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# Continuous-time term structure models: Forward measure approach*

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**Abstract.** The problem of term structure of interest rates modelling is considered in a continuous-time framework. The emphasis is on the bond prices, forward bond prices and so-called LIBOR rates, rather than on the instantaneous continuously compounded rates as in most traditional models. Forward and spot probability measures are introduced in this general set-up. Two conditions of no-arbitrage between bonds and cash are examined. A process of savings account implied by an arbitrage-free family of bond prices is identified by means of a multiplicative decomposition of semimartingales. The uniqueness of an implied savings account is established under fairly general conditions. The notion of a family of forward processes is introduced, and the existence of an associated arbitrage-free family of bond prices is examined. A straightforward construction of a lognormal model of forward LIBOR rates, based on the backward induction, is presented.

**Key words:** Term structure of interest rates, forward measure, martingale measure, LIBOR rate

**JEL classification:** E43, E44

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## 1. Introduction

The Heath-Jarrow-Morton term structure methodology (see Heath et al. [11]) is based on an arbitrage-free dynamics of the instantaneous continuously compounded forward rates. The existence of such rates, however, requires a certain degree of smoothness with respect to the tenor of the bond prices and their volatilities, and thus working with such models may be inconvenient. An alternative construction of an arbitrage-free family of bond prices, making no reference to the instantaneous rates, is in some circumstances more suitable. The first step in this direction was done by Sandmann and Sondermann [18], who focused on the effective annual interest rate. This approach was further developed in the following series of papers: Sandmann et al. [19], Goldys et al. [9], Musiela [16], Miltersen et al. [15]. It is interesting to observe that Brace et al. [4] parametrize their version of the lognormal forward LIBOR model introduced by Miltersen, Sandmann and Sondermann in [15] with a piecewise constant volatility function, however, they need to consider smooth volatility functions in order to analyse the model in the Heath-Jarrow-Morton framework.

In the present paper, the problem of continuous-time modelling of term structure of interest rates is considered in a general manner. We describe certain properties which are valid for wide classes of term structure models, so that a basis for the discussion of any specific model is developed. Three such special systems are put forward, and their properties are discussed (we refer to them as to the set-ups (BP), (FP) and (LR) in what follows). The paper proceeds as follows. In Section 2, we deal with the question of existence and uniqueness of a savings account implied by a given (weakly) arbitrage-free continuous-time family of bond prices. The next section is devoted to the problem of a construction of an arbitrage-free family of bond prices given a family of stochastic volatilities of forward processes and an initial term structure. Finally, in Section 4 a construction of the lognormal model of forward LIBOR rates is presented.

Let us comment briefly on the existence of a short-term rate of interest. In the traditional models, in which the instantaneous continuously compounded short-term rate $r$ is well defined, the *savings account* process, $B^*$ say, satisfies

$$B^*_t = \exp\left(\int_0^t r_u \, du\right) \tag{1}$$

so that it represents the amount generated at time $t$ by continuously reinvesting \$1 in the short-term rate $r$ (in such a framework, the absence of arbitrage is related to the non-negativity of the short-term rate). Though we will deal sometimes with an (implied) savings account, we will not assume that its sample paths are absolutely continuous with respect to the Lebesgue measure, therefore, term structure models in which the instantaneous rate of interest is not well-defined will be also covered by our subsequent analysis. In this more general setting, the absence of arbitrage between all bonds with different maturities implies the existence of a savings account which follows a process of finite variation. If, in addition, the absence of arbitrage between bonds and cash is assumed, the savings account is shown to follow an increasing process.

## 2. Bond price models

We assume that we are given a probability space $(\Omega, (\mathcal{F}_t)_{t\in[0,T^*]}, \mathbf{P})$ equipped with a filtration $(\mathcal{F}_t)_{t\in[0,T^*]}$ which satisfies the 'usual conditions'. For the ease of exposition, we make the following standing assumption (see Björk et al. [2]–[3] for more general term structure models).

*Assumption (A).* The process $W$ is a $d$-dimensional standard Wiener process defined on a filtered probability space $(\Omega, (\mathcal{F}_t)_{t\in[0,T^*]}, \mathbf{P})$. The underlying filtration $(\mathcal{F}_t)_{t\in[0,T^*]}$ coincides with the usual $\mathbf{P}$-augmentation of the natural filtration of $W$.

We write $\mathcal{M}_{loc}(\mathbf{P})$ and $\mathcal{M}(\mathbf{P})$ to denote the class of all real-valued local martingales and the class of all real-valued martingales, respectively. The subscript $_c$ will indicate that we consider processes with continuous sample paths, and the superscript $^+$ will denote the collection of all strictly positive processes which belong to a given class of processes. For instance, $\mathcal{M}_c^+(\mathbf{P})$ stands for the class of strictly positive $\mathbf{P}$-martingales with continuous sample paths. We denote by $\mathcal{V}$ (by $\mathcal{A}$, respectively) the class of all real-valued adapted (predictable, respectively) processes of finite variation. We write $\mathcal{S}_p(\mathbf{P})$ to denote the class of all real-valued special semimartingales, i.e., those processes $X$ which admit a decomposition $X_t = X_0 + M_t + A_t$, where $M \in \mathcal{M}_{loc}(\mathbf{P})$ and $A \in \mathcal{A}$. Abusing slightly our convention, we denote by $\mathcal{S}_p^+(\mathbf{P})$ the class of those special semimartingales $X \in \mathcal{S}_p(\mathbf{P})$ which are strictly positive, and such that, in addition, the process of left hand limits, $X_{t-}$, is also strictly positive (all the processes considered here are assumed to be càdlàg – that is, with almost all sample paths being right-continuous functions, with finite left-hand limits). Notice that the class $\mathcal{S}_p(\mathbf{P})$ (as well as $\mathcal{S}_p^+(\mathbf{P})$) is invariant with respect to an equivalent change of the underlying probability measure. More precisely, $\mathcal{S}_p(\mathbf{Q}) = \mathcal{S}_p(\mathbf{P})$ and $\mathcal{S}_p^+(\mathbf{Q}) = \mathcal{S}_p^+(\mathbf{P})$ if $\mathbf{Q}$ and $\mathbf{P}$ are mutually equivalent probability measures on $(\Omega, \mathcal{F}_{T^*})$ such that the Radon-Nikodým density

$$\Lambda_t = \frac{d\mathbf{Q}}{d\mathbf{P}}_{\,|\mathcal{F}_t}, \quad \forall\, t \in [0, T^*],$$

follows a locally bounded process (see [5], p.258). For brevity, we write $\mathbf{Q} \sim \mathbf{P}$ to denote that two probability measures $\mathbf{Q}$ and $\mathbf{P}$ are mutually equivalent. Since the filtration $(\mathcal{F}_t)_{t\in[0,T^*]}$ is generated by a Wiener process, the Radon-Nikodým density $\Lambda$ will always follow a continuous exponential martingale, hence a locally bounded process. Therefore, we may and do write simply $\mathcal{S}_p$ and $\mathcal{S}_p^+$ in what follows.

*2.1. Family of bond prices*

Let us fix a strictly positive horizon date $T^* > 0$, and let $B(t,T)$ stand for the price at time $t \le T$ of a zero-coupon bond which matures at time $T \le T^*$. By a *family of bond prices* we mean an arbitrary family of strictly positive real-valued adapted processes $B(t,T)$, $t \in [0,T]$, with $B(T,T) = 1$ for every $T \in [0,T^*]$. Notice that, for convenience, the assumption that the bond price $B(\cdot,T)$ follows a semimartingale is not imposed in the definition of a family of bond prices. In this section, a family $B(t,T)$ of bond prices is assumed to be given – that is, already constructed by means of a certain procedure. We shall usually make the following assumptions.

**(BP.1)** For any date $T \in [0,T^*]$, the bond price $B(t,T)$, $t \in [0,T]$, belongs to the class $\mathscr{S}_p^+$.

**(BP.2)** For any fixed $T \in [0,T^*]$, the *forward process*

$$F_B(t,T,T^*) \stackrel{\text{def}}{=} \frac{B(t,T)}{B(t,T^*)}, \quad \forall\, t \in [0,T],$$

follows a martingale under $\mathbf{P}$, or equivalently,

$$B(t,T) = \mathbf{E}_{\mathbf{P}}\Big(\frac{B(t,T^*)}{B(T,T^*)}\,\Big|\,\mathscr{F}_t\Big), \quad \forall\, t \in [0,T]. \tag{2}$$

In view of (BP.1)–(BP.2), the process $F_B(t,T,T^*)$, $t \in [0,T]$, follows under $\mathbf{P}$ a strictly positive continuous martingale with respect to the filtration of a Wiener process, so that $F_B(\cdot,T,T^*)$ is in $\mathscr{M}_c^+(\mathbf{P})$. Consequently, for any fixed $T \in [0,T^*]$ there exists a $\mathbf{R}^d$-valued predictable process $\gamma(t,T,T^*)$, $t \in [0,T]$, integrable with respect to the Wiener process $W$, and such that

$$F_B(t,T,T^*) = F_B(0,T,T^*)\,\mathscr{E}_t\Big(\int_0^{\cdot} \gamma(u,T,T^*)\cdot dW_u\Big),$$

or more explicitly,

$$F_B(t,T,T^*) = F_B(0,T,T^*)\,\exp\Big(\int_0^t \gamma(u,T,T^*)\cdot dW_u - \frac{1}{2}\int_0^t |\gamma(u,T,T^*)|^2 du\Big).$$

Put another way, for any fixed maturity $T \in [0,T^*]$ the process $F_B(t,T,T^*)$ satisfies

$$dF_B(t,T,T^*) = F_B(t,T,T^*)\gamma(t,T,T^*)\cdot dW_t. \tag{3}$$

Let us now consider any two maturities $T,U \in [0,T^*]$. We define the forward process $F_B(t,T,U)$ by setting

$$F_B(t,T,U) \stackrel{\text{def}}{=} \frac{F_B(t,T,T^*)}{F_B(t,U,T^*)} = \frac{B(t,T)}{B(t,U)}, \quad \forall\, t \in [0,T\wedge U]. \tag{4}$$

Suppose first that $U > T$; then the amount

$$f_s(t,T,U) = (U-T)^{-1}(F_B(t,T,U)-1) \tag{5}$$

is the *add-on (annualized) forward rate* over the future time interval $[T,U]$ prevailing at time $t$, and

$$f(t,T,U) = \frac{\ln F_B(t,T,U)}{U-T}.$$

is the (continuously compounded) forward rate at time $t$ over this interval. On the other hand, if $U < T$ then $F_B(t,T,U)$ represents the value at time $t$ of the *forward price* of a $T$-maturity bond for the forward contract which settles at time $U$. The following lemma is a straightforward consequence of Itô's formula.

**Lemma 2.1** *For any maturities $T,U \in [0,T^*]$, the dynamics under $\mathbf{P}$ of the forward process are given by the following expression*

$$dF_B(t,T,U) = F_B(t,T,U)\,\gamma(t,T,U)\cdot\big(dW_t - \gamma(t,U,T^*)\,dt\big), \tag{6}$$

*where*

$$\gamma(t,T,U) = \gamma(t,T,T^*) - \gamma(t,U,T^*) \tag{7}$$

*for every $t \in [0,U\wedge T]$*

Combining Lemma 2.1 with Girsanov's theorem, we obtain

$$dF_B(t,T,U) = F_B(t,T,U)\,\gamma(t,T,U)\cdot dW_t^U, \tag{8}$$

where for every $t \in [0,U]$

$$W_t^U = W_t - \int_0^t \gamma(u,U,T^*)\,du. \tag{9}$$

The process $W^U$ is a standard Wiener process on the filtered probability space $(\Omega,(\mathcal{F}_t)_{t\in[0,U]},\mathbf{P}_U)$, where the probability measure $\mathbf{P}_U \sim \mathbf{P}$ is defined on $(\Omega,\mathcal{F}_U)$ by means of its Radon-Nikodým derivative with respect to the underlying probability measure $\mathbf{P}$

$$\frac{d\mathbf{P}_U}{d\mathbf{P}} = \mathcal{E}_U\Big(\int_0^{\cdot} \gamma(u,U,T^*)\cdot dW_u\Big), \quad \mathbf{P}\text{-a.s.} \tag{10}$$

It is apparent that the forward process $F_B(t,T,U)$ follows an exponential local martingale under the 'forward' probability measure $\mathbf{P}_U$, since equality (8) yields

$$F_B(t,T,U) = F_B(0,T,U)\,\mathcal{E}_t\Big(\int_0^{\cdot} \gamma(u,T,U)\cdot dW_u^U\Big)$$

for $t \in [0,U\wedge T]$. Observe also that we have $\mathbf{P}_{T^*} = \mathbf{P}$ and $W^{T^*} = W$. We thus recognize the underlying probability measure $\mathbf{P}$ as a forward martingale measure associated with the horizon date $T^*$.

