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Towards a general theory of bond markets*

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To the memory of our friend and colleague Oliviero Lessi.

Abstract. The main purpose of the paper is to provide a mathematical background for the theory of bond markets similar to that available for stock markets. We suggest two constructions of stochastic integrals with respect to processes taking values in a space of continuous functions. Such integrals are used to define the evolution of the value of a portfolio of bonds corresponding to a trading strategy which is a measure-valued predictable process. The existence of an equivalent martingale measure is discussed and HJM-type conditions are derived for a jump-diffusion model. The question of market completeness is considered as a problem of the range of a certain integral operator. We introduce a concept of approximate market completeness and show that a market is approximately complete iff an equivalent martingale measure is unique.

Key words: Bond market, term structure of interest rates, stochastic integral, Banach space-valued integrators, measure-valued portfolio, jump-diffusion model, martingale measure, arbitrage, market completeness.

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1 Introduction

In the last few years a remarkable progress has been made in the understanding of bond market phenomena. The main issues of the theory developed by a number

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of researchers in tight cooperation with practitioners are the term structure of interest rates and the pricing of derivative securities (caps, floors, swaptions, etc.), see, e.g., books [11], [15], papers [1], [7], [9], [10], [16], [18], [19], [22], [32], [38], and references therein. The standard framework is that of continuous trading which is based on a stochastic calculus for semimartingales. The great success of continuous time models for description of stock markets and valuations of options on stocks strongly influenced research in the term structure of interest rates. In the majority of papers the dynamics of prices of zero-coupon bonds with maturity θ is described by a diffusion process $P_t(\theta)$, $t \leq \theta$, where θ is considered as a *continuous* parameter. However, only a few works ([2], [5], [28], [35], and some others) deal with jump-diffusion models in spite of the evidences in favour of the latter. The subject of the absolute majority of the above references can be characterized as that of a *special theory of bond markets*: mathematical description of price evolution of basic securities and floating interest rates.

The problem of term structure of interests rates is, of course, very important (and one can even imagine that these key words are synonymous to the mathematical theory of bond markets). Given an adequate model for security prices one can use it for valuation of contingent claims and hedging positions by replication of a claim by dynamically rebalanced portfolios. Here we come to a very important difference of all widely accepted models of bond markets from that of a stock market:

in the continuous-time bond market model there is naturally a continuum of basic traded securities (zero-coupon bonds parameterized by their maturities θ) while in the standard model of a stock market there is normally only a finite number of securities.

This observation makes clear that a consistent theory must admit hedging portfolios which may contain an infinite number and even a continuum of securities. Certainly, this implies the necessity of a rigorous mathematical definition of such a portfolio. In a stock market with d underlying assets, a vector $\phi = (\phi_1, ..., \phi_d)$ representing the quantities of assets of each type kept at t in a portfolio can be identified with a linear functional (i.e. with an element of the dual space $\mathbf{R}^{d*}$ coinciding with $\mathbf{R}^d$); a portfolio value V_t is the action $\phi_t P_t$ of this functional to the price vector $P_t = (P_t^1, \ldots, P_t^d)$ (and this is just a scalar product); after the work by Harrison and Pliska [17], the most general and widely accepted model for the dynamics of the latter is a semimartingale while the time-evolution of a portfolio strategy is described by a predictable process. The classic stochastic calculus provides all necessary machinery for the model: the integration theory for semimartingales is tailor-made for mathematical analysis of stock markets.

In the context of a bond market, P_t is not a finite-dimensional vector but a price curve, i.e. an element of some functional vector space; apparently, the Banach space of continuous functions is adequate to the problem and the idea of considering the evolution of the price curve $P_t(.)$ in a such space has been exploited, e.g., in [7] and [30].

It is natural to extend the definition of a portfolio as a continuous linear functional; in this case again $V_t = \phi_t P_t$ and, by analogy, one could expect that the relevant mathematics here is an integration theory with respect to Banach space-valued semimartigales. Surprisingly, we enter here *terra incognita*: the existing literature on the infinite-dimensional stochastic integration does not meet the needs of mathematical finance; moreover, it is not clear what should be called a semimartingale in this case, see Remark on p. 18 in the recent article by Laurent Schwartz [33].

In the present paper we suggest two approaches to a stochastic integration which serves as modelling tool of the bond market theory. The first one, given in Section 2 and inspired by the book of Métivier [29], is based on the concept of controlled processes as integrators. It is important to note that our integrands are weakly predictable measure-valued processes; this not only allows us to avoid problems arising from non-separability of the space of measures in the total variation topology but also opens a way to practical applications since one can approximate "theoretical" portfolios by "realistic" strategies involving only a finite (but arbitrary large) number of securities. We prove in Section 3 that an asset giving an interest equal to the spot rate (its presence in a "zero-coupon bond market" is usually justified by some limit procedure) is a portfolio of just maturing bonds; this portfolio involves a continuum of bonds but instantaneously it contains only a single one.

In Section 4 it is considered a jump-diffusion model, where the price process of each single bond (i.e. a "section" of the price curve dynamics) is a rather general semimartingale. For this model, including the majority of those discussed in the literature, we suggest another approach to define the integral for measure-valued integrands; the integration theory is reduced via Fubini theorems to the standard stochastic calculus. We prove that, modulo a slight difference in hypotheses, the alternative construction results in the same process as the general one. Since the integration theory in this paper is intended only for financial modelling we are always trying to be on a reasonable level of generality, leaving possible extensions for the future.

In Section 5 we treat in detail the jump-diffusion model specified through the dynamics of the forward rate curves. We investigate here the problem of existence of a martingale measure and derive HJM-type conditions for the coefficients.

Section 6 is devoted to the hedging of contingent claims in a bond market. It is well-known that in the mathematical theory of security markets the problem of hedging is closely related to the completeness of a market. There is an informal principle (seems to have been formulated first by Bensoussan in [3]): to hedge against n sources of randomness one needs n non-redundant securities besides the numéraire. According to this principle, there is no completeness in a stock market model based on a Lévy process with continuous jump spectrum and hence with a continuum of sources of randomness which is too much for a market with a finite number of stocks. The absence of completeness is one of the principal objections against seemingly more adequate models driven by a Lévy process.

Fortunately, in a bond market model where there is, by definition, a continuum of securities one can construct a hedge using strategies involving a continuum of assets. Nevertheless, it turns out that, in general, one can hedge (even with measure-valued portfolios) in the most favorable situation only a dense subset in the space of contingent claims. We examine the problem by considering families of "martingale operators" and their adjoints, "hedging operators", and relate the uniqueness of a martingale measure with the injectivity of martingale operators while the market completeness requires surjectivity of hedging operators. The latter, being integral operators of the first kind, may have, at best, a dense image and only in the "degenerate case" of a finite Lévy measure are surjective (iff the martingale operators are injective). This reasoning leads to the conclusion that the fundamental concept is *the approximate completeness* which is equivalent to the uniqueness of the martingale measure.

In our paper [6] addressed to readers which are mostly interested in the financial counterpart of the theory (and which deals with technically simpler models) we provide some more specific results on a market completeness and the structure of hedgeable claims.

It is worth to note that in stock market models a continuum of derivative securities is also implicitly present, say, call options parameterized by the maturity time and/or strikes, and, therefore, our approach can also be applied to such security markets. Moreover, the theory developed here gives a hint why real-world financial markets generate an enormous amount of various derivative securities: typically they are not driven by a finite number of sources of randomness and the risk averse agents, preferring at least an "approximately" complete market create a corresponding demand.

At last, Appendix contains stochastic versions of the Fubini theorem for continuous martingales and random measures.

2 Integration with respect to $C_\mathbf{T}$-valued processes

Let $(\Omega, \mathscr{F}, \mathbf{F} = (\mathscr{F}_t), \mathbf{P})$ be a stochastic basis (filtered probability space) satisfying the usual conditions, and let $P = (P_t)$, $t \in \mathbf{R}_+$, be an adapted process on it with values in the Banach space of continuous functions $C_\mathbf{T}$ (with the uniform norm denoted by $\|.\|$) where, if $\mathbf{T}$ is a compact subset of $[0, \infty]$, $C_\mathbf{T}$ is the space of all continuous functions, otherwise it is $C^0_{\mathbf{R}_+}$, the space of continuous functions converging to zero at infinity.

We denote by $\mathscr{P}$ the predictable σ-algebra in $\Omega \times \mathbf{R}_+$ generated by all real left-continuous adapted processes.

Let $\mathbf{M_T}$ be the space of signed measures on $\mathbf{T}$ equipped with the total variation norm $\|.\|_V$. For $m \in \mathbf{M_T}$ and $f \in C_\mathbf{T}$ put

$$mf := \int_\mathbf{T} f(\theta) m\,(d\theta)\,.$$

Let $\mathscr{M}_\mathbf{T}$ be the σ-algebra generated by the weak topology[1]. Recall that the space $\mathbf{M_T}$ with the weak topology is separable.

Our aim is to define a stochastic integral

$$\phi \cdot P_t := \int_0^t \phi_s dP_s \tag{2.1}$$

for (weakly) predictable measure-valued processes ϕ, i.e. for measurable mappings

$$\phi : (\Omega \times \mathbf{R}_+, \mathscr{P}) \to (\mathbf{M_T}, \mathscr{M}_\mathbf{T}).$$

Let $\mathscr{E}^b$ be the set of elementary integrands, i.e. of processes

$$\phi_t(\omega) = \sum_{i=1}^n I_{\Gamma_i \times]t_i, t_{i+1}]}(\omega, t) m_i \tag{2.2}$$

where $m_i \in \mathbf{M_T}$, $0 \le t_1 < t_2 < \ldots < t_{n+1} < \infty$, $\Gamma_i \in \mathscr{F}_{t_i}$.

For $\phi \in \mathscr{E}^b$ we set, as usual,

$$\phi \cdot P_t := \sum_{i=1}^n (m_i P_{t_{i+1} \wedge t} - m_i P_{t_i \wedge t}) I_{\Gamma_i}. \tag{2.3}$$

To ensure the path regularity of $\phi \cdot P$ (in other words, to be càdlàg) for elementary integrands we impose on P the following

Assumption 2.1 *The process P is weakly regular: there is a set Ω_1 with $\mathbf{P}(\Omega_1) = 1$ such that for any $\omega \in \Omega_1$ and $m \in \mathbf{M_T}$ the real function $mP_.(\omega) := \int P_.(\theta, \omega) m(d\theta)$ is right-continuous and with left limits.*

To extend the integral to a reasonably large class of integrands we need

Assumption 2.2 *There exist a predictable random measure $\kappa(dt, du) = l_t(du)dt$ given on $(\mathbf{R}_+ \times U, \mathscr{B}_+ \otimes \mathscr{U})$ where $(U, \mathscr{U})$ is some Lusin space, $K_t := 1 + \kappa([0,t] \times U) < \infty$ for finite t, and a measurable function*

$$p : (\Omega \times \mathbf{R}_+ \times U \times \mathbf{M_T}, \mathscr{P} \otimes \mathscr{U} \otimes \mathscr{M}_\mathbf{T}) \to (\mathbf{R}_+, \mathscr{B}_+)$$

with the following properties:

(a) $p(\omega, t, u, .)$ is a seminorm on $\mathbf{M_T}$,

(b) $p(\omega, t, u, .)$ is weakly continuous,

(c) $p(\omega, t, u, m) \le \|m\|_V$,

(d) for any $T \in \mathbf{R}_+$ there is a constant C_T such that for any stopping time $\tau \le T$ and any $\phi \in \mathscr{E}^b$

$$E \sup_{t \le \tau} |\phi \cdot P_t|^2 \le C_T E K_\tau \int_0^\tau \int_U p^2(s, u, \phi_s) \kappa(ds, du). \tag{2.4}$$

We shall say that (κ, p) in the above condition is a *control pair* for P and that a process P satisfying Assumption 2.2 is a *controlled process*.

Clearly, a linear combination of controlled processes is again a controlled process.

