Finance Stoch (2018) 22:97–126
DOI 10.1007/s00780-017-0348-0

Financial equilibrium with asymmetric information and random horizon

Umut Çetin[1]

Received: 22 April 2016 / Accepted: 13 September 2017 / Published online: 10 November 2017

Abstract We study in detail and explicitly solve the version of Kyle's model introduced in a specific case in Back and Baruch (Econometrica 72:433–465, 2004), where the trading horizon is given by an exponentially distributed random time. The first part of the paper is devoted to the analysis of time-homogeneous equilibria using tools from the theory of one-dimensional diffusions. It turns out that such an equilibrium is only possible if the final payoff is Bernoulli distributed as in Back and Baruch (Econometrica 72:433–465, 2004). We show in the second part that the signal the market makers use in the general case is a time-changed version of the one they would have used had the final payoff had a Bernoulli distribution. In both cases, we characterise explicitly the equilibrium price process and the optimal strategy of the informed trader. In contrast to the original Kyle model, it is found that the reciprocal of the market's depth, i.e., Kyle's lambda, is a uniformly integrable supermartingale. While Kyle's lambda is a potential, i.e., converges to 0, for the Bernoulli-distributed final payoff, its limit in general is different from 0.

Keywords Kyle's model · Financial equilibrium · One-dimensional diffusions · h-transform · Potential theory

Mathematics Subject Classification (2010) 60J45 · 60J60 · 91B44

JEL Classification D82 · G12 · G14

1 Introduction

The canonical model of markets with asymmetric information is due to Kyle [13]. Kyle studies a market for a single risky asset whose price is determined in equilib-

✉ U. Çetin
u.cetin@lse.ac.uk

1 Department of Statistics, London School of Economics and Political Science, 10 Houghton St., London, WC2A 2AE, UK

rium. There are mainly three types of agents that constitute the market: a strategic risk-neutral informed trader with a private information regarding the future value of the asset, non-strategic uninformed noise traders, and a number of risk-neutral market makers competing for the net demand from the strategic and non-strategic traders. The key feature of this model is that the asset value becomes public knowledge at a fixed deterministic date and the market makers cannot distinguish the informed trades from the uninformed ones, but 'learn' from the net demand by 'filtering' what the informed trader knows, which is 'corrupted' by the noise demand. The private information of the informed trader is static, i.e., it is obtained at the beginning of trading and does not change over time. The nature of this information is not really important: It could be inside information about the future payoff of the asset or an unbiased estimator of the future payoff. The latter is a more suitable interpretation when the strategic informed trader is a big investment bank with a good research division producing sophisticated research that is not shared with the public.

Kyle's model is in discrete time and assumes that the noise traders follow a random walk and the future payoff of the asset has a normal distribution. This has been extended to a continuous-time framework with general payoffs by Back [1]. In this extension, the total demand of the noise traders is given by a Brownian motion and the future payoff of the asset has a general continuous distribution, while the informed trader's private information is still static. Kyle's model and its continuous-time extension by Back have been further extended to allow multiple informed traders [11, 3], to include default risk [6] or to the case when the single informed trader receives a continuous signal as private information [4, 7].

Our first goal in this paper is to study a time-homogeneous version of this model introduced by Back and Baruch in [2]. The time-homogeneity refers to the SDE corresponding to the market price having time-homogeneous coefficients, and to the insider's optimal decisions depending only on the price of the traded asset, not on time. This is in part due to the assumption that the asset value Γ is announced at a random exponential time τ with mean r^{-1} for some $r > 0$ and independent from all other variables in the model. The informed trader knows the asset value but not τ. Thus, she has an informational advantage over the other traders; however, this can end at any time since τ will come as a surprise to the informed trader as it does to the others. Back and Baruch compute the market depth and the informed trader's strategy as a function of the price, which is only characterised implicitly using an inverse operation. Moreover, their method does not allow an explicit characterisation of the distribution of the equilibrium price process.

In Sect. 3.1, we analyse the model of Back and Baruch using tools from the theory of one-dimensional diffusions. We show that in equilibrium the market makers construct a transient Ornstein–Uhlenbeck process Y that they use for pricing. A particular consequence of this is that the pricing rule becomes a *scale function* of Y. In addition, we identify the value function of the strategic trader with an *r-excessive function* of a certain *h-transform* of the Ornstein–Uhlenbeck process.

Back and Baruch postulate the equilibrium controls for the market makers and the informed trader and then verify that these indeed constitute an equilibrium. We, on the other hand, study in detail the admissible pricing rules for the market makers given a large class of admissible strategies of the informed trader, and we characterise the ones that can appear in equilibrium. It turns out that the market makers can choose from a continuum of controls in equilibrium; however, every such choice will lead to the same SDE for the equilibrium price process.

Consistent with what was observed earlier by Back and Baruch, we show that the process measuring the price impact of trades, i.e., *Kyle's lambda*, is a uniformly integrable supermartingale converging to 0, i.e., a potential. As a result, the market gets more liquid on average as time passes. This is a deviation from Kyle [13] who predicted that Kyle's lambda must follow a martingale preventing 'systematic changes' in the market depth as explained in [3] and [9]. In the absence of such systematic changes, the insider cannot acquire a large position when the depth is low to liquidate at a later date when the liquidity is higher to obtain unbounded profits. In the model we study, even if the market gets more liquid as time passes, there are no opportunities for the informed trader to make infinite profits since the market can end on any time interval $[t, t + dt]$ with probability $r\,dt$.

One of the advantages of the approach used in this paper is that it identifies the distribution of the price process explicitly. The explicit form of the solutions, however, also indicate that one cannot have an equilibrium where the price process is a time-homogeneous diffusion if the asset value has a non-Bernoulli distribution, as explained in Remark 3.11. A recent work by Collin-Dufresne et al. [8], on the other hand, hints at the direction that one should follow for a general payoff distribution. Collin-Dufresne et al. [8], using linear Kalman filtering, study a version of the Kyle model where the announcement date is a jump time of a Poisson process with non-constant intensity and the asset value has a Gaussian distribution. It turns out in their model that the diffusion coefficient of the SDE for the equilibrium price should be time-dependent.

Motivated by their result, we study in Sect. 3.2 an extension of the model of Back and Baruch to a large class of payoff distributions. It turns out that the signal the market makers use when the payoff has a general distribution is a time-changed transient Ornstein–Uhlenbeck process. The time change V is deterministic with $V(\infty) < \infty$. The finiteness of the time change implies that the limiting distribution of the market makers' signal is a non-degenerate normal distribution. This particular feature allows us to extend the model of Back and Baruch to a much more general setting, including the normally distributed case considered in [8].

As in the other works on the Kyle model, we establish that the informed trader's trades are *inconspicuous* in equilibrium. That is, the equilibrium distribution of the total demand is the same as that of cumulative noise trades. An essential difference, on the other hand, from the earlier works on this subject is that the equilibrium prices exhibit a jump at the announcement date τ. This is only natural since τ is unknown to the informed trader, and thus there is no strategy to ensure that the market price converges almost surely to Γ as time approaches τ, which is a totally inaccessible stopping time even for the informed. However, what the informed trader can do is the following strategy, which will in fact turn out to be her equilibrium strategy: She can make sure that P_t, the market price conditioned on *survival*, i.e., $\{\tau > t\}$, converges to Γ as $t \to \infty$. That is, conditioned on indefinite survival, market prices converge to the true payoff. Note that indefinite survival has 0 probability due to the finiteness of τ; hence a jump in prices at time τ occurs with probability 1.

The outline of the paper is as follows. Section 2 introduces the model, defines the admissible controls for the market makers, as well as the informed trader, and ends with a characterisation of the market makers' pricing choice. The equilibrium when the final payoff has a Bernoulli and a general distribution is solved in Sects. 3.1 and 3.2, respectively. Finally, Sect. 4 concludes.

2 The setup

Let $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t\geq 0}, \mathbb{P})$ be a filtered probability space satisfying the usual conditions of right-continuity and $\mathbb{P}$-completeness. We suppose that $\mathcal{F}_0$ is not trivial and there exists an $\mathcal{F}_0$-measurable random variable Γ taking values in $\mathbb{R}$. Moreover, the filtered probability space also supports a standard Brownian motion B with $B_0 = 0$, and thus B is independent of Γ. We also assume the existence of an $\mathcal{F}$-measurable random variable τ, which is independent of $\mathcal{F}_\infty$ and has an exponential distribution with mean $0 < r^{-1} < \infty$. In particular, τ is independent of Γ and B.

The measure induced by Γ on $\mathbb{R}$ will be denoted by ν, i.e., $\nu(A) := \mathbb{P}[\Gamma \in A]$ for any Borel subset of $\mathbb{R}$. We further assume the existence of a family of probability measures $(\mathbb{P}^v)_{v\in\mathbb{R}}$ such that we have the disintegration formula

$$\mathbb{P}[E] = \int_{\mathbb{R}} \mathbb{P}^v[E]\nu(dv), \qquad \forall E \in \mathcal{F}. \tag{2.1}$$

The existence of such a family is easily justified when we consider $\Omega = \mathbb{R} \times C(\mathbb{R}_+,\mathbb{R})$, where $C(\mathbb{R}_+, \mathbb{R})$ is the space of real-valued continuous functions on $\mathbb{R}_+$. We set $\mathbb{P}^v = \mathbb{P}$ if $v \notin \operatorname{supp}(\nu)$. We also assume that Γ is square-integrable, that is,

$$\mathbb{E}[\Gamma] = \int_{\mathbb{R}} v^2 \nu(dv) < \infty. \tag{2.2}$$

We consider a market in which the risk-free interest rate is set to 0 and a single risky asset is traded. The fundamental value of this asset equals Γ, which will be announced at the random time τ.

There are three types of agents that interact in this market:

(i) Liquidity traders who trade for reasons exogenous to the model and whose cumulative demand at time t is given by B_t.

(ii) A single informed trader, who knows Γ from time $t = 0$ onwards, and is risk-neutral. We call the informed trader *insider* in what follows and denote her cumulative demand at time t by θ_t. The filtration $\mathcal{G}^I$ of the insider is generated by observing the price Γ of the risky asset and whether the announcement has been made, that is, whether $\tau > t$ or not for each $t \geq 0$. The filtration is also assumed to be completed with the nullsets of $(\mathbb{P}^v)_{v\in\mathbb{R}}$.

(iii) Market makers observe only the net demand $X = B + \theta$ of the risky asset, whether $\tau > t$ or not for each $t \geq 0$, and Γ when it is made public at τ. Thus, their filtration $\mathcal{G}^M$ is the minimal right-continuous filtration generated by X, $(\mathbf{1}_{\{\tau>t\}})_{t\geq 0}$ and $(\mathbf{1}_{\{t\geq\tau\}}\Gamma)_{t\geq 0}$, and completed with the $\mathbb{P}$-nullsets. In particular, τ is a $\mathcal{G}^M$-stopping time. The price process chosen by the market makers is denoted by S. Obviously,

$S_t = \Gamma$ on the set $\{t \geq \tau\}$. Prior to the announcement date, we assume that the market price is determined according to a *Bertrand competition:* Market makers make their price offers and the investors trade with the one offering the best quote. This mechanism results in S being a $\mathcal{G}^M$-martingale, i.e., $S_t = \mathbb{E}[\Gamma|\mathcal{G}_t^M]$. Consequently, S_∞ exists and equals Γ since $\mathbb{P}[\tau < \infty] = 1$.