*2.2. Spot and forward martingale measures*

The concept of a *forward martingale measure* is introduced in terms of the behaviour of relative bond prices. In the present context, we find it convenient to make use of the notion of a forward process.

**Definition 2.1** *Let $U$ be a fixed maturity date. A probability measure $\mathbf{Q}_U \sim \mathbf{P}$ on $(\Omega, \mathcal{F}_U)$ is called a* **forward martingale measure** *for the date $U$ if for any maturity $T \in [0, T^*]$ the forward process $F_B(t, T, U), t \in [0, T \wedge U],$ follows a local martingale under $\mathbf{Q}_U$.*

It follows immediately from assumption (BP.2), that the underlying probability measure $\mathbf{P}$ is indeed a forward probability measure for the date $T^*,$ in the sense of Definition 2.1. Let us now introduce the notion of a spot martingale measure within the present framework. Intuitively speaking, a spot measure is a forward measure associated with the initial date $T = 0.$ Its formal definition relates to a very specific kind of discounting, however. It should be stressed that neither a forward measure for the date $T^*$, nor a spot measure, are uniquely defined, in general.

**Definition 2.2** *A* **spot martingale measure** *for the set-up (BP.1)–(BP.2) is any probability measure $\mathbf{P}^* \sim \mathbf{P}$ on $(\Omega, \mathcal{F}_{T^*})$ for which there exists a process $B^* \in \mathcal{A}^+$, with $B_0^* = 1,$ and such that for any maturity $T \in (0, T^*]$ the bond price $B(t, T)$ satisfies*

$$B(t, T) = \mathbf{E}_{\mathbf{P}^*}(B_t^*/B_T^* \,|\, \mathcal{F}_t), \quad \forall\, t \in [0, T]. \tag{11}$$

*2.3. Arbitrage-free properties*

We shall study two forms of absence of arbitrage. The first, weaker notion refers to a pure bond market. The second form assumes, in addition, that *cash* is also present. Note that by cash we mean here money which can be carried over at no cost, rather then a savings account yielding a positive interest. We shall formulate now a sufficient condition for the absence of arbitrage between bonds with different maturities (as well as between bonds and cash).

**Definition 2.3** *A family $B(t, T)$ of bond prices is said to satisfy the* **weak no-arbitrage condition** *if and only if there exists a probability measure $\mathbf{Q} \sim \mathbf{P}$ on $(\Omega, \mathcal{F}_{T^*})$ such that for any maturity $T < T^*$ the forward process $F_B(t, T, T^*) = B(t, T)/B(t, T^*)$ belongs to $\mathcal{M}_{loc}(\mathbf{Q})$. We say that the family $B(t, T)$ satisfies the* **no-arbitrage condition** *if, in addition, inequality $B(T, U) \le 1$ holds for any maturities $T, U \in [0, T^*]$ such that $T \le U$.*

Assumption (BP.2) is manifestly sufficient for the family $B(t, T)$ to satisfy a weak no-arbitrage condition as we may take $\mathbf{Q} = \mathbf{P}.$ As mentioned, if a family $B(t, T)$ satisfies a weak no-arbitrage condition then it is possible to construct a model of the securities market with the absence of arbitrage across bonds with different maturities (let us stress that the weak no-arbitrage condition makes no explicit reference to the presence of *cash* or a *savings account*). We shall now focus on the absence of arbitrage between all bonds and cash. Under (BP.2), inequality $B(T,U) \leq 1$, which is equivalent to $F_B(T,U,T) \leq 1$, gives immediately

$$F_B(t,U,T) = \mathbf{E_P}(F_B(T,U,T) \,|\, \mathcal{F}_t) \leq 1 \tag{12}$$

for every $t \in [0,T]$. Since almost all sample paths of the forward process $F_B(t,U,T)$ are continuous functions, we may reformulate this condition in the following way.

**(BP.3)** For any two maturities $T \leq U$, the following inequality holds with probability 1

$$B(t,U) \leq B(t,T), \quad \forall\, t \in [0,T]. \tag{13}$$

Suppose, on the contrary, that $B(t,U) > B(t,T)$ for certain maturities $U > T$. In such a case, by issuing at time $t$ a bond of maturity $U$, and purchasing a $T$-maturity bond, one could lock in in a risk-free profit if, in addition, cash were present in the market. Indeed, to meet the liability at time $U$ it would be enough to carry over the period $[T,U]$ (at no cost) one unit of cash received at time $T$. At the intuitive level, the following three conditions are equivalent: (i) the bond price $B(t,T)$ is a non-increasing function of maturity $T$; (ii) the forward process $F_B(t,T,U)$ is never less than one; and (iii) the bond price $B(t,T)$ is never strictly greater than 1 (cf., Corollary 2.3 below). Not surprisingly, the absence of arbitrage between bonds and cash appears to be closely related to the question of existence of an increasing savings account implied by the family $B(t,T)$. We adopt the following definition.

**Definition 2.4** *A* **savings account implied by the family** $B(t,T)$ *of bond prices is an arbitrary process $B^*$ which belongs to $\mathcal{A}^+$, with $B^*_0 = 1$, and such that there exists a probability measure $\mathbf{P}^* \sim \mathbf{P}$ on $(\Omega, \mathcal{F}_{T^*})$ such that the relative bond price*

$$Z^*(t,T) \stackrel{\text{def}}{=} B(t,T)/B^*_t, \quad \forall\, t \in [0,T], \tag{14}$$

*is a $\mathbf{P}^*$-martingale for any maturity $T \in [0,T^*]$.*

It is clear that $Z^*(t,T)$ is a $\mathbf{P}^*$-martingale if for any maturity $T$ the bond price $B(t,T)$ satisfies

$$B(t,T) = \mathbf{E_{P^*}}(B^*_t/B^*_T \,|\, \mathcal{F}_t), \quad \forall\, t \in [0,T]. \tag{15}$$

In particular, we have

$$B(0,T) = \mathbf{E_{P^*}}(1/B^*_T), \quad \forall\, t \in [0,T^*], \tag{16}$$

so that an implied savings account $B^*$ matches also the initial term structure $B(0,T)$, $T \in [0,T^*]$. It is also clear that the probability measure $\mathbf{P}^*$ of Definition 2.4 is a spot martingale measure for the family $B(t,T)$, in the sense of Definition 2.2. One might wonder if the normalized bond price process

$B_t^* \stackrel{\text{def}}{=} B(t,T^*)/B(0,T^*)$ would be a plausible choice of an implied savings account (corresponding to $\mathbf{P}^* = \mathbf{P}$). In view of (BP.2), we have

$$\mathbf{E_P}(B_t^*/B_T^* \,|\, \mathcal{F}_t) = \mathbf{E_P}(B(t,T^*)/B(T,T^*) \,|\, \mathcal{F}_t) = B(t,T), \quad \forall\, t \in [0,T], \tag{17}$$

for any maturity date $T$. In a typical continuous-time model of the term structure, the sample paths of the bond price process $B(t,T^*)$ are of infinite variation, however. In such a case, a normalized bond price cannot simultaneously play the role of a savings account. Consequently, the spot and forward martingale measures are usually distinct.

### *2.4. Implied savings account*

In this section, we take up the issue of existence of an implied savings account. We shall show that, under (BP.1)–(BP.3), there exist an increasing process $B^*$ which represents an implied savings account for the family $B(t,T)$. We start with an auxiliary result which deals with the behaviour of the *terminal discount factor* $D_t = B^{-1}(t,T^*)$, $t \in [0,T^*]$. Note that the process $D$ belongs to the class $\mathcal{S}_p^+$ since $B(\cdot,T^*)$ does.

**Lemma 2.2** *Under the assumptions* (BP.1)–(BP.3)*, the terminal discount factor $D$ follows a strictly positive supermartingale under the forward martingale measure* $\mathbf{P}$.

*Proof.* Combining (2) with (13), we obtain

$$B(t,U) = \mathbf{E_P}\Big(\frac{B(t,T^*)}{B(U,T^*)}\,\Big|\,\mathcal{F}_t\Big) \le \mathbf{E_P}\Big(\frac{B(t,T^*)}{B(T,T^*)}\,\Big|\,\mathcal{F}_t\Big) = B(t,T),$$

so that $\mathbf{E_P}(D_U|\mathcal{F}_t) \le \mathbf{E_P}(D_T|\mathcal{F}_t)$ for $t \le T \le U \le T^*$. Setting $t = T$ in the last inequality, we find that

$$\mathbf{E_P}(D_U|\mathcal{F}_T) \le \mathbf{E_P}(D_T|\mathcal{F}_T) = D_T$$

for every $T \le U \le T^*$, so that $D$ is a $\mathbf{P}$-supermartingale. □

To show the existence of an implied savings account, we shall make use of the following standard result of Itô stochastic calculus (see, for instance, Theorem 6.19 in [12]).

**Proposition 2.1** *Suppose that $X$ belongs to the class $\mathcal{S}_p^+$, with $X_0 = 1$. There exists a unique pair $(M,A)$ of stochastic processes such that $X = MA$, the process $M$ belongs to $\mathcal{M}_{loc}^+(\mathbf{P})$, with $M_0 = 1$, and $A$ belongs to $\mathcal{A}^+$, with $A_0 = 1$. If, in addition, $X$ is a supermartingale then $A$ is a decreasing process.*

It is well known that if a strictly positive special semimartingale $X$ follows a supermartingale, then the process of left hand limits $X_{t-}$ is also strictly positive (see Proposition 6.20 in [12]), hence $X$ belongs to the class $\mathcal{S}_p^+$. Assume that the process $M$ in the decomposition above has continuous sample paths – this holds in our case, since the underlying filtration is generated by a Wiener process. Then the process $A$ is easily seen to belong to $\mathcal{S}_p^+$. We find it convenient to identify the implied savings account using a multiplicative decomposition of the terminal discount factor $D$. To this end, let us formulate a corollary to Proposition 2.1.

**Corollary 2.1** *Under* (BP.1)–(BP.2), *there exists a predictable process $\xi$ integrable with respect to the Wiener process $W$, and such that the terminal discount factor $D$ admits the unique decomposition*

$$D_t = D_0 \tilde{A}_t \tilde{M}_t = D_0 \, \tilde{A}_t \, \mathcal{E}_t\Big(\int_0^{\cdot} \xi_u \cdot dW_u\Big), \quad \forall\, t \in [0, T^*], \tag{18}$$

*where $\tilde{M}$ is in $\mathcal{M}^+_{c,loc}(\mathbf{P})$ and $\tilde{A}$ belongs to $\mathcal{A}^+$, with $\tilde{A}_0 = \tilde{M}_0 = 1$. If, in addition, condition* (BP.3) *is met then $\tilde{A}$ is a decreasing process.*

*Proof.* All assertions are immediate consequences of Lemma 2.2, combined with Proposition 2.1 and the representation theorem for strictly positive martingales with respect to the natural filtration of a Wiener process. □

Note that if there exists a savings account $B^*$ then $N_t = B^*_t / B(t, T^*)$ should follow a $\mathbf{P}$-local martingale. Consequently, $D_t = 1/B(t, T^*) = (B^*_t)^{-1} N_t$, which is essentially the multiplicative decomposition of $D$. We find it convenient to rewrite (18) as follows

$$B^*_t \overset{\text{def}}{=} 1/\tilde{A}_t = \frac{B(t, T^*)}{B(0, T^*)} \, \mathcal{E}_t\Big(\int_0^{\cdot} \xi_u \cdot dW_u\Big), \quad \forall\, t \in [0, T^*]. \tag{19}$$

To show the existence of an implied savings account, it is enough to check that the process $B^*$ given by (19) satisfies Definition 2.4. We formulate the next result under the hypotheses (BP.1)–(BP.3). Under (BP.1)–(BP.2), all claims of Proposition 2.2 remain valid, except that $B^*$ is not necessarily an increasing process if the assumption (BP.3) is relaxed.