Let τ be a bounded stopping time such that $EK_\tau^2 < \infty$. Let us introduce the linear space $L_\tau^2 = L_\tau^2(p,\kappa)$ of all predictable processes ϕ with values in $\mathbf{M_T}$ such that $q_\tau(\phi) < \infty$, where $q_\tau = q_\tau(.;\kappa,p)$ is a seminorm on L_τ^2 defined by

$$q_\tau^2(\phi) := EK_\tau \int_0^\tau \int_U p^2(s,u,\phi_s)\kappa(ds,du). \qquad (2.5)$$

Lemma 2.3 *The linear space $\mathscr{E}^b$ is dense in L_τ^2 in the topology given by q_τ.*

Proof. The inclusion $\mathscr{E}^b \subset L_\tau^2$ holds due to Assumption 2.2.(c). To show that it is dense, notice that the ball $B_c := \{m : \|m\|_V \le c\}$ is compact in the weak topology of $\mathbf{M_T}$ (the Banach–Alaoglu theorem) and metrizable. Hence, a measurable mapping

$$\phi : (\Omega \times \mathbf{R}_+, \mathscr{P}) \to (B_c, \mathscr{M}_\mathbf{T}|_{B_c})$$

can be approximated in the sense of the weak convergence by $\mathscr{P}$-measurable step functions, i.e. by processes of the form $\sum I_{A_i}(\omega,t)m_i$ where A_i are predictable sets. The properties (b) and (c) ensure that the approximating sequence converges to ϕ in the seminorm q_τ.

Since the real-valued predictable processes

$$\sum a_i I_{\Gamma_i \times]t_i,t_{i+1}]}(\omega,t)$$

(which are a generating set for $\mathscr{P}$, see [13]) are dense in $L^2(\mathscr{P}, d\mathbf{P}dK^\tau)$, we get that the elements of $\mathscr{E}^b$ are dense in the set of norm-bounded predictable processes in the topology given by q_τ and hence in L_τ^2. □

Let Π_τ be the vector space of real adapted processes with regular trajectories equipped with the seminorm $\pi_\tau(Y) = (E \sup_{t\le\tau} |Y_t|^2)^{1/2}$ (as usual, we identify indistinguishable processes). It is well-known that Π_τ is complete with respect to this seminorm.

Thus, the linear mapping $\phi \mapsto I_{[0,\tau]}\phi \cdot P$ defined on $\mathscr{E}^b$ and taking values in Π_τ, which is continuous by (2.4), can be extended to the unique continuous linear mapping from L_τ^2 into Π_τ.

Standard localization arguments allow us to extend the definition of the integral $\phi \cdot P$ to all predictable processes ϕ such that

$$\int_0^t \int_U p^2(s,u,\phi_s)\kappa(ds,du) < \infty \quad a.s. \qquad (2.6)$$

for all finite t.

Let (κ,p') be another control pair. Then $(\kappa,p'') := (\kappa,p+p')$ is again a control pair. Assume that ϕ satisfies (2.6) together with the corresponding relation for (κ,p') and hence for (κ,p'') also.

Since the seminorm $q_\tau(.;\kappa,p+p')$ is stronger than $q_\tau(.;\kappa,p)$, the integral $\phi \cdot P$ defined using (κ,p) coincides with that based on (κ,p'') and, by symmetry, on (κ,p'). Thus, the integral does not depend on the particular choice of p, and, by similar arguments, on the particular choice of κ. Thus, the definition of the integral (which is a class of indistinguishable processes) is independent on the particular choice of a control pair (κ, p). We denote the class of processes for which the integral in the above sense exists by $L^2_{loc}(P)$ (the set of weakly predictable processes satisfying (2.6)). One can notice that the integral $\phi \cdot P$ is a process which can be approximated uniformly in probability by "elementary" integrals of the form (2.3).

As usual, for any stopping time σ we have $\phi \cdot P_\sigma = I_{[0,\sigma]}\phi \cdot P_\infty$.

Some properties of the stochastic integral are summarized in the following

Theorem 2.4 *Let $\phi \in L^2_{loc}(P)$. Then*
(a) the process $\phi \cdot P$ is a (real) semimartingale and for any stopping time τ (2.4) holds;
(b) the process $\phi \cdot P$ is continuous if P is weakly continuous;
(c) if P is a martingale then $\phi \cdot P$ is a locally square integrable martingale.

Proof. (a) Inequality (2.4) holds for all $\phi \in L^2_{loc}(P)$ by definition.

Let H be a real bounded elementary integrand given by $H = \sum \xi_j I_{]t_j, t_{j+1}]}$ where ξ_j are $\mathcal{F}_{t_j}$-measurable. For the adapted right-continuous process $X := \phi \cdot P$ we put $H \cdot X := \sum \xi_j (X_{t_{j+1}} - X_{t_j})$. It follows from (2.4) that

$$E \sup_{t \le \tau} |H \cdot X_t|^2 \le \|H\|^2 EK_\tau \int_0^\tau \int_U p^2(s, u, \phi_s)\kappa(ds, du).$$

We easily infer from this inequality that for a sequence of bounded elementary integrands H^n uniformly converging to zero the sequence of integrals $H^n \cdot X_\infty$ tends to zero in probability. Thus, X is a semimartingale by the Dellacherie–Bichteler–Mokobodzki theorem, see, [13].

(b) The property is evident for elementary integrands. In the general case the integral is defined through uniform convergence which preserves continuity.

(c) From the definition it follows that for an elementary integrand ϕ^n the process $\phi^n \cdot P$ is a martingale and $E|\phi^n \cdot P_\tau|^2 < \infty$ for any τ such that $EK^2_\tau < \infty$. If, moreover, τ is such that the right-hand side of (2.4) is finite we conclude, by the approximation, that the stopped process $\phi \cdot P^\tau$ is a square integrable martingale. □

Proposition 2.5 *Let P_1, P_2 be two controlled processes and ϕ be a process integrable with respect to P_1 and P_2, i.e. $\phi \in L^1(P_1) \cap L^1(P_2)$. Then $\phi \in L^1(P_1 + P_2)$ and $\phi \cdot (P_1 + P_2) = \phi \cdot P_1 + \phi \cdot P_2$.*

Proof. Let (κ_i, p_i) be a control pair for P_i such that, with terms indexed by i, the relation (2.6) holds. Without loss of generality we can assume that U_1 and U_2 are distinct. Put $U := U_1 \cap U_2$ and define $p := p_1 I_{U_1} + p_2 I_{U_2}$, $\kappa := \kappa_1 I_{U_1} + \kappa_2 I_{U_2}$. Clearly, (κ, p) is a control pair for $P_1 + P_2$ and since ϕ satisfies (2.6) the result holds. □

Remark. The above construction of the stochastic integral goes well without any changes for an arbitrary Banach space. Certainly, the definition of the control pair can be modified and generalized in various ways (e.g., one can modify (2.4) by taking the supremum not over $[0, \tau]$ but over $[0, \tau[$ as was done in [29]). A more general integration theory merits a special study which is beyond of the scope of the present paper.

3 General model of a bond market

1. The forward and spot rates

In the mathematical description of a bond market it is usually assumed that for fixed $\theta \in \mathbf{T}$ the process $P_t(\theta),\ t \in [0,\theta]$, gives the dynamics of the default-free zero-coupon bond (with unit nominal value) maturing at time θ. Evidently, this process must be strictly positive and $P_\theta(\theta) = 1$.

It would be quite natural to impose also the constraint $P_t(\theta) \le 1$ but, following the tradition, we do not persist on this requirement since it excludes some easily treated models leading to explicit formulae.

In the continuous-time modelling of bond markets (in contrast with that of stock markets) the straightforward specification of the evolution of asset prices as a diffusion or jump-diffusion is not convenient. The main methodology is to start with a model for interest rates; the general opinion now is in favour of the forward rate though the models based on the spot rate have their own advantages. We shall follow the same mainstream of ideas adapting it to our approach which emphasizes the evolution of the whole price curve in the space of continuous functions in contrast with the traditional point of view that considers as primary object a family of individual price processes parameterized by bond maturities.

Assumption 3.1 *The price curve dynamics is given by a controlled process $P = (P_t)$. There exists a $C_{\mathbf{T}}$-valued adapted process $f = f(t)$ such that for any $\theta \in \mathbf{T}$*

$$P_t(\theta) = \exp\left\{ -\int_t^\theta f(t,s)ds \right\}, \quad t \le \theta. \tag{3.1}$$

The random variable $f(t,\theta)$ is called the *instantaneous forward spot rate* (at time t of the bond maturing at θ) or simply the *forward rate*. By definition, $r_t := f(t,t)$ is the *instantaneous spot rate* or simply the *spot rate* (called in the literature also the *short rate*, *instantaneous riskless rate* etc.).

Remark. One can assume that $P(t,\theta)$ is continuously differentiable in θ and define the forward rate in an equivalent way as

$$f(t,\theta) = -\frac{\partial}{\partial\theta} \ln P_t(\theta). \tag{3.2}$$

In almost all known models (quite often implicitly) it is assumed that there is a traded asset that pays interest r_t, i.e. the unit of money invested at time zero in this asset results at t in the amount

$$R_t^{-1} := \exp\left\{ \int_0^t r_s ds \right\}$$

(one can think about a bank account with the floating rate r_t).

It is convenient to take this asset as a *numéraire*, that is to express all other values in the units of this particular security. Prices calculated in units of the numéraire are called the *discounted prices*; in our case the discounting factor is R_t. This means that the discounted price process Z is given by the formula

$$Z_t(\theta) := R_t P_t(\theta) = \exp\Big\{ -\int_0^t r_s\,ds\Big\} P_t(\theta). \tag{3.3}$$

It is instructive to understand a possible reasoning explaining the "existence" of this numéraire. Let us split the interval $[0,t]$ into small subintervals $]t_i, t_{i+1}]$ and consider the strategy to invest at time zero a unit amount of money into the bond maturing at t_1, at the moment t_1 to reinvest the obtained value (which is equal to $P_0^{-1}(t_1-)$) into the bond maturing at t_2, and so on. Clearly, under a mild condition of equicontinuity, at time t the resulting amount is

$$\exp\Big\{ \sum_{i=0}^N \int_{t_i}^{t_{i+1}} f(t_i,s)ds \Big\} \approx \exp\Big\{ \sum_{i=0}^N r_{t_i}(t_{i+1}-t_i)\Big\},$$

and it approximates R_t^{-1}. In other words, the existence of the asset with the interest r_t means that we are allowed to execute a roll-over strategy on just-maturing bonds which leads to a portfolio involving a continuum of securities.

Up to now the bond price $P_t(\theta)$ has been given only for $t \le \theta$. To work with processes defined for all $t \in \mathbf{R}_+$ we put $P_t(\theta) = R_t^{-1}R_\theta$ for $t \ge \theta$. One can think that after maturity the bond is transferred automatically into the unit of money in the bank account.

There is another option: reparameterize the model by considering θ as time *to maturity*.

2. Portfolios of bonds

We define a (feasible) *portfolio* or *trading strategy* as a pair (ϕ,η) where ϕ is a P-integrable predictable measure-valued process, η is a real predictable process with

$$\int_0^t |\eta_s|ds < \infty \tag{3.4}$$

for finite t.

The *value process* of such a portfolio is given by

$$V_t(\phi,\eta) = \phi_t P_t + \eta_t \beta_t \tag{3.5}$$

with $\beta := R^{-1}$.

We shall consider as *admissible* only strategies with value processes bounded from below.

A portfolio is said to be *self-financing* if its increments are caused by price movements only, i.e.

$$V_t(\phi,\eta) = x + \phi \cdot P_t + \eta \cdot \beta_t \tag{3.6}$$

where x is an initial endowment.

We show now that the roll-over "strategy" of permanent reinvestment of the whole current value V_{t-} in the just maturing bond (without involving the "bank account") is an admissible portfolio $(\phi, \eta) = (V_-\delta, 0)$ where $V_- := (V_{t-})$ and δ_t is a unit mass on **T** concentrated at the point t, and this portfolio gives rise to an asset with interest rate r. Formally, this means that the linear equation

$$V = 1 + V_-\delta \cdot P \tag{3.7}$$

has a solution, the solution is unique and coincides with β. The result (under certain additional hypotheses) is a corollary of the following two lemmas.

Lemma 3.2 *The equation*

$$V = 1 + V_-\phi \cdot P \tag{3.8}$$

where ϕ is a locally bounded predictable $\mathbf{M_T}$-valued process, has a unique solution in the class of locally bounded processes with regular paths.