The way that the price process is determined yields the following since Γ is integrable.

Proposition 2.1 *In view of the condition* (2.2), *for any* $s \geq 0$,

$$\lim_{t\to\infty} \mathbb{E}\big[|S_t - \Gamma|\big|\mathcal{G}_s^M\big] = 0. \tag{2.3}$$

Proof Since $\lim_{t\to\infty} S_t = \Gamma$ and Γ satisfies (2.2), we deduce that the family $(S_t)_{t\geq 0}$ is uniformly integrable (see [12, Problem 1.3.20]), and so is $(\Gamma - S_t)_{t\geq 0}$. Since $\lim_{t\to\infty} |S_t - \Gamma| = 0$, the claim follows. □

In this paper, as in all other past work on Kyle's model, we are interested in Markovian Nash equilibria. In line with the current literature, we assume that the price chosen by the market makers is a deterministic function of some process Y, which solves $dY_t = w(t, Y_t)\,dX_t + b(t, Y_t)\,dt$ for some weighting function w and drift function b chosen by the market makers (see e.g. [1, 4, 7] among others for the use of a weighting function in the construction of the market makers' signal).

Before defining what we mean precisely by an equilibrium in this model, we introduce the class of admissible controls for the market makers and the informed trader.

Definition 2.2 The pair $((w, b, y), h)$ is an *admissible pricing rule* for the market makers if for some interval $(\ell, u) \subset \mathbb{R}$, $y \in (\ell, u)$ and $w : \mathbb{R}_+ \times (\ell, u) \to (0, \infty)$, $b : \mathbb{R}_+ \times (\ell, u) \to \mathbb{R}$ and $h : \mathbb{R}_+ \times (\ell, u) \to \mathbb{R}$ are measurable functions such that $h(t, \cdot)$ is strictly increasing for every $t > 0$, w is bounded away from 0 on compact subsets of $\mathbb{R}_+ \times (\ell, u)$, and for any Brownian motion β, there exists a unique strong solution without explosion[1] to

$$Y_t = z + \int_0^t w(s, Y_s)\,d\beta_s + \int_0^t b(s, Y_s)\,ds, \qquad \forall z \in (\ell, u).$$

Given an admissible pricing rule $((w, b, y), h)$, the market makers set the price on $\{\tau > t\}$ to be $h(t, Y_t)$, where Y follows

$$Y_t = y + \int_0^t w(s, Y_s)(dB_s + d\theta_s) + \int_0^t b(s, Y_s)\,ds, \tag{2.4}$$

whenever θ is an admissible strategy for the insider.

Since h is strictly monotone in y, Y is perfectly observable by the traders, in particular by the insider. In conjunction with the assumption that w is bounded away from 0 on compact subsets of $\mathbb{R}_+ \times (\ell, u)$, this entails that the insider can observe B since

[1] That is, the boundary points ℓ and u are not reached in finite time.

she clearly knows her own strategy θ. This assumption, therefore, also ensures that the insider's filtration is well defined and is generated by B, Γ and $(\tau \wedge t)_{t\geq 0}$. This is worth noting since otherwise, we may run into problems with the well-posedness of the model as the insider's trading strategy may depend on the observations of the price process that is adapted to the filtration generated by X, which depends crucially on the insider's trading strategy.

Definition 2.3 Given an admissible pricing rule $((w,b,y),h)$, θ is an *admissible strategy for the insider* if θ is of finite variation and the following conditions are satisfied:

1. θ is absolutely continuous, i.e.,

$$\theta_t = \int_0^t \alpha_s \, ds,$$

for some $\mathcal{G}^I$-adapted α, where $\mathcal{G}^I$, as discussed above, is the minimal right-continuous filtration generated by B, Γ and $(\tau \wedge t)_{t\geq 0}$, and completed with the $(\mathbb{P}^v)_{v\in\mathbb{R}}$-nullsets.

2. There exists a unique strong solution to

$$Y_t = z + \int_0^t w(s, Y_s)\, dX_s + \int_0^t b(s, Y_s)\, ds, \qquad \forall z \in (\ell, u).$$

3. We have the integrability condition

$$\mathbb{E}^v\left[\int_0^\tau h^2(t, Y_t)\, dt\right] < \infty, \qquad \forall v \in \operatorname{supp}(\nu), \tag{2.5}$$

where Y is the unique strong solution of (2.4).

The assumption that the strategy is absolutely continuous is without any loss of generality, since any strategy with positive quadratic variation is necessarily suboptimal due to the price impact of the trades (see [1] for a proof of this fact).

Faced with an admissible pricing rule $((w,b,y),h)$, the insider employs an admissible strategy θ and achieves

$$W_\tau = \int_{[0,\tau]} \theta_s \, dh(s, Y_s) + \theta_\tau \bigl(\Gamma - h(\tau, Y_\tau)\bigr)$$

as total profit when the public announcement is made and the trading possibilities end, where Y is the strong solution to (2.4), which exists and is unique since θ is admissible. Note that the term $\theta_\tau(\Gamma - h(\tau, Y_\tau))$ is due to a potential jump in the price when the true value is revealed. Integrating by parts, we obtain the more convenient representation

$$W_\tau = \int_0^\tau \bigl(\Gamma - h(s, Y_s)\bigr)\alpha_s \, ds.$$

Because she is risk-neutral, the informed trader's goal is to maximise $\mathbb{E}^v[W_\tau]$ within the class of admissible strategies.

Definition 2.4 $((w^*, b^*, y^*), h^*, \alpha^*)$ is an *equilibrium* if

1. $((w^*, b^*, y^*), h^*)$ is an admissible pricing rule for the market makers;
2. θ^* defined by $\theta^*_t = \int_0^t \alpha^*_s\, ds$ is an admissible trading strategy for the insider given $((w^*, b^*, y^*), h^*)$;
3. $S_t = \mathbf{1}_{\{\tau>t\}} h^*(t, Y^*_t) + \mathbf{1}_{\{\tau\le t\}} \Gamma$ is a $\mathcal{G}^M$-martingale, where Y^* is the unique strong solution of (2.4) with $w = w^*$ and $b = b^*$;
4. θ^* maximises the expected profits for the insider, that is, for all $v \in \mathrm{supp}(\nu)$,

$$\mathbb{E}^v\left[\int_0^\tau \big(v - h^*(s, Y^*_s)\big)\alpha^*_s\, ds\right] = \sup_{\alpha\in\mathcal{A}} \mathbb{E}^v\left[\int_0^\tau \big(v - h^*(s, Y^*_s)\big)\alpha_s\, ds\right],$$

   where $\mathcal{A}$ is the class of all admissible strategies given the admissible pricing rule $((w^*, b^*, y^*), h^*)$.

Since τ is independent of Γ and B, the market makers, as well as the insider, will not use controls that depends on τ in equilibrium. In this case, the expected value of the final wealth of the insider will equal

$$\mathbb{E}^v[W_\tau] = \int_0^\infty e^{-rs}\mathbb{E}^v\big[\big(v - h(s, Y_s)\big)\alpha_s\big]\, ds. \tag{2.6}$$

Let us denote by $\mathcal{F}^X$ the minimal right-continuous filtration containing the $\mathbb{P}$-null-sets with respect to which X is adapted. When the insider uses strategies that are independent of τ, this will render X independent of τ as well. In this case, one should expect that

$$S_t = P_t\mathbf{1}_{\{t<\tau\}} + \Gamma\mathbf{1}_{\{t\ge\tau\}},$$

where P is a semimartingale adapted to $\mathcal{F}^X$. The following proposition shows that this is indeed the case and gives a characterisation of P in terms of Γ.

Proposition 2.5 *Suppose that X is independent of τ and all $\mathcal{F}^X$-martingales are continuous. Then $P_t = \mathbb{E}[\Gamma|\mathcal{F}^X_t]$, i.e., P is the $\mathcal{F}^X$-optional projection of Γ. In particular, P is continuous.*

The proof of the above result is delegated to Appendix A. However, it is worth mentioning here that the proof does not rely on market makers making their pricing decision in a Markovian manner, i.e., $P_t = h(t, Y_t)$, where Y follows (2.4). That is, whatever the pricing decision is, it must satisfy $P_t = \mathbb{E}[\Gamma|\mathcal{F}^X_t]$ whenever X is independent of τ and all $\mathcal{F}^X$-martingales are continuous.[2]

In view of the above characterisation of the pricing decision P of the market makers, we search for an equilibrium in the next section by studying the optimal trading choices of the insider. As our focus is not on the uniqueness of the equilibrium, we follow our intuition and consider trading strategies that are independent of τ. Accordingly, the following section establishes the existence of an equilibrium where X is independent of τ.

[2] As we shall see in the next section, all $\mathcal{F}^X$-martingales are continuous in equilibrium. In general, a sufficient condition for this property is $\mathbb{E}[\int_0^t \alpha_s^2\, ds] < \infty$ for all $t > 0$ (see [16, Corollary 8.10]).

3 Insider's optimisation problem and the equilibrium

3.1 Bernoulli-distributed liquidation value

In this section, we assume that Γ takes values in $\{0, 1\}$ as in [2] and set $p := \nu(\{1\})$, i.e., the probability that the liquidation value equals 1.

Under this assumption, we shall next see that one can obtain an equilibrium where the coefficients in (2.4), as well as the pricing function h, do not depend on time. Consequently, the solution of the optimisation problem for the insider will be time-homogeneous as well.

We first try to formally obtain the Bellman equations associated to the value function of the insider. To this end, as observed above, suppose that X is independent of τ and the market makers set the price at time t on $\{t < \tau\}$ to be $h(Y_t)$ for some sufficiently smooth h, where

$$dY_t = a(Y_t)\,dX_t + \phi(Y_t)\,dt.$$

Also recall that $dX_t = dB_t + \alpha_t\,dt$, where $\alpha_t\,dt$ represents the infinitesimal trades of the insider.

In view of (2.6), we may consider for the insider's problem given that $\Gamma = v$ the value function

$$J(x) = \sup_\alpha \mathbb{E}^v\left[\int_0^\infty e^{-rs}\big(v - h(Y_s)\big)\alpha_s\,ds\,\middle|\,Y_0 = x\right].$$

Formally, J solves

$$\sup_\alpha\left\{\frac{1}{2}a^2 J'' + J'(a\alpha + \phi) + \alpha(v - h) - rJ\right\} = 0.$$

Thus, if J is three times continuously differentiable, we expect to have

$$aJ' = h - v, \tag{3.1}$$

$$\frac{1}{2}a^2 J'' + J'\phi - rJ = 0.$$

The first identity yields

$$J' = \frac{h - v}{a},\quad J'' = \frac{h'}{a} - \frac{h - v}{a^2}a',\qquad J''' = \frac{h''}{a} - 2\frac{h'a'}{a^2} - \frac{(h - v)a''}{a^2} + 2\frac{(h - v)(a')^2}{a^3}.$$

Plugging the above into the second identity yields

$$0 = \frac{a^2}{2}h'' + h'\phi - (h - v)\left(\frac{aa''}{2} + \frac{a'}{a}\phi - \phi' + r\right).$$

Since h cannot depend on Γ, we obtain for the candidate pricing rule h and the coefficients a and ϕ, which will be chosen by the market makers to construct the price, the conditions

$$\frac{1}{2}a^2h'' + h'\phi = 0, \tag{3.2}$$

$$\frac{1}{2}a^2a'' + a'\phi + (r - \phi')a = 0. \tag{3.3}$$

Looking at these equations, we may guess that in equilibrium,

$$dY_t = a(Y_t)\,dB_t^Y + \phi(Y_t)\,dt, \tag{3.4}$$

where B^Y is a $\mathcal{G}^M$-Brownian motion and a and ϕ solve (3.3). The following result shows that all such processes can be obtained from an Ornstein–Uhlenbeck process.

