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Scenario Simulation: Theory and methodology*

Farshid Jamshidian[1], **Yu Zhu**[2]

[1] New Products and Strategic Trading, Sakura Global Capital, 42 New Broad Street, London EC2M 1JX, United Kingdom
[2] Risk Assessment and Control, Sakura Global Capital, 65 East 55 Street, New York, NY 10022, USA

Abstract. This paper presents a new simulation methodology for quantitative risk analysis of large multi-currency portfolios. The model discretizes the multivariate distribution of market variables into a limited number of scenarios. This results in a high degree of computational efficiency when there are many sources of risk and numerical accuracy dictates a large Monte Carlo sample. Both market and credit risk are incorporated. The model has broad applications in financial risk management, including value at risk. Numerical examples are provided to illustrate some of its practical applications.

Key words: Risk analysis, Monte Carlo studies, approximations to distributions

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1. Introduction

In recent years, value-at-risk (VaR) has been considered as one of the best market risk measures by banks and other financial institutions. VaR is defined as the expected loss from an adverse market movement with a specified probability over a period of time. For each financial institution, a certain amount of capital must be set aside so that the probability that the institution will not

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survive adverse market conditions remains very small. In December 1995, the Bank for International Settlement (BIS) finalized its rules for allocating the capital to market risk, allowing banks to use their internal model to measure the market risk exposure, expressed in terms of value-at-risk.[1]

There are several commonly applied methods to calculate VaR. One simplest method is "delta approximation". It uses variance-covariance matrix of market variables and the portfolio's sensitivities with each of the market variables (delta) to approximate the potential loss of the portfolio value. This method critically depends on two dubious assumptions: the normality assumption of portfolio value, and the linearity assumption of the relationship between transactions' prices and market variables.

In general, however, financial transactions, derivatives in particular, have non-linear price characteristics. Thus, Monte Carlo simulation is a more appropriate method to estimate their market exposures. A common implementation is based on the joint lognormality assumption of market variables to generate a large number of market scenarios using their historical variance-covariance matrix. Then, transactions values for each scenario are calculated and aggregated. From the obtained distribution of the portfolio value, VaR can be easily estimated. The difficulty with the Monte Carlo approach is its computational burden. In order to obtain a reliable estimation, the sample size has to be large. In the case of large multi-currency portfolios, the required huge sample size often makes the approach impractical. Choosing a smaller sample size would result in a distorted distribution, defeating the purpose of adopting the Monte Carlo approach.

The Scenario Simulation model described in this paper is a new approach to estimate VaR. The model approximates a multidimensional lognormal distribution of interest rates and exchange rates by a multinomial distribution of key factors. While it allows very large samples, the number of portfolio evaluations is limited. As a result, a great computational efficiency has been obtained in comparison with conventional Monte Carlo methods. In addition to the efficiency, it is our view that the Scenario Simulation model fits the needs of risk management better than conventional Monte Carlo method. In the next two sections, we describe the basic assumptions and mathematical models of the scenario simulation model under the single currency yield curve setting. In Sect. 4, the model is extended into a multi-currency setting. We also describe how to incorporate other market risk variables such as volatility and basis risk.

The concept of value-at-risk is not restricted to the market risk. For the risk management of a financial institution, it is important to examine not only the market risk of the portfolio, but also its overall risk exposures. Section 5 discusses the model's application in estimating a portfolio's overall risk profile of joint market risk, counterparty default and country risk events. Section 6 applies the model to analyze the risk profile of a multi-currency swap portfolio. Section 7 concludes the paper.

2. Yield curve modelling

We make the following assumptions:
a1. A yield curve is described by a vector of key zero coupon rates:

(1) $$\{r_1, \ldots, r_i, \ldots, r_n\}$$

a2. Each key rate is lognormally distributed:

(2) $$\frac{dr_i}{r_i} = \boldsymbol{\mu}_i(t)\,dt + \boldsymbol{\sigma}_i\,dz_i$$

where $z_i(t) \sim \mathbf{N}(0, \sqrt{t}),\ i = 1, \ldots, n$. Equivalently, we have

(3) $$r_i(t) = u_i(t)\,e^{\boldsymbol{\sigma}_i z_i(t)}$$

Note in (3) $u_i(t)$ depends on the expectation hypothesis. For example, we can assume that

(4a) $$u_i(t) = F_i(t)$$

where $F_i(t)$ is the corresponding forward rate. Another possibility is to choose

(4b) $$u_i(t) = F_i(t)e^{-\frac{1}{2}\boldsymbol{\sigma}_i^2 t}$$

so that the expected value of $r_i(t)$ equals the forward rate.[2]

a3. $z_i(t), i = 1, \ldots, n$, *are correlated in the following way*:

(5) $$\langle dz_i, dz_j \rangle = \boldsymbol{\rho}_{ij}dt$$

$i, j = 1, \ldots, n$. Thus we shall have a $n \times n$ correlation matrix $R = \{\boldsymbol{\rho}_{ij}, i, j = 1, \ldots, n\}$.

Based on assumptions $(a1) \sim (a3)$, the distribution of yield curves at a future time t is determined. To simulate the distribution, one method is to apply the Monte Carlo method, which can be briefly described as follows. First, a set of random numbers $\{z_1, \ldots, z_n\}$ is selected according to their joint distribution. By applying Eq. (3), we can generate one future yield curve $\{r_1(t), \ldots, r_n(t)\}$. In order to simulate the distribution, a large number of future yield curve have to be generated.

Suppose that 12 key rates are used to describe the yield curve. In other words, the yield curve model has 12 factors. For each random variable z_i to have three possible values, the total number of future curves we need would

Table 1. Cumulative percentage of variations explained by principal components

	By correlation matrix					By covariance matrix				
No. of factors	1	2	3	4	5	1	2	3	4	5
DEM	67.7	83.9	93.7	97.1	98.5	71.5	84.8	92.9	96.7	98.4
JPY	75.8	91.2	94.3	96.6	98.1	69.7	90.7	95.6	97.7	99.2
USD	75.8	85.1	93.0	97.0	98.9	84.4	90.9	94.4	97.3	98.9

be $3^{12} = 531,441$, about half a million. If each random variable takes five outcomes, the total number would be increased to above 200 million! Of course, Monte Carlo simulation does not require all the permutations. A smaller number of trials would be sufficient in order to obtain reasonable value of moments and expectations such as mean and variance. Even then, the required number of trials will still be very large, especially in the case of multi-currency portfolio where the correlations among currencies and currency yield curves are usually low. Obviously, even with a large sample, the coverage of all "extreme" cases is not guaranteed in the Monte Carlo method.

Fortunately, to simulate the future yield curve movements, we do not have to use the "brute-force" Monte Carlo method. It has been documented that yield curve movements can be largely explained by two or three factors.[3] These studies show that the three factors can be interpreted as shift, twist, and butterfly (bend) of yield curves. The principal component analysis is one of the common methods to extract these key factors from yield curve movement data. Table 1 is an example of principal component analysis which shows the cumulative percentage of variations explained by the first five principal components of yield curves of Germany, Japan and United States.[4]

It can be seen that the first three principal components can explain about 93–95% of the variations. The principal component analysis can be preformed to either the covariance matrix or the correlation matrix. When the volatilities of all key interest rates are of the same order of magnitude, the results from the two methods are very similar. For simplicity, we shall use the second method in our discussion.

