq. (4) (or Eq. (6)) leads to the “fundamental differential equation” as in Theorem 4.7.

When the percentage forward swap volatility $\psi_i(t)$ is deterministic, the corresponding arbitrage-free price system B can be constructed explicitly by “backward induction” as in Example 4.1. But now, we modify slightly that construction to enforce the constraint $B_i(T_i) = 1$.

Example 1: *Swap market model*. Let $\psi_i(t)$ be $n-1$ deterministic, bounded, measurable d-dimensional vector-valued functions, and z^n be a Brownian motion in an equivalent measure Q. Set $S_{n-1} = S_{n-1}(0)\exp(\int \psi_i\, dz^n - |\psi_i|^2 dt/2)$. Having inductively defined and S_k for $k > i$, define $s_{ij} (j \geq i)$ and s_i by Eq. (3), and set

$$S_i = S_i(0)\exp\left(\int \left(-\frac{|\psi_i|^2}{2} - \sum_{j=i+1}^{n-1} \frac{\delta_{j-1} s_{ij} S^j \psi_i \psi_j^t}{(1+\delta_{j-1} S_j) s_i} \right) dt + \int \psi_i\, dz^n \right) \,. \tag{7}$$

One shows by induction that s_i and $S_i s_i$ are Q martingales for all i. Let $B_n \in \mathscr{E}_+$ be any process such that $B_n(0)$ is as given, $B_n(T_n) = 1$, and $B_n(T_i) = 1/(1 + S_i(T_i) s_i(T_i))$ (and if desired, $B_n < 1$). Set $B_i = B_n(1 + S_i s_i)$, and $\xi_t = B_n(0) E_t[dQ/dP]/B_n(t)$. Then, $B_i(T_i) = 1$, and ξB_i are P martingales because $1 + S_i s_i$ are Q martingales, and B is consistent with constructed S. $\square$

It is easy to show that the forward LIBOR and swap rates are related by

$$S_i = \frac{\prod_{j=i}^{n-1}(1+\delta_j L_j) - 1}{\sum_{j=i}^{n-1} \delta_j \prod_{k=j+1}^{n-1}(1+\delta_k L_k)}; \quad L_i = \delta_i^{-1}\left(\frac{1+s_i S_i}{1+s_{i+1}S_{i+1}} - 1\right) . \tag{8}$$

It follows that the LIBOR and swap market models are inconsistent with each other. (S_i and L_i cannot simultaneously have deterministic volatilities.) One chooses one or the other model as appropriate to each particular product.

9. Path-dependent LIBOR derivatives

In this and the following sections, we discuss application of the theory developed in the previous sections to some primary examples of LIBOR and swap derivatives. We will describe different model choices and implementation algorithms as appropriate for each option. Throughout, B will be an arbitrage-free price system in $\mathscr{E}_+^n$. We assume a tenor structure $\{T_i\}$ is given and $B_i(T_i) = 1$.

We consider options with payouts C_i at T_{i+1} for one or more i, with C_i measurable with respect to the sigma algebra $\mathfrak{J}_i$ generated by $L(t)$, $t \leq T_i$. For example, a cap with strike rate K has, for each i, a payout at T_{i+1} of $\delta_i \max(0, L_i(t_i) - K)$, with $t_i \leq T_i$. The payout of C_i at T_{i+1} is equivalent to the "forward transported" payout of $C_i/B_n(T_{i+1})$ at time $T = T_n$, if we assume that instead of cash, an equal amount worth of T_n maturity zero-coupon bonds is paid at T_{i+1}. Aggregating, we can assume there is a single $\mathfrak{J} = \mathfrak{J}_{n-1}$ measurable payoff at T. By Theorem 5.2, C_T can be attained by an SFTS, and the option price is $C_t = B_n(t)E_t^n[C_T]$. Generating random paths for $L(t)$ (or $S(t)$), C_0 can be computed by simply taking an average.