Proposition 3.1 *Suppose that Y is a regular one-dimensional diffusion on (ℓ, u) defined by the generator*

$$\frac{1}{2}a^2\frac{d^2}{dx^2} + \phi\frac{d}{dx},$$

where $a > 0$ and ϕ are two functions satisfying (3.3) on (ℓ, u). Assume further that for any $y \in (\ell, u)$, there exists a unique weak solution of (3.4) and we have for some $c \in (\ell, u)$ the integrability condition

$$f(x) := \int_c^x \frac{1}{a(y)} < \infty, \qquad \forall x \in (\ell, u).$$

If $R_t = f(Y_t)$, then R is an Ornstein–Uhlenbeck process with generator

$$A = \frac{1}{2}\frac{d^2}{dx^2} + (rx + \delta)\frac{d}{dx}, \tag{3.5}$$

for some $\delta \in \mathbb{R}$.

Proof Clearly, Y solves (3.4) with some Brownian motion B^Y. Thus, it follows from Itô's formula that

$$dR_t = dB_t^Y + \left(\frac{\phi(f^{-1}(R_t))}{\sigma(f^{-1}(R_t))} - \frac{1}{2}\sigma'\big(f^{-1}(R_t)\big)\right)dt.$$

Using (3.3), one can easily check that the derivative of the function

$$y \mapsto \frac{\phi(f^{-1}(y))}{\sigma(f^{-1}(y))} - \frac{1}{2}\sigma'\big(f^{-1}(y)\big)$$

equals r, which yields the claim. □

In view of the above proposition, we expect the equilibrium price process to be a function of an Ornstein–Uhlenbeck process. Thus, there is no harm in choosing $a \equiv 1$ and $\phi = rx + \delta$ for some $\delta \in \mathbb{R}$. This means that the pricing rule h will be a scale function of the diffusion with generator (3.5). Recall that the choice of a and ϕ are made by the market makers. Thus, the market makers construct their signal Y such that it is a transient process in equilibrium ($|Y_t| \to \infty$ when Y is defined by (3.5)).

Lemma 3.2 *Define*

$$s(x) = \sqrt{\frac{r}{\pi}} \int_{-\infty}^{x} \exp\left(-r\left(y + \frac{\delta}{r}\right)^2\right) dy. \tag{3.6}$$

Then s is a scale function for the diffusion defined by (3.5) *and satisfies the property* $s(-\infty) = 1 - s(\infty) = 0$.

Proof It is straightforward to check that $As = 0$. Moreover, s is the cumulative distribution function for a normal random variable, which implies the boundary conditions. □

Theorem 3.3 *Let $E \in \mathcal{F}_0$ and consider the SDE*

$$dY_t = dB_t + \left(rY_t + \delta + \mathbf{1}_E \frac{s'(Y_t)}{s(Y_t)} - \mathbf{1}_{E^c} \frac{s'(Y_t)}{1 - s(Y_t)}\right) dt, \qquad Y_0 = y \in \mathbb{R},$$

where s is as given by (3.6). *Then there exists a unique strong solution to the above. The solution in particular satisfies*

$$\lim_{t\to\infty} Y_t(\omega) = \begin{cases} \infty, & \text{if } \omega \in E; \\ -\infty, & \text{if } \omega \in E^c. \end{cases}$$

Moreover, if $\mathbb{P}[E] = s(y)$, we have

$$\mathbb{P}[E|\mathcal{F}_t^Y] = s(Y_t),$$

and consequently

$$dY_t = dB_t^Y + (rY_t + \delta)\, dt$$

in its own filtration, where B^Y is a Brownian motion.

Proof Consider an Ornstein–Uhlenbeck process R with generator (3.5) on some probability space $(\Omega, \mathcal{G}, (\mathcal{G}_t)_{t\geq 0}, P)$ such that there exists a set $F \in \mathcal{G}_0$ with $P[F] = \mathbb{P}[E]$. Clearly, $M := \mathbf{1}_F \frac{s(R)}{s(y)} + \mathbf{1}_{F^c} \frac{1-s(R)}{1-s(y)}$ is a bounded martingale with $E[M_0] = 1$. Thus, defining Q on $\mathcal{G}$ by $\frac{dQ}{dP} = M_\infty$ yields the existence of a weak solution. Moreover, the weak solution is unique in law since $M_t > 0$ for $t > 0$. The limiting condition for Y_t as $t \to \infty$ follows from this construction of the weak solution since the construction is nothing but the h-transform that achieves $R_\infty = \infty$ (resp. $R_\infty = -\infty$) on F (resp. F^c) (see [5, II.32]).

The SDE above in fact possesses a unique strong solution. Indeed, since $\sigma \equiv 1$, Lemma IX.3.3, Corollary IX.3.4 and Proposition IX.3.2 in [15] imply that pathwise uniqueness holds for this SDE. Thus, in view of the celebrated result of Yamada and Watanabe (see [12, Corollary 5.3.23]), there exists a unique strong solution.

Moreover, using the law of the solutions obtained via the weak construction performed above, we have

$$\mathbb{P}[E|\mathcal{F}_t^Y] = \frac{E^y[M_\infty \mathbf{1}_F|\mathcal{F}_t^R]}{E^y[M_\infty|\mathcal{F}_t^R]},$$

where E^y is the expectation with respect to the product measure $P^y \otimes \nu$, where P^y is the law of R with $R_0 = y$ and ν is the distribution of F. Since F and R are independent, under the assumption that $\mathbb{P}[E] = s(y)$, we obtain

$$\frac{E^y[M_\infty \mathbf{1}_F|\mathcal{F}_t^R]}{E^y[M_\infty|\mathcal{F}_t^R]} = \frac{\frac{s(R_t)}{s(y)} P[F]}{E^y[M_t|\mathcal{F}_t^R]} = s(R_t),$$

since $P[F] = \mathbb{P}[E] = s(y)$ and $E^y[M_t|\mathcal{F}_t^R] = 1$. Thus, a standard result from nonlinear filtering (e.g. [14, Theorem 8.1]) yields

$$dY_t = dB_t^Y + (rY_t + \delta)\,dt$$

for some $\mathcal{F}^Y$-Brownian motion. □

We now turn to the computation of the insider's value function. We in fact verify in Proposition 3.5 that it is given by

$$J(x) := \int_{s^{-1}(v)}^x \big(s(y) - v\big)\,dy, \tag{3.7}$$

which satisfies (3.1) by construction for $h = s$. Moreover,

$$\begin{aligned} AJ(x) - rJ(x) &= \frac{1}{2}s'(x) + \big(s(x) - v\big)\phi(x) - r\int_{s^{-1}(v)}^x \big(s(y) - v\big)\,dy \\ &= \frac{1}{2}s'(x) + \int_{s^{-1}(v)}^x \phi(y)s'(y)\,dy \\ &= \frac{1}{2}s'(x) - \frac{1}{2}\int_{s^{-1}(v)}^x s''(y)\,dy \\ &= 0, \end{aligned} \tag{3.8}$$

since $s'(s^{-1}(v)) = 0$.

Lemma 3.4 *For any $v \in \{0, 1\}$, consider the SDE*

$$Y_t = y + B_t + \int_0^t \left(rY_s + \delta + \mathbf{1}_{\{v=1\}}\frac{s'(Y_t)}{s(Y_t)} - \mathbf{1}_{\{v=0\}}\frac{s'(Y_t)}{1 - s(Y_t)} \right) ds, \tag{3.9}$$

and define

$$A^v := A + \left(\mathbf{1}_{\{v=1\}}\frac{s'}{s} - \mathbf{1}_{\{v=0\}}\frac{s'}{1-s} \right)\frac{d}{dx}.$$

Then J is an r-excessive function for A^v and $\mathbb{E}^{Q_v}[e^{-rt}J(Y_t)] \to 0$ as $t \to \infty$, where $\mathbb{E}^{Q_v}$ is the expectation operator with respect to the law of the solution of the above SDE for the given value of v.

Proof Recall from (3.8) that $AJ - rJ = 0$ and observe that

$$A^v J - rJ = AJ - rJ + \left(\mathbf{1}_{\{v=1\}}\frac{s'}{s} - r\mathbf{1}_{\{v=0\}}\frac{s'}{1-s}\right)(s-v)$$
$$= \left(\mathbf{1}_{\{v=1\}}\frac{s'}{s} - \mathbf{1}_{\{v=0\}}\frac{s'}{1-s}\right)(s-v) \le 0.$$

Define the nonnegative function

$$g_v := \left(\mathbf{1}_{\{v=1\}}\frac{s'}{s} - \mathbf{1}_{\{v=0\}}\frac{s'}{1-s}\right)(v-s).$$

J will be r-excessive once we show that g_v is integrable with respect to the speed measure of the diffusion defined by A^v. It follows from [5, II.31] that this measure is given by

$$m^v(dx) := \left(\mathbf{1}_{\{v=1\}}s^2(x) + \mathbf{1}_{\{v=0\}}\big(1-s(x)\big)^2\right)\frac{2}{s'(x)}\,dx.$$

Thus,

$$\int_{-\infty}^{\infty} g_v(x)m^v(dx) = 2\int_{-\infty}^{\infty} s(x)\big(1-s(x)\big)\,dx < \infty$$

since s is the cumulative distribution function of a normal random variable. Due to the integral representation formula for r-excessive functions (see [5, II.30]), we thus have

$$J(x) = \int_0^{\infty} e^{-rs}Q_s^v g_v(x)\,ds + c_1\psi_r(x) + c_2\phi_r(x),$$

where $(Q_t^v)_{t\ge 0}$ is the transition function for the solutions of (3.9) for a given value of v, and ϕ_r and ψ_r are decreasing and increasing solutions of $A^v u - ru = 0$, respectively. We claim that $c_1 = c_2 = 0$. Indeed, suppose $v = 0$. Then $J'(\infty) = 1$ and $J(-\infty) = 0$. However, since ℓ and u are natural boundaries, $\phi_r(-\infty) = \infty = \psi_r'(\infty)$ (see [5, II.10]), which in turn yields $c_1 = c_2 = 0$. Similar considerations yield the same conclusion when $v = 1$.