Suppose $R = \{\rho_{ij}\}_{i,j=1,\ldots,n}$ is the correlation matrix of the n key rates $\{r_1, \ldots, r_n\}$. Let the j-th eigenvector of R be defined as

$$\beta_j = (\beta_{ij}, \ldots, \beta_{nj})^T \tag{6}$$

From the definition of the eigenvector, we have

$$R\,\beta_j = \lambda_j \beta_j \quad j = 1, \ldots, n \tag{7}$$

where λ_j is the j-th eigenvalue. Since R is a symmetric non-negative definite matrix $\lambda_j \geqq 0$, $j = 1, \ldots, n$, and the eigenvectors are orthogonal to each other. We normalize all β_j such that

$$|\beta_j|^2 = \sum_{i=1}^{n} \beta_{ij}^2 = \lambda_j$$

Assume that

$$\lambda_1 \geqq \lambda_2 \geqq \cdots \geqq \lambda_n, \tag{8}$$

then β_1 is called the first principal component, β_2 is the second principal component, and β_j is the j-th principal component. As we discussed above, the empirical analysis of the historical yield curve data demonstrates that all but the first three principal components are small in magnitude. Now, define the principal factors dw_j by

$$dw_j = \frac{1}{\boldsymbol{\lambda}_j}\sum_{k=1}^{n} \boldsymbol{\beta}_{kj}\, dz_k \qquad j = 1, \ldots, n \tag{9a}$$

Then, it is easy to see that

$$dz_i = \sum_{j=1}^{n} \boldsymbol{\beta}_{ij}\, dw_j \qquad i = 1, \ldots, n \tag{9b}$$

and $\langle dw_k, dw_j \rangle = 0$ $(k \neq j)$, and $dw_k \cdot dw_k = \mathrm{dt}$.[5]

If we assume that

$$\boldsymbol{\lambda}_4, \ldots, \boldsymbol{\lambda}_n \ll \boldsymbol{\lambda}_1, \boldsymbol{\lambda}_2, \boldsymbol{\lambda}_3$$

then Eq. (9b) can be approximated by

$$dz_i \approx \boldsymbol{\beta}_{i1}\, dw_1 + \boldsymbol{\beta}_{i2}\, dw_2 + \boldsymbol{\beta}_{i3}\, dw_3$$

The residual should be very small. Based on this approximation, Eq. (2) can be rewritten as

$$\frac{dr_i(t)}{r_i(t)} = \boldsymbol{\mu}_i(t)dt + \boldsymbol{\delta}_{i1}dw_1 + \boldsymbol{\delta}_{i2}dw_2 + \boldsymbol{\delta}_{i3}dw_3 \tag{10}$$

where

$$\boldsymbol{\delta}_{ij} = \boldsymbol{\sigma}_i \boldsymbol{\beta}_{ij}; \quad i = 1, \ldots, n; \; j = 1, 2, 3.$$

Similarly, in the integrated form, Eq. (3) becomes

$$r_i(t) = u_i(t) e^{\boldsymbol{\delta}_{i1} w_1(t) + \boldsymbol{\delta}_{i2} w_2(t) + \boldsymbol{\delta}_{i3} w_3(t)} \tag{11}$$

The dimensionality of the problem is thus reduced from n to 3. Note also, w_i, $i = 1, 2, 3$, are now independent random variables.

3. Scenario simulation model

Equation (11) is a three-factor yield curve model. To simulate the future yield curve movements, we can apply the Monte Carlo method. Say we select 100 possible outcome for each random variable w_i, $i = 1, 2, 3$. The total number of possible yield curve at one future point in time will be 10^6. This generates an adequate distribution of the future yield curves, but it requires a large number of portfolio evaluations. The question is how to reduce the computational burden further, while achieving the objective of obtaining a satisfactory representation of the distribution of a portfolio's risk exposure.

Suppose W is a random variable with distribution $P(w)$. In the Monte Carlo method, hundreds of possible w are selected to simulate the random variable W, and each w has the same probability. The number of w's which fall in range between w_A and w_B is proportional to the probability $P(w_A < w \leqq w_B)$. For the risk simulation, we can select a state w_k which represents the region $(w_A, w_B]$, and assign the probability of the region to that state (scenario). Thus a limited number of states can be selected. With appropriately assigned probabilities, these states (scenarios) provide a good representation of the distribution.

It is well known that the multinomial distribution constitutes a good approximation to a multinormal distribution.[6] If $m+1$ states (ordered from 0 to m) are selected, the probability of state i of a binomial distribution can be expressed as

$$\text{Probability}(i) = 2^{-m} \cdot \frac{m!}{i\,!\,(m-i)!} \quad i = 0, \dots, m \tag{12}$$

The distance between two adjacent states is $2/\sqrt{m}$ standard deviations, and the furthest state is $\sqrt{m}$ standard deviation away from the center. For example, if we select five states ($m = 4$) to describe w, the corresponding probabilities of the five states are

$$\frac{1}{16}, \frac{1}{4}, \frac{3}{8}, \frac{1}{4}, \frac{1}{16}$$

In this case the furthest points is two standard deviations away from the center. The distance between two adjacent states is one standard deviation. Similarly, if we select seven states ($m = 6$), the corresponding probabilities are

$$\frac{1}{64}, \frac{3}{32}, \frac{15}{64}, \frac{5}{16}, \frac{15}{64}, \frac{3}{32}, \frac{1}{64}$$

The furthest state is 2.45 standard deviations away from the center. The distance between the two adjacent states is 0.82 standard deviations.

This method allows us to discretize the three factor yield curve model (11). We can select, say, seven states for the first factor, five states for the second factor, and three states for the third factor. The states ($w_1 = i, w_2 = j, w_3 = k$) constitutes a scenario. Thus the total number of scenarios in the above setting is $7 \times 5 \times 3 = 105$.[7] Since these factors are independent, their joint probability is simply the products of the three marginal probabilities as given by Eq. (12).

3.1 Comparison with traditional Monte Carlo simulation

The potential exposure of a security (or a portfolio) can be described by its price distribution at a horizon date. The Scenario Simulation model has been applied to calculate the potential exposures of derivative securities. The first three examples in Table 2a shows the simulation results for interest rate swaps in three different currencies: German Mark, Japanese Yen, and U.S. dollar. All three swaps are of 10 year maturity, and with one billion notional amount in their original currencies. The other two examples are both in dollars. One is a 4.75 year 3-month Libor floor with strike of 5%. The other is a 5-year receiver's swaption into a five-year swap (5×5 swaption). In all calculations we use the swap curves as of 6/30/1995. The historical volatilities and correlation matrices are from JPM RiskMetrics monthly data as of 5/30/1995. The results

Table 2a. Comparison of scenario simulation vs Monte Carlo simulation

	30-day Horizon value (in thousands)			
Method	Mean	STD	Maximum	Minimum
(1) *DEM 1 billion 10-year market rate swap*				
2-factor (5 × 3)	−8,411	23,295	49,524	−64,478
3-fac. (7 × 5 × 3)	−8,405	23,527	71,260	−84,430
4-fac. (9 × 7 × 5 × 3)	−8,406	23,555	88,735	−100,104
5-fac. (11 × 9 × 7 × 5 × 3)	−8,406	23,560	103,704	−113,233
Monte Carlo (20,000)	−8,408	23,426	90,042	−100,056
(2) *JPY 1 billion 10-year market rate swap*				
2-factor (5 × 3)	956	27,534	71,526	−59,797
3-fac. (7 × 5 × 3)	993	27,669	97,352	−76,906
4-fac. (9 × 7 × 5 × 3)	1,043	28,307	128,198	−95,210
5-fac. (11 × 9 × 7 × 5 × 3)	1,047	28,333	153,160	−108,487
Monte Carlo (20,000)	975	28,234	138,345	−100,311
(3) *USD 1 billion 10-year market rate swap*				
2-factor (5 × 3)	−25,378	27,233	37,294	−85,443
3-fac. (7 × 5 × 3)	−25,386	27,273	55,700	−101,799
4-fac. (9 × 7 × 5 × 3)	−25,389	27,281	71,038	−115,210
5-fac. (11 × 9 × 7 × 5 × 3)	−25,386	27,369	87,359	−129,098
Monte Carlo (20,000)	−25,378	27,106	108,494	−145,155
(4) *USD 20 million 4.75-year floor* (strike=5%)				
2-factor (5 × 3)	179.84	66.77	351.09	78.19
3-fac. (7 × 5 × 3)	180.79	69.06	455.18	60.28
4-fac. (9 × 7 × 5 × 3)	182.17	70.33	566.39	48.68
5-fac. (11 × 9 × 7 × 5 × 3)	187.49	71.59	672.67	36.85
Monte Carlo (20,000)	183.39	69.58	677.44	26.08
(5) *USD 100 million 5 × 5 receiver swaption* (strike=5%)				
2-factor (5 × 3)	2,349	859	4,397	893
3-fac. (7 × 5 × 3)	2,349	861	5,060	662
4-fac. (9 × 7 × 5 × 3)	2,350	865	5,773	473
5-fac. (11 × 9 × 7 × 5 × 3)	2,354	877	6,627	309
Monte Carlo (20,000)	2,366	872	7,896	164

Note: Yield curves are as of 6/30/1995, JPM RiskMetrics as of 5/30/1995.