Proof. Let W be the difference of two solutions. Then $W = W_-\phi \cdot P$. By localization, we can assume that $|W|$, ϕ, and K are bounded by some constant. It follows from Assumption 2.2 that

$$E \sup_{s\le t} |W_-\phi \cdot P|^2$$

$$\le C_T E K_t \int_0^t \int_U p^2(s, u, W_-\phi)\kappa(ds, du) \le C \int_0^t E \sup_{v\le s} |W_v|^2 l_s(U) ds.$$

Thus, W is zero by the Gronwall–Bellman lemma. □

Lemma 3.3 *Assume that the following conditions are satisfied:*

i) the spot rate r is a regular process (càdlàg);

ii) for any finite T we have

$$\lim_{\theta\downarrow t} \sup_{t\le T} |f(t,\theta) - f(t,t)| = 0; \tag{3.9}$$

iii) in the control pair (κ, p) for the price process P the function p has the form

$$p(\omega, t, u, m) = |mg(\omega, t, u)| \tag{3.10}$$

where $g(\omega, t, \theta, u)$ is bounded by a constant and right-continuous in t.

Then for any continuous process G we have

$$G\delta \cdot P_t = \int_0^t G_s r_s ds. \tag{3.11}$$

Proof. Standard localization arguments reduce the problem to the case when G and K are bounded. Let us consider the approximation of $\phi := G\delta$ by the processes

$$\phi^n_s(d\theta) = \sum_{i=0}^{n} G_{t_i}\delta_{t_{i+1}}(d\theta)I_{]t_i,t_{i+1}]}(s) \tag{3.12}$$

with $t_i = it/n$. It is rather obvious that

$$\phi^n \cdot P_t = \sum_{i=0}^{n} G_{t_i}(P_{t_{i+1}}(t_{i+1}) - P_{t_i}(t_{i+1})) = \sum_{i=0}^{n} G_{t_i}\Big(1 - \exp\Big\{-\int_{t_i}^{t_{i+1}} f(t_i,s)ds\Big\}\Big)$$

$$= \sum_{i=0}^{n} G_{t_i}r_{t_i}(t_{i+1}-t_i) + o(1) \to \int_0^t G_s r_s ds \tag{3.13}$$

due to i) and ii)

On the other hand, making use iii) we have:

$$q^2_t(\phi_s - \phi^n_s) \le CE\int_0^t\int_U |(\phi_s - \phi^n_s)g(s,u)|^2 l_s(du)ds$$

$$= CE\int_0^t\int_U\sum_{i=0}^{n}|G_s g(s,s,u) - G_{t_i}g(s,t_{i+1},u)|^2 I_{]t_i,t_{i+1}]}(s)l_s(du)ds \to 0. \tag{3.14}$$

Hence the left-hand side of (3.13) converges in probability to the stochastic integral $\phi \cdot P_t$ and (3.11) holds. □

As a corollary of (3.11) we have that

$$\beta\delta \cdot P_t = \int_0^t \beta_s r_s ds = \beta_t - 1. \tag{3.14}$$

Thus, under the assumptions of Lemma 3.2 the process β is the solution of (3.7) (which is unique at least in the class of locally bounded processes).

Remark. One may think that the above reasoning is not correct in some sense since we extended the bond prices after maturity using the process $R_\theta\beta$. However, the approximation (3.12) is chosen in such a way that the corresponding integral sum does not involve values of the bonds after maturities. Of course, the arguments can be repeated for the case when θ is the time to maturity.

3. Classification of portfolios

Now we consider discounted bond prices $Z_t(\theta) := R_tP_t(\theta)$ and discounted values of a portfolio $V^Z_t(\phi_t,\eta_t) := R_tV_t(\phi_t,\eta_t)$ which correspond to a choice of the roll-over strategy as the numéraire. Clearly, $Z_t(\theta) = 1$ for $t \ge \theta$ and $V^Z_t(\phi,\eta) = \phi_tZ_t + \eta_t$. For a self-financing portfolio we have $V^Z_t(\phi,\eta) = x + \phi \cdot Z_t$. From now on we shall consider only self-financing strategies. Since in this case the value process (hence, the process η) is uniquely defined by the ϕ-component we omit η in notations.

For particular models of bond prices one can expect a redundancy of traded assets. It may happen that a certain value process corresponds to different trading strategies. It is important to distinguish also portfolios that instantaneously involve only a finite number of assets. To study different possible situations we introduce the following definitions.

We say that two trading strategies ϕ and ϕ' are *equivalent* if they have the same value processes: $V(\phi) = V(\phi')$ (**P**-a.s.). A strategy ϕ is called an *n-dimensional* if for any t and almost all ω the measure $\phi(\omega, t, d\theta)$ is concentrated at most in n points of $\mathbf{T}$. We say that a strategy ϕ is *n-reducible* if there exists an n-dimensional equivalent strategy but there is no k-dimensional strategy with $k < n$. The definitions of *countably dimensional* and *countably reducible* strategies follow the above patterns. Some results concerning the problem of reducibility are given in [6].

4 Jump-diffusion model

1.

In this section we consider more specific integrators by assuming that for every fixed θ the real-valued process $P(\theta) = (P_t(\theta))$ is a semimartingale of a rather general form. This hypothesis leads to a setting which covers the majority of existing models of bond price processes and provides an important example of application of the theory developed above. Making use of the imposed particular structure we suggest as alternative a more explicit construction of the integral for measure-valued processes and show that it results in the same object.

Let $P = (P_t)$ be a $C_{\mathbf{T}}$-valued process such that for any $\theta \in \mathbf{T}$ the real process $P(\theta) = (P_t(\theta))$ admits the representation

$$P_t(\theta) = x(\theta) + \int_0^t a_s(\theta)ds + \int_0^t \sigma_s(\theta)dw_s + \int_0^t \int_X g(s,x,\theta)(\mu(ds,dx) - \nu(ds,dx)) \tag{4.1}$$

where w is a Wiener process with values in $\mathbf{R}^n$, $\mu(\omega, dt, dx)$ is a $\mathscr{P}\otimes\mathscr{X}$-$\sigma$-finite integer-valued random measure (adapted to the filtration), $\nu(\omega, dt, dx) = \lambda_t(\omega, x)dx$ is its compensator (dual predictable projection), $(X, \mathscr{X})$ is a Lusin space (in applications, usually, $X = \mathbf{R}^n$, or $X = \mathbf{N}$, or a finite set), $g(., \theta)$ is a $\mathscr{P}\otimes\mathscr{X}$ -measurable function ($\mathscr{P}$ is the predictable σ-algebra in $\Omega \times \mathbf{R}_+$). The coefficients must be such that all integrals are well-defined and this requirement is met, of course, by the following

Assumption 4.1 *The coefficients of (4.1) are continuous in θ, $a(\theta)$ and $\sigma(\theta)$ are predictable processes with values in $\mathbf{R}$ and $\mathbf{R}^n$ such that for finite t*

$$\int_0^t \|a_s\|ds < \infty, \quad \int_0^t \|\sigma_s\|^2 ds < \infty \qquad a.s., \tag{4.2}$$

$g(.,\theta)$ is a $\mathscr{P}\otimes\mathscr{X}$-measurable real-valued function such that for finite t

$$\int_0^t \int_X \|g(s,x)\|^2 \nu(ds,dx) < \infty \qquad a.s. \tag{4.3}$$

Put $\Lambda_t = t$. In the standard notations of the stochastic calculus for semimartingales (4.1) can be written as follows:

$$P_t(\theta) = x(\theta) + a(\theta) \cdot \Lambda_t + \sigma(\theta) \cdot w_t + g(\theta) * (\mu - \nu)_t. \tag{4.4}$$

Remark. The definition (4.1) includes as a particular case the process generated by the Gaussian–Poisson model:

$$P_t(\theta) = x(\theta) + \int_0^t a_s(\theta)ds + \int_0^t \sigma_s(\theta)dw_s + \sum_{i=1}^m \int_0^t g_s(i,\theta)(dN_s^i - \lambda_s^i ds) \tag{4.5}$$

where N^i are independent Poisson processes with intensities λ_t^i.

2.
Let ϕ be a predictable $\mathbf{M_T}$-valued process such that for all finite t

$$\int_0^t |\phi_s a_s| ds < \infty, \tag{4.6}$$

$$\int_0^t |\phi_s \sigma_s|^2 ds < \infty, \tag{4.7}$$

and

$$\int_0^t \int_X |\phi_s g(s,x)|^2 \nu(ds,dx) < \infty \tag{4.8}$$

where

$$\phi_s a_s = \int_{\mathbf{T}} a_s(\theta)\phi_s(d\theta)$$

etc. For ϕ satisfying (4.6) – (4.8) we put

$$\phi \circ P_t := \int_0^t \phi_s a_s ds + \int_0^t \phi_s \sigma_s dw_s + \int_0^t \int_X \phi_s g(s,x)(\mu(ds,dx) - \nu(ds,dx)) \tag{4.9}$$

where the first integral in the right-hand side is the ordinary Lebesgue integral and the second and the third ones are the usual stochastic integrals. In abbreviated notations one can write (4.9) as

$$\phi \circ P_t := (\phi a) \cdot \Lambda_t + (\phi\sigma) \cdot w_t + (\phi g) * (\mu - \nu)_t. \tag{4.10}$$

Proposition 4.2 *(a) Under Assumption 4.1 the process $P(\theta)$ is controlled and $\phi\cdot P_t = \phi\circ P_t$ for $\phi\in\mathscr{E}^b$.*

(b) If, moreover, for finite t

$$\int_0^t \|a_s\|^2 ds < \infty \tag{4.11}$$

and ϕ is a predictable process such that (4.6) – (4.8) are fulfilled and also

$$\int_0^t |\phi_s| ds < \infty \tag{4.12}$$

for $t<\infty$ then $\phi\in L^2_{loc}(P)$ and $\phi\cdot P = \phi\circ P$.

Proof. (a) Notice that for $\phi\in\mathscr{E}^b$ of the form $\phi = I_{\Gamma\times]t_1,t_2]}m$ we have by the definitions and the Fubini theorems for ordinary and stochastic integrals (see Appendix) that

$$\begin{aligned}
\phi\cdot P &= I_\Gamma\int_{\mathbf{T}}\left(\int_{t_1}^{t_2} a_s(\theta)ds\right)m(d\theta) + I_\Gamma\int_{\mathbf{T}}\left(\int_{t_1}^{t_2}\sigma_s(\theta)dw_s\right)m(d\theta)\\
&\quad + I_\Gamma\int_{\mathbf{T}}\left(\int_{t_1}^{t_2}\int_X g(s,x,\theta)(\mu(ds,dx)-\nu(ds,dx)\right)m(d\theta)\\
&= I_\Gamma\int_{t_1}^{t_2}\left(\int_{\mathbf{T}} a_s(\theta)m(d\theta)\right)ds + I_\Gamma\int_{t_1}^{t_2}\left(\int_{\mathbf{T}}\sigma_s(\theta)m(d\theta)\right)dw_s\\
&\quad + I_\Gamma\int_{t_1}^{t_2}\int_X\left(\int_{\mathbf{T}} g(s,x,\theta)m(d\theta)\right)(\mu(ds,dx)-\nu(ds,dx)) = \phi\circ P.
\end{aligned}$$

To show that P is a controlled process it is sufficient to check that each integral in (4.1) defines a controlled process.

Let $\phi\in\mathscr{E}^b$. For any stopping time τ we have by the Cauchy–Schwarz inequality that

$$E\sup_{t\le\tau}\left(\int_0^t \phi_s a_s ds\right)^2 \le EK^a_\tau\int_0^\tau p_a^2(s,\phi_s)dK^a_s \tag{4.13}$$

where

$$K^a_t := 1+\int_0^t \|a_s\|ds, \quad p_a(s,\phi_s) := |\phi_s a_s|\|a_s\|^{\oplus},$$

and $^{\oplus}$ is the "pseudoinverse": $b^{\oplus} = b^{-1}$ for $b\ne 0$ and $0^{\oplus} = 0$.

By the Doob inequality

$$E\sup_{t\le\tau}\left(\int_0^t \phi_s\sigma_s dw_s\right)^2 \le 4E\int_0^\tau p_\sigma^2(s,\phi_s)\|\sigma_s\|^2 ds \le 4EK^\sigma_\tau\int_0^t q_s^2(\phi_s)dK^\sigma_s$$

where

$$K^\sigma_t := 1+\int_0^t \|\sigma_s\|^2 ds, \quad p_\sigma(s,\phi_s) := |\phi_s\sigma_s|\|\sigma_s\|^{\oplus}.$$

Similarly,

$$E \sup_{t\le\tau}\Big(\int_0^t\int_X \phi_s g(s,x)(\mu(ds,dx)-\nu(ds,dx))\Big)^2$$

$$\le 4E\int_0^\tau\int_X p_g^2(s,x,\phi_s)\|g(s,x)\|^2\nu(ds,dx)$$

$$\le 4EK_\tau^g\int_0^\tau\int_X p_g^2(s,x,\phi_s)\kappa(ds,dx)$$

where

$$\kappa(ds,dx) := \|g(s,x)\|^2\nu(ds,dx),\quad K_t^g := 1+\kappa([0,t]\times X),$$

$$p_g(s,x,\phi_s) := |\phi_s g(s,x)|\|g(s,x)\|^{\oplus}.$$

Thus P is a controlled process.