Therefore,

$$e^{-rt}Q_t^v J(y) = \int_t^{\infty} e^{-rs}Q_s^v g_v(y)\,ds.$$

But the above converges to 0 as $t \to \infty$. Indeed, since

$$Q_t^v g_v = \mathbf{1}_{\{v=1\}}\frac{P_t(sg_v)}{s} + \mathbf{1}_{\{v=0\}}\frac{P_t((1-s)g_v)}{1-s},$$

where $(P_t)_{t\geq 0}$ is the transition function for the Ornstein–Uhlenbeck process with generator (3.5), we have

$$Q_t^v g_v(y) \leq \max\left\{\frac{P_t s'(y)}{s(y)}, \frac{P_t s'(y)}{1-s(y)}\right\} \leq \frac{\sqrt{r}}{\sqrt{\pi}} \max\left\{\frac{1}{s(y)}, \frac{1}{1-s(y)}\right\}. \qquad \square$$

In view of the above lemma, we can now construct the optimal strategy of the insider when the market makers use s as their pricing rule.

Proposition 3.5 *Suppose that $P_t = s(Y_t)$, where s is given by (3.6) and*

$$Y_t = y + X_t + \int_0^t (rY_s + \delta)\, ds,$$

with $y \in \mathbb{R}$ being chosen so that $s(y) = p = \mathbb{P}[\Gamma = 1]$. Then an optimal strategy for the insider is

$$\alpha_t = \Gamma \frac{s'(Y_t)}{s(Y_t)} - (1-\Gamma)\frac{s'(Y_t)}{1-s(Y_t)}. \tag{3.10}$$

The expected profit of the insider who uses this strategy equals $J(y)$, where J is the function defined in (3.7).

Proof Let J be given by (3.7) and recall from (3.8) that $AJ - rJ = 0$. Thus, Itô's formula together with integration by parts yields

$$e^{-rt} J(Y_t) = J(y) + \int_0^t e^{-rs}\big(h(Y_s) - \Gamma\big)\{dB_s + \alpha_s\, ds\},$$

where α is an arbitrary admissible strategy of the insider. Since h and v are bounded,

$$\int_0^t e^{-rs}\big(h(Y_s) - \Gamma\big)\, dB_s$$

is a uniformly integrable martingale. Thus, since $J \geq 0$,

$$\mathbb{E}^v\left[\int_0^\infty e^{-rs}\big(v - h(Y_s)\big)\alpha_s\, ds\right] \leq J(y),$$

i.e., $J(y)$ is an upper bound for the expected profit of the insider. Observe that the distribution of $J(Y_t)$ under $\mathbb{P}^v$ depends on the choice of α. Thus, if we can find an admissible α for which $\mathbb{E}^v[\lim_{t\to\infty} e^{-rt} J(Y_t)] = 0$, it will be optimal. However, Lemma 3.4 shows that if α is given by (3.10), then this limit indeed equals 0. Moreover, since h is bounded, the admissibility of α will follow as soon as

$$\int_0^t |\alpha_s|\, ds < \infty \qquad \mathbb{P}^v\text{-a.s.}$$

Indeed, on $\{\Gamma = 1\}$, for every $T > 0$, we have

$$\int_0^T \mathbb{E}^v\left[\frac{s'(Y_t)}{s(Y_t)}\right] dt = \int_0^T E^y\left[\frac{s'(R_t)}{s(R_t)} s(R_t)\right] dt = \int_0^T E^y[s'(R_t)]\, dt$$

in view of the absolute continuity relationship between the solutions of (3.9) and the Ornstein–Uhlenbeck process with generator (3.5), as observed in the proof of Theorem 3.3. Moreover, since $0 < s < 1$ and s' is bounded, the admissibility on the set $\{\Gamma = 1\}$ is verified. It can be shown similarly that α is admissible on the set $\{\Gamma = 0\}$. This completes the proof. □

A Markovian equilibrium for the market under consideration is given in the following result.

Theorem 3.6 *Let $a \equiv 1$, $\phi(x) = rx + \delta$ for some $\delta \in \mathbb{R}$, s be the function defined in (3.6), y the unique solution of $s(y) = p$, and α the process given by (3.10). Then $((1, \phi, y), s, \alpha)$ is an equilibrium.*

Proof Given this choice of σ, ϕ and s, we have seen in Proposition 3.5 that α is optimal. Thus it remains to show, in view of Proposition 2.5, that

$$s(Y_t) = \mathbb{E}[\Gamma|\mathcal{F}_t^X] = \mathbb{E}[\Gamma|\mathcal{F}_t^Y],$$

as all $\mathcal{F}^X$- (or, equivalently, $\mathcal{F}^Y$-) martingales are continuous due to the fact that the filtration is Brownian (see Corollary 3.8 for another manifestation of this fact). However, this assertion follows easily from Theorem 3.3 since $s(y) = \mathbb{P}[\Gamma = 1]$. □

Although the theorem above gives the impression that there is a continuum of equilibria indexed by δ, in fact all these choices of ϕ lead to the same SDE for the price. Moreover, the insider's optimal strategy does not depend on δ when expressed in terms of P.

Corollary 3.7 *Let δ, ϕ and α be as in Theorem 3.6 and let P^* be the equilibrium price associated to $((1, \phi, y), s, \alpha)$. Then P^* is a uniformly integrable $\mathcal{F}^X$-martingale with $P^*_\infty = \Gamma$ and*

$$P_t^* = \mathbb{E}[\Gamma] + \int_0^t \lambda(P_s^*)\left(dB_s + \left(\Gamma\frac{\lambda(P_s^*)}{P_s^*} - (1-\Gamma)\frac{\lambda(P_s^*)}{1-P_s^*}\right)ds\right), \tag{3.11}$$

where $\lambda(x) = s_0'(s_0^{-1}(x))$ and s_0 is the function in (3.6) with $\delta = 0$. In particular, the function λ does not depend on δ. Moreover, the insider's strategy in equilibrium has the form

$$\alpha_t^* = \Gamma\frac{\lambda(P_t^*)}{P_t^*} - (1-\Gamma)\frac{\lambda(P_t^*)}{1-P_t^*}. \tag{3.12}$$

Proof Let $\delta \in \mathbb{R}$ be fixed and $Y = B + \int_0^\cdot (\phi(Y_t) + \alpha_t)\,dt$ so that $P_t^* = s(Y_t)$, where s is given by (3.6). Then

$$dP_t^* = s'(Y_t)\left(dB_t + \left(\Gamma\frac{s'(Y_t)}{s(Y_t)} - (1-\Gamma)\frac{s'(Y_t)}{1-s(Y_t)}\right)dt\right).$$

Observe that $P_0^* = s(y) = \mathbb{E}[\Gamma]$ by the choice of y.

Next, note that

$$s(x) = s_0\left(x + \frac{\delta}{r}\right),$$

implying $s'(x) = s_0'(x + \frac{\delta}{r})$, as well as $s^{-1}(x) = s_0^{-1}(x) - \frac{\delta}{r}$. Combining these two observations, we then deduce that

$$s'\big(s^{-1}(x)\big) = s_0'\big(s_0^{-1}(x)\big), \qquad \forall \delta \in \mathbb{R}.$$

Therefore,

$$dP_t^* = s_0'\big(s_0^{-1}(P_t^*)\big)\left(dB_t + \left(\Gamma \frac{s_0'(s_0^{-1}(P_t^*))}{P_t^*} - (1-\Gamma)\frac{s_0'(s_0^{-1}(P_t^*))}{1-P_t^*}\right)dt\right).$$

This yields, in view of the definition of λ, the dynamics of P^* given by (3.11).

That P^* is uniformly integrable follows from the boundedness of s. Its limiting property $P_\infty^* = s(Y_\infty) = \Gamma$ is due to Theorem 3.3.

The form of the insider's strategy follows immediately from the corresponding change of variable. □

In Kyle's model with risk-neutral market makers, it is in general observed that in equilibrium, the insider's trades are inconspicuous, i.e., the distribution of the equilibrium demand process equals that of the noise trades. We observe the same phenomenon here.

Corollary 3.8 *Consider the equilibrium described in Theorem 3.6 or Corollary 3.7 and let X^* denote the equilibrium level of demand. Then X^* is a Brownian motion in its own filtration.*

Proof Note that

$$X_t^* = B_t + \int_0^t \alpha_s^*\, ds,$$

where α^* is given by (3.12). Recall that the relationship between X and Y entails they generate the same filtration. Moreover, since the pricing rule is a strictly increasing function, we deduce that the filtrations generated by P^* and X^* coincide. Therefore,

$$\mathbb{E}[\alpha_t^* | \mathcal{F}_t^{X^*}] = \mathbb{E}[\alpha_t^* | \mathcal{F}_t^{P^*}] = \mathbb{E}[\Gamma | \mathcal{F}_t^{P^*}]\frac{\lambda(P_t)}{P_t} - (1 - \mathbb{E}[\Gamma | \mathcal{F}_t^{P^*}])\frac{\lambda(P_t)}{1-P_t} = 0$$

since P^* is a uniformly integrable $\mathcal{F}^{Y^*}$-martingale with $P_\infty^* = \Gamma$ by Corollary 3.7. Thus, X^* is a Brownian motion in its own filtration by [14, Theorem 8.1]. □

Since we are able to characterise the equilibrium price process as a function of an Ornstein–Uhlenbeck process, this allows us to observe a deviation in equilibrium from the original model in Kyle. In [13], as explained in [3] and [9], the reciprocal of the market depth follows a martingale preventing 'systematic changes' in the market depth. In the absence of such systematic changes, the insider cannot acquire a large position when the depth is low to liquidate at a later date when the liquidity is higher to obtain unbounded profits. The reciprocal of the market depth is called *Kyle's lambda*, and it is given by the process $\lambda(P^*)$ in our model, where λ and P^* are as in Corollary 3.7. The next result shows that in contrast to Kyle's findings, the reciprocal of the market depth follows a supermartingale, which was also observed by Back and Baruch in [2] using different arguments.

Corollary 3.9 *Let λ and P^* be as in Corollary 3.7. Then $\lambda(P^*)$ is a $\mathcal{G}^M$-supermartingale such that* $\lim_{t\to\infty}\mathbb{E}[\lambda(P^*_t)]=0$, *i.e., $\lambda(P^*)$ is a $\mathcal{G}^M$-potential.*

Proof As observed earlier, suppose without loss of generality that $\delta=0$ so that $\phi(x)=rx$ and the market makers' signal Y^* solves in its own filtration

$$dY^*_t=d\beta_t+rY^*_t\,dt,$$

where β is an $\mathcal{F}^{Y^*}$-Brownian motion. Since $P^*=s_0(Y^*)$, it suffices to show that $s_0'(Y^*)$ is an $\mathcal{F}^{Y^*}$-supermartingale. Independence of τ and P^* will then imply that $s_0'(Y^*)$, hence also $\lambda(Y^*)$, is a $\mathcal{G}^M$-supermartingale.