Table 2b. 30-day value-at-risk of three swaps

	Confidence level	
	97.5%	99.0%
(1) *DEM 70mm 10-year swap* (6.9% fixed vs libor)		
3-fac. (7 × 5 × 3)	4,087,572	4,807,697
Monte Carlo (20,000)	4,077,033	4,793,734
(2) *JPY 5 billion 10-year swap* (3.10% fixed vs libor)		
3-fac. (7 × 5 × 3)	336,153,389	382,393,775
Monte Carlo (20,000)	336,625,614	388,177,770
(3) *USD 100mm 10-year swap* (6.50% fixed vs libor)		
3-fac. (7 × 5 × 3)	6,763,594	9,020,414
Monte Carlo (20,000)	7,153,692	8,682,214

Note: Yield curves are as of 10/06/1995, JPM RiskMetrics as of 9/15/1995.

from the Scenario Simulation model are compared with Monte Carlo simulation with 20,000 trials using the same input data.

These examples demonstrate that the Scenario Simulation model with a modest number of scenarios, generates exposure estimation very similar to that of Monte Carlo simulation. If we use three factors and 105 scenarios, the differences in mean and standard deviations of the 30-day horizon values are less than 1% for DEM and USD swaps. In all other examples, the differences are less than 2%.[8] The computational efficiency, on the other hand, was increased dramatically.

Table 2a illustrates another difference between Monte Carlo simulation and Scenario Simulation. As we discussed above, in Monte Carlo, each trial is selected randomly (according to a given distribution), thus one has little control over the maximum and minimum values of the simulation outcomes. For example, when simulating the 30-day horizon value of the JPY swap, the maximum value $(138{,}345)$ was reached within the first 13,000 trials, while the minimum $(-100{,}311)$ was reached within the first 3,000 trials. In contrast, we can have a much better control over these extreme values in the Scenario Simulation. The range between the maximum and minimum increases with the number of factors and number of scenarios in each factor. This is a desirable feature which allows risk managers to conduct "what if" analyses under a more controllable environment.

After obtaining the distribution of a transaction's horizon value, the value-at-risk for a given confidence level can be easily calculated. Table 2b shows that in calculating VaR, the results from the Scenario Simulation method and the Monte Carlo method are very similar. To calculate VaR of a portfolio containing all three swaps, an extension of the model into a multi-currency setting is required, which will be discussed later in the paper.

3.2 Mean reversion of interest rates

In the above discussion, yield curves are assumed to follow lognormal distribution as expressed in Eq. (2). It produces reasonable results when simulating derivative transaction's price distribution over a short period of time. The assumption's deficiency becomes apparent, however, when the model is extended to estimate potential exposures over a long time horizon. Based on the assumption of geometric Brownian motion, the interest rate variance is increasing over time by the factor t, where t is the time to a horizon date. For example, the Japanese three-year interest rate monthly volatility in June-July of 1995 was extremely high, at around 18% (annualized volatility of 62%). The geometric Brownian motion model implies that, at 10-year horizon, the standard deviation of 3-year interest rate would become $18\% \times \sqrt{(10 \times 12)} = 197.2\%$. If the 3-year forward rate is 3.86%, one standard deviation away from the forward rate would be $3.86 \times \exp(1.972) = 28\%$, two standard deviation away would be $3.86 \times \exp(2 \times 1.972) = 199\%$, and three standard deviation away would be as high as 1,431%.

A more reasonable model of describing the evolution of yield curve over a long period of time should incorporate mean reversion of interest rates. In-

Table 3. The effects of mean reversion factor k
Assumptions: $\boldsymbol{\sigma} = 18\%$ (monthly); 5-year forward 3-year rate= 4.17%; 10-year forward 3-year rate = 3.86%

	$k = 0.00$	0.10	0.20	0.30	0.40	0.50
Three-year JPY interest rate at 5-year horizon (%)						
STD	139.43	110.85	91.68	78.47	69.07	62.14
@1 STD	16.81	12.64	10.43	9.13	8.34	7.76
@2 STD	67.80	38.28	26.10	20.02	16.60	14.43
@3 STD	273.34	116.01	65.26	43.91	33.10	26.90
Three-year JPY interest rate at 10-year horizon (%)						
STD	197.18	129.65	97.68	80.40	69.70	62.35
@1 STD	27.73	14.11	10.25	8.62	7.75	7.20
@2 STD	199.19	51.61	27.23	19.27	15.56	13.43
@3 STD	1430.92	188.70	72.32	43.06	31.24	25.06

Notes:
1. STD: Standard deviation of log of 3-year interest rate at a horizon date.
2. @i STD: The simulated 3-year interest rate i ($i = 1, 2, 3$) standard deviations away from the forward rate.

stead of assuming that each key rate follows a geometric Brownian motion, we assume that

$$\begin{aligned} \frac{dr_i}{r_i} &= \boldsymbol{\mu}_i(t)dt + \boldsymbol{\sigma}_i dz_i \\ dz_i &= -kz_i(t)dt + dB_i \end{aligned} \tag{13}$$

where $k \geqq 0$, and $B_i(t)$ is a Brownian motion. $z_i(t)$ is called Orestein–Uhlenbeck process.[9] If $k = 0$, $dz_i = dB_i$, (13) reduces to the same equation as (2). If $k > 0$, then the distribution of $z_i(t)$ is no longer $N(0, \sqrt{t})$. Rather, its distribution is

$$N\left(0, \sqrt{\frac{1 - e^{-2kt}}{2k}}\right) \tag{14}$$

It can be seen that at a given time t, z_i is normally distributed, equivalently, $r_i(t)$ is lognormally distributed. Therefore, all previous analyses of the Scenario Simulation model hold without modification. In particular, the principal factors $w_i(t)$ defined in Eq. (9a) are also Orestein–Uhlenbeck processes with the same mean reversion parameter k. Over time, the mean reversion factor k reduces the variance of $w_i(t)$. But, since $w_i(t)$ is still normal, it can be discretized by the same method as before.

Table 3 uses the aforementioned example of Japanese interest rates to demonstrate the effect of the mean reversion factor k. It seems that a mean reversion factor of $0.30 \sim 0.50$ is warranted for this very volatile case. If k is 0.50, the standard deviation of 3-year interest rate would be reduced by about half for a 5-year horizon, and by 2/3 for a 10-year horizon. This results in much more reasonable simulated interest rates, even though they are still quite high.

An interesting issue is how to select the mean reversion factor k for each currency yield curve. Empirical estimation based on historical data is one possible method. Another more practical approach is to select k such that the mean reversion will be “kicked in” only if volatilities are much higher than the historical norms for that currency yield curve.

4. Joint probability model of Scenario Simulation

For a single currency, the principal factors of the yield curve are independent, and their joint probability density is simply the product of the individual densities. This furnishes the mean, standard deviation, histogram, and percentiles of future portfolio value for each currency. However, in order to aggregate the portfolios and obtain similar statistics for the U.S. dollar value of the total portfolio, we need a discrete joint probability model for the totality of the principal factors of all countries as well as all the exchange rates.

One approach would be to perform the principal component factorization of the international economy, i.e., of the combined correlation matrix of all key rates and exchange rates, and choose a sufficiently high number of factors (say 10) that explain most of the total variation. Then, the principal factors are independent and the joint probability is again a simple product. However, these principal factors are difficult to interpret intuitively. Moreover, this approach breaks the symmetry and stratification of the single currency models, which carry important information for risk management, and dramatically increases computational time.