(b) Notice that, under (4.11), (4.12), one can write instead of (4.13) that for $\tau \le T < \infty$

$$E\sup_{t\le\tau}\Big(\int_0^t \phi_s a_s ds\Big)^2 \le TE\int_0^\tau |\phi_s a_s|^2 ds \le TE\tilde{K}_\tau^a\int_0^\tau p_a^2(s,\phi_s)d\tilde{K}_s^a$$

where

$$\tilde{K}_t^a := 1+\int_0^t \|a_s\|^2 ds.$$

In view of Proposition 2.5 it is sufficient to consider the case when there is only one integral in the representation (4.1) of P. E.g., assume that $P_t(\theta)$ is simply the integral with respect to w. Let τ be the minimum of $N>0$ and the hitting time of the level N by the process $\int_0^\cdot \|\sigma_s\|^2 ds$. Then the process ϕ satisfying (4.7) is in $L_\tau^2(p_\sigma, K_\sigma)$ and the convergence of integrands ϕ^n to ϕ in this space means exactly that

$$E\int_0^\tau |\phi_s^n\sigma_s-\phi_s\sigma_s|^2 ds \to 0.$$

Hence, for the approximating sequence of elementary integrands we have that $\phi^n\cdot P^T=\phi^n\circ P^T$ approach simultaneously $\phi\cdot P^T$ and $\phi\circ P^T$. □

Remark. The definition (4.9) does not require neither continuity of P_t and of the coefficients of (4.1) in θ nor the integrability conditions (4.2) – (4.3).

5 Existence of an equivalent martingale measure for the jump-diffusion model

1. From forward rates to price curves

Suppose that a bond price process is specified through forward rates, i.e. for $\theta \in \mathbf{R}_+$

$$P_t(\theta) = \exp\Big\{ -\int_t^\theta f(t,s)ds \Big\}, \quad t \le \theta, \tag{5.1}$$

where $f(t,.)$ is a $C_{\mathbf{R}_+}$-valued adapted process and hence the price curves are continuously differentiable with

$$f(t,\theta) = -\frac{\partial}{\partial\theta} \ln P_t(\theta). \tag{5.2}$$

Assumption 5.1 *The dynamics of the forward rates is given by*

$$df(t,\theta) = \alpha(t,\theta)dt + \sigma(t,\theta)dw_t + \int_X \delta(t,x,\theta)(\mu(dt,dx) - \nu(dt,dx)) \tag{5.3}$$

where w is a standard Wiener process in $\mathbf{R}^n$, μ is a $\mathscr{P}\otimes\mathscr{X}$-$\sigma$-finite random measure (one can think that it is the jump measure of a semimartingale) with the continuous compensator $\nu(dt,dx)$, the coefficients are continuous in θ, the functions $\alpha(t,\theta)$ and $\sigma(t,\theta)$ are $\mathscr{P}\otimes\mathscr{B}_+$-measurable, and $\delta(t,x,\theta)$ is $\mathscr{P}\otimes\mathscr{X}\otimes\mathscr{B}_+$-measurable.

For all finite t and $\theta \ge t$

$$\int_0^\theta \int_t^\theta |\alpha(u,s)|dsdu < \infty, \quad \int_0^\theta \int_t^\theta |\sigma(u,s)|^2 dsdu < \infty, \tag{5.4}$$

and

$$\int_0^\theta \int_X \int_t^\theta |\delta(u,x,s)|^2 ds\nu(du,dx) < \infty. \tag{5.5}$$

It is convenient to extend the definitions of the coefficients by putting them equal to zero for $\theta < t$.

To abbreviate the formulae we shall use sometimes the notation $\bar{\mu} := \mu - \nu$. The relation (5.3) means that

$$f(t,\theta) = f(0,\theta) + \int_0^t \alpha(u,\theta)du + \int_0^t \sigma(u,\theta)dw_u + \int_0^t \int_X \delta(u,x,\theta)\bar{\mu}(du,dx). \tag{5.6}$$

In particular, for the spot rate $r_t := f(t,t)$ we have

$$r_t = f(0,t) + \int_0^t \alpha(u,t)du + \int_0^t \sigma(u,t)dw_u + \int_0^t \int_X \delta(u,x,t)\bar{\mu}(du,dx). \tag{5.7}$$

Notice that the integrability conditions (5.4) – (5.5) are fulfilled if the coefficients are bounded for t and θ from a bounded set (by a constant depending on ω and the set) and $\nu([0,t]\times X) < \infty$ for finite t.

Put

$$A_t(\theta) = -\int_t^\theta \alpha(t,s)ds, \tag{5.8}$$

$$S_t(\theta) = -\int_t^\theta \sigma(t,s)ds, \tag{5.9}$$

$$D(t,\theta,x) = -\int_t^\theta \delta(t,x,s)ds. \tag{5.10}$$

The dynamics of the price curve is given by the following

Proposition 5.2 *The discounted bond price process $Z_t(\theta)$ on $[0,\theta]$ has the form*

$$Z_t(\theta) = Z_0(\theta)\exp\Big\{\int_0^t A_s(\theta)ds + \int_0^t S_s(\theta)dw_s + \int_0^t\int_X D(s,x,\theta)\overline{\mu}(ds,dx)\Big\} \tag{5.11}$$

and satisfies the linear stochastic differential equation

$$\frac{dZ_t(\theta)}{Z_{t-}(\theta)} = a_t(\theta)dt + S_t(\theta)dw_t + \int_X D(t,x,\theta)\overline{\mu}(dt,dx) + \int_X (e^{D(t,x,\theta)} - 1 - D(t,x,\theta))\mu(dt,dx) \tag{5.12}$$

with

$$a_t(\theta) = A_t(\theta) + \frac{1}{2}|S_t(\theta)|^2. \tag{5.13}$$

Proof. Applying the Fubini theorem and its stochastic versions we get from (5.1) and (5.6) that

$$\ln P_t(\theta) = -\int_t^\theta f(t,s)ds = -\int_t^\theta f(0,s)ds$$

$$-\int_0^t\int_t^\theta \alpha(u,s)dsdu - \int_0^t\int_t^\theta \sigma(u,s)dsdw_u - \int_0^t\int_X\int_t^\theta \delta(u,x,s)ds\overline{\mu}(du,dx)$$

$$= -\int_0^\theta f(0,s)ds - \int_0^t\int_u^\theta \alpha(u,s)dsdu - \int_0^t\int_u^\theta \sigma(u,s)dsdw_u - \int_0^t\int_X\int_u^\theta \delta(u,x,s)ds\overline{\mu}(du,dx)$$

$$+\int_0^t f(0,s)ds + \int_0^t\int_u^t \alpha(u,s)dsdu + \int_0^t\int_u^t \sigma(u,s)dsdw_u + \int_0^t\int_X\int_u^t \delta(u,x,s)ds\overline{\mu}(du,dx)$$

$$= \ln P_0(\theta) + \int_0^t A_u(\theta)du + \int_0^t S_u(\theta)dw_u + \int_0^t\int_X D(u,x,\theta)\overline{\mu}(du,dx) + \int_0^t r_s ds$$

according to our definitions (5.8) – (5.10) and since the sum of the four integrals in the left-hand side of the last equality (again by the Fubini theorems) coincides with the expression for the integrated spot rate

$$\int_0^t f(0,s)ds + \int_0^t \int_0^s \alpha(u,s)duds$$
$$+ \int_0^t \int_0^s \sigma(u,s)dw_u ds + \int_0^t \int_0^s \int_X \delta(u,x,s)\bar{\mu}(du,dx)ds.$$

Thus, (5.11) is proved. By the Ito formula we get from (5.11) that

$$dZ_t(\theta) = Z_{t-}(\theta)\Big[A_t(\theta)dt + S_t(\theta)dw_t + \int_X D(t,x,\theta)\bar{\mu}(dt,dx)$$
$$+\frac{1}{2}|S_t(\theta)|^2 dt + \int_X (e^{D(t,x,\theta)} - 1 - D(t,x,\theta))\mu(dt,dx)\Big]$$

and (5.12) holds. □

2. Absence of arbitrage and dynamics under a martingale measure

As usual, we shall use the notation $\mathbf{P}_t := \mathbf{P}|\mathscr{F}_t$ (the restriction of $\mathbf{P}$ to the σ-algebra $\mathscr{F}_t$).

Let $\mathscr{Q}$ be the set of all probability measures $\tilde{\mathbf{P}}$ with $\tilde{\mathbf{P}}_t \sim \mathbf{P}_t$ for all finite t and such that the discounted bond price process $Z_t := R_t P_t(\theta)$ is a local $\tilde{\mathbf{P}}$-martingale for every $\theta \in \mathbf{T}$.

We say that a *model has the* EMM-*property* if the set $\mathscr{Q}$ is nonempty.

We begin with a comment concerning terminology. In the literature on the term structure of interest rates the EMM-property (or its slight modification) is quite often referred to as absence of arbitrage. This is rather confusing since it would be more consistent, as it is usually done in the theory of stock markets, to separate the "no-arbitrage" or "no-free lunch" properties which have a transparent economical meaning (impossibility to obtain "profits" without "risk") from the more mathematically convenient but difficult to interpret EMM-property. We use the quotation marks above since the mentioned concepts should be rigorously defined; one can find different variants in [14] where the problem of no-arbitrage is solved for a continuous-time market model with a finite number of assets; see also [26] for an approach based on the notion of a large financial market. Of course, EMM-property always implies no-arbitrage.

The EMM-property implies that the coefficients of the model are interrelated and cannot be chosen in an arbitrary way. The following result (generalizing the well-known observation of Heath–Jarrow–Morton [18] for the diffusion case) reveals this fact in a remarkably simple way when the model is specified under a (local) martingale measure.

Proposition 5.3 *The probability* $\mathbf{P} \in \mathscr{Q}$ *iff the following two conditions hold:*

$$\int_0^t \int_X (e^{D(s,x,\theta)} - 1 - D(s,x,\theta))\nu(ds,dx) < \infty, \tag{5.14}$$

$$\int_0^t a_s(\theta)ds + \int_0^t \int_X (e^{D(s,x,\theta)} - 1 - D(s,x,\theta))\nu(ds,dx) = 0 \tag{5.15}$$

for any $t \in \mathbf{R}_+$.

In the particular case when $\nu(dt,dx) = \lambda_t(dx)dt$, *the probability* $\mathbf{P} \in \mathcal{Q}$ *iff (5.14) holds for any* $t \in \mathbf{R}_+$ *and*

$$a_t(\theta) + \int_X (e^{D(t,x,\theta)} - 1 - D(t,x,\theta))\lambda_t(dx) = 0 \tag{5.16}$$

($d\mathbf{P}dt$*-a.e.).*

Proof. ($\Leftarrow$) Under (5.14) the representation (5.12) can be rewritten in the following way:

$$\frac{dZ_t(\theta)}{Z_{t-}(\theta)} = a_t(\theta)dt + S_t(\theta)dw_t + \int_X (e^{D(t,x,\theta)} - 1)(\mu(dt,dx) - \nu(dt,dx))$$
$$+ \int_X (e^{D(t,x,\theta)} - 1 - D(t,x,\theta))\nu(dt,dx). \tag{5.17}$$

It follows from (5.15) that the process $[Z_-(\theta)]^{-1} \cdot Z(\theta)$ is a local martingale, hence $Z(\theta)$ is also a local martingale, i.e. $\mathbf{P} \in \mathcal{Q}$.

($\Rightarrow$) In this case the process $M := [Z_-(\theta)]^{-1} \cdot Z(\theta)$ is a local martingale. Let μ^M be the jump measure of M and ν^M be its compensator. By II.2.29 in [21] we have that $|x| \wedge |x|^2 * \nu_t^M < \infty$ for finite t. Hence

$$|e^D - 1| \wedge |e^D - 1|^2 * \nu_t = |x| \wedge |x|^2 * \nu_t^M < \infty.$$

Since $|D|^2 * \nu_t < \infty$ the property (5.14) holds by virtue of the elementary inequality

$$e^D - 1 - D \le C(|e^D - 1| \wedge |e^D - 1|^2 + D^2)$$

where C is a constant. Using (5.17) we infer that M is a local martingale only if the process given by the left-hand side of (5.15) is indistinguishable from zero. □

Remark. One can observe that the hypothesis $\nu(dt,dx) = \lambda_t(dx)dt$ is not a restriction since (5.15), actually, implies this structure on the set where μ has an effect on the price curve dynamics. We leave the formal statement to the reader.