Now we have

$$s_0'(x)=\sqrt{\frac{r}{\pi}}e^{-rx^2}.$$

Thus an application of Itô's formula yields

$$ds_0'(Y^*_t)=-s_0'(Y^*_t)(2rY^*_t\,d\beta_t+r\,dt),$$

which in turn implies that $s_0'(Y^*)$ has the required supermartingale property. Moreover, as a consequence of the dominated convergence theorem, we obtain

$$\lim_{t\to\infty}\mathbb{E}[s_0'(Y^*_t)]=\mathbb{E}\Big[\lim_{t\to\infty}s_0'(Y^*_t)\Big]=0$$

since $\lim_{t\to\infty}|Y^*_t|=\infty$. □

Remark 3.10 In fact, the actual marginal price impact that is observed in the market is given by $\lambda(P_t)\mathbf{1}_{\{\tau>t\}}$ since the price is constant from τ onwards. Note that this is again a $\mathcal{G}^M$-supermartingale in view of Lemma A.1.

A simple consequence of the above result is that the market gets more liquid on average as time passes. The reason for this deviation from Kyle is due to the random deadline for the whole trading activities, which is independent of everything else. The insider is aware of the fact that her informational advantage is going to end at a totally inaccessible stopping time, which will come as a surprise. She nevertheless chooses not to trade aggressively revealing her information quickly since, as observed by Back and Baruch [2], the price impact is decreasing in time so that the risk of waiting can be compensated by lower execution costs.

Remark 3.11 The computations made in this section suggest that there is no time-homogeneous equilibrium unless Γ has a Bernoulli distribution. Indeed, if X is independent of τ, Proposition 3.1 implies that Y is a transient Ornstein–Uhlenbeck process. On the other hand, $P_\infty = \Gamma$ and the equilibrium price is given by a scale function of Y. Consequently, P_∞ can take only two different values since $Y_\infty \in \{-\infty, \infty\}$.

3.2 More general liquidation value

In this section, we consider more general distributions for Γ and suppose[3] that we have $\Gamma \stackrel{d}{=} f(\eta)$ for some *continuous and strictly increasing* f and a standard normal random variable η. We can incorporate atoms and consider more general distributions for Γ. However, this will only result in more complicated coefficients for Y^* in the filtration of the insider and will not significantly alter the qualitative inferences that one can make within this model. Thus, for simplicity of exposition and model, we make the following assumption.

Assumption 3.12 *There exists a continuous, strictly increasing function* $f:\mathbb{R}\to\mathbb{R}$ *such that* $\Gamma \stackrel{d}{=} f(\eta)$, *where* η *is a standard normal random variable.*

Remark 3.11 suggests that one cannot go beyond a Bernoulli-distributed payoff using a time-homogeneous SDE for Y. Collin-Dufresne et al. [8] consider a similar problem when $\Gamma \stackrel{d}{=} \eta$ and obtain an equilibrium where the coefficients of Y^* depend on time by using ideas from Kalman filtering.

We next show that an equilibrium exists for more general payoff distributions. In fact, the market makers' equilibrium signal process Y^* will turn out to be just a time change of the Ornstein–Uhlenbeck process that appears in the equilibrium when Γ had a Bernoulli distribution as in the last section. To wit, let us suppose

$$dY_t = \sigma(t)a(Y_t)\,dX_t + \sigma^2(t)\phi(Y_t)\,dt,$$

where $\sigma:\mathbb{R}_+\to\mathbb{R}_+$. Following similar arguments as in the beginning of Sect. 3.1, we obtain after obvious modifications

$$h_t + \frac{1}{2}a^2\sigma^2 h_{yy} + \sigma^2\phi h_y = 0, \tag{3.13}$$

$$\frac{1}{2}a\sigma^2 a'' + \phi\sigma^2\frac{a'}{a} - \sigma^2\phi' + \frac{\sigma'}{\sigma} + r = 0.$$

It is easy to check that when $\sigma \equiv 1$, the above equations reduce to (3.2) and (3.3).

As in the previous section, we choose $a \equiv 1$ and $\phi(x) = rx$. This implies

$$\frac{\sigma'}{\sigma(1-\sigma)(1+\sigma)} = -r. \tag{3.14}$$

[3] $\stackrel{d}{=}$ stands for equality in distribution.

Since

$$\frac{1}{x(1-x)(1+x)} = \frac{1}{x} + \frac{1}{2(1-x)} - \frac{1}{2(1+x)},$$

integrating (3.14) yields

$$\frac{\sigma^2(t)}{1-\sigma^2(t)} = C^2 e^{-2rt}$$

for some constant C to be determined later. Thus we can solve the above to deduce

$$\sigma^2(t) = \frac{C^2 e^{-2rt}}{1+C^2 e^{-2rt}}. \tag{3.15}$$

The next lemma defines a function which later turns out to be the value function for the insider.

Lemma 3.13 *Let* $h : \mathbb{R}_+ \times \mathbb{R} \to \mathbb{R}$ *satisfy* (3.13) *and be such that* $h(t,\cdot)$ *is strictly increasing for each* $t \geq 0$. *Consider*

$$J(t,y) := \int_{h^{-1}(t,v)}^{y} \frac{h(t,x)-v}{\sigma(t)}\,dx + \frac{1}{2}e^{rt}\int_t^\infty e^{-rs}\sigma(s)h_y\big(s,h^{-1}(s,v)\big)\,ds,$$

where $h^{-1}(t,\cdot)$ *represents the inverse of* $h(t,\cdot)$ *for every* $t \geq 0$. *Then we have*

$$J_t + \frac{1}{2}\sigma^2(t)J_{yy} + \sigma^2(t)ryJ_y - rJ = 0,$$

provided that

$$\int_t^\infty e^{-rs}\sigma(s)h_y\big(s,h^{-1}(s,v)\big)\,ds < \infty, \qquad \forall t \geq 0.$$

Proof Direct differentiation yields

$$\begin{aligned}
J_t &= \int_{h^{-1}(t,v)}^{y} \frac{h_t(t,x)}{\sigma(t)}\,dx - \int_{h^{-1}(t,v)}^{y} \frac{h(t,x)-v}{\sigma^2(t)}\sigma'(t)\,dx \\
&\quad + \frac{re^{rt}}{2}\int_t^\infty e^{-rs}\sigma(s)h_y\big(s,h^{-1}(s,v)\big)\,ds - \frac{1}{2}\sigma(s)h_y\big(t,h^{-1}(t,v)\big) \\
&= -\sigma(t)\int_{h^{-1}(t,v)}^{y}\left(\frac{1}{2}h_{yy}(t,x) + h_y(t,x)rx\right)dx - \int_{h^{-1}(t,v)}^{y} \frac{h(t,x)-v}{\sigma^2(t)}\sigma'(t)\,dx \\
&\quad + \frac{re^{rt}}{2}\int_t^\infty e^{-rs}\sigma(s)h_y\big(s,h^{-1}(s,v)\big)\,ds - \frac{1}{2}\sigma(s)h_y\big(t,h^{-1}(t,v)\big) \\
&= \frac{\sigma(t)}{2}\Big(h_y\big(t,h^{-1}(t,v)\big) - h_y(t,y)\Big) - \int_{h^{-1}(t,v)}^{y} \frac{h(t,x)-v}{\sigma^2(t)}\sigma'(t)\,dx \\
&\quad - \sigma(t)\int_{h^{-1}(t,v)}^{y} h_y(t,x)rx\,dx + \frac{re^{rt}}{2}\int_t^\infty e^{-rs}\sigma(s)h_y\big(s,h^{-1}(s,v)\big)\,ds \\
&\quad - \frac{1}{2}\sigma(t)h_y\big(t,h^{-1}(t,v)\big).
\end{aligned}$$

Thus,

$$
\begin{aligned}
J_t + AJ - rJ &= \sigma(t) r y \big(h(t,y) - v\big) - \int_{h^{-1}(t,v)}^{y} \frac{(h(t,x)-v)(r\sigma + \sigma'(t))}{\sigma(t)}\, dx \\
&\quad - \sigma(t) \int_{h^{-1}(t,v)}^{y} h_y(t,x) r x \, dx \\
&= \sigma(t) r y \big(h(t,y) - v\big) - r\sigma(t) \int_{h^{-1}(t,v)}^{y} \big(h(t,x) - v\big)\, dx \\
&\quad - \sigma(t) \int_{h^{-1}(t,v)}^{y} h_y(t,x) r x \, dx \\
&= 0,
\end{aligned}
$$

where the last equality follows from integration by parts. □

The PDE (3.13) satisfied by h indicates that the market makers's signal in equilibrium will be a time-changed Ornstein–Uhlenbeck process, where the time change is given by

$$
V(t) := \int_0^t \sigma^2(s)\, ds = \frac{1}{2r} \log \frac{1 + C^2}{1 + C^2 e^{-2rt}}. \tag{3.16}
$$

Indeed, any solution of (3.13) can be obtained by a time change as we see in the next lemma.

Lemma 3.14 *Suppose that $a \equiv 1$ and $\phi(x) = rx$. Then h is a solution of (3.13) if and only if g defined by*

$$
g(t,y) := h\big(V^{-1}(t), y\big),
$$

where V is the absolutely continuous function given in (3.16), solves

$$
g_t + \frac{1}{2} g_{yy} + r y g_y = 0.
$$

Proof Note that $h(t,y) = g(V(t), y)$. Since $dV(t) = \sigma^2(t)\, dt$, the claim follows. □

Consistently with the findings of the previous section, the insider should construct a bridge process Y^* such that $\lim_{t\to\infty} h^*(t, Y_t^*) = \Gamma$ and Y^* in its own filtration follows

$$
dY_t^* = \sigma(t)\, dB_t^Y + r\sigma^2(t) Y_t^*\, dt,
$$

i.e., a time-changed version of the Ornstein–Uhlenbeck process with generator (3.5), where $\delta = 0$. As observed in the previous lemma, the time change is given by the function $V(t)$ which converges to $V(\infty) = \frac{1}{2r}\log(1 + C^2) < \infty$. This implies that the distribution of Y_∞^* equals that of the Ornstein–Uhlenbeck process defined by (3.5) at $V(\infty)$ with $\delta = 0$. This distribution is Gaussian and allows us to go beyond a Bernoulli distribution for Γ.

Let $p(t,x,y)$ be the transition density of the Ornstein–Uhlenbeck process defined by (3.5) with $\delta=0$. It is well known that

$$p(t,x,y)=q\left(\frac{e^{2rt}-1}{2r},\, y-xe^{rt}\right),$$

where $q(t,x)=\frac{1}{\sqrt{2\pi t}}e^{-\frac{x^2}{2t}}$. The next theorem defines the bridge process that is the key to the insider's strategy in equilibrium.

Theorem 3.15 *Let f be the function in Assumption 3.12 and assume $C^2=2r$. Then for any $v\in\mathbb{R}$, there exists a unique strong solution to*

$$Y_t=\int_0^t \sigma(s)\,dB_s+r\int_0^t \frac{f^{-1}(v)-Y_s\cosh(\frac{1}{2}\log(1+2re^{-2rs}))}{\sinh(\frac{1}{2}\log(1+2re^{-2rs}))}\sigma^2(s)\,ds,$$

where $\sigma>0$ is defined via (3.15). Moreover, $\lim_{t\to\infty}Y_t=f^{-1}(v)$ $\mathbb{Q}^v$-a.s., where $\mathbb{Q}^v$ is the law of the solution. Moreover,

$$\mathbb{E}^{\mathbb{Q}^v}[F(Y_s;s\le t)]=\frac{\mathbb{E}^{\mathbb{Q}}[p(V(\infty)-V(t),Y_t,f^{-1}(v))F(Y_s;s\le t)]}{p(V(\infty),0,f^{-1}(v))},$$

where F is a bounded measurable function and $\mathbb{Q}$ is the law of the unique solution to

$$Y_t=\int_0^t \sigma(s)\,dB_s+r\int_0^t \sigma^2(s)Y_s\,ds. \tag{3.17}$$

Proof The existence and uniqueness of the solution follow immediately since the SDE has Lipschitz coefficients in every compact interval $[0,T]$.