An example is a *knockout cap*. In one variation, caplet i gets knocked out when only the spot LIBOR $L_i(T_i)$ at T_i is below (or above) a certain level. This is path independent, actually a combination of a LIBOR cap and a digital. In the path-dependent variation, at the first fixing T_i such that $L_i(T_i) \leq K_0$ all remaining caplets for futures fixing dates T_j, $j \geq i$, get knocked out. So the payout at time T_{i+1} is

$$C_{T_{i+1}} = \delta_i \max(0, L_i(T_i) - K)\ 1_{\{\min(L_1(T_1),\ldots,L_i(T_i)) > K_0\}} .$$

Another example is an *Asian cap*, with a single payout at time $T = T_n$ of

$$C_T = \max\left(0, \sum_{i=1}^{n-1} \delta_i (L_i(T_i) - K)\right) .$$

Note, unlike currency or equity markets which check for knockout daily or continuously, in the swap markets the check is done (quarterly) at the fixing dates T_i. An example borrowed from the mortgage market is the *periodic cap*, embedded in a periodically capped floating-rate note, where the floating rate coupon K_i (for payment at T_{i+1}) is set at spot LIBOR, subject to it not exceeding the previously set coupon by a prescribed amount x_i. Thus, $K_i = \min(L_i(T_i), K_{i-1} + x_i)$. A more complex structure is a *ratchet* where $K_i = \max(\min(L_i(T_i) + y_i, K_{i-1} + x_i), K_{i-1})$. Clearly, in both cases K_i is a function of $L_j(T_j)$, $j \leq i$. A new structure is the *flexible cap*. Here, a number $m < n - 1$ is specified, and the cap knocks out as soon as m of the caplets end up in the money. Forward transporting, in all these examples the payout C_T is a function of past spot LIBOR fixings:

$$C_T = C(L_1(T_1), \ldots, L_{n-1}(T_{n-1})) .$$

The present value $C(0)$ of the option is given by either of the equivalent formulae

$$C_0 = B_n(0)E^n[C(L_1(T_1), \ldots, L_{n-1}(T_{n-1}))] = E^* \left[\frac{C(L_1(T_1), \ldots, L_{n-1}(T_{n-1}))}{\prod_{i=1}^{n-1}(1 + \delta_i L_i(T_i))} \right] .$$

Clearly, these formulae also hold when C_T is expressed as a function of spot swap rates $S_i(T_i)$. The first formula "discounts outside the path" and takes the "forward-risk-adjusted average", while the second formula "discounts along the path" and takes the "spot LIBOR average". When simulating, both types of average are just the ordinary average. The difference comes from the choice of SDE employed to generate random paths: the SDE (5.4) describing the dynamics in the P_n measure, versus the SDE (6.5) for the dynamics in the P^* measure. As shown in Examples 4.1 and 8.1, the P_n dynamics can be explicitly constructed for LIBOR and swap market models. (This can be used to construct explicitly the P^* dynamics as well.) This enables the computation of the dt integrals by the trapezoid rule, rather than by a step function as is usual when dealing with a general SDE. Our experiments with the P_n dynamics have shown that high accuracy is achieved if the Brownian motion is sampled only quarterly to compute the dt integrals.

10. Bermudan swaptions

A T_i-start ($i \leq n - 1$) receiver swap with coupon K is a contract to receive (fixed) $\delta_j K$ and pay (floating) $\delta_j L_j(T_j)$ at each time T_{j+1}, $i \leq j \leq n - 1$. Given a subsequence T_{i_k} of start dates, and expirations $t_k \leq T_{i_k}$, $k = 1, \ldots, m$, a *Bermudan receiver swaption* is an option which at each time t_k gives the holder the right to enter a T_{i_k}-start swap, provided this right has not already been exercised at a previous time t_p, $p \leq k$. When $m = 1$, this is a European swaption. Otherwise, usually $i_k = q + k$, where $q = n - m - 1$, i.e., the swaption is exercisable after the first q "noncall" periods $T_1, \ldots, T_q$. Often, t_k is at the LIBOR fixing date (two business days before T_{i_k}). Other times, early notice is to be given, and t_k may be 20 to 40 days before T_{i_k}. Bermudan swaptions frequently arise as embedded options in cancellable (callable) swaps, which in turn often originate from new issue swapping or asset packaging of callable bonds.