We seek an approach that is consistent with the single currency model in the sense that the marginal distribution for each currency’s yield curve has the same scenarios (with the same probabilities) as those constructed for that currency separately. To this end, we define a discretization of the multivariate normal distribution such that each coordinate is discretized into a binomial distribution exactly as in the one-dimensional case. To our knowledge, this construction has not appeared in the literature before. We will show that the multivariate discretization converges with probability one to the original multivariate normal variate. Further, we will describe a procedure for Monte Carlo simulation of the discrete distribution.

To proceed, for each integer $m > 0$, we define an increasing sequence numbers of a_i by $a_0 = -\infty$, $a_{m+1} = \infty$, and the relations,

$$\frac{1}{\sqrt{2\pi}} \int_{a_i}^{a_{i+1}} exp\left(-\frac{x^2}{2}\right) dx = \frac{2^{-m} m!}{i!(m-i)!}, \quad i = 0, \ldots, m. \tag{15}$$

Thus, a_i are the points on the real line such that the area under the normal curve on the segment $[a_i, a_{i+1}]$ equals the binomial probability of state i. For example, for $3, 5$, and 7 states, one computes:

$m = 2$: $a_1 = -0.67449$, $a_2 = 0.67449$.

$m = 4$: $a_1 = -1.53412$, $a_2 = -0.48878$, $a_3 = 0.48878$, $a_4 = 1.53412$.

$m = 6$: $a_1 = -2.15387$, $a_2 = -1.22986$, $a_3 = -0.40225$, $a_4 = 0.40225$, $a_5 = 1.22986$, $a_6 = 2.15387$.

Now define the function $B^{(m)}$ on the real line with values in the set $\{0,\ldots,m\}$ as follows:

$$B^{(m)}(z) = i \text{ if } a_i \leqq z < a_{i+1}.$$

By construction the function $B^{(m)}$ is so designed that if Z is a normally distributed random variable with mean zero and variance 1, then $B^{(m)}(Z)$ is binomially distributed with $m+1$ state.

If X is a k-dimensional normal variate with correlation matrix Q,

$$X = (X_1,\ldots,X_k) \sim N(0,Q),$$

each X_i having mean zero and variance 1, we define its discretization to be the multivariate discrete variate

$$B = B^{(m)} = (\mathbf{B}^{(m)}(X_1),\ldots,\mathbf{B}^{(m)}(X_k)). \tag{16}$$

Note, each coordinate of B is binomially distributed and equals the discretization of the corresponding component of X. It is in this sense that we say this discretization preserves the stratification.

From this definition, we see that it is easy to generate random samples of the multidimensional discrete variate B. We simply generate a multivariate $N(0,Q)$ normal deviate $x = (x_1,\ldots,x_k)$, then simply find between which a_i and a_{i+1} each x_j lies, and thus form $(B^{(m)}(x_1),\ldots,B^{(m)}(x_k))$.

The joint probability density of B is clearly given by

$$prob[B = (i_1,\ldots,i_k)] = \int_{a_{i_1}}^{a_{i_1+1}} \cdots \int_{a_{i_k}}^{a_{i_k+1}} p(x_1,\ldots,x_k)dx_1\cdots dx_k, \tag{17}$$

where

$$p(x) = p(x_1,\ldots,x_k) = (2\pi)^{-k/2}\det(Q)^{-1/2}\exp\left(-\frac{1}{2}x^tQ^{-1}x\right)$$

is the multivariate normal density function of X.

We now show that the standardized $B^{(m)}$ converges to X with probability one. In particular, it follows that the correlation matrix of $B^{(m)}$ approaches Q for large m. But even for small values m, it is close enough to Q for practical purposes.

Define $\boldsymbol{\beta}^{(m)}(z) = (2B^{(m)}(z) - m)/\sqrt{m}$. Since the mean of the binomial distribution is $m/2$ and its variance is $m/4$, the standardized $B^{(m)}$ equals $(\boldsymbol{\beta}^{(m)}(X_1),\ldots,\boldsymbol{\beta}^{(m)}(X_k))$. We claim that for each number z, $\boldsymbol{\beta}^{(m)}(z)$ approaches z. Then it follows at once that each $\boldsymbol{\beta}^{(m)}(\mathrm{X_i})$ approaches X_i with probability one, and with this, that $(\boldsymbol{\beta}^{(\mathrm{m})}(X_1),\ldots,\boldsymbol{\beta}^{(m)}(X_k))$ approaches $(X_1,\ldots,X_k)$ with probability one.

To show $\boldsymbol{\beta}^{(m)}(z)$ approaches z, let $\boldsymbol{\Phi}$ and F denote respectively the normal and binomial cumulative distribution functions, and set $i = B^{(m)}(z)$. Then $\boldsymbol{\beta}^{(m)}(z) = (2i - m)/\sqrt{m}$, and it follows from the deMoivre–Laplace theorem that $F(i) \sim \boldsymbol{\Phi}(\boldsymbol{\beta}^{(m)}(z))$.[10] But by Eq. (15), $\boldsymbol{\Phi}(a_i) = F(i)$. Hence, $\boldsymbol{\Phi}(a_i) \sim \boldsymbol{\Phi}(\boldsymbol{\beta}^{(m)}(z))$. But, z lies between a_i and a_{i+1}, and $a_{i+1} - a_i \to 0$, implying $a_i \to z$, and so $\boldsymbol{\Phi}(a_i) \to \boldsymbol{\Phi}(z)$. Thus $\boldsymbol{\Phi}(\boldsymbol{\beta}^{(m)}(z)) \to \boldsymbol{\Phi}(z)$, which implies $\boldsymbol{\beta}^{(m)}(z) \to z$, as claimed.

We mention in passing that the above technique can also be used to define a multivariate version of any *continuous* distribution F. Indeed, if Z is normal $N(0,1)$, then $F^{-1}(\mathbf{\Phi}(Z))$ has distribution function equal F. The multivariate version can be defined as $(F^{-1}(\mathbf{\Phi}(X_1)),\ldots,F^{-1}(\mathbf{\Phi}(X_k)))$, where X is $N(0,Q)$ as before.

4.1 The distribution of a multi-currency portfolio

Consider a portfolio containing interest-rate instruments in several currencies. We wish to determine the distribution of the portfolio value at a given horizon date. We take the US dollar (USD) as the base currency. Transactions denominated in other currencies are to be converted into USD and aggregated.

We assume that key interest rates of all countries are jointly lognormally distributed, and perform a principal component decomposition of each and discretize them as discussed before. In addition, we assume a joint lognormal distribution with all exchange rates. Each exchange rate X = X(t) is lognormally distributed as

(18) $$X(t) = f(t)e^{\boldsymbol{\sigma}_x w_x(t)},$$

where $f(t)$ is the forward exchange rate for the horizon t. As before, we can incorporate mean reversion if appropriate, or choose the mean, rather than median, of $X(t)$ to be $f(t)$.

We discretize $w_x(t)$, and with it $X(t)$, by a binomial variate as before. If, for example, seven states are used to describe X, combining with the assumed 105 yield curve scenarios, we have $105 \times 7 = 735$ USD equivalent scenarios for that currency. If the exchange rate X and interest rates are uncorrelated, the 735 joint probabilities are simply given by

$$prob\,(X = x,\ YCV = y) = prob\,(X = x) \cdot prob\,(YCV = y).$$

(Above, YCV stands for yield curve.) The two items on the right hand side are computed by the binomial probabilities of Eq. (12). In general, in case of non-zero correlations, we can use the joint probability formula given in equation (17).