**Proposition 2.2** *Let the family $B(t, T)$ of bond prices satisfy* (BP.1)–(BP.3). *Assume, in addition, that the process $\tilde{M}$, defined by the multiplicative decomposition* (18) *of the terminal discount factor $D$ is a martingale, and not only a local martingale under $\mathbf{P}$. Let $B^* = 1/\tilde{A}$ be an increasing predictable process uniquely determined by* (18). *Then* (i) *$B^*$ represents a savings account implied by the family $B(t, T)$.* (ii) *$B^*$ is associated with the spot martingale measure $\mathbf{P}^*$ which equals*

$$\frac{d\mathbf{P}^*}{d\mathbf{P}} \overset{\text{def}}{=} \tilde{M}_{T^*} = B^*_{T^*} \, B(0, T^*), \quad \mathbf{P}\text{-a.s.} \tag{20}$$

(iii) *The relative price process $B^*_t / B(t, T^*)$ follows a martingale under the forward martingale measure $\mathbf{P}$ for the date $T^*$.*

*Proof.* Let $\mathbf{P}^*$ be an arbitrary probability measure on $(\Omega, \mathcal{F}_{T^*})$ equivalent to $\mathbf{P}$. Then the Radon-Nikodým density of $\mathbf{P}^*$ with respect to $\mathbf{P}$ restricted to the $\sigma$-field $\mathcal{F}_t$ equals

$$\frac{d\mathbf{P}^*}{d\mathbf{P}}_{\,|\mathcal{F}_t} = \mathcal{E}_t\Big(\int_0^\cdot \tilde{\xi}_u \cdot dW_u\Big), \quad \mathbf{P}\text{ -a.s.,} \tag{21}$$

for some predictable process $\tilde{\xi}$. We start by considering a zero-coupon bond of maturity $T^*$. In view of (18), the relative bond price $Z^*(t,T^*) = B(t,T^*)/B_t^*$ satisfies under $\mathbf{P}$

$$Z^*(t,T^*) = B(0,T^*)/M_t = B(0,T^*)\,\mathcal{E}_t^{-1}\Big(\int_0^\cdot \xi_u \cdot dW_u\Big) \tag{22}$$

for $t \in [0,T^*]$. Consequently, under $\mathbf{P}^*$ we have

$$Z^*(t,T^*) = B(0,T^*)\exp\Big(-\int_0^t \xi_u \cdot dW_u^* - \frac{1}{2}\int_0^t \xi_u \cdot (2\tilde{\xi}_u - \xi_u)\,du\Big),$$

where $W_t^* = W_t - \int_0^t \tilde{\xi}_u\,du$ follows a Wiener process under $\mathbf{P}^*$. It is thus evident that the relative bond price $Z^*(t,T^*)$ is a local martingale under $\mathbf{P}^*$ provided that $\tilde{\xi} = \xi$. Under this assumption we have

$$Z^*(t,T^*) = B(0,T^*)\,\mathcal{E}_t\Big(-\int_0^\cdot \xi_u \cdot dW_u^*\Big). \tag{23}$$

We are in a position to define a candidate for a spot probability measure by setting $\tilde{\xi} = \xi$ in (21). In view of Definition 2.4, we have to check that for any maturity $T < T^*$ the relative bond price $Z^*(t,T)$ follows a martingale under $\mathbf{P}^*$. It is enough to show that for any maturity $T < T^*$ we have

$$B(t,T) = \mathbf{E}_{\mathbf{P}^*}\Big(\frac{B_t^*}{B_T^*}\,\Big|\,\mathcal{F}_t\Big), \quad \forall\, t \in [0,T]. \tag{24}$$

For this purpose, observe first that equality $\tilde{\xi} = \xi$, combined with (21)–(22), gives

$$\eta_t = \frac{d\mathbf{P}^*}{d\mathbf{P}}_{\,|\mathcal{F}_t} = \frac{B(0,T^*)}{Z^*(t,T^*)} = \frac{B_t^* B(0,T^*)}{B(t,T^*)}, \quad \forall\, t \in [0,T^*]. \tag{25}$$

Consequently, using the abstract Bayes rule we obtain

$$I_t \stackrel{\text{def}}{=} \mathbf{E}_{\mathbf{P}^*}\Big(\frac{B_t^*}{B_T^*}\,\Big|\,\mathcal{F}_t\Big) = \mathbf{E}_{\mathbf{P}}\Big(\frac{B_t^*\eta_T}{B_T^*\eta_t}\,\Big|\,\mathcal{F}_t\Big) = \mathbf{E}_{\mathbf{P}}\Big(\frac{B(t,T^*)}{B(T,T^*)}\,\Big|\,\mathcal{F}_t\Big),$$

where the last equality follows from (25). Using the assumed equality (2), we find that $I_t = B(t,T)$, as required. We have thus shown that the process $B^* = 1/\tilde{A}$ satisfies all conditions of the definition of an implied savings account, and $\mathbf{P}^*$ is the associated spot martingale measure (this follows immediately from (25)). □

Notice that the probability measure $\mathbf{P}^*$ given by (20) is the spot martingale measure associated with the forward martingale measure $\mathbf{P}$ for the date $T^*$. More generally, the forward measure $\mathbf{P}$ and the associated spot measure $\mathbf{P}^*$ are related to each other through the formula

$$\frac{d\mathbf{P}^*}{d\mathbf{P}} = B^*_{T^*}\, B(0,T^*), \quad \mathbf{P}\text{-a.s.} \tag{26}$$

Therefore, both measures coincide if and only if the random variable $B^*_{T^*}$ is constant. As mentioned earlier, the uniqueness of a spot and forward measure is not a universal property. Summarizing, for any forward measure $\mathbf{Q}$ for the date $T^*$, the probability measure $\mathbf{Q}^*$ which is defined on $(\Omega, \mathscr{F}_{T^*})$ by the formula

$$\frac{d\mathbf{Q}^*}{d\mathbf{Q}} = B^*_{T^*}\, B(0,T^*), \quad \mathbf{Q}\text{-a.s.}, \tag{27}$$

is a spot measure for the family $B(t,T)$. Conversely, if $\mathbf{Q}^*$ is a spot measure, then the probability $\mathbf{Q}$ given by (27) is a forward measure for the date $T^*$.

### *2.5. Uniqueness of an implied savings account*

The aim of this section is to establish uniqueness of an implied savings account. We start by an auxiliary result.

**Proposition 2.3** *Let $\tilde{B}$ and $\hat{B}$ be two processes from $\mathscr{A}^+$ such that for every $T \in [0,T^*]$*

$$\mathbf{E}_{\tilde{\mathbf{P}}}(\tilde{B}_t/\tilde{B}_T \,|\, \mathscr{F}_t) = \mathbf{E}_{\hat{\mathbf{P}}}(\hat{B}_t/\hat{B}_T \,|\, \mathscr{F}_t), \quad \forall\, t \in [0,T], \tag{28}$$

*where $\tilde{\mathbf{P}} \sim \hat{\mathbf{P}}$ are two probability measures on $(\Omega, \mathscr{F}_{T^*})$. If $\tilde{B}_0 = \hat{B}_0$ then $\tilde{B} = \hat{B}$.*

Before we proceed to the proof of Proposition 2.3, let us quote the following result from Dellacherie and Meyer [5] (p.231).

**Lemma 2.3** *Let $A$ be an increasing process, defined on a filtered probability space $(\Omega, (\mathscr{F}_t)_{t\in[0,T^*]}, \mathbf{P})$ which satisfies the usual conditions, and such that the random variable $A_{T^*}$ is $\mathbf{P}$-integrable. Denote by $A^p$ the dual predictable projection of $A$. Then*

$$A^p_t = \lim_{n\to\infty} \sum_{k=0}^{2^n-1} \mathbf{E}_{\mathbf{P}}(A_{(k+1)2^{-n}t} - A_{k2^{-n}t} \,|\, \mathscr{F}_{k2^{-n}t}), \quad \forall\, t \in [0,T^*],$$

*where the convergence is in the sense of the weak $L^1$ norm. If $A$ has no predictable jumps then the convergence is in the sense of (strong) $L^1$ norm. Moreover, for any bounded predictable process $H$ we have*

$$\int_0^t H_{u-}\, dA^p_u = \lim_{n\to\infty} \sum_{k=0}^{2^n-1} H_{k2^{-n}t}\, \mathbf{E}_{\mathbf{P}}(A_{(k+1)2^{-n}t} - A_{k2^{-n}t} \,|\, \mathscr{F}_{k2^{-n}t}), \quad \forall\, t \in [0,T^*].$$

*Proof of Proposition 2.3.* We introduce predictable processes of finite variation $\tilde{A} = 1/\tilde{B}$ and $\hat{A} = 1/\hat{B}$. Assume first that $\tilde{\mathbf{P}} = \hat{\mathbf{P}}$, so that we have

$$Y_t \stackrel{\text{def}}{=} \hat{A}_t\, \mathbf{E}_{\tilde{\mathbf{P}}}(\tilde{A}_{T^*} \,|\, \mathscr{F}_t) = \tilde{A}_t\, \mathbf{E}_{\tilde{\mathbf{P}}}(\hat{A}_{T^*} \,|\, \mathscr{F}_t), \quad \forall\, t \in [0,T^*].$$

Equality, $\hat{A} = \tilde{A}$ follows immediately from the uniqueness of a multiplicative decomposition of strictly positive semimartingale $Y$ (it is clear that $Y$ belongs to $\mathscr{S}_p^+$). We now consider the general case. Since $\tilde{\mathbf{P}} \sim \hat{\mathbf{P}}$, the process $\Lambda$ defined by

$$\Lambda_t = \frac{d\hat{\mathbf{P}}}{d\tilde{\mathbf{P}}}_{\,|\mathscr{F}_t} = \mathbf{E}_{\tilde{\mathbf{P}}}(\Lambda_{T^*} \,|\, \mathscr{F}_t), \quad \forall\, t \in [0, T^*],$$

follows a strictly positive continuous (hence predictable) martingale under $\tilde{\mathbf{P}}$. Equality (28) combined with the Bayes rule yields

$$\mathbf{E}_{\tilde{\mathbf{P}}}(\tilde{A}_T/\tilde{A}_t \,|\, \mathscr{F}_t) = \mathbf{E}_{\hat{\mathbf{P}}}(\hat{A}_T/\hat{A}_t \,|\, \mathscr{F}_t) = \mathbf{E}_{\tilde{\mathbf{P}}}(\Lambda_T \hat{A}_T/(\Lambda_t \hat{A}_t) \,|\, \mathscr{F}_t)$$

for every $T \in [0, T^*]$, and thus

$$\mathbf{E}_{\tilde{\mathbf{P}}} \left( \Lambda_t (\tilde{A}_T - \tilde{A}_t)/\tilde{A}_t \,\middle|\, \mathscr{F}_t \right) = \mathbf{E}_{\tilde{\mathbf{P}}} \left( \Lambda_T (\hat{A}_T - \hat{A}_t)/\hat{A}_t \,\middle|\, \mathscr{F}_t \right) \tag{29}$$

for every $t \in [0, T]$. We wish to show that processes $\tilde{A}$ and $\hat{A}$ admit the same dual predictable projection, and thus coincide. Let us fix an arbitrary $t \in [0, T^*]$. It follows from (29), that

$$\mathbf{E}_{\tilde{\mathbf{P}}} \left( \Lambda_{t_k^n} (\tilde{A}_{t_{k+1}^n} - \tilde{A}_{t_k^n})/\tilde{A}_{t_k^n} \,\middle|\, \mathscr{F}_{t_k^n} \right) = \mathbf{E}_{\tilde{\mathbf{P}}} \left( \Lambda_{t_{k+1}^n} (\hat{A}_{t_{k+1}^n} - \hat{A}_{t_k^n})/\hat{A}_{t_k^n} \,|\, \mathscr{F}_{t_k^n} \right),$$