In order to model such options, one assumes that the holder acts optimally, in that the swaption will be exercised at time t_k if the value at time t_k of the T_{i_k}-start swap is not less than the swaption value. American options are usually posed as optimal stopping problems. But, we can cast the Bermudan swaption in the setting of an ordinary path-dependent by assuming that if the swaption is exercised at t_k, then instead of actually entering the T_{i_k}-start swap, the counterparties settle by the holdor receiving at time t_k, T_n-maturity zero-coupon bonds worth the value at time t_k of the T_{i_k}-start swap. Obviously, this assumption does not affect the price and hedge of the Bermudan swaption at time $t = 0$. The T_{i_k}-start swap is worth $B_n(t_k)V^k$ at time t_k, where

$$V^k = \frac{B^{i_k,n}(t_k)}{B_n(t_k)}(K - S_{i_k}(t_k)) .$$

Thus, if the holder exercises at t_k, he will receive face value V^k of T_n-maturity zero-coupon bonds, so he will have a payoff of V^k at time $T = T_n$. Define the random variables $C^m, \ldots, C^1$ inductively by

$$C^m = \max(V^m, 0)\,; \quad C^k = 1_{\{V^k \geq E^n_{t_k}[C^{k+1}]\}}V^k + 1_{\{V^k < E^n_{t_k}[C^{k+1}]\}}C^{k+1}\,, \quad k \leq m-1\,.$$

The optimality assumption implies that the payoff at time T of the Bermudan swaption is C^1. As such, we formally define a Bermudan swaption to be the asset which pays C^1 at time T. Since C^1 is $\mathfrak{J}$-measurable, by Theorem 5.2 this payoff can be attained by an SFTS and its price is $B_n(t)E^n_t[C^1]$.

Using the recursive relation above, we easily find that the price $C(t)$ of the Bermudan swaption is

$$\begin{aligned} C_t &= B_n(t)E^n_t[\max(V^1, E^n_{t_1}[\max(V^2, \ldots \\ &\quad E^n_{t_{m-2}}[\max(V^{m-1}, E^n_{t_{m-1}}[\max(V^m, 0)])] \ldots)])],\ t \leq t_1\,. \end{aligned}$$

(A similar formula is obtained by using B^* as the numeraire.) Another Bermudan product is the *callable capped floating rate swap*, in which a counter party receives LIBOR and pays min(K, LIBOR + spread) at each T_i, and has the right to cancel this swap at a fixed date (European) or at every T_i after a fixed date (Bermudan). These arise from *callable capped floating rate notes*, where the issuer offers a spread over the market rate in exchange for a cap on the floating coupon plus a call option. We can formalize their payout structure as above. The European ones are akin to captions (options on caps), already having a compound option character.

Numerical evaluation of captions, European callable capped floating rate swaps, and Bermudan swaption having only two call dates, presents no difficulty. We would use the LIBOR market model for the first two, and the swap market model for the latter. As shown in Example 3.2, these models price caps and European swaptions by a closed-form Black-Scholes formula (where, in the formulae of Example 3.1, T is now to be replaced by t_2). Therefore, only a single unconditional expectation needs to be calculated, and this can be done accurately by Monte Carlo simulation as described in the previous section. This is valuable, as it can be used as a benchmark to attest the accuracy of other numerical techniques which can handle more call dates.

However, application of conventional Monte Carlo simulation to the general Bermudan case is formidable. The difficulty is the recursive nature of the payoff and the multitude of conditional expectations. Except for the last conditional expectation which is available in closed-form, to compute the k-th conditional expectation, each path up to time t_k must branch into several paths, say N_k paths, to time t_{k+1} before an average can be taken. This means a total of $N_1 \ldots N_{m-1}$ paths, a number that explodes very quickly with the number of call dates m. Some recent Monte Carlo techniques, such as those proposed by Broadie and Glasserman (1994) appear promising, if they can be adapted to LIBOR and swap market models.

A more standard approach is using "bushy trees" constructed from a binomial or multinomial discretization of the underlying Brownian motion, as described by Heath et al. (1990), and, in the context of the LIBOR market model, by Gatarek (1996). Here, 10, or possibly even 20, call dates can be incorporated without difficulty. The problem is that coarse time steps must be taken for long-dated swaptions, e.g., a semiannual time step for a 10-year Bermudan swaption. Coarse time steps result in sparse sampling of the distributions for the earlier call dates. But, it is the earlier call dates that contribute more significantly to the Bermudan premium. Unequal time steps may remedy this to some extent. Another possibility is generating, say, 200 antithetical random paths for the first year, followed by a busy tree along each path. Other tricks may be possible along these lines, but, we are not aware of any substantiated published account.