Assume there are a total of k currencies, including USD. Then, there will be a total of $105 \times 735^{k-1}$ scenarios. Equation (17) again provides the probability of each state. The calculation of the portfolio mean and variance, however, does not require all these states. Let

$$V_i = \text{the i-th currency portfolio value;}$$
$$X_i = \text{the exchange rate of the i-th currency.}$$

Thus, the total value of the multi-currency portfolio is equal to

(19) $$V = X_1V_1 + X_2V_2 + \cdots + X_kV_k$$

The computation of expected value only requires joint probabilities of X_i and V_i. To calculate the variance, by definition, we have

$$Var\,[V] = E[V^2] - [E\,V]^2.$$

Since

$$V^2 = \sum_{i=1}^{k}\sum_{j=1}^{k} X_i\ X_j\ V_i\ V_j,$$

$$E[V^2] = \sum_{i=1}^{k}\sum_{j=1}^{k} E[X_i\ X_j\ V_i\ V_j].$$

The above equation indicates that joint probabilities of $\{X_i, X_j, V_i, V_j\}$ are required to calculate $E[V^2]$. Generally, this is a non-trivial computation. However, if it is assumed that exchange rates and interest are uncorrelated and, yields of different countries are correlated only through, say, their first principal factors, then the above expression factors into expectations of several bivariate distributions, which can be easily computed using Eq. (17).

In some applications, it is plausible to assume that, for a diversified portfolio, V is normally distributed, so that the risk exposure can be determined just by the expected value $E[V]$ and the variance $\mathrm{Var}\,[V]$. In general, however, the normality assumption of portfolio value V is not plausible. For example, the portfolio may contain a disproportionate number of options, or, skewness may result from truncations brought by considerations of counterparty default. In this case, we need the entire distribution which requires all possible states. If the number of currencies k equals 2 or 3, this may well be computationally manageable. But, for $k = 6$ for instance, the number of scenarios $(105 \times 735^{k-1})$ is more than 10^{16}, i.e., 10,000 trillion. We therefore use Eq. (16) to Monte Carlo simulate the discrete distribution.

In this regard, it cannot be over-emphasized that our approach to simulation fundamentally differs from the usual Monte Carlo approach applied in risk management. The usual approach does not employ stratified discretization as in our theory. If N is the number of Monte Carlo trials, it generates N yield curves and exchange rates for each currency. This means that each transaction must be evaluated in N states. For the computation to be feasible with a large portfolio, then of necessity N must be rather small. On the other hand, the high dimensionality of the problem dictates a large N for an adequately accurate sample.

By contrast, in our approach each transaction (like an interest-rate swap or a swaption) is only calculated in the 105 scenarios of the corresponding currency yield curve. Transactions in each currency i are then simply aggregated for each of the 105 scenarios, yielding V_i. At this stage, N joint scenarios are randomly generated using Eq. (16). Then, all that needs be done is the addition in Eq. (19) for each of the N samples.

In short, our approach requires only 105 evaluation of each transaction regardless of the number trials N. For this reason, even for very large portfolios,

Table 4. Correlation matrices of yield curve principal factors (Based on 9/15/1995 JPM RiskMetrics monthly data)

	CHF	DEM	GBP	JPY	USD
First principal factor					
CHF	1.000	0.809	0.666	0.101	0.322
DEM	0.809	1.000	0.721	0.133	0.379
GBP	0.666	0.721	1.000	0.020	0.412
JPY	0.101	0.133	0.020	1.000	0.017
USD	0.322	0.379	0.412	0.017	1.000
Second principal factor					
CHF	1.000	0.286	0.121	0.074	0.067
DEM	0.286	1.000	0.273	0.228	0.190
GBP	0.121	0.273	1.000	−0.062	0.050
JPY	0.074	0.228	−0.062	1.000	0.023
USD	0.067	0.190	0.050	0.023	1.000
Third principal factor					
CHF	1.000	0.130	0.081	0.048	0.220
DEM	0.130	1.000	0.136	0.033	0.003
GBP	0.081	0.136	1.000	0.123	0.228
JPY	0.048	0.033	0.123	1.000	−0.144
USD	0.220	0.003	0.228	−0.144	1.000

we can comfortably take N to be as large as, say, a million, and thereby achieve stable and accurate results.

We should also mention that the scenario simulation methodology does take account of rare events and tails of the distribution. For example, although the single currency yield curve has only 105 scenarios, some of them occur with a probability of 0.024%, well below the 1% threshold commonly considered as the probability of a rare event. These low probabilities compound for joint scenarios.

4.2 Further details of model implementation

Although by no means required, certain reasonable assumptions can be made to further exploit the symmetries of the model and increase its computational efficiency. Namely, we assume:

a4. Exchange rates and interest rates are uncorrelated. In fact, the historical correlations of interest rates and exchange rates are generally low, and empirical evidence suggests that their signs are often unstable and depend on the time period and the length of the sample period.

a5. The first principal factor of one country's yield curve can be correlated with the first principal factor of another country, and the second principal factors can be correlated, and so on, but the first principal factors are not

Table 5. Correlation matrix and volatilities of exchange rates (Based on 9/15/1995 JPM RiskMetrics monthly data)

	CHF	DEM	GBP	JPY	Volatility%
CHF	1.000	0.978	0.768	0.771	6.995
DEM	0.978	1.000	0.783	0.777	6.215
GBP	0.768	0.783	1.000	0.552	3.884
JPY	0.771	0.777	0.552	1.000	7.830

correlated with the second or third principal factors, *etc.* We argue that it seems unlikely that, say, parallel shift and yield curve twist within the same country be independent, yet across different countries be dependent. The data for the five major currency yield curves in Table 4 show that the correlation between interest rate levels (first principal factor) accounts for most of correlation between yield curve of different countries.[11] Coupled with the fact that the first principal components have the largest magnitude and contribute greatest to the variance, it seems that assumption (a5) is very innocuous. Table 5 shows the correlation matrix and volatilities of the four exchange rates.

By enriching the structure of the model, these assumptions actually increase the complexity of programming. But the result is well worth it in terms of gained efficiency. The primary benefit is that the problem basically reduces from one of dimensionality $4k$, as it were, to 4 problems, each with dimensionality k. Thus, we deal with the $k \times k$ correlation matrix of the first factors, and those of the second and third factors, and the $(k-1) \times (k-1)$ correlation matrix of exchange rates. This reduces by a factor of 4 certain calculations needed to form a 4k-dimensional random vector. It also enables use of antithetical symmetry for each of the four groups, effectively reducing the total number of random vector generations by a factor of 16.

The Scenario Simulation model can also be extended to cover other market risks, such as volatility or basis spread. Consider for example the risk associated to caps, floors and swaptions as a result of a change in the overall level of implied volatilities employed for the valuation of these options.[12] Let us denote dv_i as the change in the level of implied volatility for currency i, $i = 1, \ldots, k$. For currency i, we can introduce dv_i as the fourth independent factor, assuming for example it takes three values, $0, +1\%$ and -1% with probabilities 1/2, 1/4 and 1/4 respectively. Then, there will be a total of $105*3$ =405 joint yield-curve/implied-volatility scenarios for each currency, with given probabilities. Thus, securities that depend on volatility have to be evaluated 405 times, while 105 evaluations still suffices for swaps or cross-currency swaps. The joint international distribution of yield-curve/implied-volatility is discretized and Monte-Carlo sampled as before, one million times. Moreover, we can incorporate given correlations between implied volatility factors dv_i of different currencies i, in the same manner as we have allowed for correlation between exchange rates, or between the first principal factors, etc. In short, implied volatility simply enters as another factor for each currency.