Next observe that

$$\frac{1}{2}\log(1+2re^{-2rt})=r\bigl(V(\infty)-V(t)\bigr).$$

Thus, if we define $R_t:=Y_{V^{-1}(t)}$, we obtain

$$R_t=\int_0^{V^{-1}(t)}\sigma(s)\,dB_s+r\int_0^t \frac{f^{-1}(v)-R_s\cosh(r(V(\infty)-s))}{\sinh(r(V(\infty)-s))}\,ds.$$

On the other hand, $\beta_t:=\int_0^{V^{-1}(t)}\sigma(s)\,dB_s$ is a local martingale with respect to the filtration $(\mathcal{G}_t)_{t\ge0}$, where $\mathcal{G}_t:=\mathcal{F}_{V^{-1}(t)}$. Moreover, $[\beta,\beta]_t=t$ for each $t\ge0$. Therefore, β is a $\mathcal{G}$-Brownian motion. Consequently,

$$R_t=\beta_t+r\int_0^t \frac{f^{-1}(v)-R_s\cosh(r(V(\infty)-s))}{\sinh(r(V(\infty)-s))}\,ds,\qquad t<V(\infty).$$

However, the above is the SDE for

$$\rho_t=W_t+r\int_0^t \rho_s\,ds$$

conditioned on the event $\{\rho_{V(\infty)} = f^{-1}(v)\}$. Indeed, the SDE representation of this Markovian bridge follows from [10, Example 2.3], which coincides with the above SDE since $F(t) = e^{rt}$ and $\Sigma(s,t) = \frac{e^{2rt}-1}{2r}$, where F and Σ are the functions defined in [10, Example 2.3]. Therefore, $R_t \to f^{-1}(v)$ as $t \to V(\infty)$, which is equivalent to $Y_t \to f^{-1}(v)$ as $t \to \infty$.

The absolute continuity relationship is a consequence of [10, Theorem 2.2] since the solution of (3.17) is a Markov process with transition density $p(V(t) - V(s), y, z)$. □

We are now ready to state the existence of an equilibrium in the next theorem, whose proof is postponed to Appendix B.

Theorem 3.16 *Suppose $C^2 = 2r$ and assume that there exist positive constants $K > 0$ and $k < \frac{1}{1+2r}$ such that*

$$|f(y)| \le K e^{\frac{ky^2}{4}}.$$

Define

$$g(t,y) := \int_{-\infty}^{\infty} f(z) p\big(V(\infty) - t, y, z\big)\, dz$$

and let $h^*(t,y) = g(V(t), y)$. Then $((\sigma^*, \phi^*, 0), h^*, \alpha^*)$ is an equilibrium, where σ^* is the positive square root of (3.15) with $C^2 = 2r$, $\phi^*(y) = ry$, and

$$\alpha^*(t) = r\sigma^*(t)\left(\frac{f^{-1}(\Gamma) - Y_t^* \cosh(\frac{1}{2}\log(1+2re^{-2rt}))}{\sinh(\frac{1}{2}\log(1+2re^{-2rt}))} - Y_t^*\right).$$

Moreover,

$$Y_t^* = \int_0^t \sigma^*(s)\, dB_s^* + r\int_0^t \big(\sigma^*(s)\big)^2 Y_s^*\, ds, \tag{3.18}$$

where B^* is an $\mathcal{F}^{Y^*}$-Brownian motion.

As in the time-homogeneous case, Kyle's lambda will be a uniformly integrable supermartingale. However, in contrast to the time-homogeneous case, it does not disappear as $t \to \infty$, i.e., it is not a potential in general.

Proposition 3.17 *Consider the equilibrium given in Theorem 3.16 and define the process $\lambda_t^* := h_y^*(t, Y_t^*)$. Then λ^* is a uniformly integrable $(\mathcal{F}^{Y^*}, \mathbb{P})$-supermartingale and*

$$\lim_{t\to\infty} \mathbb{E}[\lambda_t^*] = \int_{-\infty}^{\infty} f'(z) \frac{1}{\sqrt{2\pi}} e^{-\frac{z^2}{2}}\, dz. \tag{3.19}$$

Proof Define $u(t,y) = h_y^*(t,y)$. Differentiating

$$h_t^* + \frac{\sigma^2(t)}{2} h_{yy}^* + \sigma^2(t) r y h_y^* = 0$$

with respect to y yields

$$u_t + \frac{\sigma^2(t)}{2} u_{yy} + \sigma^2(t) r y u_y = -r\sigma^2(t) u.$$

Applying Itô's formula to u and Y^*, which satisfies (3.18), yields

$$d\lambda_t^* = u_y(t, Y_t^*)\sigma^*(t)\, dB_t^* - r\big(\sigma^*(t)\big)^2 \lambda_t^*\, dt.$$

Since $\lambda^* \geq 0$, the stochastic integral in the above decomposition is a supermartingale, which leads to the desired supermartingale property of λ^*.

It follows from (B.1) that

$$u(t, y) = \sqrt{1 + 2re^{-2rt}} \int_{-\infty}^{\infty} f'(z) p\big(V(\infty) - V(t), y, z\big)\, dz.$$

Thus, $\lambda_\infty^* := \lim_{t\to\infty} \lambda_t^* = f'(Y_\infty^*)$ $\mathbb{P}$-a.s. and

$$\mathbb{E}[u(t, Y_t^*)] = \sqrt{1 + 2re^{-2rt}} \int_{-\infty}^{\infty} f'(z) p\big(V(\infty), 0, z\big)\, dz$$

by the Chapman–Kolmogorov equation. Therefore,

$$\lim_{t\to\infty} \mathbb{E}[u(t, Y_t^*)] = \int_{-\infty}^{\infty} f'(z) p\big(V(\infty), 0, z\big)\, dz = \mathbb{E}[f'(Y_\infty^*)] = \mathbb{E}[\lambda_\infty^*].$$

This implies that λ^* is a uniformly integrable supermartingale. (3.19) follows from the fact that Y_∞^* is standard normal. □

Remark 3.18 As in Corollary 3.8, one can show that the insider's trades are inconspicuous, i.e., X^* is a Brownian motion in its own filtration. This directly follows from the equilibrium level of Y^* which satisfies (3.18).

4 Conclusion

Using tools from potential theory for one-dimensional diffusions, we have solved a version of the Kyle model with general payoffs when the announcement date has an exponential distribution and is independent of all other parameters of the model. It is shown that a stationary equilibrium exists only if the payoff has a Bernoulli distribution, which corresponds to the special case considered in [2]. The approach considered here is novel in its study and characterisation of the optimal strategies of the insider in terms of excessive functions of an associated diffusion process. In particular, the so-called Kyle's lambda, which is a measurement of liquidity, is identified with a *potential*.

As in the earlier literature on the Kyle model, we have shown that the total demand for the asset in equilibrium has the same distribution as that of the noise traders, i.e., the insider's trades are inconspicuous. What is different from the earlier models, however, is that the equilibrium prices no longer converge to the payoff Γ as time approaches the announcement date. That is, there is a jump in the price when Γ becomes public knowledge. This is due to the fact that the announcement comes as a surprise even for the insider and therefore, she is not able to construct a bridge of random length τ for the demand process in order for the prices to converge to Γ (cf. the bridge construction in [7]). However, in equilibrium, she trades in such a way that the price process conditioned on *no announcement*, i.e., P, converges to Γ.

It will be interesting to see how these conclusions, especially the last one, change if the announcement date τ is no longer assumed to be independent of the other variables. However, this would require a framework beyond the scope of the current paper and is left for future study.

Appendix A: Proof of Proposition 2.5

Recall that we are searching for a decomposition

$$S_t = P_t \mathbf{1}_{\{t<\tau\}} + \Gamma \mathbf{1}_{\{t\geq\tau\}},$$

where P is a semimartingale adapted to $\mathcal{F}^X$. In order for P to be a candidate price process, one must have that S is a $\mathcal{G}^M$-martingale. To this end, the following lemma will be crucial.

Lemma A.1 *Define*

$$\begin{aligned} N_t &:= \Gamma \mathbf{1}_{\{t\geq\tau\}} - r\int_0^t \mathbf{1}_{\{s<\tau\}} \mathbb{E}[\Gamma|\mathcal{F}_s^X]\,ds, \\ M_t &:= \mathbf{1}_{\{t\geq\tau\}} - r\int_0^t \mathbf{1}_{\{s<\tau\}}\,ds. \end{aligned}$$

Then N and M are $\mathcal{G}^M$-martingales.

Proof Note that for $s<t$,

$$\begin{aligned} \mathbb{E}[N_t|\mathcal{G}_s^M] &= \mathbf{1}_{\{\tau\leq s\}}\mathbb{E}\left[\Gamma - r\int_0^\tau \mathbb{E}[\Gamma|\mathcal{F}_u^X]\,du\middle|\mathcal{G}_s^M\right] \\ &\quad + \mathbf{1}_{\{\tau>s\}}\mathbb{E}\left[\Gamma\mathbf{1}_{\{t\geq\tau\}} - r\int_0^t \mathbf{1}_{\{u<\tau\}}\mathbb{E}[\Gamma|\mathcal{F}_u^X]\,du\middle|\mathcal{G}_s^M\right] \\ &= \mathbf{1}_{\{\tau\leq s\}}N_s + \mathbf{1}_{\{\tau>s\}}\left(\mathbb{E}[\Gamma|\mathcal{F}_s^X](1-e^{-r(t-s)}) - r\int_0^s \mathbb{E}[\Gamma|\mathcal{F}_u^X]\,du\right) \\ &\quad - \mathbf{1}_{\{\tau>s\}}\mathbb{E}\left[\int_s^t r\mathbf{1}_{\{u<\tau\}}\mathbb{E}[\Gamma|\mathcal{F}_u^X]\,du\middle|\mathcal{G}_s^M\right] \\ &= \mathbf{1}_{\{\tau\leq s\}}N_s + \mathbf{1}_{\{\tau>s\}}\left(\mathbb{E}[\Gamma|\mathcal{F}_s^X](1-e^{-r(t-s)}) - r\int_0^s \mathbb{E}[\Gamma|\mathcal{F}_u^X]\,du\right) \\ &\quad - \mathbf{1}_{\{\tau>s\}}\int_s^t re^{-r(u-s)}\mathbb{E}[\Gamma|\mathcal{F}_s^X]\,du \\ &= N_s, \end{aligned}$$

where the second and third equalities are due to the independence of X and τ.