The approach we favour most is approximation of the model by a convenient non-arbitrage-free model, and numerical implementation of the latter. The simplest, and possibly best, candidate is taking the spot swap (or LIBOR) rate as the state variable, while treating zero-coupon bonds as deterministic for the purpose of discounting. What makes this attractive is that we know that it already prices simultaneously all t_k-expiry European swaptions consistently with the market model.

Bermudan swaptions are then evaluated along an ordinary binomial lattice or grid for the state variable. Without presenting details, we just report that comparison with the Monte Carlo technique in the case of two exercise dates indicated surprisingly high accuracy. An approximation by a "Gaussian core" on a non-bushy multinomial tree has been proposed by Brace (1996). Another interesting possibility is approximation by a log-Gaussian short-rate model.

11. LIBOR in arrears and CMS convexity adjustment

As we have assumed, LIBOR is fixed at the beginning of the interest accrual period, and paid at the end. In rare cases, known as *LIBOR in Arrears*, it is fixed just before it is paid. This can also apply to caps. Given $t \le T_i$ and a payoff $C(L_i(t))$ for payment at T_i, the problem is thus to calculate its present value $B_i(0)E^i[C(L_i(t))]$, and relate it to the usual case. This is done simply by

$$C_s = B_i(s)E_s^i[C(L_i(t))] = B_{i+1}(s)E_s^{i+1}[1 + \delta_i L_i(t))C(L_i(t))] , \quad s \le t . \qquad (1)$$

The main case of interest is when $C(L_i(t)) = L_i(t)$, in which case we simply have

$$E_s^i[L_i(t)] = E_s^{i+1}\left[L_i(t)\frac{1+\delta_i L_i(t)}{1+\delta_i L_i(s)}\right] = L_i(s) + \frac{\delta_i \operatorname{var}_s^{i+1}[L_i(t)]}{1+\delta_i L_i(s)} \ . \tag{2}$$

The second term on the right hand side is called a convexity adjustment. In the LIBOR market $L_i(t)$ is lognormally distributed in the P_{i+1} measure with mean $L_i(s)$, providing an integral expression for Eq. (1) (which leads to explicit formulae for in Arrears caps and digitals), and (2) simplifies to

$$E^i[L_i(t)] = L_i(0) + \frac{\delta_i L_i^2(0)(e^{\langle \log L_i \rangle_t} - 1)}{1+\delta_i L_i(0)} \ .$$

The process $y = 1 - B_{i+1}/B_i$ is a P_i martingale, and in the Ito case follows

$$dy = \lambda_i y(1-y)dz_i \ , \quad y \equiv 1 - \frac{B_{i+1}}{B_i} \ .$$

When $\lambda_i(t)$ is deterministic, this process is described in Karlin and Taylor (1981, Chap. 15.15) as a model of gene frequency fluctuations. Using (1), we can calculate its probability transition function

$$p(s,x,t,y) = \frac{(1-x)e^{-\left(\log\frac{y(1-x)}{x(1-y)}\right)+v_i^2(s,t)/2)^2/(2v_i^2(s,t))}}{\sqrt{2\pi}y(1-y)^2 v_i(s,t)} \ , \quad v_i^2(s,t) \equiv \int_s^t |\lambda_i(u)|^2 du \ .$$

A more important convexity adjustment is associated with the *constant maturity swap* (CMS). In this swap at each payment date T_{i+1} spot LIBOR $L_i(T_i)$ is received and an amount equal to the spot swap rate $S_i(T_i)$, for a fixed length swap is paid. We therefore wish to calculate $E^{i+1}[S_i(t)]$. Since both S_i and $B_{i+1}/B_{i,n}$ are $P^{i,n}$ martingales, we obtain from the definition of covariance

$$\begin{aligned} E_s^{i+1}[S_i(t)] &= \frac{B_{i,n}(s)}{B_{i+1}(s)} E_s^{i,n}\left[S_i(t)\frac{B_{i+1}(t)}{B_{i,n}(t)}\right] \\ &= S_i(s) + \frac{B_{i,n}(s)}{B_{i+1}(s)} \operatorname{cov}_s^{i,n}\left[S_i(t), \frac{B_{i+1}(t)}{B_{i,n(t)}}\right] \end{aligned} \tag{3}$$