For the Monte Carlo sampling of the scenarios, we use L'Ecuyer algorithm for uniform (0,1) deviates, supplemented by "shuffling". By combining two linear congruential generators, it results in a very long period, as desired for this problem, and induces practically no serial correlation. From two or more uniform deviates, we generate a single Gaussian deviate, using the Box-Muller method combined with acceptance/rejection within the unit disk. To generate a k-dimensional $N(0, Q)$ deviate, we first generate k i.i.d Gaussian deviates, put them in a k-vector Y, and pre-multiply Y by some matrix A satisfying $AA^t = Q$.[13] For the choice of A, we can take the principal component factorization of Q, the symmetric square root of Q, or the Cholesky factor of Q. We choose the latter, because A is then a triangular matrix and multiplying it by a vector can be made more efficient. Finally, we invoke the discretization function $B^{(m)}$, i.e., form $B^{(m)}(AY)$. The entire sequence is generated by a single seed.[14]

It is instructive to point out what happens if, instead of first multiplying Y by the Cholesky factor A and then descretizing, we first discretize and then multiply by A. In the latter case, the resulting deviate $A(B^{(m)}(Y))$ will have correlation matrix exactly Q. However, the stratification is destroyed: whereas each component of $B^{(m)}(AY)$ has only $m + 1$ states, each component of $A(B^{(m)}(Y))$ has in general m^k states. So, we revert to the case of the usual Monte Carlo where an extortionate number of portfolio valuations is required. Another technique known as rank correlation also preserves the stratification. But, we regard our approach to be more appropriate and theoretically complete.

4.3 Example of simulation on a five currency portfolio

As an example, we construct a portfolio with five interest rate swaps in five different currencies. For simplicity, the tensor is 10 years, and the pay and receive frequencies are semi-annual for all swaps in the portfolio. The total portfolio value as of the valuation date (9/15/1995) is $-\$315,637$. The details of these swaps are listed below:

Currency	Notional		Receive	Pay	Market value	
CHF	sfr	90 mm	4.8%	Libor	sfr	−502,430
DEM	DM	70 mm	Libor	6.90%	DM	−719,400
GBP	£	50 mm	Libor	8.42%	£	386,582
JPY	¥	5 bil	Libor	3.10%	¥	−23,863,598
USD	$	100 mm	6.5%	Libor	$	256,980

We ran the Scenario Simulation model for this five currency portfolio to estimate the portfolio value distribution for a 30-day horizon. The historical volatility and correlation information of the yield curve and exchange rates are based on 9/15/1995 JPM RiskMetrics monthly data.

The simulation takes two steps. First, we simulate each transaction in its own currency within the framework of single currency Scenario Simulation

Table 6. 30-day portfolio horizon value
Portfolio statistics and percentiles ($1,000)

No. of trials	Mean	STD	1%	2%	98%	99%	Maximum	Minimum
1,000,000	−319	5,013	−11,918	−10,630	9,809	11,055	20,784	−21,366
2,000,000	−319	5,028	−11,971	−10,665	9,838	11,096	22,148	−23,156
3,000,000	−319	5,027	−11,974	−10,665	9,836	11,096	22,185	−23,156
4,000,000	−319	5,028	−11,972	−10,668	9,838	11,092	22,185	−23,156

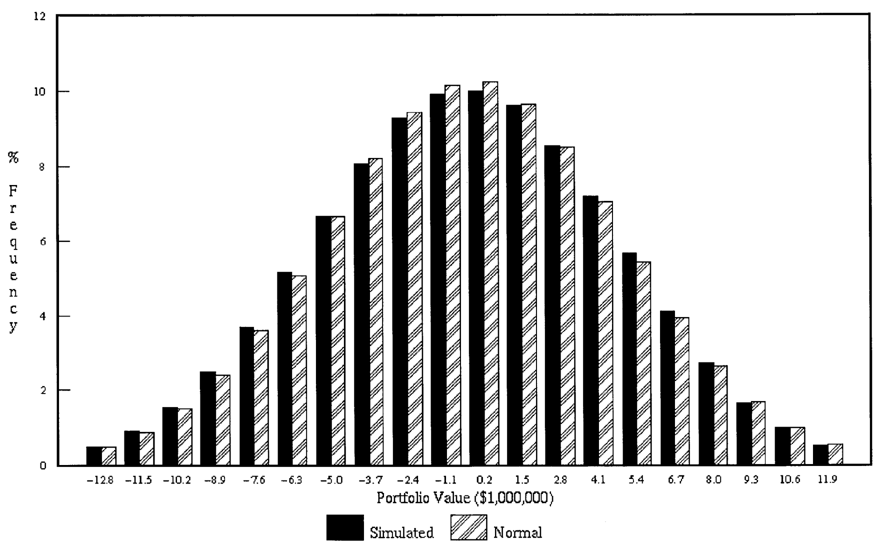

Fig. 1. Five currency portfolio horizon value (Histogram of 1,000,000 trials)

model described in Sect. 3. As before, we apply three principal factors and $7 \times 5 \times 3 = 105$ scenarios for each currency yield curve. For this portfolio, the single currency simulation generates 105 prices at the horizon date for each swap. That is, the swap valuation model would be called $105 \times 5 = 525$ times. In the second step, we perform Monte Carlo to simulate the distribution of the portfolio's horizon value. The simulation results are shown in Table 6. Clearly, the sample statistics become quite stable with one million trials. Comparing the histogram of the simulated potential horizon values to the normal distribution with the same mean and standard deviation, it can be seen that the sample distribution is well behaved and has a fatter tail (Fig. 1).

Even though the number of trials in our example is significantly larger than that in the usual Monte Carlo simulation applied in risk management, our approach does not take too much computation time due to the efficiency of the methodology as well as its implementations. In fact, running on a Sun SPARC Station 20, the second step takes 54, 96, 140 and 180 seconds to complete one, two, three and four million trials, respectively.

5. Estimating credit exposures with Scenario Simulation model

The credit exposure reflects the potential loss one transacting party may incur if its counterparty defaults. The credit risk is related to but different from the market risk. When a counterparty defaults, all transactions with that counterparty will be terminated, and the transactions' value by the end of the termination period is called the termination value (or replacement value).

For a given market scenario s, let $V(s)$ be the net potential termination value of a counterparty's portfolio at the end of one period. If $p(s)$ is the probability of the market scenario, then the distribution of the termination value can be described by

$$(V(s), p(s),\ s = 1, \ldots, n)$$

where n is the total number of market scenarios. If the counterparty defaults, the transacting party will suffer a loss only if the transactions with the counterparty have net gains. Otherwise, the credit loss would be zero. Thus, the distribution of the potential credit loss amount with respect to the counterparty is

$$(max[V(s), 0], p(s),\ s = 1, \ldots, n)$$

In order to apply the Scenario Simulation model to estimate the credit exposure, additional assumptions are required. We assume:

a6. The default incident of a counterparty is independent of the market variables by which a market scenario is defined, and it is also independent of other counterparties. This is a strong assumption as adverse market conditions and financial situations of related counterparties inevitably affect counterparties' economic well being. However, it is a convenient assumption because not only it simplifies the analysis considerably, but there is hardly any data available to deduct their correlations.

a7. The default probability of a counterparty is solely determined by its credit ratings.

a8. The loss ratio $(0 < \mathrm{L} \leqq 1)$ *of a defaulting party is assumed to be one.* Though this assumption is for simplicity and by no means necessary, it results in a more conservative estimation of credit risk as empirical evidence shows that, depending on the seniority of a transaction, the actual loss amount can be a fraction of total default amount.[15]

Assume q is a counterparty's marginal default probability for one period. Several one period credit risk measures can be readily calculated using the Scenario Simulation model. For example, the expected credit loss for the period can be defined as

$$E[D] = \sum_{s=1}^{n} \max[V(s), 0] \cdot p(s) \cdot q \tag{20}$$

One can also define the maximum average credit loss with probability α as follows:

$$D_\alpha = \min[V \geqq 0 : prob.(V(s) > v) \leqq \alpha] \cdot q \tag{21}$$

D_α could be interpreted as the average credit loss among a group of counterparties with the same ratings under certain unfavorable market conditions. Even though the economic meaning is not as clear, D_α can be used for some practical applications such as credit risk control and allocation. These one period average credit exposure measures can be easily extended to measure credit exposures in a multiperiod setting.[16]

In many risk management applications, however, the above average credit exposure measurement is inadequate. Instead, we need to model default incidents directly in the Scenario Simulation model.