where for every natural $n$ and every $k = 0, \dots, 2^n - 1$, we set $t_k^n = k2^{-n}t$. By virtue of Lemma 2.3, for the left-hand side of the last equality we get

$$\lim_{n\to\infty} \sum_{k=0}^{2^n-1} \Lambda_{t_k^n} \tilde{A}_{t_k^n}^{-1} \, \mathbf{E}_{\tilde{\mathbf{P}}}(\tilde{A}_{t_{k+1}^n} - \tilde{A}_{t_k^n} \,|\, \mathscr{F}_{t_k^n}) = \int_0^t \Lambda_u \tilde{A}_{u-}^{-1} \, d\tilde{A}_u,$$

since the process $H_t = \Lambda_t/\tilde{A}_{t-}$ is predictable, and manifestly $\tilde{A}^p = \tilde{A}$. To show that

$$\lim_{n\to\infty} \sum_{k=0}^{2^n-1} \hat{A}_{t_k^n}^{-1} \, \mathbf{E}_{\tilde{\mathbf{P}}} \left( \Lambda_{t_{k+1}^n} (\hat{A}_{t_{k+1}^n} - \hat{A}_{t_k^n}) \,\middle|\, \mathscr{F}_{t_k^n} \right) = \int_0^t \Lambda_u \hat{A}_{u-}^{-1} \, d\hat{A}_u, \quad \forall\, t \in [0, T^*],$$

it is enough to verify that

$$\lim_{n\to\infty} \sum_{k=0}^{2^n-1} \hat{A}_{t_k^n}^{-1} \, \mathbf{E}_{\tilde{\mathbf{P}}} \left( (\Lambda_{t_{k+1}^n} - \Lambda_{t_k^n})(\hat{A}_{t_{k+1}^n} - \hat{A}_{t_k^n}) \,\middle|\, \mathscr{F}_{t_k^n} \right) = \int_0^t \hat{A}_{u-}^{-1} \, d\, \langle \Lambda, \hat{A} \rangle_u = 0.$$

The last equality follows from the fact that the predictable quadratic covariation $\langle \Lambda, \hat{A} \rangle$ vanishes ($\hat{A}$ being a predictable process of finite variation has null continuous martingale component). □

The following corollary to Proposition 2.3 establishes the uniqueness of an implied savings account.

**Corollary 2.2** *Under* (BP.1)–(BP.2), *the uniqueness of an implied savings account holds.*

*Proof.* Let $\tilde{B}$ and $\hat{B}$ be two arbitrary savings accounts implied by the family $B(t,T)$. Definition 2.4 yields for every $T \in [0,T^*]$

$$\mathbf{E}_{\tilde{\mathbf{P}}}(\tilde{B}_t/\tilde{B}_T \,|\, \mathcal{F}_t) = B(t,T) = \mathbf{E}_{\hat{\mathbf{P}}}(\hat{B}_t/\hat{B}_T \,|\, \mathcal{F}_t), \quad \forall\, t \in [0,T],$$

where $\tilde{\mathbf{P}}$ and $\hat{\mathbf{P}}$ are mutually equivalent probability measures on $(\Omega, \mathcal{F}_{T^*})$. Also, $\tilde{B}$ and $\hat{B}$ are predictable processes of finite variation, hence, equality $\tilde{B} = \hat{B}$ is a straightforward consequence of Proposition 2.3. □

The next result examines a relationship between spot and forward measures (for the proof, see Musiela and Rutkowski [17]).

**Proposition 2.4** *Under the hypotheses* (BP.1)–(BP.2), *the class of forward measures for the date $T^*$ and the class of spot measures admit a common element if and only if the implied savings account satisfies $\tilde{B}_{T^*} = \mathbf{E}_{\tilde{\mathbf{P}}}(\tilde{B}_{T^*})$, that is, if the random variable $\tilde{B}_{T^*}$ is degenerate.*

In the next corollary we deal with the equivalence of various forms of no-arbitrage with cash condition. It should be noticed that in the proof the implication (iii) ⇒ (iv) we make use, in particular, of Assumption (A).

**Corollary 2.3** *Under* (BP.1)–(BP.2), *the following are equivalent.* (i) *The bond price $B(t,T)$ is a non-increasing function of maturity date $T$.* (ii) *The forward process $F_B(t,T,U)$, $t \le T \le U$, is never strictly less than one.* (iii) *The bond price $B(t,T)$ is never strictly greater than 1.* (iv) *The implied savings account follows an increasing process.*

*Proof.* Equivalence of (i), (ii) and (iv) is trivial. Also it is obvious that (iv) implies (iii). It remains check that (iv) follows from (iii). Let $B^*$ be the unique savings account associated with the family $B(t,T)$. Condition $B(t,T) \le 1$ implies immediately that the process $1/B^*$ follows a strictly positive supermartingale under the spot measure $\mathbf{P}^*$. Since it is a process of finite variation, its martingale part, being continuous martingale by virtue of Assumption (A), vanishes identically. Therefore, $B^*$ is an increasing process. □

### *2.6. Bond price volatility*

Throughout this section, we assume that a family $B(t,T)$ of bond prices satisfies (BP.1)–(BP.3). In the present set-up, we find convenient to introduce the notion of a bond price volatility by means of the following definition.

**Definition 2.5** *An $\mathbf{R}^d$-valued adapted process $b(t,T)$ is called a* **bond price volatility** *for maturity $T$ if the bond price $B(t,T)$ admits the representation*

$$dB(t,T) = B(t,T)b(t,T) \cdot dW_t + dC_t^T, \tag{30}$$

*where $C^T$ is a predictable process of finite variation.*

Under (BP.1)–(BP.2), the existence and uniqueness of bond price volatility $b(t,T)$ for any maturity $T$ is a simple consequence of the canonical decomposition of the special semimartingale $B(\cdot,T) \in \mathscr{S}_p^+$, combined with the predictable representation theorem. Also, it is not hard to check that the bond price volatility, as defined above, is invariant with respect to an equivalent change of probability measure. More precisely, if (30) holds, then under any probability measure $\tilde{\mathbf{P}} \sim \mathbf{P}$ we have

$$dB(t,T) = B(t,T)b(t,T) \cdot d\tilde{W}_t + d\tilde{C}_t^T \tag{31}$$

for some predictable process of finite variation $\tilde{C}_t^T$, where $\tilde{W}$ follows a Wiener process under $\tilde{\mathbf{P}}$. Since we have assumed that conditions (BP.1)–(BP.3) are satisfied, there exists a unique savings account $B^*$ associated with a spot probability measure $\mathbf{P}^*$. For any maturity $T$, the relative bond price $Z^*(t,T) = B(t,T)/B_t^*$ follows a local martingale under $\mathbf{P}^*$ so that

$$Z^*(t,T) = B(0,T)\, \mathscr{E}_t\Big(\int_0^{\cdot} b(t,T) \cdot dW_u^*\Big). \tag{32}$$

By comparing the last equality with (23), we find that $b(t,T^*) = -\xi_t$, i.e., the volatility of a $T^*$-maturity bond is determined by the multiplicative decomposition (18). Upon setting $T = t$ in (32), we obtain the following representation for a savings account $B^*$ in terms of bond price volatilities

$$B_t^* = B^{-1}(0,t) \exp\Big(-\int_0^t b(u,t) \cdot dW_u^* + \frac{1}{2}\int_0^t |b(u,t)|^2\, du\Big) \tag{33}$$

for every $t \in [0,T^*]$.

*Remark 2.1* Observe that for any maturities $T, U \in [0,T^*]$ we have

$$\gamma(t,T,U) = b(t,T) - b(t,U), \quad \forall\, t \in [0, T \wedge U], \tag{34}$$

where $\gamma(t,T,U)$ is the volatility of the forward process $F_B(t,T,U)$. Therefore, the *forward volatilities* $\gamma(t,T,U)$ are uniquely specified by the bond price volatilities $b(t,T)$. It is thus natural to ask if the converse implication holds; that is, whether the bond price volatilities are uniquely determined by the forward volatilities.

*Example 2.1* Let us focus on a special case when processes $C^T$ are absolutely continuous; that is, when for any maturity $T \le T^*$ we have

$$\frac{dB(t,T)}{B(t,T)} = a(t,T)\, dt + b(t,T) \cdot dW_t \tag{35}$$

for some adapted processes $a(t,T)$ and $b(t,T)$. We assume, for simplicity, that $a$ and $b$ are uniformly bounded – that is, $|a(t,T)| + |b(t,T)| \le K$, for some constant $K$. Our goal is to show that (35), combined with the weak no-arbitrage condition, implies the existence of an absolutely continuous savings account. It leads also, under mild additional assumptions, to the existence of continuously compounded forward rates. Note that forward process $F_B(t,T,T^*)$ follows under **P**

$$dF_B(t,T,T^*) = F_B(t,T,T^*)\Big(\big(c(t,T) - c(t,T^*)\big)\,dt + \gamma(t,T,T^*)\cdot dW_t\Big),$$

where

$$c(t,T) \stackrel{\text{def}}{=} a(t,T) - b(t,T)\cdot b(t,T^*), \quad \forall\, t \in [0,T].$$

Suppose first that a family $B(t,T)$ satisfies the weak no-arbitrage condition. More specifically, assume that all forward processes $F_B(t,T,T^*)$ follow martingales under a probability measure **Q** equivalent to **P** (notice that the underlying probability measure **P** is not assumed to be a forward martingale measure). In particular, the expected value $\mathbf{E_Q}(B^{-1}(T,T^*))$ is finite for every $T \le T^*$. Then there exists an adapted process, $h$ say, such that

$$\frac{d\mathbf{Q}}{d\mathbf{P}} = \mathscr{E}_{T^*}\Big(\int_0^{\cdot} h_u \cdot dW_u\Big), \quad \mathbf{P}\text{-a.s.},$$

and for every $T \le T^*$

$$c(t,T) - c(t,T^*) + h_t \cdot \big(b(t,T) - b(t,T^*)\big) = 0, \quad \forall\, t \in [0,T].$$

This implies that the quantity

$$N(t,T) \stackrel{\text{def}}{=} c(t,T) + h_t \cdot b(t,T), \quad \forall\, t \in [0,T],$$

is in fact independent of variable $T$, meaning that for any maturity $T \le T^*$ we have

$$r_t \stackrel{\text{def}}{=} c(t,T^*) + h_t \cdot b(t,T^*) = c(t,T) + h_t \cdot b(t,T), \quad \forall\, t \in [0,T].$$

In the formula above, the process $r$ is adapted to the filtration $(\mathscr{F}_t)_{t\in[0,T^*]}$, with almost all sample paths integrable on $[0,T^*]$. Furthermore, the bond price satisfies under **Q**

$$\frac{dB(t,T)}{B(t,T)} = \big(r_t + b(t,T)\cdot b(t,T^*)\big)\,dt + b(t,T)\cdot d\hat{W}_t,$$

where $\hat{W}_t = W_t - \int_0^t h_u\,du$. Let us put

$$\eta_t = \mathscr{E}_t\Big(-\int_0^{\cdot} b(t,T^*)\cdot d\hat{W}_t\Big), \quad \forall\, t \in [0,T^*],$$

and let us assume that $\mathbf{E_Q}(\eta_{T^*}) = 1$. Also, let $B^*$ be an adapted continuous process of finite variation given by the right-hand side of (1). It is easily seen that the process $Y_t = B^*_t/B(t,T^*)$ also follows a martingale under **P**, since $Y$ satisfies the SDE

$$dY_t = -Y_t\, b(t,T^*)\cdot d\hat{W}_t$$

with $Y_0 = 1/B(0,T^*)$. We deduce easily that $Y_t = \eta_t/B(0,T^*)$ for $t \in [0,T^*]$. It is also useful to observe that we have

$$\eta_t = B_t^* B^{-1}(t,T^*)B(0,T^*), \quad \forall\, t \in [0,T^*]. \tag{36}$$

Let us define a probability measure $\mathbf{P}^* \sim \mathbf{Q}$ by setting $d\mathbf{P}^* = \eta_{T^*}\, d\mathbf{Q}$. In view of (36), we obtain

$$\mathbf{E}_{\mathbf{P}}\Big(\frac{1}{B(T,T^*)}\,\Big|\,\mathcal{F}_t\Big) = \mathbf{E}_{\mathbf{P}^*}\Big(\frac{\eta_t}{\eta_T B(T,T^*)}\,\Big|\,\mathcal{F}_t\Big) = \mathbf{E}_{\mathbf{P}^*}\Big(\frac{B_t^*}{B(t,T^*)B_T^*}\,\Big|\,\mathcal{F}_t\Big).$$

By combining this with the martingale property of $F_B(t,T,T^*)$ under $\mathbf{P}$, we find that

$$B(t,T) = B(t,T^*)\,\mathbf{E}_{\mathbf{P}}(F_B(T,T,T^*)\,|\,\mathcal{F}_t) = B_t^*\,\mathbf{E}_{\mathbf{P}^*}\Big(\frac{B_t^*}{B_T^*}\,\Big|\,\mathcal{F}_t\Big). \tag{37}$$

It is now clear that for any maturity $T$ the discounted process $Z^*(t,T) = B(t,T)/B_t^*$ is a martingale under $\mathbf{P}^*$. We conclude that $B^*$ is the unique savings account implied by the family $B(t,T)$.