3. A jump-diffusion model in the Brace–Musiela parameterization

Quite recently Brace and Musiela [7] (see also [30]) observed that in some aspects it is more natural to describe the forward rate in the Heath–Jarrow–Morton model using another parameterization: not in terms of *maturity time* but in terms of *time to maturity*. In particular, in their version the dynamics of the forward rate curve under an equivalent martingale measure is given by a very simple stochastic differential equation in the space of continuous functions.

Assume that the $C_{\mathbf{R}_+}$-valued adapted process $r(t,.) := f(t,t+.)$ is such that for any u the scalar process $r(.,u)$ admits the representation

$$r(t,u) = r(0,u) + \int_0^t \beta_s(u)ds + \int_0^t \tau_s(u)dw_s + \int_0^t \int_X \eta(s,x,u)\bar{\mu}(ds,dx) \quad (5.18)$$

where the coefficients satisfy the integrability conditions: for all finite t

$$\int_0^t \int_0^t |\beta_s(u)|dsdu < \infty, \quad \int_0^t \int_0^t |\tau_s(u)|^2 dsdu < \infty, \quad (5.19)$$

and

$$\int_0^t \int_X \int_0^t |\eta(s,x,u)|^2 ds\nu(du,dx) < \infty. \quad (5.20)$$

Proposition 5.4 *For the forward rates given by (5.18) the discounted bond price process $Z_t(\theta)$ on $[0,\theta]$ satisfies the linear stochastic differential equation*

$$\frac{dZ_t(\theta)}{Z_{t-}(\theta)} = \left[r(t,\theta-t) - r(t,0) + B_t(\theta-t) + \frac{1}{2}|T_t(\theta-t)|^2\right]dt + T_t(\theta-t)dw_t$$

$$+ \int_X H(t,x,\theta-t)\bar{\mu}(dt,dx) + \int_X (e^{H(t,x,\theta-t)} - 1 - H(t,x,\theta-t))\mu(dt,dx) \quad (5.21)$$

where

$$B_t(\theta) = -\int_0^\theta \beta_t(u)du, \quad (5.22)$$

$$T_t(\theta) = -\int_0^\theta \tau_t(u)du, \quad (5.23)$$

$$H(t,x,\theta) = -\int_0^\theta \eta(t,x,u)du. \quad (5.24)$$

Proof. Put $F_t(\theta) := Z_t(\theta+t)$. From the definitions and the Fubini theorems it follows that

$$\ln F_t(\theta) = -\int_0^t r_s ds - \int_0^\theta r(t,u)du = -\int_0^t r(s,0)ds - \int_0^\theta r(0,u)du$$

$$-\int_0^t \int_0^\theta \beta_s(u)duds - \int_0^t \int_0^\theta \tau_s(u)dudw_s - \int_0^t \int_X \int_0^\theta \eta(s,x,u)du\bar{\mu}(ds,dx).$$

Applying the Ito formula we easily get the representation

$$\frac{dF_t(\theta)}{F_{t-}(\theta)} = \left[-r(t,0) + B_t(\theta) + \frac{1}{2}|T_t(\theta)|^2\right]dt + T_t(\theta)dw_t$$

$$+ \int_X (e^{H(t,x,\theta)} - 1)\bar{\mu}(dt,dx) + \int_X (e^{H(t,x,\theta)} - 1 - H(t,x,\theta))\nu(dt,dx). \quad (5.25)$$

Since

$$dZ_t(\theta) = dF_t(\theta-t) - \frac{\partial F_t(\theta-t)}{\partial x}dt = dF_t(\theta-t) + Z_t(\theta)r(t,\theta-t)dt,$$

the equation (5.25) implies (5.21). □

Similarly to Proposition 5.3 we get as a corollary a certain relation between the coefficients for the case when the basic probability is a martingale measure.

Proposition 5.5 *Assume that* $\nu(dt, dx) = \lambda_t(dx)dt$. *Then the probability* $\mathbf{P} \in \mathcal{Q}$ *iff*

$$\int_0^t \int_X (e^{H(t,x,u)} - 1 - H(t,x,u))\lambda_t(dx)dt < \infty \tag{5.26}$$

for finite t *and* u, *and*

$$r(t,.) = r(t,0) - B_t(.) - \frac{1}{2}|T_t(.)|^2 - R_t(.) \tag{5.27}$$

($d\mathbf{P}dt$*-a.e.) where*

$$R_t(u) := \int_X (e^{H(t,x,u)} - 1 - H(t,x,u))\lambda_t(dx) \tag{5.28}$$

and the functions $B_t(u)$, $T_t(u)$, *and* $H(t,x,u)$ *are defined by (5.22) – (5.24).*

Remark. The relation (5.27) implies (under a mild integrability assumption) that $r(t,.)$ is an absolutely continuous function and

$$\beta_t(.) = \frac{\partial}{\partial u} r(t,.) + \frac{1}{2}\tau_t(.) \int_0^{.} \tau_t(v)dv + \frac{\partial}{\partial u} R_t(.) \tag{5.29}$$

with

$$\frac{\partial}{\partial u} R_t(u) = -\int_X (e^{H(t,x,u)} - 1)\eta(t,x,u))\lambda_t(dx).$$

One can deduce from (5.18), (5.29) that if the model is specified under a martingale measure then the dynamics of the forward rate curve is given by the following stochastic evolution equation

$$dr(t,.) = [\mathcal{A} r(t,.) + C(t,.)]dt + \tau_t(.)dw_t + \int_X \eta(t,x,.)\overline{\mu}(dt,dx), \tag{5.30}$$

where $\mathcal{A} := \partial/\partial u$,

$$C(t,.) := \frac{1}{2}\tau_t(.) \int_0^{.} \tau_t(v)dv - \int_X (e^{H(t,x,.)} - 1)\eta(t,x,.)\lambda_t(dx). \tag{5.31}$$

4. Modeling under the objective probability

For the case when $\mathbf{P}$ is a martingale measure the relations between coefficients of the model for forward rates are simple and easy to treat. Certainly, the objective probability need not to be a martingale measure and now we investigate consequences of the EMM-property for this general case assuming for simplicity that $\nu(dt, dx) = \lambda_t(dx)dt$.

Proposition 5.6 *Let* $\tilde{\mathbf{P}} \in \mathcal{Q}$. *Then there exist a predictable process* φ *with values in* $\mathbf{R}^n$ *and a* $\mathcal{P} \otimes \mathcal{X}$ *-measurable function* $Y = Y(\omega, t, x) > 0$ *with*

$$\int_0^t |\varphi_s|^2 ds < \infty, \tag{5.32}$$

$$\int_0^t \int_X (\sqrt{Y(s,x)} - 1)^2 \lambda_s(dx)ds < \infty \tag{5.33}$$

for finite t such that:

1) the process

$$\tilde{w}_t := w_t - \int_0^t \varphi_s ds \tag{5.34}$$

is Wiener with respect to $\tilde{\mathbf{P}}$;

2) the random measure $\tilde{\nu} := Y\nu$ is the $\tilde{\mathbf{P}}$-compensator of μ;

3) the following integrability condition is satisfied:

$$\int_0^t \int_X (e^{D(s,x,\theta)} - 1) I_{\{D(s,x,\theta) > \ln 2\}} Y(s,x) \lambda_s(dx)ds < \infty \tag{5.35}$$

for finite t and θ;

4) it holds that for any θ

$$a_t(\theta) + S_t(\theta)\varphi_t + \int_X [(e^{D(t,x,\theta)} - 1)Y(t,x) - D(t,x,\theta)]\lambda_t(dx) = 0 \tag{5.36}$$

$d\mathbf{P}dt$-a.e.

Proof. Existence of φ with the property 1) and satisfying (5.32) is given by the classical Girsanov theorem (see Theorem III.3.24 in [21] for a general version). Existence of $Y \geq 0$ with the property 2) follows from the Girsanov theorem for random measures (Theorem III.3.17 in [21]). Since $\tilde{\mathbf{P}}$ and $\mathbf{P}$ are locally equivalent one can choose Y to be strictly positive. The property (5.33) holds because by Theorem IV.3.39 in [21] the process $(\sqrt{Y} - 1)^2 * \nu$ is dominated by the Hellinger process $h(1/2, \mathbf{P}, \tilde{\mathbf{P}})$ which is finite $\tilde{\mathbf{P}}$-a.s. (and hence $\mathbf{P}$-a.s.) according to Theorem IV.2.1 in [21].

Let $\mu^{M,\theta}$ be the jump measure of the semimartingale $M := [Z_-(\theta)]^{-1} \cdot Z(\theta)$ having the representation (5.12). Notice that $\Delta M_t = \int_X (e^{D(t,x,\theta)} - 1)\mu(\{t\}, dx)$ and

$$\int \int_{\mathbf{R}} f(t,u)\mu^{M,\theta}(dt,du) = \int \int_X f(t, e^{D(t,x,\theta)} - 1)\mu(dt,dx)$$

for any positive measurable function f. Evidently, for the $\tilde{\mathbf{P}}$ -compensator of $\mu^{M,\theta}$ we have the similar property:

$$\int \int_{\mathbf{R}} f(t,u)\tilde{\nu}^{M,\theta}(dt,du) = \int \int_X f(t, e^{D(t,x,\theta)} - 1)Y(t,x)\nu(dt,dx).$$

The process M is a special semimartingale with respect to $\tilde{\mathbf{P}}$. Hence, by Proposition II.2.29 in [21] $uI_{\{|u|>1\}} * \tilde{\nu}_t^{M,\theta} < \infty$ for finite t and (5.35) holds.

Now we get from (5.34) that

$$\int_0^t S_s(\theta)dw_s = \int_0^t S_s(\theta)d\tilde{w}_s + \int_0^t S_s(\theta)\varphi_s ds,$$

and, furthermore, by simple transformations,

$$(e^D - 1)I_{\{D>\ln 2\}} * \mu = (e^D - 1)I_{\{D>\ln 2\}} * (\mu - Y\nu)_t + (e^D - 1)I_{\{D>\ln 2\}}Y * \nu,$$

$$DI_{\{|D|\le 1\}} * (\mu - \nu) = DI_{\{|D|\le 1\}} * (\mu - Y\nu)_t + DI_{\{|D|\le 1\}}(Y - 1) * \nu,$$

$$\begin{aligned}&[(e^D - 1)I_{\{D\le \ln 2\}} - DI_{\{|D|\le 1\}}] * \mu \\ &= F * (\mu - Y\nu) + [(e^D - 1)I_{\{D\le \ln 2\}} - DI_{\{|D|\le 1\}}]Y * \nu\end{aligned}$$

where $F := (e^D - 1)I_{\{D\le \ln 2\}} - DI_{\{|D|\le 1\}}$. The right-hand sides of these identities are well-defined and give the canonical decompositions with respect to $\tilde{\mathbf{P}}$ of special semimartingales. Substitution to (5.12) shows that the predictable process in the canonical decomposition of M with respect to $\tilde{\mathbf{P}}$ is equal to

$$\int_0^t [a_s(\theta) + S_s(\theta)\varphi_s]ds + \int_0^t \int_X [(e^{D(t,x,\theta)} - 1)Y(t,x) - D(t,x,\theta)]\nu(dt,dx).$$

But it must be zero and we get (5.36). □

The above proposition means that if $\tilde{\mathbf{P}} \in \mathcal{Q}$ then the "integral" equations (5.36) for almost all (ω, t) have a nonempty set of solutions (φ, Y) where $\varphi \in \mathbf{R}^n$, $Y \ge 0$, $\sqrt{Y} - 1 \in L^2(X, \lambda_t)$. Moreover, one can chose in these sets a certain measurable selector such that the integrability properties (5.32), (5.33), and (5.35) are fulfilled.

Remark. It follows from (5.5) and (5.10) that for θ finite we have

$$\int_X D^2(t,x,\theta)\lambda_t(dx) < \infty. \tag{5.37}$$

In the case when $\lambda_t(X) < \infty$ this implies that

$$\int_X |D(t,x,\theta)|\lambda_t(dx) < \infty \tag{5.38}$$

and, thus, one can transform (5.36) to the simpler form

$$a_t(\theta) - \int_X D(t,x,\theta)\lambda_t(dx) + S_t(\theta)\varphi_t + \int_X (e^{D(t,x,\theta)} - 1)Y(t,x)\lambda_t(dx) = 0 \tag{5.39}$$

which will be used later.