Let d_j be the default random variable for counterparty j:

$$\begin{aligned} d_j &= 1 \quad \text{counterparty } j \text{ defaults with probability } q_j \\ &= 0 \quad \text{no default, probability } 1 - q_j, j = 1, \ldots, J. \end{aligned}$$

Recall we use k binomial random variables to describe one market scenario (see Eq. (17)). At each trial, we randomly generate a market scenario $s = (i_1, i_2, \ldots, i_k)$, and J independent Bernoulli random variables d_j $(j = 1, \ldots, J)$ according to their respective default probabilities. Thus the joint market and default scenario becomes

$$s^* = (i_1, i_2, \ldots, i_k; d_1, d_2, \ldots, d_J)$$

For counterparty j, if netting is enforceable, the scenario value with respect to the counterparty can be obtained by aggregating all transactions with the counterparty. Assume there are $g = 1, \ldots, G$ transactions with counterparty j and let $V_j(s)$ be the scenario value without default. Then,

$$V_j(s) = \sum_{g=1}^{G} V_{j,g}(s)$$

where $V_{j,g}$ is the market value of transaction g with counterparty j. The joint scenario value becomes

$$V_j(s^*) = \min[V_j(s), V_j(s)(1 - d_j)] \tag{22}$$

Equation (22) means that if $d_j = 0$, i.e., no default, then the joint scenario value is equal to the market value at scenario $s : V_j(s^*) = V_j(s)$. If the counterparty j defaults, $d_j = 1$, then $V_j(s^*) = \min[V_j(s), 0]$.

If netting is not enforceable, then

$$V_j(s^*) = \sum_{g=1}^{G} \min[V_{j,g}(s), V_{j,g}(s)(1 - d_j)] \tag{23}$$

It is easy to see that if the counterparty does not default, the joint scenario value $V_j(s^*)$ is simply the summation of all transactions' market values. If the counterparty defaults, none of the transaction gains with respect to the counterparty can be used to offset the losses with the same counterparty, i.e., $V_j(s^*) = \sum_g \min[V_{j,g}(s), 0]$.

The portfolio value V_p at the joint scenario s^* is

$$V_p(s^*) = \sum_{j=1}^{J} V_j(s^*) \tag{24}$$

Using the Monte Carlo method with a large number of joint scenarios, we can simulate the distribution of the termination values of each counterparty as well as the whole portfolio, taking into account both market risk and credit risk.

We can also incorporate the country risk event (e.g., a country defaults or it imposes foreign exchange control) into the Scenario Simulation model. When such a risk event happens in a country, we assume that all counterparties in that country default simultaneously and the transacting party's exposure is the netted amount of these counterparties.[17]

Let $c_1, \ldots, c_H$, be H independent Bernoulli random variables, representing country risk events for H countries we are dealing with. Assume that c_h equals one (zero) when the country does (not) default. The joint scenario of market variables, counterparty default event and country risk event now becomes

$$s^{**} = (i_1, i_2, \ldots, i_k; d_1, d_2, \ldots, d_J; c_1, \ldots, c_H)$$

The portfolio value with respect to all counterparties in country h under scenario s^{**} is

$$V_h(s^{**}) = \min\left[\sum_j V_{j,h}(s^*), \sum_j V_{j,h}(s^*) \cdot (1 - c_h)\right] \tag{25}$$

where $V_{j,h}(s^*)$ is the joint scenario (market and counterparty default) value of counterparty j within country h. If country risk events are incorporated, the portfolio value at the joint scenario s^{**} is

$$V_p(s^{**}) = \sum_{h=1}^{H} V_h(s^{**}) \tag{26}$$

6. Joint exposure distribution of a swap portfolio

We construct a swap sample portfolio to illustrate the methodology. The portfolio consists of 305 interest rate swaps and cross currency swaps with 71 counterparties. The credit ratings of all counterparties are investment grade.

These transactions are in five different currencies and the total notional amount is $11.8 billion, among which 16% is cross currency swaps. The average remaining tenor is 2.5 years. The basis statistics of the portfolio are listed below:

Ratings	Number of counterparties
AAA	6 (8.4%)
AA	19 (26.8%)
A	40 (56.4%)
BBB	6 (8.4%)

Currency	Proportion (%)
CHF	11.2
DEM	5.4
GBP	4.1
JPY	43.3
USD	36.0

The Scenario Simulation model uses JPM RiskMetrics monthly data as of 10/20/1995. The simulation is to calculate the potential exposure 30 days from 11/10/1995, the valuation date. The total number of trials is one million.

For illustration purposes, we apply the following extremely stressed one-month default probabilities: for AAA, 1.3524%; Aa/AA, 1.7294%; A/A, 2.6866%; and Baa/BBB, 5.3596%.[18] The simulation results are shown in Table 7 and Figs. 2–4.

It is interesting to observe that while the market exposure of the swap portfolio (Fig. 2) looks like "bell-shaped" or approximately normally distributed, the joint distribution of market risk and credit risk shown in Fig. 3 is noticeably skewed because of the possibilities of counterparty defaults. Figure 4 shows that there is a possibility of very large losses due to counterparty default. But for diversified portfolios and counterparties, the probability of these losses is very small.

In this example, if the market risk is the only concern, the portfolio's 30-day VaR would be $34 mm at 99% confidence level. On the other hand, the potential loss due to counterparty default is $55 mm at the same confidence level. Such a large potential credit loss is a result of using exaggerated default probabilities. However, the probability that in 30-days, the "true" potential loss due to adverse market movements and/or counterparty default exceeds $66 mm is less than 1%. In other words, VaR calculated by the joint distribution is not the simple summation of the potential market loss and the default loss, and it is perhaps a better measure of portfolio's overall risk.

Table 7. Portfolio 30-day exposure distribution ($1,000) using stressed default probabilities

Portfolio exposure Type	Mean	Std	Maximum	Minimum	Potential loss percentile 0.1%	0.5%	1.0%	2.5%	5.0%
Market	30,825	13,139	86,262	−21,905	−42,322	−36,842	−34,120	−29,766	−26,055
Credit	−8,756	12,537	0	−143,981	−79,616	−62,469	−55,113	−45,541	−38,132
Joint	22,079	17,893	85,264	−119,565	−90,148	−73,522	−65,903	−55,120	−45,549