To show the existence of instantaneous forward rates $f(t,T)$ we shall follow [1]. We assume, in addition, that

$$\mathbf{E}_{\mathbf{P}^*}\Big(\int_0^{T^*} |r_t| B_t^{-1}\, dt\Big) < \infty,$$

and we denote by $G(t,u)$ the jointly measurable version of the martingale (we refer to [1] for the existence of such a version of $G(t,u)$)

$$G(t,u) = \mathbf{E}_{\mathbf{P}^*}(r_u B_u^{-1}\,|\,\mathcal{F}_t), \quad \forall\, t \in [0,u].$$

The conditional version of Fubini's theorem yields

$$\int_0^T G(t,u)\,du = \mathbf{E}_{\mathbf{P}^*}\Big(\int_0^T r_u B_u^{-1}\,du\,\Big|\,\mathcal{F}_t\Big) = 1 - \mathbf{E}_{\mathbf{P}^*}(B_T^{-1}\,|\,\mathcal{F}_t) \tag{38}$$

since $dB_t^{-1} = -r_t B_t^{-1}\,dt$. By combining (37) with (38), we obtain

$$B(t,T) = B_t\Big(1 - \int_0^T G(t,u)\,du\Big). \tag{39}$$

It follows immediately from (39) that $B(t,T)$ is differentiable in $T$. Furthermore, for any fixed $T \leq T^*$ the implied instantaneous forward interest rate $f(t,T)$ equals

$$f(t,T) = -\frac{\partial \ln B(t,T)}{\partial T} = B_t B^{-1}(t,T)G(t,T), \tag{40}$$

or equivalently,

$$f(t,T) = B_t B^{-1}(t,T)\,\mathbf{E}_{\mathbf{P}^*}(r_T B_T^{-1}\,|\,\mathcal{F}_t).$$

It is now easy to check that

$$f(t,T^*) = \mathbf{E}_{\mathbf{P}}(r_{T^*}\,|\,\mathcal{F}_t), \quad \forall\, t \in [0,T^*],$$

so that the forward rate $f(\cdot,T^*)$ is a martingale under the forward measure $\mathbf{P}$. It follows that for any fixed maturity $T$, the process $f(\cdot,T)$ follows a continuous semimartingale with an absolutely continuous component of finite variation. More explicitly, we have

$$f(t,T) = f(0,T) + \int_0^t \alpha(u,T)\,du + \int_0^t \sigma(u,T)\cdot dW_u^* \tag{41}$$

for some adapted processes $\alpha$ and $\sigma$, where $W_t^* = \hat{W}_t - \int_0^t b(u,T^*)\,du$ is a Wiener process under $\mathbf{P}^*$. Moreover, for any $T \le T^*$ we have

$$\sigma(t,T) = -\frac{\partial \ln b(t,T)}{\partial T}, \quad \alpha(t,T) = -\sigma(t,T)\cdot b(t,T). \tag{42}$$

To check the first equality in (42) it is enough to show that the bond price volatilities are absolutely continuous with respect to $T$, or more specifically, that for any maturity $T$ the bond price volatility $b(t,T)$ satisfies

$$b(t,T) = -\int_t^T \sigma(t,u)\,du, \quad \forall\, t \in [0,T].$$

This can be done by a straightforward application of Fubini's theorem to the formula

$$B(t,T) = \exp\Big(-\int_t^T f(t,u)\,du\Big), \quad \forall\, t \in [0,T].$$

The second equality in (42) now follows from Girsanov's theorem. For $T = T^*$, it is enough to examine first the martingale $f(t,T^*)$ under the forward measure $\mathbf{P}$, and then to derive the dynamics of $f(t,T^*)$ under the spot measure $\mathbf{P}^*$.

## 3. Forward processes

In this section, we examine a method of bond price modelling based on the exogenous specification of forward volatilities – that is, the volatilities of forward processes. It should be stressed that we no longer assume that we are given a family of bond prices. We make instead the following assumptions.

**(FP.1)** For any $T \in [0,T^*)$ we are given an adapted $\mathbf{R}^d$-valued process $\gamma(t,T,T^*)$, $t \in [0,T]$, such that

$$\mathbf{P}\Big(\int_0^T |\gamma(u,T,T^*)|^2\,du < +\infty\Big) = 1. \tag{43}$$

By convention, $\gamma(t,T^*,T^*) = 0 \in \mathbf{R}^d$ for every $t \in [0,T^*]$.

**(FP.2)** We are given a deterministic function $P(0,T)$, $T \in [0,T^*]$, with $P(0,0) = 1$, which represents an initial term structure of interest rates.

Notice that $P(0,T)$ is an exogenously given initial term structure, which should be matched by a family of bond prices, which we are going to construct. Let us introduce the notion of a family of forward processes implied by the set-up (FP.1)–(FP.2).

**Definition 3.1** *Given the set-up* (FP.1)–(FP.2)*, for any maturity* $T \in [0, T^*]$ *we define the* **forward process** ***F(t,T,T*** $^*$***)***, $t \in [0, T]$, *by specifying its dynamics under* **P**

$$dF(t, T, T^*) = F(t, T, T^*)\, \gamma(t, T, T^*) \cdot dW_t, \tag{44}$$

*and the initial condition*

$$F(0, T, T^*) = \frac{P(0, T)}{P(0, T^*)}, \quad \forall\, T \in [0, T^*]. \tag{45}$$

For any $T \le T^*$, the unique solution of (44) is given by the standard exponential formula

$$F(t, T, T^*) = \frac{P(0, T)}{P(0, T^*)}\ \mathcal{E}_t\Big(\int_0^{\cdot} \gamma(u, T, T^*) \cdot dW_u\Big), \tag{46}$$

where $t \in [0, T]$. We postulate that the process $F(t, T, T^*)$ has a financial interpretation as the ratio of bond prices, more exactly, we require that

$$F(t, T, T^*) = \frac{B(t, T)}{B(t, T^*)}, \quad \forall\, t \in [0, T], \tag{47}$$

where bond prices $B(t, T)$ remain yet unspecified. Indeed, our goal is to construct a family $B(t, T)$ which would be consistent with the dynamics (46) of forward processes, and would match the initial term structure $P(0, T)$; that is, $B(0, T) = P(0, T)$ for any maturity $T \le T^*$. Note that in this section the bond price is not required a priori to be a semimartingale. Nevertheless, in some circumstances we shall make reference to the volatility of a bond price, which is defined only for the bond price which follows a semimartingale.

In view of assumptions (FP.1)–(FP.2) and (47), to find a family $B(t, T)$, it is sufficient to specify the price of $T^*$-maturity bond. When searching for a candidate for the process $B(t, T^*)$, we need to take into account the terminal condition $B(T^*, T^*) = 1$ and the initial condition $B(0, T^*) = P(0, T^*)$. A family $B(t, T)$ is then defined by setting $B(t, T) \stackrel{\text{def}}{=} F(t, T, T^*)B(t, T^*)$ for every $t \le T \le T^*$. Such a family is easily seen to match a prespecified initial term structure, the terminal condition $B(T, T) = 1$ is not necessarily satisfied, however, unless a judicious choice of the process $B(t, T^*)$ is made. Let us introduce a counterpart of condition (BP.3). We find it convenient to introduce the family of processes $F(t, T, U)$ by setting

$$F(t, T, U) \stackrel{\text{def}}{=} \frac{F(t, T, T^*)}{F(t, U, T^*)}, \quad \forall\, t \in [0, T \wedge U]. \tag{48}$$

**(FP.3)** For any maturities $T, U \in [0, T^*]$ such that $T \le U$ we have

$$F(t, T, U) \ge 1, \quad \forall\, t \in [0, T]. \tag{49}$$

Notice that (FP.3) implies, in particular, that $P(0, U) \le P(0, T)$ for $T \le U$. A family of bond prices associated with the set-up (FP.1)–(FP.2) is defined as follows.

**Definition 3.2** *We say that a family $B(t,T)$ of bond prices* **is associated with (FP.1)–(FP.2)** *if the following holds.*
*(a) Processes $F(t,T,T^*)$ given by (46) coincide with processes $F_B(t,T,T^*)$ which are given by the formula*

$$F_B(t,T,T^*) \overset{\text{def}}{=} \frac{B(t,T)}{B(t,T^*)}, \quad \forall\, t \in [0,T^*]. \tag{50}$$

*(b) Equality $B(0,T) = P(0,T)$ is satisfied for every $T \in [0,T^*]$.*

To show that any family of forward processes $F(t,T,T^*)$ admits an associated family of bond prices, we shall use the notion of a savings account implied by the set-up (FP.1)–(FP.2). Formally, a *savings account implied by* (FP.1)–(FP.2) is any process which represents an implied savings account for some family of bond prices associated with (FP.1)–(FP.2).

It is clear that we may represent the volatility $\gamma(t,T,T^*)$ as follows

$$\hat{b}(t,T) = \gamma(t,T,T^*) + \hat{b}(t,T^*), \quad \forall\, t \in [0,T]. \tag{51}$$

for some family of processes $\hat{b}(t,T)$, $t \le T \le T^*$. Given a family of forward volatilities $\gamma(t,T,T^*)$, in order to determine uniquely all processes $\hat{b}(t,T)$ it suffices to specify the process $\hat{b}(t,T^*)$. The bond price volatilities $b(t,T)$ of any associated family $B(t,T)$, if well-defined, necessarily satisfy relationship (51); that is, for any maturity $T \le T^*$ we have

$$b(t,T) = \gamma(t,T,T^*) + b(t,T^*), \quad \forall\, t \in [0,T]. \tag{52}$$

This does not mean, of course, that arbitrary processes $\hat{b}(t,T)$ which satisfy (51) are indeed price volatilities of some family $B(t,T)$ of bond prices associated with (FP.1)–(FP.2). On the other hand, it follows immediately from (51)–(52) that for an arbitrary choice of the process $\hat{b}(t,T^*)$, there exists a unique process $\psi$ such that the 'true' bond price volatility $b(t,T)$ satisfies

$$b(t,T) = \hat{b}(t,T) + \psi_t, \quad \forall\, t \in [0,T],$$

for any maturity $T \le T^*$. Indeed, it is enough to set $\psi_t = b(t,T^*) - \hat{b}(t,T^*)$ for every $t \in [0,T^*]$. For the sake of expositional simplicity, we shall assume from now on that the forward volatilities $\gamma(t,T,T^*)$ are bounded. Our goal is to find explicitly a family of bond prices associated with (FP.1)–(FP.2). First, we take an arbitrary bounded adapted $\mathbf{R}^d$-valued process $\hat{b}(t,T^*)$, and we define the probability measure $\hat{\mathbf{P}} \sim \mathbf{P}$ on $(\Omega, \mathcal{F}_{T^*})$ by setting

$$\frac{d\hat{\mathbf{P}}}{d\mathbf{P}} = \mathcal{E}_{T^*}\Big(-\int_0^{\cdot} \hat{b}(u,T^*) \cdot dW_u\Big), \quad \mathbf{P}\text{-a.s.} \tag{53}$$

The process $\hat{W}_t$ given by the formula $\hat{W}_t = W_t + \int_0^t \hat{b}(u,T^*)\,du$ is a Wiener process under $\hat{\mathbf{P}}$. In the second step, we introduce a candidate for the savings account process $\hat{B}_t$

$$\hat{B}_t = P^{-1}(0,t)\,\exp\Big(-\int_0^t \hat{b}(u,t)\cdot d\hat{W}_u + \frac{1}{2}\int_0^t |\hat{b}(u,t)|^2\,du\Big), \qquad (54)$$

where $\hat{b}(t,T)$ is defined by (51).