Now we discuss the reciprocal assertion to Proposition 5.6. Starting from ϕ and $Y > 0$ satisfying the integrability conditions one can define the local martingale $\rho = (\rho_t)$ with

$$\begin{aligned}\ln \rho_t &= \int_0^t \varphi_s dw_s - \frac{1}{2}\int_0^t |\varphi_s|^2 ds \\ &+ \int_0^t \int_X \ln Y(s,x)\mu(ds,dx) + \int_0^t \int_X (1 - Y(s,x))\nu(ds,dx).\end{aligned} \tag{5.40}$$

As usual in the Girsanov theory, it may not be a true martingale (even if the pair (φ, Y) originates from $\tilde{\mathbf{P}}$ by Proposition 5.6 !); including this property as an additional hypothesis, i.e. assuming that $E\rho_t = 1$, $t \in \mathbf{R}$, we can define the probability measures $\widehat{\mathbf{P}}^t := \rho_t \mathbf{P}$ for finite t. However, a measure $\widehat{\mathbf{P}}$ such that $\widehat{\mathbf{P}}^t_t = \widehat{\mathbf{P}}_t$ still may not exist and one must exclude this unpleasant situation related to "noncompactness" of the stochastic basis.

We say that a stochastic basis is *sufficiently rich* if for any family of probability measures $\{\widehat{\mathbf{P}}^t\}$ with the property $\widehat{\mathbf{P}}^t_s = \widehat{\mathbf{P}}^s_s$ for all $s \le t$ there exists a measure $\widehat{\mathbf{P}}$ on $\mathscr{F}$ such that $\widehat{\mathbf{P}}^t_t = \widehat{\mathbf{P}}_t$.

Since under the probability measure $\widehat{\mathbf{P}}$, which is locally equivalent to $\mathbf{P}$ with the density process ρ, the process $Z(\theta)$ can be written as follows:

$$
\begin{aligned}
dZ_t(\theta) = Z_{t-}(\theta)\Bigg[a_t(\theta) - \int_X D(s,x,\theta)\lambda_t(dx) + S_t(\theta)\varphi_t \\
+ \int_X (e^{D(s,x,\theta)} - 1)Y(s,x)\lambda_t(dx)\Bigg] dt + \\
+ Z_{t-}(\theta)S_t(\theta)d\tilde{w}_t + Z_{t-}(\theta)\int_X (e^{D(s,x,\theta)} - 1)(\mu(dt,dx) - \tilde{\nu}(dt,dx)), \qquad (5.41)
\end{aligned}
$$

the arguments above lead to the following

Proposition 5.7 *Suppose that the stochastic basis is sufficiently rich and that the measurable functions* φ *and* $Y(t,x) > 0$ *satisfy (5.32), (5.37), (5.35), (5.36), and* $E\rho_t = 1$ *for all finite* t. *Then the set* $\mathscr{Q}$ *is nonempty.*

Remark. To avoid the condition on the stochastic basis (which is not very esthetic) one can work with a set of density processes or "martingale densities" (see [6]) imposing instead the more restrictive assumption that $E\rho_\infty = 1$ (then $\mathscr{Q}$ will contain a probability which is absolutely continuous with respect to $\mathbf{P}$).

6 Uniqueness of the martingale measure and market completeness

1.
Now we study the relation between uniqueness of the martingale measure (this means that the set $\mathscr{Q}$ is a singleton) and market completeness. The model is the same as in Section 5 but the following additional hypotheses will be assumed throughout the end of the section:

Assumption 6.1 *(Predictable representation property.) Any local martingale* M *with respect to* $\mathbf{P}$ *has the form*

$$
M_t = M_0 + \int_0^t \psi_s dw_s + \int_0^t \int_X \Psi(s,x)(\mu(ds,dx) - \nu(ds,dx)) \qquad (6.1)
$$

where ψ *is a predictable process,* Ψ *is a* $\mathscr{P} \otimes \mathscr{X}$*-measurable function, and*

$$
\int_0^t |\psi_s|^2 ds < \infty, \qquad (6.2)
$$

$$\int_0^t \int_X \frac{\Psi^2(s,x)}{1+|\Psi(s,x)|}\nu(ds,dx) < \infty \tag{6.3}$$

for finite t.

Let the filtration $\mathbf{F}$ be generated by w and μ. Then there are two important and well-known cases when the predictable representation property holds:
(a) μ is a Poisson random measure, i.e. ν is deterministic;
(b) μ is the measure associated with a multivariate point process in the sense of [20] (or [21] with an extra requirement that $\nu([0,t]\times X) < \infty$ for finite t) and $n = 0$ (no Wiener process).

It turns out that in the latter case the representation property holds for arbitrary n. To prove this, one can use the criteria Theorem III.4.29 of [21] and, arguing with the conditional distributions of μ given $w = y$, show the uniqueness of a measure on $\mathscr{F}_\infty$ such that w is a Wiener process and μ has ν as compensator.

Notice that the predictable representation property is preserved under a locally absolute continuous change of the probability measure, see Ch. III of [21] for an extended discussion.

Now all density processes have the form given by (5.40) (hence, they are uniquely defined by the Girsanov transformation parameters φ and Y) and one can combine Propositions 5.6 and 5.7 in the following

Proposition 6.2 *Suppose that Assumption 6.1 is fulfilled and the stochastic basis is sufficiently rich. Then $\mathcal{Q} \neq \emptyset$ iff there are measurable functions φ and $Y(t,x) > 0$ satisfying (5.32), (5.37), (5.35), (5.36), and $E\rho_t = 1$ for all finite t.*

Under the measure $\tilde{P}$ defined by the density process ρ, the properties 1) and 2) of Proposition 5.6 hold.

2. Martingale operators and uniqueness of the martingale measure
One can observe that the existence results involve “space-time” integrability conditions and also “instantaneous identities” (5.36) or (5.39). Regarding the latter as integral equations it is easy to formulate the uniqueness results in terms of injectiveness of the corresponding operators.

We investigate the problem under

Assumption 6.3 *(a) The process $\lambda_t(X)$ is finite.*
(b) For almost all ω, t and N there exists $c_N(\omega,t) < \infty$ such that $|D(\omega,t,x,\theta)| \leq c_N(\omega,t)$ for all $x \in X$ and $\theta \leq N$.

Let us consider the family of continuous linear operators

$$\mathscr{K}_t(\omega) : \mathbf{R}^n \times L^1(X, \mathscr{X}, \lambda_t(\omega, dx)) \to C_{\mathbf{R}_+} \tag{6.4}$$

defined by

$$\mathscr{K}_t(\omega) : (\varphi, Y) \mapsto S(\omega,t,.)\varphi + \int_E Y(x)(e^{D(\omega,t,x,.)} - 1)\lambda_t(\omega,dx). \tag{6.5}$$

We shall refer to $\mathscr{K}$ as “the martingale operators”.

In view of Proposition 5.6 the following result is almost evident.

Proposition 6.4 *Under Assumptions 6.1 and 6.3 suppose that $\mathcal{Q} \neq \emptyset$. Then $\mathcal{Q}$ is a singleton iff dPdt-a.e.*

$$\operatorname{Ker} \mathscr{K}_t(\omega) = 0. \tag{6.6}$$

Corollary 6.5 *Suppose that the model coefficients $\alpha(t,T)$, $\sigma(t,T)$, $\delta(t,x,T)$, and $\lambda_t(dx)$ are deterministic and the martingale measure Q is unique. Then the Girsanov transformation parameters φ and Y are deterministic functions, i.e. under Q the process $\tilde{W}$ is a Wiener process with drift and μ is a Poisson measure.*

Proof. The operators $\mathscr{K}_t$ do not depend of ω and hence (outside the exclusive $dPdt$-null set) the values of the Girsanov transformation parameters corresponding to a fixed t but different ω must satisfy the *same* equation (5.39) which has a unique solution by (6.6). □

Notice that the operators $\mathscr{K}_t(\omega)$ are integral operators of the first kind.

Corollary 6.6 *Suppose, in addition to the hypotheses of Corollary 6.5, that $\alpha(t,T) = \alpha(T-t)$, $\sigma(t,T) = \sigma(T-t)$, $\delta(t,x,T) = \delta(T-t,x)$, and $\lambda(t,dx) = \lambda(dx)$. Then the Girsanov transformation parameters φ and Y do not depend also on t, i.e. under the unique measure $Q \in \mathcal{Q}$ the process $\tilde{W}$ is a Wiener process with a constant drift and μ is a Poisson measure invariant under time translations.*

The definition (6.5), being very simple, fits well the above claims. However, it has a certain drawback because it involves the space $C_{\mathbf{R}_+}$ with the unpleasant dual. As we shall see below, it is rather natural to modify a bit the definition of the martingale operators and impose the following constraint on the model:

Assumption 6.7 *There exists a positive predictable process $C_t = C_t(\omega)$ such that for almost all (ω, t) its sections $C_t(\omega, .): \mathbf{T} \to \mathbf{R}_+$ are bounded functions,*

$$Z_{t-}(.)|e^{D(t,.,.)} - 1| \le C_t \quad a.e.,$$

and

$$\lim_{\theta\to\infty} Z_{t-}(\theta)S_t(\theta) = 0, \qquad \lim_{\theta\to\infty} Z_{t-}(\theta)(e^{D(t,x,\theta)} - 1) = 0. \tag{6.7}$$

Let $C^0_{\mathbf{R}_+}$ be the space of continuous functions on $\mathbf{R}_+$ converging to zero at infinity. Notice that $C^{0*}_{\mathbf{R}_+} = \mathbf{M}_{\mathbf{R}_+}$, the space of measures on $\mathbf{R}_+$ with finite total variation.

The formula

$$\mathscr{K}^Z_t(\omega) : (\varphi, Y) \mapsto Z_{t-}(\omega, .)S(\omega, t, .)\varphi + Z_{t-}(\omega, .)\int_X Y(x)(e^{D(\omega,t,x,.)} - 1)\lambda_t(\omega, dx) \tag{6.8}$$

defines a family of linear operators

$$\mathscr{K}^Z_t(\omega): \mathbf{R}^n \times L^2(X, \mathscr{X}, \lambda_t(\omega, dx)) \to C^0_{\mathbf{R}_+}. \tag{6.9}$$

In other words, $\mathscr{K}^Z_t(\omega)$ is the product of the operator $\mathscr{K}_t(\omega)$ and the operator $\mathscr{Z}_t(\omega)$ of multiplication by the function $Z_{t-}(\omega, .)$, so, one can write that $\mathscr{K}^Z_t =$

$\mathscr{Z}_t \mathscr{K}_t$. Clearly, the above results hold also with $\mathscr{K}$ substituted by $\mathscr{K}^Z$ but the modified definition allows to exploit a duality arising in the problem of market completeness.

3. Hedging operators and market completeness
Using financial terminology, we say that a *bounded (contingent) T-claim Ξ* (which is just a random variable $\Xi \in L^\infty(\mathscr{F}_T)$ is *hedgeable* (or *replicable*) if there is a **bounded** discounted value process V^Z such that $\Xi = V^Z_T$, i.e. there exist a strategy ϕ and an initial endowment x such that $\Xi = x + \phi \circ Z_T$ and the integral $\phi \circ Z_T$ is bounded on $[0, T]$.

The bond market is said to be *complete* if all bounded T-claims are hedgeable for every $T \in \mathbf{R}_+$ and *approximately complete* if for any bounded T-claim Ξ there exists a sequence of hedgeable T-claims Ξ^n converging to Ξ in $L^2(Q)$ for some $Q \in \mathscr{Q}$.

We deliberately restrict ourselves to bounded claims in the above definitions since the space L^∞ (as well as L^0) is invariant under an equivalent change of probability measure (recall that convergence in probability can be expressed in terms of convergence a.s. of subsequences). We may thus assume from now on to the end of this subsection (mainly for notational convenience) that the model is specified **under a martingale measure**, i.e. $\mathbf{P} \in \mathscr{Q}$, and, moreover, this is exactly the measure which is involved in the definition of the approximate completeness.

Remark. Notice that integrability assumptions under a martingale measure, made in the definitions of completeness on claims to be hedged (which one can observe in the literature), are rather awkward and even inconsistent in the context of the problem considered here that deals with properties of $\mathscr{Q}$.