Hence, the covariance term above is the CMS convexity adjustment. Note, for $s = 0$ it also equals $E^{i,n}[\langle S_i, B_{i+1}/B_{i,n}\rangle_t]$. Recalling $B_{i+1}/B_n = (1 + s_{i+1}S_{i+1})$ with s_i as in (8.3), another expression is

$$E_s^{i+1}[S_i(t)] = \frac{B_n(s)}{B_{i+1}(s)} E_s^n[S_i(t)(1 + S_{i+1}(t)s_{i+1}(t))] \ . \tag{4}$$

Unlike LIBOR in arrears, neither (3) nor (4) have a closed-form solution, even in the market models. However, (4) can easily be calculated by Monte Carlo simulation of the P_n dynamics of the swap market given by Eq. (8.7). (The LIBOR market model can also be used, but it is more natural to utilize swaption volatilities here.) It may also be possible to use an approximation based on Eq. (3). If the volatility of B_{i+1}/B_n were deterministic, then B_{i+1}/B_n would be a log-Gaussian process in the $P^{i,n}$ measure, because it is a martingale there. Obviously, this assumption is inconsistent with $S_i(t)$ also having deterministic volatility. But, if we accept it as an approximation, then Eq. (3) is the covariance of two lognormal random variables, which can be easily calculated.

Another convexity adjustment arises from what is sometimes called *extended* LIBOR. Here, the floating payment $L_i(T_i)$, instead of being paid at T_{i+1}, is paid at a later date T_j, and we calculate $E^j[L_i(T_i)]$. Now the convexity adjustment will be negative. Again, there is no closed-form solution, but valuation can be done easily by Monte Carlo simulation of the LIBOR market model.

12. Options depending on both LIBOR and swap rates

The products discussed so far have been basically either a LIBOR/cap product or a swap/swaption product, but not both. But some products explicitly involve both. An example is the *spread option*, a series of payoffs of the form $\max(S_i(t)-L_i(t)-K,0)$, where K can be zero, positive or negative. Another is the *LIBOR trigger swap*. A start date T_i, end date T_n, strike K and coupon K_s are specified, and if $L_i(T_i) \geq K$, then counterparties enter a T_i-start swap with coupon K_s and end date T_n. In another variation the swap is triggered at the first $j \geq i$ such that $L_j(T_j) \geq K$. The most important product in this category is the *index amortizing swap*. It is an interest-rate swap with a stochastic decreasing notional modelled after mortgage prepayment functions. The basic idea is that if rates fall, the notional is reduced. A range of LIBOR rates and corresponding percentage amounts are specified by which percentage the notional drops from its previous level at every T_i for which LIBOR $L_i(T_i)$ is within one of the ranges. There are other variations, including a longer rate for the index and the index amortizing cap.

In order to incorporate both caplet and swaptions volatilities for valuation of these products, we need to construct a swap (or LIBOR) market model that is root-search calibrated to caplets (swaptions) prices. This can be done as follows. We assume that Black-model caplet volatilities, and hence prices $C_i(0)$ are available for all expirations T_i. For example, they are derived by appropriate interpolation of market quoted cap volatilities. Similarly, we assume all T_i-start swaption volatilities υ_i are available. Start with a 2-factor swap market model, and let ψ_{i1} and ψ_{i2} be the two components of (percentage) volatility of S_i, taken to be independent of t. Then, $\upsilon_i = (\psi_{i1}^2 + \psi_{i2}^2)^{1/2}$. We can find another equation by a backward induction. For $i = n-1$, the caplet and swaption coincide, so we arbitrary fix the one degree of freedom, e.g., by setting $\psi_{n-1,1} = \upsilon_{n-1}$, $\psi_{n-1,2} = 0$. Having inductively determined ψ_{j1} and ψ_{j2} for $j > i$, begin with an initial guess for ψ_{i1} (e.g., use $\psi_{i+1,1}$). This determines ψ_{i2} via υ_i. Use these to construct paths S_j, $j \geq i$, according to Eq. (8.7). By Eq. (8.8), we can express the T_n payoff of the T_i caplet in terms of S_j, $j \geq i$. We iterate ψ_{i1} until $C_i(0)/B_n(0)$ equals the average of the payoff over all paths.

The above construction is feasible numerically. Once completed, then the above options can be computed by Monte Carlo simulation. However, calibrating market models by root search seems to defeat their very purpose. Moreover, unless the input cap and swaption volatilities are highly coherent, the solution may be unstable or not exist at all. Also, it does not incorporate directly given correlations between forward LIBOR and swap rates. For these reasons, we think some practical improvisation is called for.

