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ERRATUM

Erratum to: Utility maximization in incomplete markets with random endowment

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Abstract K. Larsen, M. Soner and G. Žitković kindly pointed out to us an error in our paper (Cvitanić et al. in Finance Stoch. 5:259–272, 2001) which appeared in 2001 in this journal. They also provide an explicit counterexample in Larsen et al. (https://arxiv.org/abs/1702.02087, 2017).

In Theorem 3.1 of Cvitanić et al. (Finance Stoch. 5:259–272, 2001), it was incorrectly claimed (among several other correct assertions) that the value function $u(x)$ is continuously differentiable. The erroneous argument for this assertion is contained in Remark 4.2 of Cvitanić et al. (Finance Stoch. 5:259–272, 2001), where it was claimed that the dual value function $v(y)$ is strictly concave. As the functions u and v are mutually conjugate, the continuous differentiability of u is equivalent to the strict convexity of v. By the same token, in Remark 4.3 of Cvitanić et al. (Finance Stoch. 5:259–272, 2001), the assertion on the uniqueness of the element $\hat{y}$ in the supergradient of $u(x)$ is also incorrect.

Similarly, the assertion in Theorem 3.1(ii) that $\hat{y}$ and x are related via $\hat{y} = u'(x)$ is incorrect. It should be replaced by the relation $x = -v'(\hat{y})$ or, equivalently, by requiring that $\hat{y}$ is in the supergradient of $u(x)$.

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To the best of our knowledge, all the other statements in Cvitanić et al. (Finance Stoch. 5:259–272, 2001) are correct.

As we believe that the counterexample in Larsen et al. (https://arxiv.org/abs/1702.02087, 2017) is beautiful and instructive in its own right, we take the opportunity to present it in some detail.

Keywords Utility maximization · Incomplete markets · Random endowment · Duality

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1 Discussion

We sketch the counterexample in [4] in a slightly modified and self-contained way. In the sequel, we suppose that the reader is familiar with the paper [1] as well as with [3, Example 5.1 bis]. We briefly recall the notation of that example which is the basis of the counterexample in [4].

The stock price process $S = (S_0, S_1)$ is defined by $S_0 = 1$ and by letting S_1 assume the value $x_0 = 2$ with probability $p_0 = 1 - \alpha$ and, for $n \geq 1$, the value $x_n = \frac{1}{n}$ with probability $p_n = \alpha 2^{-n}$, for $0 < \alpha < 1$ sufficiently small.

For logarithmic utility $U(x) = \ln x$, we obtain, for given endowment $x > 0$, that it is optimal to invest the entire endowment into the stock S so that we end up at time $t = 1$ with the random wealth $\hat{X}(x) = xS_1$. For the value function $u(x)$, we thus obtain (see [3, Example 5.1 bis] for details) the expected utility of $U(\hat{X}(x))$, i.e.,

$$u(x) = \mathbb{E}\big[U\big(\hat{X}(x)\big)\big] = \ln x + \sum_{n=0}^{\infty} p_n \ln x_n. \tag{1.1}$$

Using the notation from [1], the point of this example is that the dual optimizer $\hat{Q}(y)$ is not an element of L^1, but only of its bidual $(L^\infty)^*$. In other words, $\hat{Q}(y)$ defines a finitely additive probability measure on $(\Omega, \mathcal{F})$ which fails to be countably (or σ-) additive. We write $\hat{Q}(y) = \hat{Q}^r(y) + \hat{Q}^s(y)$ for the decomposition of $\hat{Q}(y)$ into its regular and singular parts.

To verify that $\|\hat{Q}^r(y)\| < \|\hat{Q}(y)\|$, fix $x = y = 1$. Writing $\hat{X}$ for $\hat{X}(1)$, $\hat{Q}$ for $\hat{Q}(1)$ and q_n for $\hat{Q}[S_1 = x_n]$, we have the relation [3, Theorem 2.2]

$$U'(\hat{X}) = \frac{d\hat{Q}^r}{d\mathbb{P}}, \tag{1.2}$$

so that

$$(x_n)^{-1} = \frac{q_n}{p_n}, \quad n \geq 0,$$

which yields

$$\sum_{n=0}^{\infty} q_n = \frac{1-\alpha}{2} + \alpha \sum_{n=1}^{\infty} n p_n.$$

This term is smaller than 1 (recall that $0 < \alpha < 1$ is small), which readily shows that the regular part $\hat{Q}^r$ has a smaller mass than $\hat{Q}$.