*Remark 3.1* It is not known a priori whether the process $\hat{B}$ is of finite variation (or even if it follows a semimartingale). It appears that $\hat{B}$ is of finite variation if and only if it represents an implied savings account for a family $B(t,T)$ of bond prices defined by formula (55) below. In the opposite case, neither the process $\hat{B}$, nor the bond prices are semimartingales.

We are in a position to introduce a family $B(t,T)$ by setting

$$B(t,T) = P(0,T)\hat{B}_t\,\mathcal{E}_t\Big(\int_0^{\cdot} \hat{b}(u,T)\cdot d\hat{W}_u\Big), \quad \forall\, t \in [0,T], \qquad (55)$$

for any maturity $T \in [0,T^*]$. We claim that $B(t,T)$ is a family of bond prices associated with (FP.1)–(FP.2). To check this, we analyse the forward process $F_B(t,T,T^*)$ associated with the family $B(t,T)$. It is clear that

$$F_B(t,T,T^*) = \frac{P(0,T)}{P(0,T^*)}\,\exp\Big(\int_0^t \gamma(u,T,T^*)\cdot d\hat{W}_u - \frac{1}{2}\int_0^t \delta_u(T,T^*)\,du\Big),$$

where

$$\delta_u(T,T^*) = |\hat{b}(u,T)|^2 - |\hat{b}(u,T^*)|^2$$

for every $u \in [0,T]$. Let us check that the condition (a) of Definition 3.2 is satisfied. To this end, notice that making using of (51) and (55), we get after simple manipulations

$$F_B(t,T,T^*) = \frac{P(0,T)}{P(0,T^*)}\,\mathcal{E}_t\Big(\int_0^{\cdot} \gamma(u,T,T^*)\cdot dW^*_u\Big).$$

Condition (b) of Definition 3.2 is an immediate consequence of (54)–(55). Family $B(t,T)$ of bond prices introduced above manifestly satisfies the weak no-arbitrage condition. Furthermore, if assumption (FP.3) is met, family $B(t,T)$ satisfies the no-arbitrage condition. Recall that we assume throughout, for simplicity of exposition, that the volatilities $\gamma(t,T,T^*)$ of forward processes are bounded.

**Proposition 3.1** *For any bounded adapted process* $\hat{b}(t,T^*)$, *processes* $B(t,T)$ *given by* (53)–(55) *represent a family of bond prices associated with* (FP.1)–(FP.2). *This family satisfies the weak no-arbitrage condition (it satisfies the no-arbitrage condition if* (FP.3) *holds). The process* $\hat{B}$ *given by* (54) *represents a savings account implied by the family* $B(t,T)$ *if and only if it follows a predictable process of finite variation.*

*Proof.* In view of previous considerations, only the last claim is not obvious. The ‘only if’ clause follows directly from the definition of a savings account. The ‘if’ clause is a consequence of results of the previous section. In fact, for any maturity $T$ the relative process

$$\hat{Z}(t,T) \stackrel{\text{def}}{=} B(t,T)/\hat{B}_t = P(0,T)\,\mathcal{E}_t\Big(\int_0^{\cdot} \hat{b}(u,T)\cdot d\hat{W}_u\Big), \quad \forall\, t \in [0,T],$$

is evidently in $\mathcal{M}_{loc}(\hat{\mathbf{P}})$. If the volatility of a $T^*$-maturity bond equals $\hat{b}(u,T^*)$ then, of course, the process $\hat{b}(t,T)$ given by (51) is the bond price volatility for maturity $T$. To conclude, it is enough to compare (54) with (33). □

*Example 3.1* Let us now consider a simple example (we take $d = 1$, for convenience). Assume that the forward volatilities $\gamma(t,T,T^*)$ are constant, more precisely, there exists a non-zero real $\gamma$ such that $\gamma(t,T,T^*) = \gamma$ for every $T \in [0,T^*)$ and $t \in [0,T]$. Furthermore, we have as usual $\gamma(t,T^*,T^*) = 0$ for every $t \in [0,T^*]$. This implies that for any maturity $T \in [0,T^*)$

$$F(t,T,T^*) = \frac{P(0,T)}{P(0,T^*)}\,\exp\Big(\gamma W_t - \frac{1}{2}\gamma^2 t\Big), \quad \forall\, t \in [0,T]. \tag{56}$$

On the other hand, we assume that the deterministic function $P(0,t)$ representing the initial term structure belongs to $\mathcal{S}_p^{+}$. Let us choose $\hat{b}(t,T^*) = 0$ for every $t \in [0,T^*]$ so that for any maturity $T \in [0,T^*)$ we have (cf. (51))

$$\hat{b}(t,T) = \gamma(t,T,T^*) = \gamma, \quad \forall\, t \in [0,T]. \tag{57}$$

Notice also that the probability measure $\hat{\mathbf{P}}$ defined by (53) satisfies $\hat{\mathbf{P}} = \mathbf{P}$, so that $\hat{W} = W$. The process $\hat{B}$, given by (54), thus equals[1]

$$\hat{B}_t = P^{-1}(0,t)\,\exp\Big(-\gamma W_t + \frac{1}{2}\gamma^2 t\Big), \quad \forall\, t \in [0,T^*), \tag{58}$$

and $\hat{B}_{T^*} = P^{-1}(0,T^*)$. Let us first find the bond price $B(t,T^*)$. By virtue of (55), it is clear that $\hat{B}(t,T^*) = P(0,T^*)\hat{B}_t$ for every $t \in [0,T^*]$. More explicitly,

$$B(t,T^*) = \frac{P(0,T^*)}{P(0,t)}\,\exp\Big(-\gamma W_t + \frac{1}{2}\gamma^2 t\Big), \quad \forall\, t \in [0,T^*),$$

and $B(T^*,T^*) = 1$. Let us now consider a bond of maturity $T < T^*$. In view of (55), we have

$$B(t,T) = P(0,T)\hat{B}_t\,\exp\Big(\int_0^t \hat{b}(u,T)\,dW_u - \frac{1}{2}\int_0^t \hat{b}^2(u,T)\,du\Big), \quad \forall\, t \in [0,T].$$

Combining (57) with (58), we find that for any maturity $T < T^*$ we have $B(t,T) = P(0,T)/P(0,t)$ for every $t \in [0,T]$. This completes the construction of a family $B(t,T)$ associated with (FP.1)–(FP.2). Let us now investigate basic properties of this family. First, observe that for any maturity $T < T^*$ we have

$$F_B(t,T,T^*) \stackrel{\text{def}}{=} \frac{B(t,T)}{B(t,T^*)} = \frac{P(0,T)}{P(0,T^*)}\,\exp\Big(\gamma W_t - \frac{1}{2}\gamma^2 t\Big), \quad \forall\, t \in [0,T],$$

[1] Notice that $\hat{B}$ is predictable, since any optional process with respect to a filtration of a Wiener process is predictable. On the other hand, $\hat{B}$, being obviously a semimartingale, does not follow a process of finite variation as it admits a non-zero continuous martingale component.

so that the forward processes $F_B(t,T,T^*)$ and $F(t,T,T^*)$ coincide. We shall now check that the process $B^*$ which equals $B^*_t = P^{-1}(0,t)$ for $t \in [0,T^*)$, and

$$B^*_{T^*} = P^{-1}(0,T^*)\,\exp\Big(\gamma W_{T^*} - \frac{1}{2}\gamma^2 T^*\Big)$$

is the unique implied savings account for the family $B(t,T)$. It is clear that $B^*$ belongs to $\mathscr{A}^+$; it is thus enough to check that all relative bond prices $Z^*(t,T) = B(t,T)/B^*_t$ follow local martingales under some probability measure $\mathbf{P}^* \sim \mathbf{P}$. For any maturity $T < T^*$ we have $Z^*(t,T) = P(0,T)$ for every $t \in [0,T]$, hence $Z^*(t,T)$ follows trivially a martingale under any probability measure equivalent to $\mathbf{P}$. For $T^*$ we have

$$Z^*(t,T^*) = P(0,T^*)\,\exp\Big(-\gamma\,(W_t - \gamma t) - \frac{1}{2}\gamma^2 t\Big), \quad \forall\, t \in [0,T^*],$$

so that $Z^*(t,T^*)$ is a martingale under the probability measure $\mathbf{P}^* \sim \mathbf{P}$ which is given by the formula

$$\frac{d\mathbf{P}^*}{d\mathbf{P}} = \exp\Big(\gamma W_{T^*} - \frac{1}{2}\gamma^2 T^*\Big), \quad \mathbf{P}\text{-a.s.}$$

Observe that due to the jump at time $T^*$, the savings account $B^*$ is not increasing, even if the initial term structure $P(0,t)$ is a strictly decreasing function. Let us now determine the bond price volatilities. It is apparent that $b(t,T) = 0$ for any maturity $T < T^*$, while $b(t,T^*) = -\gamma$ (it seems interesting to compare this with the initial guess: $\hat{b}(t,T) = \gamma$ for $T < T^*$, and $\hat{b}(t,T^*) = 0$). This example, though rather simplistic, provides some insight into the features of the proposed procedure. First, the process $\hat{B}_t$ given by (54) does not necessarily represent the savings account implied by the family $B(t,T)$. Second, the implied savings account may follow a discontinuous process; in our example, this feature is related to the fact that the forward volatilities $\gamma(t,T,U)$ are discontinuous in $U$.

Let us now examine the problem of uniqueness of a family of bond prices associated with a given collection of forward processes. Since any family $B(t,T)$ of bond prices associated with the set-up (FP.1)–(FP.2) satisfies

$$F(t,T,T^*) = F_B(t,T,T^*) = \frac{B(t,T)}{B(t,T^*)}, \quad \forall\, t \in [0,T],$$

we have (see formula (48))

$$B(t,T) = \frac{B(t,T)}{B(t,t)} = \frac{F(t,T,T^*)}{F(t,t,T^*)} = F(t,T,t), \quad \forall\, t \in [0,T]. \tag{59}$$

Therefore, a family of bond prices associated with a given collection $F(t,T,T^*)$ of forward processes is uniquely determined; this implies in turn the uniqueness of the savings account implied by forward processes. Notice, however, that formula (59) is not very useful in practice, since the dynamics of $F(t,T,t)$ are not easily available.