We begin with the spread option. Its price is $B_{i+1}(0)E^{i+1}[\max(S_i(t) - L_i(t) - K, 0)]$. The improvisation we suggest is to assume that for all i both $L_i(t)$ and $S_i(t)$ are P_{i+1} lognormal, although we know this is at odds with the no-arbitrage principle. Then, for $K = 0$, we have a robust Margrabe/Black-Scholes formula for the price. For positive (negative) K, conditioning on $L_i(S_i)$, we get an integral involving the Black-Scholes formula, which can be quickly integrated numerically. For $E^{i+1}[S_i(t)]$, one should incorporate the CMS convexity adjustment, but $E^{i+1}[L_i(t)]$ remains $L_i(0)$.

Consider now the LIBOR-trigger swap. Since the events $\{L_i(t) > K\}$ and $\{B_i(t) > (1 + \delta_i K)B_{i+1}(t)\}$ are the same, a LIBOR trigger swap is a portfolio of trigger swaps as in Example 3.3. But the pricing formula there assumes B_i have deterministic volatilities, which is applicable to the Gaussian model, but not to market models. However, the example still provides a formula for $E[1_{X>Y}Z]$ with X, Y and Z jointly lognormal, on which we can base our improvisation. Indeed, the price of the trigger swap is $B_{i,n}(0)E^{i,n}[1_{L_i(t)>K}(S_i(t) - K_S)]$. Therefore, if we assume that both $S_i(t)$ and $L_i(t)$ are $P^{i,n}$ lognormally distributed, we can at once write down a simple Black-Scholes type formula.

For index amortizing swaps an improvisation consistent with the preceding two would basically regard the spot rates $L_i(T_i)$ and $S_i(T_i)$ as jointly lognormal state variables, while treating discount factors deterministically. But, since Monte Carlo simulation is still to be used, one may as well use the “proper” calibrated market model constructed above if the solution is adequately stable.

Finally, let us mention *callable reverse floaters*, which fits into none of the categories discussed so far. These are (European or Bermudan) callable notes whose coupon paid at T_{i+1} is $\max(K - L_i(T_i), 0)$. The swap version is more general: one pays $\max(K - L_i(T_i), K')$, receives $L_i(T_i)$, and has the right to cancel. When $K - K'$ is much larger than $L_i(0)$ (e.g., twice as large), this is essentially a swaption with a coupon of $(K - K')/2$ and twice the notional. In this case, which is fortunately the usual case, it can be priced using the swap market model. However, in general, this product exhibits both cap and swaption characteristics, but unlike the above examples, they cannot be “separated”. We have not yet come up with a sensible improvisation for this product in general.

13. Conclusion

The general theory that we described is consistent and reasonably elegant, but it does not provide a perfect solution to all the issues and product range that practitioners are faced with day-to-day. For one thing, we ignored transaction costs, default risk, and process jumps. To be sure, there are theories that address these.[11] But, the problem is not just theoretical imperfections. The very principle of arbitrage by dynamic trading can be questioned as a practical proposition. This had significant bearing on our attitude to modelling, and leads us to treat the theory only as a guide, making sensible improvisations when useful.

We started out based firmly on the principle of no-arbitrage, but when it came to actual securities, we violated it by recommending mutually inconsistent models for different securities, some of which were not even arbitrage-free. We defend this stance on the grounds that all option models are at best rough approximations of reality, from their assumption on the market mechanism (frictionless and perfect markets, continuous trading, infinitely divisible prices, etc.) to their statistical specifications (number of factors, distribution, estimated parameters, etc.). Assumptions and approximations are to be judged by their reasonableness and usefulness, and this depends on the product and the trading environment.

For example, according to the theory, the event that the underlying price ends up right at the option strike has zero probability. This is innocuous for most options. But for an at the money digital or trigger swap near expiration, it is the event of most concern to traders. Model choice also depends on the purpose of the trade. Market makers in LIBOR and swap derivatives are normally interested in preserving the initial margin by hedging all market risk. It is therefore important for their valuation to be calibrated to liquid caps and European swaptions, which serve as natural hedge instruments. However, for proprietary trading, one intends to keep certain exposures unhedged, and, it may therefore be better to use more stationary and equilibrium-like models, which imply what cap and swaption prices (or even, the rates themselves) should be in the first place.