We consider the family

$$\mathscr{K}_t^{Z*}(\omega) : \ \mathbf{M}_{\mathbf{R}_+} \to \mathbf{R}^n \times L^2(X, \mathscr{X}, \lambda_t(\omega, dx)) \tag{6.10}$$

of *hedging* operators acting on measures in the following way:

$$\mathscr{K}_t^{Z*}(\omega) : \ m \mapsto \begin{bmatrix} \int_0^\infty Z_{t-}(\omega, \theta) S_t(\omega, \theta) m(d\theta) \\ \\ \int_0^\infty Z_{t-}(\omega, \theta)(e^{D(\omega, t, \cdot, \theta)} - 1) m(d\theta) \end{bmatrix}. \tag{6.11}$$

Evidently, the operator $\mathscr{K}_t^{Z*}(\omega)$ is adjoint to $\mathscr{K}_t^Z(\omega)$.

We recall that, due to Assumption 6.1, for any $\Xi \in L^2(\mathscr{F}_T, \mathbf{P})$ the martingale $M_t := E(\Xi | \mathscr{F}_t)$, $t \le T$, admits the predictable representation

$$M_t = M_0 + \int_0^t \psi_s \, dw_s + \int_0^t \int_X \Psi(s, x)(\mu(ds, dx) - \nu(ds, dx)) \tag{6.12}$$

with $M_0 = E\Xi$ and, since it is square integrable, it follows easily that

$$E \int_0^T |\psi_s|^2 ds < \infty, \tag{6.13}$$

$$E \int_0^T \int_X \Psi^2(s,x)\nu(ds,dx) < \infty. \tag{6.14}$$

The coefficients of this representation are uniquely defined. More precisely, $\Xi \mapsto (\psi, \Psi)$ is a continuous linear mapping from $L^2(\mathscr{F}_T, \mathbf{P})$ onto $L^2(\mathscr{P}, d\mathbf{P}dt) \times L^2(\mathscr{P} \otimes \mathscr{X}, d\mathbf{P}\lambda_t(dx)dt)$.

Proposition 6.8 *The claim $\Xi \in L^\infty(\mathscr{F}_T)$ is hedgeable iff there exists a predictable measure-valued process $h = h(t, d\theta)$ which satisfies the integrability conditions*

$$E \int_0^T \left| \int_{\mathbf{R}_+} Z_t(\theta) S_t(\theta) h(t, d\theta) \right|^2 dt < \infty, \tag{6.15}$$

$$E \int_0^T \int_X \left| \int_{\mathbf{R}_+} Z_{t-}(\theta)(e^{D(\omega,t,x,\theta)} - 1) h(t, d\theta) \right|^2 \nu(dt, dx) < \infty, \tag{6.16}$$

and solves on $[0, T]$ ($d\mathbf{P}dt$-a.e.) the equation

$$\mathscr{K}_t^{Z*} h = \begin{bmatrix} \psi_t \\ \Psi(t,.) \end{bmatrix}. \tag{6.17}$$

Proof. Since $\mathbf{P} \in \mathscr{Q}$ we have by Propositions 5.2 and 5.3 that

$$dZ_t(\theta) = Z_{t-}(\theta) S_t(\theta) dw_t + Z_{t-}(\theta) \int_X (e^{D(\omega,t,x,\theta)} - 1)(\mu(ds, dx) - \nu(ds, dx)). \tag{6.18}$$

Thus, the discounted value process V^Z is of the form

$$V_t^Z = x + \int_0^t \left(\int_{\mathbf{R}_+} Z_s(\theta) S_s(\theta) h(s, d\theta) \right) dw_s$$

$$+ \int_0^t \int_X \left(\int_{\mathbf{R}_+} Z_{s-}(\theta)(e^{D(\omega,s,x,\theta)} - 1) h(s, d\theta) \right) (\mu(ds, dx) - \nu(ds, dx)) \tag{6.19}$$

Comparison of (6.12) and (6.19) yields the result. □

As a corollary, we get

Proposition 6.9

1. *The martingale measure is unique iff the mappings $\mathscr{K}^Z$ are injective (a.e.).*
2. *The market is complete iff the mappings $\mathscr{K}^{Z*}$ are surjective (a.e.).*

The proof of a natural extension of the second assertion which we give below involves a measurable selection technique. The operator $\mathscr{K}_t^{Z*}(\omega)$ is a mapping to $\mathbf{R}^n \times L^2(X, \mathscr{X}, \lambda_t(\omega, dx))$ and "cl" means the closure in this space.

Proposition 6.10 *The following conditions are equivalent.*

(i) The market is approximately complete.
(ii) $\mathrm{cl}\,(\mathrm{Im}\,\mathscr{K}_t^{Z*}(\omega)) = \mathbf{R}^n \times L^2(X, \mathscr{X}, \lambda_t(\omega, dx))$ *(a.e.).*

Proof. (**i**) $\Leftarrow$ (**ii**) Let Ξ be a bounded discounted contingent T-claim to be approximated. For $\varepsilon > 0$ put

$$F^{\varepsilon}(t,m) := |\mathscr{K}_t^{Z*,1}(m) - \psi_t|^2 + \|\mathscr{K}_t^{Z*,2}(m) - \Psi(t,.)\|^2_{L^2(\lambda_t(dx))}$$

where we use superscripts to denote the first and the second "coordinates" in (6.11). Recall that balls in $\mathbf{M}_{\mathbf{R}_+}$ are metrizable compacts, hence, $(\mathbf{M}_{\mathbf{R}_+}, \mathscr{M}_{\mathbf{R}_+})$ is a Lusin space as a countable union of Polish spaces. The function F^{ε}, being $\mathscr{P}$-measurable in (ω, t) and continuous in m, is jointly measurable. Thus, the set-valued mapping

$$(\omega, t) \mapsto \{m \in \mathbf{M}_{\mathbf{R}_+} : \ F^{\varepsilon}(\omega, t, m) \le \varepsilon\}$$

has a $\mathscr{P} \otimes \mathbf{M}_{\mathbf{R}_+}$-measurable graph and, by assumption, non-empty values (a.e.). Therefore, it admits a $\mathscr{P}$-measurable a.e.-selector $m^{\varepsilon}(t, d\theta)$ (see, e.g., [13]), which "almost" solves the problem. Indeed, for the value process $V^Z(h^{\varepsilon}) = E\Xi + h^{\varepsilon} \circ Z$ corresponding to the strategy $h_t^{\varepsilon}(d\theta) = I_{[0,t]}(\theta) m^{\varepsilon}(t, d\theta)$ we have

$$E|V_T^Z(h^{\varepsilon}) - \Xi|^2 \le E\int_0^T F(t, m^{\varepsilon})dt \le \varepsilon T \to 0, \quad \varepsilon \to 0.$$

However, the construction is not accomplished since these strategies generate a value processes which are not bounded (and even admissible). Notice that the predictable process C_t from Assumption 6.7 is locally bounded, i.e. there exists a sequence of stopping times $\sigma_n \uparrow \infty$ a.s. and such that $C_t \le n$ for $t \le \sigma_n$. Put

$$h^{\varepsilon,n}(t,.) := h^{\varepsilon} I_{\{\|(h^{\varepsilon}(t,.)\|_V \le n\}} I_{\{t \le \sigma_n\}}.$$

Clearly, $E|V_T^Z(h^{\varepsilon,n}) - V_T^Z(h^{\varepsilon})|^2 \to 0$ as $n \to \infty$. By Assumption 6.7, we have that

$$\left|\int_{\mathbf{R}_+} Z_{t-}(\theta)(e^{D(\omega,t,x,\theta)} - 1)h^{\varepsilon,n}(t, d\theta)\right| \le \int_{\mathbf{R}_+} C_t h^{\varepsilon,n}(t, d\theta) \le n^2.$$

Hence, the value process corresponding to $h^{\varepsilon,n}$ has the bounded jumps. Let $\tilde{h}^{\varepsilon,n} := h^{\varepsilon,n} I_{[0,\sigma'_n]}$ where σ'_n is the exit time of $V^Z(h^{\varepsilon,n})$ from $[-n, n]$. Then $V_T^Z(\tilde{h}^{\varepsilon,n})$ is a sequence of headgeable claims converging in L^2 to $V_T^Z(h^{\varepsilon})$. This leads to the desired goal.

(**i**) $\Rightarrow$ (**ii**) Assume that the market is approximately complete, i.e. an arbitrary bounded T-claim can be approached by a sequence of hedgeable claims converging in L^2. Then there exists a countable set $H = \{\Xi^j\}$ of bounded hedgeable random variables dense in the Hilbert space $L^2(\mathscr{F}_T)$ and closed under linear combinations with rational coefficients; let (ψ^j, Ψ^j) be the coefficients in the integral representation of Ξ^j given by (6.12). We continue with the case $n = 0$; the arguments can be extended easily for the general case but, in fact, there is no need in this: one can identify the product space in the right-hand side of (**ii**) with L^2 over an extension of E by n extra points. Of course, we may assume that for all (ω, t) one has $\|\Psi^n\|_{\omega,t} < \infty$ where $\|.\|_{\omega,t}$ and $(.,.)_{\omega,t}$ are, respectively, the norm and the scalar product in $L^2(X, \mathscr{X}, \lambda_t(\omega, dx))$. Let us denote by $H_{\omega,t}$ the closure in this norm of the set $\{\Psi^n(\omega,t)\}$, which is, evidently, a linear subspace, and by $H^{\perp}_{\omega,t}$ its orthogonal complement.

It is easy to show that there exists a $\mathscr{P}\otimes\mathscr{X}$-measurable function Ψ such that $\|\Psi\|_{\omega,t}=1$ if $H^{\perp}_{\omega,t}\neq 0$. Indeed, let $\{I(i)\}$ be a sequence of indicator functions generating $\mathscr{X}$ and

$$k(\omega,t) := \inf\left\{ i:\ \inf_j \|I(i)-\Psi^j(\omega,t,.)\|_{\omega,t} > 0\right\}.$$

Put $\tilde{\Psi}(\omega,t,x) := I(k(\omega,t),x)$ if $k(\omega,t) < \infty$ and $\tilde{\Psi}(\omega,t,x) := 0$ otherwise. Clearly, $\tilde{\Psi}$ meets the necessary measurability requirements. Furthermore, there is $\tilde{\Psi}^{\pi}$ which is measurable in the same way and such that all the sections $\tilde{\Psi}^{\pi}(\omega,t)$ are representatives of the projections of $\tilde{\Psi}(\omega,t)$ onto $H_{\omega,t}$ (one can orthogonalize $\{\Psi^j(\omega,t)\}$ preserving measurability and notice that in this case the Fourier coefficients are obviously predictable). Normalizing the difference $\tilde{\Psi}-\tilde{\Psi}^{\pi}$ we get Ψ with the required properties.

The function Ψ defines by (6.12) with $M_0=0$ a random variable $M_T\in L^2(\mathscr{F}_T)$ which is orthogonal, by construction, to all Ξ^j. If **(ii)** does not hold then M_T is nontrivial. This leads to an apparent contradiction. □

By experience from the theory of financial markets with finitely many assets one could expect that the market is complete if and only if the martingale measure is unique, but in our infinite dimensional setting this is no longer true. Due to the duality relation $(\mathrm{Ker}\,\mathscr{K})^{\perp}=\mathrm{cl}\,(\mathrm{Im}\,\mathscr{K}^{\star})$ we obtain instead from the above assertion

Theorem 6.11 *The market is approximately complete iff the martingale measure is unique.*

Remark. For the above theorem, Assumption 6.3 (a), is not, of course, very pleasant since it, actually, means that the set $\mathscr{Q}$ (always assumed to be non-empty) contains a measure under which the compensator has such a property. However, it automatically holds in the important case when μ is a multivariate point process with absolutely continuous compensator. We believe that Theorem 6.11 can be extended to a much more general setting.

For a model when all measures $\lambda_t(dx)$ are concentrated in a finite number of points (in particular, when the mark space X is finite) and the hedging problem is reduced to a finite-dimensional system of equations (for each (ω,t)), the duality relation is simply $(\mathrm{Ker}\,\mathscr{K})^{\perp}=\mathrm{Im}\,\mathscr{K}^{*}$, so in this case we have

Corollary 6.12 *Suppose that the measures $\lambda_t(dx)$ are concentrated in a finite number of points (a.e.). Then the bond market is complete iff the martingale measure is unique.*

In general, the “principle” that uniqueness of Q is equivalent to completeness of the market fails: the set of hedgeable claims may be a strict subset in the set of all claims $L^{\infty}(\mathscr{F}_T)$. Clearly, this is the case when D is smooth in x and bounded (so, the image contains only continuous functions); typically, $\mathscr{K}_t^{Z*}$ is a compact operator and, hence, has no bounded inverse.