The proof for the martingale property of M follows along similar lines. □

In view of the above lemma, in order for S to be a $\mathcal{G}^M$-martingale, one needs to show that

$$U_t := P_t \mathbf{1}_{\{t<\tau\}} + r\int_0^t \mathbf{1}_{\{s<\tau\}} \mathbb{E}[\Gamma|\mathcal{F}_s^X]\,ds$$

is a $\mathcal{G}^M$-martingale. This leads to

Proof of Proposition 2.5 Suppose that the semimartingale decomposition of P is given by $P_t = P_0 + Z_t + A_t$, where Z is a continuous local martingale and A is a predictable process of finite variation.[4] Let

$$U_t = P_t \mathbf{1}_{\{t<\tau\}} + r\int_0^t \mathbf{1}_{\{s<\tau\}} \mathbb{E}[\Gamma|\mathcal{F}_s^X]\,ds.$$

Then in view of the previous lemma, we have

$$\begin{aligned} dU_t &= \mathbf{1}_{\{t\le\tau\}}(dZ_t + dA_t) - P_{t-}(dM_t + r\mathbf{1}_{\{t<\tau\}}\,dt) \\ &\qquad + r\mathbf{1}_{\{t<\tau\}}\mathbb{E}[\Gamma|\mathcal{F}_t^X]\,dt - \Delta A_\tau \mathbf{1}_{\{t=\tau\}} \\ &= \mathbf{1}_{\{t\le\tau\}}\,dZ_t - P_{t-}\,dM_t + \mathbf{1}_{\{t<\tau\}}\,dA_t + r\mathbf{1}_{\{t<\tau\}}(\mathbb{E}[\Gamma|\mathcal{F}_t^X] - P_t)\,dt. \end{aligned}$$

Consequently,

$$\int_0^t \big(\mathbf{1}_{\{s<\tau\}}\,dA_s + r\mathbf{1}_{\{s<\tau\}}(\mathbb{E}[\Gamma|\mathcal{F}_s^X] - P_s)\,ds\big)$$

is a predictable local martingale of finite variation. Thus,

$$\mathbf{1}_{\{t<\tau\}}\,dA_t = \mathbf{1}_{\{t<\tau\}} r(P_t - \mathbb{E}[\Gamma|\mathcal{F}_t^X])\,dt.$$

Since τ is independent of $\mathcal{F}^X$ and $\mathbb{P}[\tau > t] > 0$ for all $t \ge 0$, this yields

$$dA_t = r(P_t - \mathbb{E}[\Gamma|\mathcal{F}_t^X])\,dt.$$

Thus A is continuous and so is P. Consider

$$\tau_n := \inf\{t \ge 0 : |Z_t| > n\}.$$

Since Z is continuous, $\tau_n \to \infty$ $\mathbb{P}$-a.s.

Next, let $\hat{\Gamma}_t = \mathbb{E}[\Gamma|\mathcal{F}_t^X]$ and consider $\sigma_n := \inf\{t \ge 0 : |P_t - \hat{\Gamma}_t| > n\}$, which converges to ∞ due to the continuity of P and $\hat{\Gamma}$. Then

$$P_{t\wedge\tau_n\wedge\sigma_m} - \hat{\Gamma}_{t\wedge\tau_n\wedge\sigma_m} - r\int_0^{t\wedge\tau_n\wedge\sigma_m} (P_s - \hat{\Gamma}_s)\,ds$$

[4] Under the assumption that all martingales are continuous, the optional and predictable σ-algebras coincide.

is a martingale for each n and m. Moreover, denoting the semimartingale local time of $P - \hat{\Gamma}$ at 0 by L, we deduce that

$$|P_{t\wedge\tau_n\wedge\sigma_m} - \hat{\Gamma}_{t\wedge\tau_n\wedge\sigma_m}| - r\int_0^{t\wedge\tau_n\wedge\sigma_m} |P_u - \hat{\Gamma}_u|\,du - L_{t\wedge\tau_n\wedge\sigma_m} \tag{A.1}$$

is a $\mathcal{G}$-local martingale and thus a submartingale, being bounded from above.

Also, note that since S is a uniformly integrable martingale as observed in the proof of Proposition 2.1, since $\mathbf{1}_{\{\tau>t\}}|P_t| \le S_t$ and since τ is independent of P, the family $(P_{t\wedge\tau_n\wedge\sigma_m})_{n\ge1,m\ge1}$ is uniformly integrable. Thus we obtain for $s < t$

$$\mathbb{E}[|P_t - \hat{\Gamma}_t||\mathcal{F}_s^X] \ge |P_s - \hat{\Gamma}_s| + r\int_s^t \mathbb{E}[|P_u - \hat{\Gamma}_u||\mathcal{F}_s^X]\,du$$

after taking limits as $n \to \infty$ and $m \to \infty$ and using the monotone convergence theorem on the integral in (A.1), as well as the fact that L is increasing. A straightforward application of Gronwall's inequality therefore implies that for any $t > s$,

$$\mathbb{E}\big[|P_t - \hat{\Gamma}_t|\big|\mathcal{F}_s^X\big] \ge |P_s - \hat{\Gamma}_s|e^{r(t-s)}. \tag{A.2}$$

On the other hand,

$$\begin{aligned}
\mathbf{1}_{\{\tau>s\}}\mathbb{E}\big[|S_t - \Gamma|\big|\mathcal{G}_s^M\big] &= \mathbf{1}_{\{\tau>s\}}\mathbb{E}\big[\mathbf{1}_{\tau>t}|P_t - \Gamma|\big|\mathcal{G}_s^M\big] \\
&= \mathbf{1}_{\{\tau>s\}}e^{-r(t-s)}\mathbb{E}\big[|P_t - \Gamma|\big|\mathcal{F}_s^X\big] \\
&\ge \mathbf{1}_{\{\tau>s\}}e^{-r(t-s)}\mathbb{E}\big[|P_t - \hat{\Gamma}_t|\big|\mathcal{F}_s^X\big],
\end{aligned}$$

where the last inequality follows from Jensen's inequality since P_t is $\mathcal{F}_t^X$-measurable. However, (A.2) then yields

$$\lim_{t\to\infty} \mathbf{1}_{\{\tau>s\}}\mathbb{E}\big[|S_t - \Gamma|\big|\mathcal{G}_s^M\big] \ge \lim_{t\to\infty} |P_s - \hat{\Gamma}_s| = |P_s - \hat{\Gamma}_s|,$$

which contradicts (2.3) unless $P_s = \hat{\Gamma}_s$ $\mathbb{P}$-a.s. Since P and $\hat{\Gamma}$ are continuous, the nullset can be chosen to be independent of s, which completes the proof. □

Appendix B: Proof of Theorem 3.16

We show that $((\sigma^*, \phi^*, 0), h^*, \alpha^*)$ is an equilibrium by checking 1) α^* is admissible and optimal given $(\sigma^*, \phi^*, 0)$, and 2) $(\sigma^*, \phi^*, 0)$ is an admissible pricing rule given α^*.

Step 1. *Insider's optimality.* In view of Lemma 3.13, let us first verify that

$$\int_t^\infty e^{-rs}\sigma^*(s)h_y^*\big(s, h^{-1}(s, v)\big)\,ds < \infty, \qquad \forall t \ge 0.$$

To see this, first observe that $h^*_y(s, h^{-1}(s, v)) = \frac{1}{\frac{dh^{-1}(s,y)}{dy}}$ and

$$\begin{aligned} y &= \int_{-\infty}^{\infty} f(z)p\big(V(\infty) - V(t), h^{-1}(t, y), z\big)\,dz \\ &= \int_{-\infty}^{\infty} f(z)p\left(\frac{1}{2r}\log(1 + 2re^{-2rt}), h^{-1}(t, y), z\right)dz \\ &= \int_{-\infty}^{\infty} f(z)q\big(e^{-2rt}, z - h^{-1}(t, y)\sqrt{1 + 2re^{-2rt}}\big)\,dz. \end{aligned} \tag{B.1}$$

Due to the bound on f, we can differentiate inside the integral to get

$$\begin{aligned} 0 &< \frac{1}{\frac{dh^{-1}(s,y)}{dy}} \\ &= \sqrt{1 + 2re^{-2rt}} \int_{-\infty}^{\infty} f(z)\frac{z - h^{-1}(t, y)\sqrt{1 + 2re^{-2rt}}}{e^{-2rt}} \\ &\qquad\qquad \times q\big(e^{-2rt}, z - h^{-1}(t, y)\sqrt{1 + 2re^{-2rt}}\big)\,dz \\ &= \sqrt{1 + 2re^{-2rt}} \int_{-\infty}^{\infty} f\big(z + h^{-1}(t, y)\sqrt{1 + 2re^{-2rt}}\big)\frac{z}{e^{-2rt}}q(e^{-2rt}, z)\,dz \\ &\le C\exp\left(\frac{k}{2}\big(h^{-1}(t, y)\big)^2(1 + 2r)\right)e^{3rt}\int_0^{\infty} \frac{z}{\sqrt{2\pi}}\exp\left(-\frac{z^2}{2}(e^{2rt} - k)\right)dz \\ &= C\exp\left(\frac{k}{2}\big(h^{-1}(t, y)\big)^2(1 + 2r)\right)\frac{e^{3rt}}{e^{2rt} - k} \\ &\sim C\exp\left(\frac{k}{2}\big(h^{-1}(t, y)\big)^2(1 + 2r)\right)e^{rt} \qquad \text{as } t \to \infty, \end{aligned} \tag{B.2}$$

where C above (and also throughout the proof) is a constant independent of t that may change from line to line. Moreover, $h^{-1}(t, y) = g^{-1}(V(t), y)$ is bounded for fixed y since $V(\infty) < \infty$ and $g(, \cdot, y)$ is strictly increasing and continuous on $[0, V(\infty)]$ for each y. Thus the above asymptotic establishes that the condition is verified since

$$\int_0^{\infty} \sigma^*(s) < \infty.$$

Thus, for any admissible α, we have

$$e^{-rt}J(t, Y_t) = J(0, 0) + \int_0^t e^{-rs}\big(h(s, Y_s) - \Gamma\big)(dB_s + \alpha_s\,ds)$$

due to the fact that $J_t + AJ - rJ = 0$ by Lemma 3.13. In view of the admissibility condition (2.5),

$$\left(\int_0^t e^{-rs} \big(h(s, Y_s) - \Gamma \big) \, dB_s \right)_{t \geq 0}$$

is a uniformly integrable martingale that converges in $L^1(\mathbb{P}^v)$. Thus, if $e^{-rt} J(t, Y_t)$ has a limit in $L^1(\mathbb{P}^v)$ as $t \to \infty$, then

$$\mathbb{E}^v \left[\int_0^\infty e^{-rs} \big(h(s, Y_s) - \Gamma \big) \alpha_s \, ds \right] = J(0, 0) - \mathbb{E}^v \Big[\lim_{t \to \infty} e^{-rt} J(t, Y_t) \Big].$$

Since $J \geq 0$, the process α will be the optimal strategy if it achieves

$$\lim_{t \to \infty} e^{-rt} J(t, Y_t) = 0.$$

To see that the above limit holds in $L^1(\mathbb{P}^v)$, first note that

$$\begin{aligned} e^{-rt} \frac{1}{\sigma^*(t)} \int_{h^{-1}(t,v)}^{y} \big(h(t, x) - v \big) \, dx &\leq \frac{\sqrt{1 + 2r}}{\sqrt{2r}} \int_{h^{-1}(t,v)}^{y} \big(h(t, x) - v \big) \, dx \\ &\leq \frac{\sqrt{1 + 2r}}{\sqrt{2r}} \big(h(t, y) - v \big) \big(y - h^{-1}(t, v) \big) \end{aligned} \tag{B.3}$$

due to (3.15) and the fact that h is increasing in y. Thus, $\lim_{t \to \infty} e^{-rt} J(t, Y_t) = 0$ $\mathbb{P}^v$-a.s. if we have $Y_t - h^{-1}(t, v) \to 0$ $\mathbb{P}^v$-a.s. However, Theorem 3.15 shows that α^* yields that $\lim_{t \to \infty} Y_t^* = f^{-1}(v) = \lim_{t \to \infty} h^{-1}(t, v)$ $\mathbb{P}^v$-a.s. Therefore if we have $\sup_{t \geq 0} \mathbb{E}^v [|h^*(s, Y_s^*)|^{2+\varepsilon} + |Y_t^*|^{2+\epsilon}] < \infty$ for some $\epsilon > 0$, we can conclude that $e^{-rt} J(t, Y_t)$ converges to 0 in $L^1(\mathbb{P}^v)$ as $t \to \infty$ in view of (B.3). Note that $\sup_{t \geq 0} \mathbb{E}^v [(h^*(t, Y_t^*))^2] < \infty$ will also imply that α^* is admissible.