The bottom line is that model choice should reflect the intended hedge instruments. For different products the model should be adjusted to keep this dependency as intimate and robust as possible. It is better to use well-adapted but mutually inconsistent models for different products, than to use a uniform model ill adapted to all. Traders use the Black-Scholes formula to inconsistently price both an option on S&P500 and options on the individual stocks. It would certainly be foolhardy to attempt to arbitrage this inconsistency by using a 500-factor option model for the index. Likewise, quarterly and semiannual tenor LIBOR market models are inconsistent, and both are inconsistent with the swap market model. But this engenders no more practical arbitrage opportunity than does the S&P500 option. The assumption that quarterly volatility is deterministic is no more empirically compelling than semiannual LIBOR volatility being deterministic. One considers using one in favour of another only because it is more convenient and natural to the product in question.

Endnotes

1. A function $f(x_1, \ldots, x_n)$ is homogeneous of degree m if $f(x_1, \ldots, x_n) = \alpha^m f(\alpha x_1, \ldots \alpha x_n)$ for all $\alpha > 0$.
2. A function $f(t, x)$ is locally Lipschitz in x if $\forall$ integer n, $\exists K_n > 0$ s.t. if $|x| < n$ and $|y| < n$, then $|f(t,x) - f(t,y)| \leq K_n|x - y|$, $\forall t$. Some textbooks assume the global Lipschitz condition for existence, but it is known that the local condition together with linear growth suffices.

3. Examples of SFTS involving forward and futures contracts can be found in Cox et al. (1981), and including exchanges rates, in Jamshidian (1994).

4. The term “arbitrage-free model” is usually reserved for models that do not admit “free lunches” in an appropriate sense. This is then related to the existence of an equivalent martingale measure (See Harrison and Pliska (1981), and for recent general results in this direction, Delbaen and Schachermayer (1997).). However, this relationship is not the central topic of this paper, and the less usual terminology adopted here is more convenient for our purposes.

5. Indeed, set $B' = B/X$, and $\xi' = \xi X/X(0)$. Then $\xi' B_i' = \xi B_i/X(0)$ is a P-martingale. In particular, if X represents a foreign a currency, so that B' is the price system in the units of the foreign currency, we see that $\xi' = \xi X/X(0)$ represents a state price deflator with respect to the foreign economy. The relative prices are however independent of the currency: $B_i'/B_j' = B_i/B_j$. So is the i-th numeraire measure P_i defined by $dP_i/dP = M_i(T)$, where $M_i \equiv \xi B_i/B_i(0)$, because $M_i' \equiv \xi' B_i'/B_i'(0) = M_i$.

6. We note that for a d-dim process σ adapted to $\mathfrak{J}_t$ with $\int |\sigma|^2 dt < \infty$, $\int_{\mathfrak{J}} \sigma dw = \int \sigma dw$, where $\int_{\mathfrak{J}}$ denote stochastic integration with respect to $\mathfrak{J}_t$. Consequently also covariation with respect to the two filtrations coincide for $(\mathfrak{J}_t, w_t)$ Ito processes. And these are also the same for the measures P and P_n. So, we need not distinguish between the four possible combinations.

7. Equation (5) is the discrete tenor version of the “forward-rate drift restriction” in Heath et al. (1992). In the continuous tenor limit, the sum is replaced by an integral and δ_j in the numerator is replaced by dT. But the denominator then becomes 1, and because of this, when the absolute forward rate volatility β_{i-1} has linear growth, the drift term will have quadratic growth (as opposed to linear growth in the discrete tenor case), and the solution explodes.

8. We could similarly construct an arbitrage-free B such that $B_i(T_i) = b_i$ for any given $\mathscr{F}_{Ti}$ measurable random variables b_i, by choosing B_n such that $B_n(T_i) = b_i/Y_i(T_i)$.

9. The linear interpolation was also mentioned by the referee, whom I thank.

10. Here we have assumed $1 + \delta_{i-1} S_i > 0$, which will be the case if for example $B_i \geq B_n$ or $\delta_{n-1} \geq \delta_{i-1}$. Otherwise, we can still write down the same formula, but in a more complex form.

11. Some recent papers on these topics are: Cvitanic and Karatzas (1996) on transaction costs; Björk et al. (1997) on jumps; Duffle and Singleton (1994) on defaultable interest rates. The latter’s framework indicates that market models may still be applicable in the presence of default risk, provided LIBOR and swap rates represent default-free rates plus a default spread.

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