## 4. Models of forward LIBOR rates

To introduce the notion of a *forward LIBOR rate*, we place ourselves within the set-up (BP.1)–(BP.2). This means that we are given a family $B(t,T)$ of bond prices, and thus also the collection $F_B(t,T,U)$ of forward processes. A strictly positive real number $\delta < T^*$ is fixed throughout. By the definition, the forward $\delta$-LIBOR rate[2] $L(t,T)$ for the future date $T \le T^* - \delta$ prevailing at time $t$ is given by the conventional market formula

$$1 + \delta L(t,T) = F_B(t,T,T+\delta), \quad \forall\, t \in [0,T]. \tag{60}$$

Comparing (60) with (5) we find that $L(t,T) = f_s(t,T,T+\delta)$ so that the forward LIBOR rate $L(t,T)$ represents in fact the add-on rate prevailing at time $t$ over the future time interval $[T, T+\delta]$. We can also re-express $L(t,T)$ directly in terms of bond prices as for any $T \in [0, T^* - \delta]$ we have

$$1 + \delta L(t,T) = \frac{B(t,T)}{B(t,T+\delta)}, \quad \forall\, t \in [0,T]. \tag{61}$$

In particular, the initial term structure of forward LIBOR rates satisfies

$$L(0,T) = f_s(0,T,T+\delta) = \delta^{-1}\left(\frac{B(0,T)}{B(0,T+\delta)} - 1\right). \tag{62}$$

Given a family $F_B(t,T,T^*)$ of forward processes, it is not hard to derive the dynamics of the associated family of forward LIBOR rates. For instance, one finds that under the forward measure $\mathbf{P}_{T+\delta}$ we have

$$dL(t,T) = \delta^{-1} F_B(t,T,T+\delta)\,\gamma(t,T,T+\delta)\cdot dW_t^{T+\delta},$$

where $W_t^{T+\delta}$ and $\mathbf{P}_{T+\delta}$ are defined by (9) and (10), respectively. This means that $L(\cdot,T)$ solves the equation

$$dL(t,T) = \delta^{-1}(1 + \delta L(t,T))\,\gamma(t,T,T+\delta)\cdot dW_t^{T+\delta} \tag{63}$$

subject to the initial condition (62). Suppose that forward LIBOR rates $L(t,T)$ are strictly positive. Then formula (63) can be rewritten as follows

$$dL(t,T) = L(t,T)\,\lambda(t,T)\cdot dW_t^{T+\delta}, \tag{64}$$

where for any $t \in [0,T]$

$$\lambda(t,T) = \frac{1 + \delta L(t,T)}{\delta L(t,T)}\,\gamma(t,T,T+\delta). \tag{65}$$

This shows that the collection of forward processes specifies uniquely the family of forward LIBOR rates. The construction of a model of forward LIBOR rates relies on the following assumptions.

[2] In practice, several types of LIBOR rates occur, e.g., 3-month LIBOR and 6-month LIBOR. For the ease of exposition, we consider a fixed maturity $\delta$.

**(LR.1)** For any maturity $T \le T^* - \delta,$ we are given a $\mathbf{R}^d$-valued bounded deterministic function[3] $\lambda(\cdot, T)$ which represents the volatility of the forward LIBOR rate process $L(\cdot, T)$.

**(LR.2)** We assume a strictly decreasing and strictly positive initial term structure $P(0, T), T \in [0, T^*],$ and thus an initial term structure $L(0, T)$ of forward LIBOR rates

$$L(0, T) = \delta^{-1}\left(\frac{P(0, T)}{P(0, T+\delta)} - 1\right), \quad \forall\, T \in [0, T^* - \delta]. \tag{66}$$

*4.1. Discrete-tenor case*

We start by studying a *discrete-tenor* version of a lognormal model of forward LIBOR rates. It should be stressed that a so-called discrete-tenor model still possesses certain continuous-time features, for instance, the forward LIBOR rates follow continuous-time processes. For the ease of notation, we shall assume that the horizon date $T^*$ is a multiple of $\delta$, say, $T^* = M\delta$ for a natural $M$. We shall focus on a finite number of dates, $T^*_{m\delta} = T^* - m\delta$ for $m = 1, \ldots, M-1$. The construction is based on backward induction, therefore, we start by defining the forward LIBOR rate with the longest maturity, $L(t, T^*_\delta)$. We postulate that the rate $L(t, T^*_\delta)$ is governed under the probability measure $\mathbf{P}$ by the following SDE (cf. (64))

$$dL(t, T^*_\delta) = L(t, T^*_\delta)\, \lambda(t, T^*_\delta) \cdot dW_t \tag{67}$$

with the initial condition

$$L(0, T^*_\delta) = \delta^{-1}\left(\frac{P(0, T^*_\delta)}{P(0, T^*)} - 1\right). \tag{68}$$

Put another way, we postulate that for every $t \in [0, T^*_\delta]$

$$L(t, T^*_\delta) = \delta^{-1}\left(\frac{P(0, T^*_\delta)}{P(0, T^*)} - 1\right) \mathscr{E}_t \left(\int_0^{\cdot} \lambda(u, T^*_\delta) \cdot dW_u\right). \tag{69}$$

Since $P(0, T^*_\delta) > P(0, T^*)$ it is clear that $L(t, T^*_\delta)$ is in $\mathscr{M}^+_c(\mathbf{P})$. Also, for any fixed $t \le T^* - \delta$ the random variable $L(t, T^*_\delta)$ has a lognormal probability law under $\mathbf{P}$. The next step is to define the forward LIBOR rate for the date $T^*_{2\delta}$, using relationship (65) with $T = T^*_\delta$, that is,

$$\gamma(t, T^*_\delta, T^*) = \frac{\delta L(t, T^*_\delta)}{1 + \delta L(t, T^*_\delta)}\, \lambda(t, T^*_\delta), \quad \forall\, t \in [0, T^* - \delta]. \tag{70}$$

Given that the volatility $\gamma(t, T^*_\delta, T^*)$ is determined by (70), the forward process $F_B(t, T^*_\delta, T^*)$ is known to solve under $\mathbf{P}$

$$dF_B(t, T^*_\delta, T^*) = F_B(t, T^*_\delta, T^*)\, \gamma(t, T^*_\delta, T^*) \cdot dW_t \tag{71}$$

[3] Volatility $\lambda$ could follow a stochastic process; we deliberately focus here on a *lognormal model* of forward LIBOR rates in which $\lambda$ is deterministic.

and initial condition is $F_B(0, T^*_\delta, T^*) = P(0, T^*_\delta)/P(0, T^*)$. The forward process $F_B(t, T^*_\delta, T^*)$ belongs to $\mathcal{M}^+_c(\mathbf{P})$, since the volatility $\gamma(t, T^*_\delta, T^*)$ follows a bounded process. We introduce a $d$-dimensional Wiener process $W^{T^*_\delta}$, which corresponds to the date $T^*_\delta$, by setting

$$W^{T^*_\delta}_t = W_t - \int_0^t \gamma(u, T^*_\delta, T^*)\, du, \quad \forall\, t \in [0, T^*_\delta]. \tag{72}$$

Due to the boundedness of the process $\gamma(t, T^*_\delta, T^*)$, the existence of the process $W^{T^*_\delta}$ and of the associated probability measure $\mathbf{P}_{T^*_\delta} \sim \mathbf{P}$, which is given by the formula

$$\frac{d\mathbf{P}_{T^*_\delta}}{d\mathbf{P}} = \mathcal{E}_{T^*_\delta}\Big(\int_0^{\cdot} \gamma(u, T^*_\delta, T^*) \cdot dW_u\Big), \quad \mathbf{P}\text{-a.s.} \tag{73}$$

is trivial. The process $W^{T^*_\delta}$ may be interpreted as the forward Wiener process for the date $T^*_\delta$. We are in a position to specify the dynamics of the forward LIBOR rate for the date $T^*_{2\delta}$ under the forward probability measure $\mathbf{P}_{T^*_\delta}$. Analogously to (67) we set

$$dL(t, T^*_{2\delta}) = L(t, T^*_{2\delta})\, \lambda(t, T^*_{2\delta}) \cdot dW^{T^*_\delta}_t, \tag{74}$$

with the initial condition

$$L(0, T^*_{2\delta}) = \delta^{-1}\Big(\frac{P(0, T^*_{2\delta})}{P(0, T^*_\delta)} - 1\Big). \tag{75}$$

Solving (74) and comparing with (65) for $T = T^*_{2\delta}$, we obtain

$$\gamma(t, T^*_{2\delta}, T^*_\delta) = \frac{\delta L(t, T^*_{2\delta})}{1 + \delta L(t, T^*_{2\delta})}\, \lambda(t, T^*_{2\delta}), \quad \forall\, t \in [0, T^*_{2\delta}]. \tag{76}$$

To find $\gamma(t, T^*_{2\delta}, T^*)$ we make use of the relationship (cf. (7))

$$\gamma(t, T^*_{2\delta}, T^*_\delta) = \gamma(t, T^*_{2\delta}, T^*) - \gamma(t, T^*_\delta, T^*), \quad \forall\, t \in [0, T^*_{2\delta}]. \tag{77}$$

Given the process $\gamma(t, T^*_{2\delta}, T^*_\delta)$, we can define the pair $(W^{T^*_{2\delta}}, \mathbf{P}_{T^*_{2\delta}})$ corresponding to the date $T^*_{2\delta}$ and so forth. By working backward to the first relevant date $T^*_{(M-1)\delta} = \delta$, we construct a family of forward LIBOR rates $L(t, T^*_{m\delta}), m = 1, \dots, M-1$. Notice that the lognormal probability law of every process $L(t, T^*_{m\delta})$ under the corresponding forward probability measure $\mathbf{P}_{T^*_{(m-1)\delta}}$ is ensured. Indeed, for any $m = 1, \dots, M-1$ we have

$$dL(t, T^*_{m\delta}) = L(t, T^*_{m\delta})\, \lambda(t, T^*_{m\delta}) \cdot dW^{T^*_{(m-1)\delta}}_t, \tag{78}$$

where $W^{T^*_{(m-1)\delta}}$ is a standard Wiener process under $\mathbf{P}_{T^*_{(m-1)\delta}}$. This completes the derivation of the *lognormal model* of forward LIBOR rates in a discrete-tenor framework. Note that we have constructed simultaneously a family of forward LIBOR rates and a family of associated forward processes.

*4.2. Implied savings account*

We shall now examine the existence and uniqueness of the implied saving account, in a discrete-time setting. The implied savings account is thus seen as a discrete-time process, $B^*_t,\ t = 0, \delta, \ldots, T^* = M\delta.$ Intuitively, the value $B^*_t$ of a savings account at time $t$ can be interpreted as cash amount accumulated up to time $t$ by rolling over a series of zero-coupon bonds with the shortest maturities available. In a discrete-tenor framework, to find the process $B^*,$ we do not have to specify explicitly all bond prices; the knowledge of forward bond prices is sufficient. Indeed, from (4) we get

$$F_B(t, T_j, T_{j+1}) = \frac{F_B(t, T_j, T^*)}{F_B(t, T_{j+1}, T^*)} = \frac{B(t, T_j)}{B(t, T_{j+1})},$$

where we write $T_j = j\delta.$ This in turn yields, upon setting $t = T_j,$

$$F_B(T_j, T_j, T_{j+1}) = 1/B(T_j, T_{j+1}), \tag{79}$$

so that the price $B(T_j, T_{j+1})$ of a one-period bond is uniquely specified for every $j$. Though the bond which matures at time $T_j$ does not physically exist after this date, it seems justified to consider $F_B(T_j, T_j, T_{j+1})$ as its forward value at time $T_j$ for the next future date $T_{j+1}.$ Put another way, the spot value at time $T_{j+1}$ of one cash unit received at time $T_j$ equals $B^{-1}(T_j, T_{j+1}).$ The discrete-time savings account $B^*$ thus equals

$$B^*_{T_k} = \prod_{j=1}^{k} F_B\big(T_{j-1}, T_{j-1}, T_j\big) = \Big(\prod_{j=1}^{k} B\big(T_{j-1}, T_j\big)\Big)^{-1}$$

for $k = 0, \ldots, M-1,$ since by convention $B^*_0 = 1$. Note that

$$F_B\big(T_j, T_j, T_{j+1}\big) = 1 + \delta L(T_j, T_{j+1}) > 1$$

for $j = 1, \ldots, M-1,$ and since

$$B^*_{T_{j+1}} = F_B(T_j, T_j, T_{j+1})\, B^*_{T_j},$$

we find that $B^*_{T_{j+1}} > B^*_{T_j}$ for every $j = 0, \ldots, M-1.$ We conclude that the implied savings account $B^*_t$ follows a strictly increasing discrete-time process. We define the probability measure $\mathbf{P}^* \sim \mathbf{P}$ on $(\Omega, \mathcal{F}_{T^*})$ by the formula (cf., (26))

$$\frac{d\mathbf{P}^*}{d\mathbf{P}} = B_{T^*} P(0, T^*), \quad \mathbf{P}\text{-a.s.} \tag{80}$$

The probability measure $\mathbf{P}^*$ appears to be a plausible candidate for a spot martingale measure. Indeed, if we set

$$B(T_l, T_k) = \mathbf{E}_{\mathbf{P}^*}(B^*_{T_l}/B^*_{T_k} \,|\, \mathcal{F}_{T_l}) \tag{81}$$

for every $l \le k \le M,$ then in the case of $l = k-1,$ equality (81) coincides with (79). It should be stressed that it is not possible to uniquely determine the continuous-time dynamics of a bond price $B(t, T_j)$ within the framework of the discrete-tenor model of forward LIBOR rates (the knowledge of forward LIBOR rates for all maturities is necessary for this).