We first show that $\mathbb{E}^v [|Y_t^*|^{2+\varepsilon}]$ is bounded. Indeed, in view of the absolute continuity relationship established in Theorem 3.15, we have

$$\begin{aligned} \mathbb{E}^v [|Y_t^*|^{2+\varepsilon}] &= \mathbb{E}^{\mathbb{Q}} \left[\frac{|Y_t^*|^{2+\varepsilon} p(V(\infty) - V(t), Y_t^*, f^{-1}(v))}{p(V(\infty), 0, f^{-1}(v))} \right] \\ &= \frac{1}{\sqrt{2\pi}\, p(V(\infty), 0, f^{-1}(v))} \\ &\quad \times \int_{-\infty}^{\infty} e^{rt} |y|^{2+\varepsilon} p\big(V(\infty) - V(t), y, f^{-1}(v) \big) \exp\left(-\frac{y^2}{2e^{-2rt}} \right) dy \\ &\leq C e^{-rt} \int_{-\infty}^{\infty} |y|^{\varepsilon} p\big(V(\infty) - V(t), y, f^{-1}(v) \big) \, dy \\ &\leq C. \end{aligned}$$

Above, the third line follows from the boundedness of xe^{-x} on $(0, \infty)$ and the last line is due to the finiteness of $V(\infty)$.

To show $\sup_{t\geq 0}\mathbb{E}^v[|h^*(s,Y_s^*)|^{2+\varepsilon}]<\infty$, first note that

$$\begin{aligned}|h(t,y)| &\leq C\exp\left(\frac{k}{2}y^2(1+2r)\right)\int_{-\infty}^{\infty}\frac{e^{rt}}{\sqrt{2\pi}}\exp\left(-\frac{z^2}{2}(e^{2rt}-k)\right)dz\\ &\leq C\exp\left(\frac{k}{2}y^2(1+2r)\right)\frac{e^{rt}}{\sqrt{e^{2rt}-k}}\\ &\leq C\exp\left(\frac{k}{2}y^2(1+2r)\right),\end{aligned}$$

in view of the exponential bound on f and similar arguments as those for (B.2). Therefore, setting $k^\varepsilon = k(1+\epsilon/2)$,

$$\begin{aligned}&\mathbb{E}^v[|h^*(s,Y_s^*)|^{2+\varepsilon}]\\ &\leq C\int_{-\infty}^{\infty}\exp\left(k^\varepsilon z^2(1+2r)\right)q\left(e^{-2rs},f^{-1}(v)-z\sqrt{1+2re^{-2rs}}\right)\\ &\qquad\times q\left(\frac{1}{1+2re^{-2rs}},z\right)dz\\ &\leq C\int_{-\infty}^{\infty}\exp\left(z^2\left((1+2r)k^\varepsilon-\frac{1+2re^{-2rs}}{2}\right)\right)\\ &\qquad\times q\left(e^{-2rs},f^{-1}(v)-z\sqrt{1+2re^{-2rs}}\right)dz\\ &\leq C\int_{-\infty}^{\infty}\exp\left(z^2e^{-2rs}\left(\frac{(1+2r)k^\varepsilon}{1+2re^{-2rs}}-\frac{1}{2}\right)\right)\exp\left(-\frac{(f^{-1}(v)e^{rs}-z)^2}{2}\right)dz\\ &\leq C\exp\left(\left(f^{-1}(v)\right)^2\frac{m(s)}{1-2e^{-2rs}m(s)}\right),\end{aligned}$$

where

$$m(s):=\frac{k^\varepsilon(1+2r)}{1+2re^{-2rs}}-\frac{1}{2}<\frac{1}{2}$$

if $\epsilon>0$ is chosen small enough. This completes the proof that $\mathbb{E}^v[|h^*(s,Y_s^*)|^{2+\epsilon}]$ is bounded. Hence, α^* is an admissible optimal strategy.

Step 2. *Market makers' best response.* Recall that

$$\begin{aligned}h^*(t,y) &= \int_{-\infty}^{\infty}f(z)q(e^{-2rt},z-y\sqrt{1+2re^{-2rt}})\,dz\\ &= \int_{-\infty}^{\infty}f(z+y\sqrt{1+2re^{-2rt}})q(e^{-2rt},z)\,dz,\end{aligned}$$

which shows that h^* is strictly increasing in y since f is. All that remains to show now is that $(h^*(t,Y_t^*))$ is an $(\mathcal{F}_t^{Y^*},\mathbb{P})$-martingale converging to $f(v)$.

It follows from Theorem 3.15 and the disintegration formula (2.1) that for any bounded and measurable F, we have

$$\mathbb{E}[F(Y_t^*)|\mathcal{F}_s^{Y^*}] = \int_{\mathbb{R}} \mathbb{E}^v[F(Y_t^*)|\mathcal{F}_s^{Y^*}]\,\nu(dv)$$
$$= \int_{\mathbb{R}} \frac{\mathbb{E}^{\mathbb{Q}}[F(Y_t)p(V(\infty)-V(t),Y_t,f^{-1}(v))|\mathcal{F}_s^Y]}{p(V(\infty)-V(t),Y_s,f^{-1}(v))}\nu(dv).$$

As we did before, let $R_t = Y_{V^{-1}(t)}$, where Y is the solution of (3.17). Thus

$$dR_t = d\beta_t + rR_t\,dt$$

for some Brownian motion β, which in particular implies that Y_t under $\mathbb{Q}$ is normal with mean 0 and variance equalling

$$\frac{e^{2rV(t)}-1}{2r} = \frac{1}{1+2re^{-2rt}}.$$

Thus,

$$\mathbb{E}[F(Y_t^*)|\mathcal{F}_s^{Y^*}]$$
$$= \int_{\mathbb{R}} \frac{\mathbb{E}^{\mathbb{Q}}[F(R_{V(t)})p(V(\infty)-V(t),R_{V(t)},f^{-1}(v))|\mathcal{F}_s^Y]}{p(V(\infty)-V(t),R_{V(s)},f^{-1}(v))}\nu(dv)$$
$$= \int_{\mathbb{R}}\int_{\mathbb{R}} f(y)\frac{p(V(\infty)-V(t),y,f^{-1}(v))p(V(t)-V(s),R_{V(s)},y)}{p(V(\infty)-V(s),R_{V(s)},f^{-1}(v))}\,dy\,\nu(dv)$$
$$= \int_{\mathbb{R}} \mathbb{E}^{\mathbb{Q}}[F(R_{V(t)})|R_{V(s)},R_{V(\infty)}=f^{-1}(v)]\,\nu(dv)$$
$$= \mathbb{E}^{\mathbb{Q}}[F(R_{V(t)})|R_{V(s)}],$$

where the last line is due to the fact that $R_{V(\infty)}$ is a standard normal random variable due to the choice of C^2, and $f^{-1}(\Gamma)$ has a standard normal distribution by Assumption 3.12. Thus the processes Y^* and $R_{V(\cdot)}$ have the same law. This implies that Y^* satisfies (3.18), and in particular, $(h^*(t,Y_t^*))$ is an $(\mathcal{F}_t^{Y^*},\mathbb{P})$-martingale. The convergence is immediate since $\lim_{t\to\infty} Y_t^* = v$ $\mathbb{P}^v$-a.s. by Theorem 3.15. □

References

1. Back, K.: Insider trading in continuous time. Rev. Financ. Stud. **5**, 387–409 (1992)
2. Back, K., Baruch, S.: Information in securities markets: Kyle meets Glosten and Milgrom. Econometrica **72**, 433–465 (2004)
3. Back, K., Cao, C.H., Willard, G.A.: Imperfect competition among informed traders. J. Finance **55**, 2117–2155 (2000)
4. Back, K., Pedersen, H.: Long-lived information and intraday patterns. J. Financ. Mark. **1**, 385–402 (1998)
5. Borodin, A.N., Salminen, P.: Handbook of Brownian Motion—Facts and Formulae, 2nd edn. Birkhäuser, Basel (2012)

6. Campi, L., Çetin, U.: Insider trading in an equilibrium model with default: a passage from reduced form to structural modelling. Finance Stoch. **11**, 591–602 (2007)
7. Campi, L., Çetin, U., Danilova, A.: Dynamic Markov bridges motivated by models of insider trading. Stoch. Process. Appl. **121**, 534–567 (2011)
8. Collin-Dufresne, P., Fos, V., Muravyev, D.: Informed trading and option prices: evidence from activist trading. Swiss Finance Institute Research Paper No. 15-55 (2015). Available online at http://ssrn.com/abstract=2675866
9. Çetin, U., Danilova, A.: Markovian Nash equilibrium in financial markets with asymmetric information and related forward–backward systems. Ann. Appl. Probab. **26**, 1996–2029 (2016)
10. Çetin, U., Danilova, A.: Markov bridges: SDE representation. Stoch. Process. Appl. **126**, 651–679 (2016)
11. Foster, F.D., Viswanathan, S.: Strategic trading when agents forecast the forecasts of others. J. Finance **51**, 1437–1478 (1996)
12. Karatzas, I., Shreve, S.E.: Brownian Motion and Stochastic Calculus, 2nd edn. Springer, Berlin (1991)
13. Kyle, A.S.: Continuous auctions and insider trading. Econometrica **53**, 1315–1335 (1985)
14. Liptser, R.S., Shiryaev, A.N.: Statistics of Random Processes, 2nd edn. Springer, Berlin (2001)
15. Revuz, D., Yor, M.: Continuous Martingales and Brownian Motion, 3rd revised edn. Springer, Berlin (1999)
16. Rogers, L.C.G., Williams, D.: Diffusions, Markov Processes, and Martingales. Vol. 1, Foundations. Cambridge University Press, Cambridge (2000). (Reprint of the second (1994) edition)