So far, we have just recalled [3, Example 5.1 bis]. For the next step, we follow [2] and distinguish between the odd and even numbers $n \in \mathbb{N}_0$ to define

$$A = \{S_1 = x_n \text{ for odd } n \geq 1\}, \qquad B = \{S_1 = x_n \text{ for even } n \geq 0\}.$$

Now comes the beautiful idea from [4]. Define the process $\tilde{S} = (\tilde{S}_0, \tilde{S}_1)$ by setting $\tilde{S}_0 = S_0 = 1$ and

$$\tilde{S}_1 = S_1 + \frac{1}{2}\mathbb{1}_A. \tag{1.3}$$

We also consider the random variable e_T which we define as

$$e_T = -\frac{1}{2}\mathbb{1}_A. \tag{1.4}$$

For the utility maximization problem, subject to the additional random endowment e_T, we define the value function $\tilde{u}(x)$ as in [1]. This boils down to the formula

$$\tilde{u}(x) = \sup_{\lambda \in \mathbb{R}} \mathbb{E}\big[\ln\big(x + \lambda(\tilde{S}_1 - \tilde{S}_0) + e_T\big)\big]. \tag{1.5}$$

For example, for $x = 1$, we again find that the optimizer $\hat{\lambda}$ in (1.5) equals 1, i.e., it again is optimal to invest the entire initial endowment $x = 1$ into the stock $\tilde{S}$; in this case, the terms $\mathbb{1}_A$ in (1.3) and (1.4) cancel out perfectly. In fact, for all $x \geq 1$, we find that the optimal $\hat{\lambda}$ in (1.5) equals $\hat{\lambda}(x) = x$, just as in (1.1). Indeed, for $x \geq 1$, the crucial constraint is that we cannot invest more than the amount $\hat{\lambda} = x$ into the stock due to the definition of $\tilde{S}$ and e_T on the set B, which corresponds to the even numbers $n \in \mathbb{N}_0$.

On the other hand, for $x < 1$, the picture changes; now the binding constraint is given by the odd numbers $n \in \mathbb{N}_0$, i.e., by the behaviour of $\tilde{S}$ and e_T on the set A. For $\frac{1}{2} < x \leq 1$, we find that the optimizer $\hat{\lambda}$ in (1.5) equals $\hat{\lambda}(x) = 1 - 2(1-x) = 2x - 1$. The remaining amount $x - (2x - 1) = 1 - x$ of the initial wealth x is kept in the bond. Note that for $x \leq \frac{1}{2}$, there is no admissible solution λ in (1.5), i.e., $\tilde{u}(x) = -\infty$ in (1.5). We thus obtain for the optimal terminal wealth $\hat{X}(x) = x + \hat{\lambda}(\tilde{S}_1 - \tilde{S}_0) + e_T$, for $\frac{1}{2} < x \leq 1$,

$$\hat{X}(x) = \begin{cases} (2x-1)x_n + (1-x) & \text{for } n \text{ even,} \\ (2x-1)(x_n + \frac{1}{2}) + (\frac{1}{2} - x) & \text{for } n \text{ odd} \end{cases} \tag{1.6}$$

and, for $1 \leq x < \infty$,

$$\hat{X}(x) = \begin{cases} x x_n & \text{for } n \text{ even}, \\ x(x_n + \frac{1}{2}) - \frac{1}{2} & \text{for } n \text{ odd}. \end{cases} \tag{1.7}$$

Note that we always have that $\hat{X}(x)$ is an a.s. strictly positive random variable whose essential infimum is zero. The latter property is obtained by considering the sets $\{S_n = x_n\}$ with n tending to infinity, where we have to consider the odd n in the case $\frac{1}{2} < x \leq 1$, and the even n in the case $x \geq 1$.