### 4.3. Continuous-tenor case

By a *continuous-tenor model* we mean a model in which all forward LIBOR rates $L(t,T)$ with $T \in [0,T^*]$ are specified. Given a discrete-tenor skeleton constructed in the previous section, to produce a continuous-tenor model it is sufficient to fill the gaps between the discrete dates. To construct a model in which each forward LIBOR rate $L(t,T)$ follows a lognormal process under the forward measure for the date $T+\delta$, we shall proceed by backward induction.

*First step.* We construct a discrete-tenor model using the previously described method.

*Second step.* We focus on the forward rates and forward measures for maturities $T \in (T^*_\delta, T^*)$. In this case we do not have to take into account the forward LIBOR rates $L(t,T)$ (such rates do not exist in the present model after the date $T^*_\delta$). From the previous step, we are given the values $B^*_{T^*_\delta}$ and $B^*_{T^*}$ of a savings account. It is important to observe that $B^*_{T^*_\delta}$ and $B^*_{T^*}$ are $\mathcal{F}_{T^*_\delta}$-measurable random variables. We start by defining, a spot martingale measure $\mathbf{P}^*$ associated with the discrete-tenor model, using formula (80), Since the model needs to match a given initial term structure, we search for an increasing function $\alpha : [T^*_\delta, T^*] \to [0,1]$ such that $\alpha(T^*_\delta) = 0,\ \alpha(T^*) = 1,$ and the process

$$\ln B^*_t = (1-\alpha(t)) \ln B^*_{T^*_\delta} + \alpha(t) \ln B^*_{T^*}, \quad \forall\, t \in [T^*_\delta, T^*],$$

satisfies $P(0,t) = \mathbf{E}_{\mathbf{P}^*}(1/B^*_t)$ for every $t \in [T^*_\delta, T^*]$. Since $0 < B^*_{T^*_\delta} < B^*_{T^*}$, and $P(0,t),\ t \in [T^*_\delta, T^*],$ is assumed to be a strictly decreasing function, a function $\alpha$ with desired properties exists and is unique.

*Remark 4.1* The second step corresponds, in a sense, to the specific choice of bond price volatility $\sigma$ made by Brace et al. [4], who work within the Heath-Jarrow-Morton framework. They assume that for each maturity $T \in [0,T^*],$ the volatility function $b(t,T)$ vanishes for every $t \in [(T-\delta)\vee 0, T]$. The construction proposed in [4] relies on forward induction, as opposed to backward induction used here. Brace et al. [4] start by postulating that the dynamics of $L(t,T)$ under the martingale measure $\mathbf{P}^*$ are governed by the following SDE

$$dL(t,T) = \mu(t,T)\,dt + L(t,T)\lambda(t,T)\cdot dW^*_t,$$

where $\lambda$ is a deterministic function, and the drift coefficient $\mu$ is yet unspecified. The arbitrage-free dynamics of the instantaneous continuously compounded forward rate $f(t,T)$ are (see Heath et al. (1994))

$$df(t,T) = \sigma(t,T)\cdot\sigma^*(t,T)\,dt + \sigma(t,T)\cdot dW^*_t.$$

On the other hand, we have (cf., (61))

$$1 + \delta L(t,T) = \exp\Big(\int_T^{T+\delta} f(t,u)\,du\Big). \tag{82}$$

Applying Itô's formula to both sides of (82), and comparing the diffusion terms, we find that

$$\sigma^*(t,T+\delta) - \sigma^*(t,T) = \int_T^{T+\delta} \sigma(t,u)\,du = \frac{\delta L(t,T)}{1+\delta L(t,T)} \quad \lambda(t,T)\,.$$

To solve the last equation for $\sigma^*$ in terms of $L$, it is necessary to impose some sort of initial condition on $\sigma^*$. For instance, by setting $\sigma(t,T) = 0$ for $0 \le t \le T \le t+\delta$, we obtain the following relationship

$$b(t,T) = -\sigma^*(t,T) = -\sum_{k=1}^{[\delta^{-1}T]} \frac{\delta L(t,T-k\delta)}{1+\delta L(t,T-k\delta)}\,\lambda(t,T-k\delta). \tag{83}$$

The existence and uniqueness result for solutions of SDEs which govern the instantaneous forward rate $f(t,T)$ and the forward LIBOR rate $L(t,T)$ for the coefficient $\sigma^*$ given by (83) can be shown by forward induction. Taking this for granted, we conclude that $L(t,T)$ satisfies under the spot martingale measure $\mathbf{P}^*$,

$$dL(t,T) = L(t,T)\sigma^*(t,T)\cdot\lambda(t,T)\,dt + L(t,T)\lambda(t,T)\cdot dW_t^*.$$

The continuous-time model of forward LIBOR rates is thus completely specified.

*Third step.* In the previous step we have constructed the savings account $B_t^*$ for every $t \in [T_\delta^*, T^*]$. Hence the forward measure for any date $T \in (T_\delta^*, T^*)$ can be defined by the formula

$$\frac{d\mathbf{P}_T}{d\mathbf{P}^*} = \frac{1}{B_T^* P(0,T)}, \quad \mathbf{P}^*\text{-a.s.} \tag{84}$$

Combining (84) with (80 ), we obtain

$$\frac{d\mathbf{P}_T}{d\mathbf{P}} = \frac{d\mathbf{P}_T}{d\mathbf{P}^*}\frac{d\mathbf{P}^*}{d\mathbf{P}} = \frac{B_{T^*}^* P(0,T^*)}{B_T^* P(0,T)}, \quad \mathbf{P}\text{-a.s.},$$

for every $T \in [T_\delta^*, T^*]$, so that

$$\frac{d\mathbf{P}_T}{d\mathbf{P}}_{\,|\mathcal{F}_t} = \mathbf{E_P}\Big(\frac{B_{T^*}^* P(0,T^*)}{B_T^* P(0,T)}\,\Big|\,\mathcal{F}_t\Big), \quad \forall\, t \in [0,T].$$

Exponential representation of the above martingale – that is, the formula

$$\frac{d\mathbf{P}_T}{d\mathbf{P}}_{\,|\mathcal{F}_t} = \frac{P(0,T^*)}{P(0,T)}\,\mathcal{E}_t\Big(\int_0^{\cdot} \gamma(u,T,T^*)\cdot dW_u^*\Big), \quad \forall\, t \in [0,T],$$

yields the forward volatility $\gamma(t,T,T^*)$ for any maturity $T \in (T_\delta^*, T^*)$. This in turn allows to define also the associated $\mathbf{P}_T$-Wiener process $W^T$. Given the forward probability measure $\mathbf{P}_T$ and the associated Wiener process $W^T$, we define the forward LIBOR rate process $L(t,T-\delta)$ for arbitrary $T \in (T_\delta^*, T^*)$ by setting (cf. (67)–(68))

$$dL(t,T_\delta) = L(t,T_\delta)\,\lambda(t,T_\delta)\cdot dW_t^T,$$

where $T_\delta = T - \delta$, with initial condition

$$L(0, T_\delta) = \delta^{-1}\Big(\frac{P(0,T_\delta)}{P(0,T)} - 1\Big).$$

Finally, we set (cf. (70))

$$\gamma(t, T^*_\delta, T^*) = \frac{\delta L(t, T^*_\delta)}{1 + \delta L(t, T^*_\delta)}\,\lambda(t, T^*_\delta), \quad \forall\, t \in [0, T^*_\delta],$$

hence we are in the position to introduce also the forward measure $\mathbf{P}_T$ for the date $T = T_\delta$. To define forward probability measures $\mathbf{P}_U$ and the corresponding Wiener processes $W^U$ for all maturities $U \in (T^*_{2\delta}, T^*_\delta)$, we put

$$\gamma(t, U, T) = \gamma(t, T_\delta, T) = \frac{\delta L(t, T_\delta)}{1 + \delta L(t, T_\delta)}\,\lambda(t, T_\delta), \quad \forall\, t \in [0, T_\delta],$$

where $U = T_\delta$ so that $T = U + \delta$ belongs to $(T^*_\delta, T^*)$. The coefficient $\gamma(t, U, T^*)$ is found from the relationship

$$\gamma(t, U, T^*) = \gamma(t, U, T) - \gamma(t, T, T^*), \quad \forall\, t \in [0, U].$$

Proceeding by backward induction, we are able to specify a fully continuous-time family $L(t, T)$ of forward LIBOR rates with desired properties. Moreover, since we determine also a family of forward volatilities $\gamma(t, T, T^*)$, $T \in (0, T^*)$, we construct in the same time a family $F(t, T, T^*)$ of forward processes, namely, each process $F(t, T, T^*)$ is assumed to solve the SDE

$$dF(t, T, T^*) = F(t, T, T^*)\gamma(t, T, T^*) \cdot dW_t.$$

From results of the preceding section, we know that such a collection of forward processes admits an associated family $B(t, T)$ of bond prices, which is formally defined by setting $B(t, T) = F(t, T, t)$. Family $B(t, T)$ obtained in such a way always satisfies the weak no-arbitrage condition. The no-arbitrage with cash property may fail to hold, however.

*Counter-example.* Assume, for the sake of expositional simplicity,[4] that $P(0, T^*_\delta) = P(0, T^*)$, or equivalently, that $B^*_{T^*_\delta} = B^*_{T^*}$. This means that in our construction we put $B^*_t = B^*_{T^*_\delta} = B^*_{T^*}$ for $t \in [T^*_\delta, T^*]$. Consequently, for every $T \in [T^*_\delta, T^*]$ the forward measure $\mathbf{P}_T$ coincides with $\mathbf{P}^*$. Moreover, $F_B(t, T, U) = 1$ for any $T, U \in [T^*_\delta, T^*]$ and every $t \in [0, T \wedge U]$. It is not hard to check that

$$F_B(t, T, U) = \frac{1 + \delta L(t, T)}{1 + \delta L(t, U)}\,F_B(t, T + \delta, U + \delta), \quad \forall\, t \in [0, T], \tag{85}$$

provided that maturities $T \le U$ belong to the same interval $((m-1)\delta, m\delta)$ for some $m = 1, \ldots, M$. In our case, (85) yields for $m = 1$ and $t = T$

[4] Though equality $P(0, T^*_\delta) = P(0, T^*)$ contradicts the general assumption that the initial term structure $P(0, T)$ is strictly decreasing, this simplification is not essential and thus may be relaxed.

$$F_B(T,T,U) = \frac{P(0,T^*) + \delta P(0,T)\,\mathcal{E}_T\left(\int_0^{\cdot} \lambda(u,T)\,dW_u\right)}{P(0,T^*) + \delta P(0,U)\,\mathcal{E}_T\left(\int_0^{\cdot} \lambda(u,U)\,dW_u\right)}.$$

Take $d = 1$, and assume that for some maturities $T, U \in (T^*_{2\delta}, T^*_\delta)$ we have $\lambda(u,T) = \lambda_1$ and $\lambda(u,U) = \lambda_2$, where $\lambda_1 < \lambda_2$ are strictly positive constants. Then

$$F_B(T,T,U) = 1/B(T,U) = \frac{P(0,T^*) + \delta P(0,T)\exp(\lambda_1 W_T - \frac{1}{2}\lambda_1^2 T)}{P(0,T^*) + \delta P(0,U)\exp(\lambda_2 W_T - \frac{1}{2}\lambda_2^2 T)}. \quad (86)$$

It follows easily from (86) that $\mathbf{P}^*(F_B(T,T,U) < 1) = \mathbf{P}^*(B(T,U) > 1) > 0$ and thus the absence of arbitrage between bonds and cash is violated. The arbitrage-free features of the continuous-time version of a forward LIBOR rates model thus depend essentially on the choice of volatilities $\lambda(t,T)$.

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