Thus, models with an infinite mark space introduce some completely new features into the theory, and we also encounter some new problems when it comes to the numerical computation of hedging portfolios. Namely, the hedging equations (6.17) are, in general, ill-posed in the sense of Hadamard, i.e. the inverse of $\mathscr{K}_t^{Z*}$ restricted to $\operatorname{Im}\mathscr{K}_t^{Z*}$ may not be bounded. Hence, a small perturbation of the right-hand side (e.g., due to a small round-off error) gives rise to large fluctuations in the solution. Thus, a simple approximation scheme for the calculation of a concrete hedge may lead to great numerical errors. Fortunately, the literature provides a number of methods to get stable solutions of ill-posed problems.

7 Conclusions

A consistent theory of the zero-coupon bond markets can be based on a setting where the price curve is considered as a point in the Banach space of continuous functions and its evolution is described by a random process in this space. In such an approach a portfolio strategy at a fixed time is identified with a linear functional which is an element of the conjugate space, i.e. a measure on maturities. The dynamics of a strategy is given by a weakly predictable measure-valued process.

The needed mathematical tool is a stochastic integration with respect to C-valued processes for which our paper suggests a certain general recipe. As a justification of the general framework, we prove that the asset paying an interest corresponding to the short term interest rate is the value process of a roll-over strategy consisting in permanent reinvestment in just maturing bonds. Traditionally, the existence of such an asset in a bond market is an auxiliary hypothesis explained by heuristic arguments.

The integration theory has a more explicit structure for models where the dynamics of any bond, i.e. evolution of each point of the price curve, is given by a jump-diffusion model. In this case, one can use a construction involving standard finite-dimensional integrals. Starting the modelling from the description of the forward rate dynamics we derive HJM-type conditions for the existence of an equivalent martingale measure.

The formal definition of a portfolio strategy allows to define other economically meaningful properties of a bond market, in particular, market completeness. For a model with a finite Lévy measure we show that the completeness is equivalent to the uniqueness of the equivalent martingale measure, a relation which is well-known for stock market models. However, in the case of an infinite Lévy measure this is no longer true; it happens that the uniqueness of the equivalent martingale measure is a property that holds iff the market is approximately complete, i.e. every contingent claim can be approached in a certain sense by a sequence of hedgeable claims. This result is deduced from duality considerations leading, moreover, to the conclusion that the hedging problem is ill-posed.

It is worth mentioning that the results of this paper open the door to a systematic use of models driven by Lévy processes that give better statistical fitting of real-world financial data but lead to theoretical difficulties related to absence of completeness. Moreover, the idea of measure-valued portfolios seems to be useful also in the context of stock markets augmented by an infinite number of derivative securities or bonds.

One can observe that a number of questions are only briefly touched here and we foresee further mathematical developments within the framework of the considered approach.

A Appendix

Stochastic Fubini theorems

We give formulations of the stochastic Fubini theorems which are used in the present paper. The proofs for integrals with respect to a martingale can be found in the textbook [31], the case of random measures is treated in the same way.

Let M be a continuous real martingale, $\mu = \mu(dt, dx)$ a $\mathscr{P} \otimes \mathscr{X}$-$\sigma$-finite integer-valued adapted random measure with compensator $\nu = \nu(dt, dx)$, and m a measure on $(\mathbf{T}, \mathscr{B}_\mathbf{T})$ with the finite total variation $|\mu|$. Let $H = H(\omega, t, \theta)$ and $\Psi = \Psi(\omega, t, x, \theta)$ be, correspondingly, $\mathscr{P} \otimes \mathscr{B}_\mathbf{T}$-measurable and $\mathscr{P} \otimes \mathscr{X} \otimes \mathscr{B}_\mathbf{T}$-measurable functions. We denote by H^θ and Ψ^θ their θ-sections, i.e. $H^\theta : (\omega, t) \mapsto H(\omega, t, \theta)$; as usual, $\mathscr{O}$ is the notation for the optional σ-algebra; mH (or $m(H)$ in ambiguous cases) stands for the integral with respect to $m = m(d\theta)$.

As in the ordinary Fubini theorem, there is a statement concerning measurability; since in the stochastic case the integral is defined up to a P-null set, the problem, in fact, is that of existence of suitably measurable versions.

Proposition A.1 *(a) Assume that for each θ the integral $(H^\theta)^2 \cdot \langle M \rangle_t$ is finite for finite t. Then there exists an $\mathscr{O} \otimes \mathscr{B}_\mathbf{T}$-measurable function $U(\omega, t, \theta)$ such that for each θ the process U^θ is a version of the stochastic integral $H^\theta \cdot M$.*

(b) Assume that for each θ the integral $(\Psi^\theta)^2 * \nu_t$ is finite for finite t. Then there exists an $\mathscr{O} \otimes \mathscr{B}_\mathbf{T}$-measurable function $V(\omega, t, \theta)$ such that for each θ the process V^θ is a version of the stochastic integral $\Psi^\theta * (\mu - \nu)$.

By virtue of these assertions the notations $H^\theta \cdot M$ and $\Psi^\theta * (\mu - \nu)$ always mean the suitable measurable versions of the integrals.

Proposition A.2 *(a) Suppose that for finite t*

$$(mH^2) \cdot \langle M \rangle_t := \int_0^t \left(\int_\mathbf{T} H^2(t, \theta) m(d\theta) \right) d\langle M \rangle_t < \infty. \tag{A.1}$$

Then the process $m(H \cdot M)$ is indistinguishable from $(mH) \cdot M$.

(b) Suppose that for finite t

$$(m\Psi^2) * \nu_t := \int_0^t \int_X \left(\int_{\mathbf{T}} \Psi^2(t,x,\theta) m(d\theta) \right) \nu(dt,dx) < \infty.$$

Then the process $m(\Psi * (\mu - \nu))$ *is indistinguishable from* $m\Psi * (\mu - \nu)$.

Comments. The measurability result has been proved in great generality in [36]. The interchangeability of the integrals under the assumptions above is almost a folklore (for this and other versions see [31] with the literature therein and also [37]; the book [12] contains an extension to Hilbert space-valued Wiener processes) although it is not easy to give a precise reference except [27] for the case of random measures. We do not consider ramifications of this result which are delicate and still of current interest. Actually, the stochastic Fubini theorem is rather unfortunate: even the usually reliable source [20] contains an erroneous formulation in Theorem 5.44 (see p. 161 in [31] for a counterexample and further remarks). The most general results are given in the recent deep study [27] where the problem is treated in the framework of vector integration theory (independently, the same approach to the stochastic Fubini theorem is used in the paper [4] submitted, however, much later).

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[1] In the language of probability theory: i.e. generated by all mappings $m \mapsto mf$ where f $\in C_{\mathbf{T}}$; in the language of functional analysis this is, of course, weak* topology.

References

1. Artzner, P., Delbaen, F.: Term structure of interest rates: the martingale approach Adv. Appl. Math. **10**, 95–129 (1989)
2. Babbs, S., Webber, N.: A theory of the term structure with an official short rate. Working paper, University of Warwick (1993)
3. Bensoussan, A.: On the theory of option pricing. Acta Appl. Math. **2**, 139–158 (1984)
4. Bichteler, K., Lin, S.J.: On the stochastic Fubini theorem. Stochastics Stochastics Rep. **54**, 271–279 (1995)
5. Björk, T.: On the term structure of discontinuous interest rates. Surv. Indust. Appl. Math. **2** (4), 626–657 (1995)
6. Björk, T., Kabanov, Y., Runggaldier, W.: Bond market structure in the presence of marked point processes. Math. Finance **7**(2) (1996)
7. Brace, A., Musiela, M.: A multifactor Gauss Markov implementation of Heath, Jarrow and Morton. Math. Finance **4**(3), 259–283 (1994)
8. Chatelain, M., Stricker, C.: On componentwise and vector stochastic integration. Math. Finance **4** (1), 57–65 (1994)
9. Cox, J., Ingersoll, J., Ross, S. The relation between forward and futures prices. J. Financ. Econ. **9**, 321–341 (1981)
10. Cox, J., Ingersoll, J., Ross, S.: A theory of the term structure of interest rates. Econometrica **53**, 385–408 (1985)
11. Dana, R.-A., Jeanblanc-Picqué, M.: Marchés Financiers en Temp Continue. Valorisation et Équilibre. Paris: Economica 1994
12. Da Prato, G., Zabczyk, J.: Stochastic Equations in Infinite Dimensions. New York: Cambridge University Press 1992

13. Dellacherie, C., Meyer, P.-A.: Probabilités et potentiel, volumes 1 and 2. Paris: Hermann 1976, 1982
14. Delbaen, F., Schachermayer, W.: A general version of the Fundamental Theorem of Asset Pricing. Math. Annal. **300**, 463–520 (1994)
15. Duffie, D.: Dynamic Asset Pricing Theory. Princeton: Princeton University Press 1992
16. El Karoui, N., Myneni, R., Viswanathan, R.: Arbitrage pricing and hedging of interest rate claims with state variables. I, II. Preprints, Laboratoire de Probabilités, Paris 6, 1992
17. Harrison, M., Pliska, S.: Martingales and stochastic integrals in the theory of continuous trading. Stochast. Processes Appl. **11**, 215–260 (1981)
18. Heath, D., Jarrow, R., Morton, A. J.: Bond pricing and the term structure of interest rates. A new methodology for contingent claim valuation. Econometrica **60** (1), 77–106 (1992)
19. Hull, J., White, A.: Pricing of interest rate derivatives. Rev. Financial Studies **3**, 573–592 (1990)
20. Jacod, J.: Calcul Stochastiques et Problèmes de Martingales. (Lect. Notes Math., vol. **714**) Berlin–Heidelberg–New York: Springer 1979
21. Jacod, J., Shiryaev, A.N.: Limit Theorems for Stochastic Processes. Berlin–Heidelberg-New York: Springer 1987
22. Jamshidian, F.: Bond and option evaluation in the Gaussian interest rate model. Res. Finance **9**, 131–170 (1991)
23. Jarrow, R., Madan, D.: Characterization of complete markets on a Brownian filtration. Math. Finance **1** (3), 31–43 (1991)
24. Jarrow, R., Madan, D.: Option pricing using term structure of interest rates to hedge systematic discontinuities in asset returns. Math. Finance **5** (3), 311–336 (1995)
25. Jarrow, R. Madan, D.: Valuing and hedging contingent claims on semimartingales. Preprint 1995
26. Kabanov, Yu., Kramkov, D.: Asymptotic arbitrage in large financial markets. Finance Stochast. **1** (Forthcoming, 1997)
27. Lebedev, V.A.: Fubini theorem for parameter-dependent stochastic integrals with respect to L^0-valued random measures. Probab. Theory Appl. **40** (2), 313–323 (1995)
28. Lindberg, H., Orszag, M., Perraudin, W.: Yield curves with jump short rates. Preprint 1995
29. Métivier, M.: Semimartingales. Berlin–New York: De Gruyter 1992
30. Musiela, M.: Stochastic PDEs and term structure models. Preprint 1993
31. Protter, Ph.: Stochastic Integration and Differential Equations. Berlin–Heidelberg–New York: Springer 1990
32. Sandmann, K., Sondermann, D.: A term structure model and the pricing of interest rate derivatives. Preprint 1992
33. Schwartz, L.: Semi-martingales banachiques: le théorème des trois opérateurs. Séminaire de Probabilités XXVIII (Lect. Notes Math., vol. **1583**) Berlin–Heidelberg–New York: Springer 1994
34. Shirakawa, H.: Security market model with Poisson and diffusion type return process processes. Technical Report, IHSS-18, Tokyo Institute of Technology, 1990
35. Shirakawa, H.: Interest rate option pricing with Poisson-Gaussian forward rate curve processes. Math. Finance **1** (4), 77–94 (1991)
36. Stricker, C., Yor, M.: Calcul stochastique dépendant d'un paramètre. Z. Wahrscheinlichkeitstheorie verw. Gebiete **45**, 109–133 (1978)
37. Szpirglas, J.: Sur l'equivalence d'equations différentielles stochastiques à valeurs mesure dans le filtrage Markovien non linéaire. Ann. Institut Henri Poincaré, Sect. B **14**, 33–59 (1978)
38. Vasiček, O.: Equilibrium characterization of the term structure. J. Financial Econ. **5** (1), 177–188 (1977)