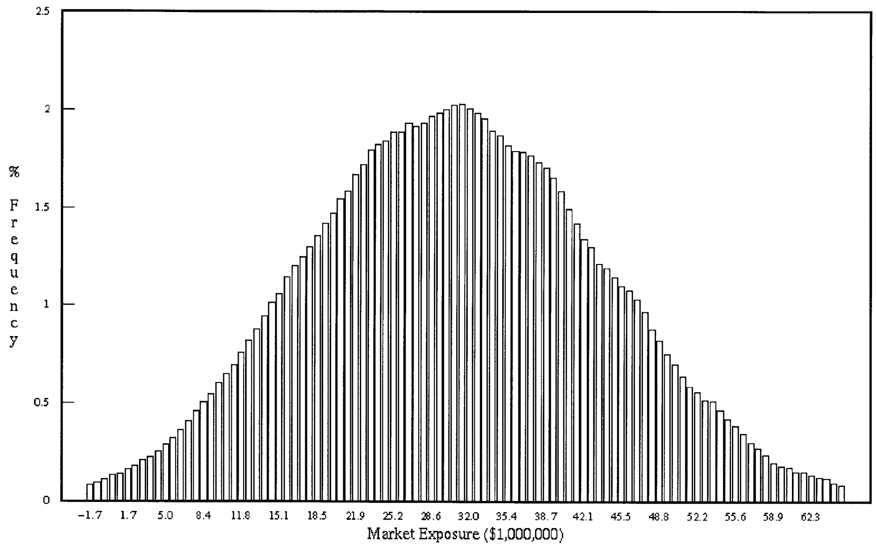

Fig. 2. 30-day market exposure distribution of swap portfolio

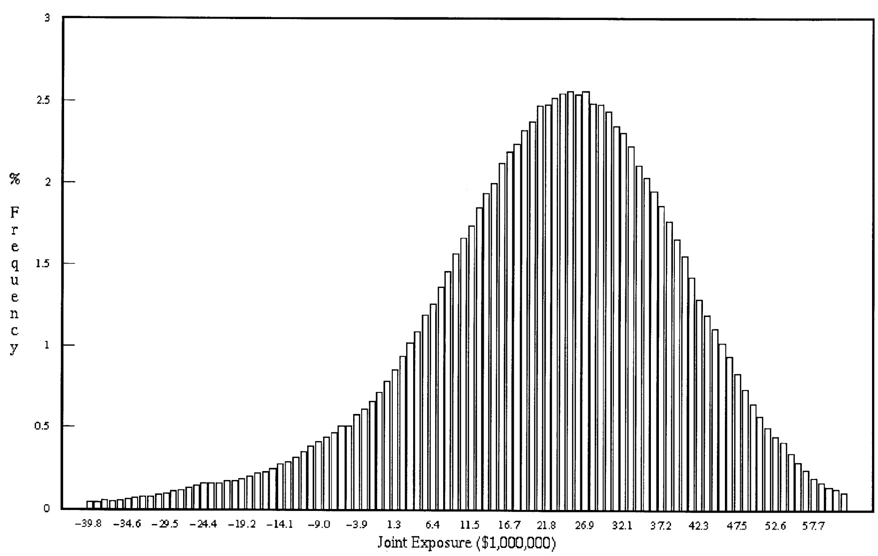

Fig. 3. Joint exposure distribution of both market risk and credit risk (Sample portfolio of swaps)

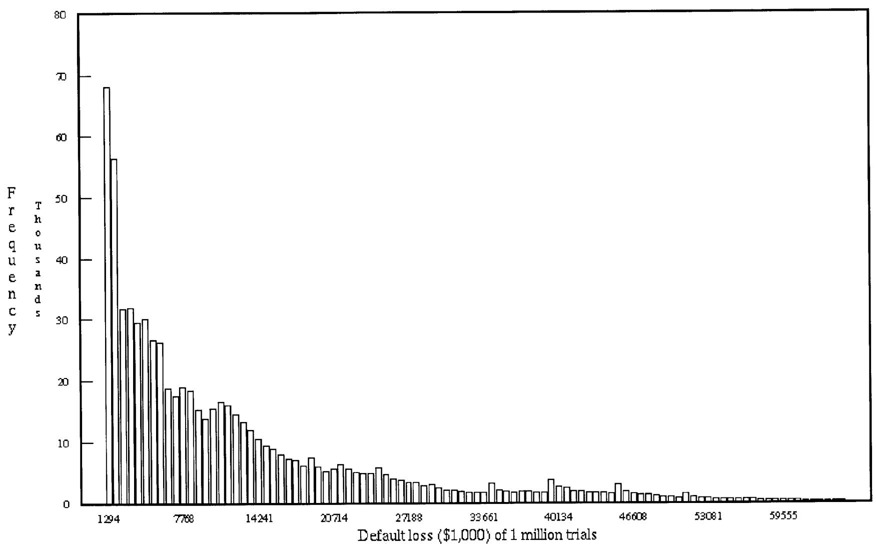

Fig. 4. Histogram of default losses for a sample portfolio (Sample portfolio of swaps)

7. Concluding remarks

The Scenario Simulation model provides a rigorous yet practical framework for quantitative analysis of multi-currency portfolio risk. The model is highly efficient computationally, and can be applied to a wide range of risk management tasks such as estimation of portfolio market risk and credit exposure, and calculation of value-at-risk for a financial institution.

The basic idea is to discretize the multivariate distribution of market variables into a manageable number of scenarios, e.g., 105 yield curve scenarios for each currency. When there are several currencies, we can take a large Monte Carlo sample, e.g., of size one million, to achieve numerical stability; yet, we only have to evaluate each transaction in the portfolio 105 times.

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[1] Value-at-risk was first introduced by G30 in 1993. See Global Derivative Study Group (1993). BIS formally adopted the concept in 1995 as the determinant of market risk capital requirement. For various methods to estimate VaR, see "Value-At-Risk", Risk Magazine Special Supplement, June 1996.

[2] We emphasize that in this model, all distributions, Brownian motions, etc., are with respect to the *actual* measure, rather than the risk-neutral measure. As such, the model is not intended for valuation of derivatives, rather, it generates yield curve scenarios for future dates to be used as input to appropriate valuation models. The model is nevertheless arbitrage free, as the uncertainty of each key rate carries its own market price of risk. The latter are implicitly determined by either of the two expectation hypotheses (4a) or (4b). In this connection, we view the later reduction to three factors as an approximation of the covariance structure, rather than exact modelling with three factors.

[3] See, e.g., Murphy et al. (1995) and Kahn (1989).

[4] The historical covariance data are from J.P. Morgan RiskMetrics (5/30/1995 monthly data) which is a trademark of J.P. Morgan. For its estimation method, see J.P. Morgan's technical

document (1995). It should be noted that the Scenario Simulation model does not depend on a particular data source.

[5]For these derivations, it is useful to note that if $\boldsymbol{\beta}$ denotes the matrix $(\boldsymbol{\beta}_{ij})$, then, $\boldsymbol{\beta}\boldsymbol{\beta}^t = R$, $\boldsymbol{\beta}^t\boldsymbol{\beta} = \text{diag}(\boldsymbol{\lambda})$, and $\boldsymbol{\beta}^{-1} = \text{diag}(1/\boldsymbol{\lambda})\boldsymbol{\beta}^t$.

[6]For example, Cox and Rubinstein (1985).

[7]To determine the appropriate number of scenarios is of course an empirical issue.

[8]In Table 2, we use $7 \times 5 \times 3$ to represent 7, 5, and 3 scenarios for the first, second and third factor respectively. The total number of scenarios is equal to $7 \times 5 \times 3 = 105$. Though not reported here, the comparison of percentiles of the two methods gives similar results.

[9]See, e.g., Karlin and Taylor (1981).

[10]See, e.g., Billingsley (1986). Note here the symbol "$\sim$" means "asymptotic to".

[11]The correlation between two principal factors of two different countries is obtained from Eq. (9a) applied to both countries.

[12]These implied volatilities are distinct from the volatilities in Eq. (2) that drive the key rates.

[13]This is because if Y is a k-dimensional variate with an identity covariance matrix, then $X = AY$ will have correlation matrix Q.

[14]A good reference for the numerical methods mentioned in this paragraph is Press et al. (1992).

[15]Moody's estimated that the average recovery rate (i.e., $1 - L$) for senior secured bonds is as high as 53.05%, and for senior unsecured bonds is 45.88%. For senior surbordinated and subordinated debt the recovery rates become 37.02% and 29.57%, respectively. For junior subordinated debt, it is merely 16.40% (see Carty and Lieberman, 1996).

[16]The expected credit loss for time period t, $E[D_t]$, can be calculated as $\sum_s \max\,[V_t(s), 0] \cdot p_t(s) \cdot q_t$, where $V_t(s)$ is a counterparty's termination value at time t in scenario s, $p_t(s)$ is the probability of scenario s, and q_t is the marginal default probability of the counterparty. If $P(t)$ is the discount bond price maturing in t, then the total expected credit loss can be expressed as $E[D] = \sum_t P(t)E[D_t]$, which can be interpreted as the fair insurance cost of the counterpart default.

[17]For detailed discussion, see Gluck and Clarkson (1993).

[18]These are in fact Standard & Poor's 5-year cumulative stressed default probabilities but used for 1-month period. By taking the 12-th root of S&P's yearly probability transition matrix we can obtain one-month default probabilities, which are respectively 0.0202, 0.0243, 0.0354, and 0.0978% for the above rating categories. In other words, the default probabilities used in Table 7 are more than 50 times larger than S&P's stressed one-month default probabilities. S&P emphasizes that stressed probabilities are advisable particularly for estimation of potential credit losses over a short period of time (see Bahar and Gold 1995).

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