Clearly, the definitions (1.6) and (1.7) coincide for $x = 1$ in which case we obtain $\hat{X}(1) = S_1$. In particular, the value function

$$\tilde{u}(x) = \mathbb{E}[\ln \hat{X}(x)]$$

is continuous at $x = 1$, as must be the case. We shall see that $\tilde{u}$ has a kink at $x = 1$. Indeed, we may calculate the derivative of $\tilde{u}(x)$, for $x \in (\frac{1}{2}, 1)$ as well as for $x \in (1, \infty)$, by using the formula

$$\frac{d}{dx}\tilde{u}(x) = \mathbb{E}\left[\frac{d}{dx} \ln \hat{X}(x)\right], \quad x \in (1/2, 1) \cup (1, \infty).$$

Hence the difference $\Delta\tilde{u}'(1) = \lim_{x \searrow 1}(\frac{d}{dx}\tilde{u}(x)) - \lim_{x \nearrow 1}(\frac{d}{dx}\tilde{u}(x))$ of the right and left derivatives of $\tilde{u}$ at $x = 1$ can be explicitly computed as

$$\Delta\tilde{u}'(1) = \sum_{n=0}^{\infty} p_n \left(\frac{1}{x_n} - \frac{2}{x_n}\right) = -\left(\frac{p_0}{2} + \sum_{n=1}^{\infty} n p_n\right),$$

which clearly shows that the function $\tilde{u}(x)$ fails to be differentiable at $x = 1$.

Summing up, following [4], we have constructed an example where the value function $\tilde{u}(\cdot)$ fails to be differentiable.

We still want to have a closer look at the dual problem associated to the above example. In particular, we want to spot precisely where the erroneous argument in [1] has occurred. Define

$$y_1 = \lim_{x \searrow 1}\left(\frac{d}{dx}\tilde{u}(x)\right) \quad \text{and} \quad y_2 = \lim_{x \nearrow 1}\left(\frac{d}{dx}\tilde{u}(x)\right).$$

As the dual value function $\tilde{v}$ (see [1] for the definition)

$$\tilde{v}(y) = \min_{Q \in \mathcal{D}}\left\{\mathbb{E}\left[V\left(y \frac{dQ^r}{d\mathbb{P}}\right)\right] + y\langle Q, e_T\rangle\right\} \tag{1.8}$$

is conjugate to $\tilde{u}$ (see [1]), we know from the fact that $\tilde{u}(x)$ has a kink at $x = 1$ that $\tilde{v}(y)$ is an affine function with slope -1 on the interval $[y_1, y_2]$, in view of the basic relation

$$\tilde{v}(y) = \sup_x\big(\tilde{u}(x) - xy\big).$$

What are the dual optimizers $\hat{Q}_y$ for $y \in [y_1, y_2]$, given by [1, Theorem 3.1 and Lemma 4.1]? We know from [1] that the regular parts $\hat{Q}^r_y$ are unique and given by the formula

$$U'\big(\hat{X}(x)\big) = y\frac{d\hat{Q}^r_y}{d\mathbb{P}},$$

as in (1.2) above. The number x is associated to y via the relation $-\tilde{v}'(y) = x$, which yields $x = 1$ for $y \in [y_1, y_2]$. This implies the amazing fact that *the regular parts* $y\hat{Q}^r_y$ *of the dual optimizers* $y\hat{Q}_y$ *are identical, for all* $y \in [y_1, y_2]$. Note that the total mass of the elements $y\hat{Q}_y \in (L^\infty)^*$ equals $\|y\hat{Q}_y\| = y$. If we pass as usual to the normalized finitely additive probability measures $\hat{Q}_y$, their regular parts $\hat{Q}^r_y$ scale by the factor y^{-1}.

As regards the singular part $\hat{Q}^s_y$ of $\hat{Q}_y$, it is clear that $\hat{Q}^s_y$ is supported by each of the sets

$$C_N = \bigcup_{n=N}^{\infty} \{S_n = x_n\}.$$

Indeed, for each $\epsilon > 0$, the singular measure $\hat{Q}^s_y$ is supported by the set $\{\hat{X}(x) < \epsilon\}$, where $-\tilde{v}'(y) = x$. This follows from the analysis in [3, Example 5.1 bis]. But now the additional aspect of the odd and even n arises: How much of this singular mass sits on $C_N \cap A$, and how much on $C_N \cap B$?

It follows from (1.6), (1.7) and the subsequent discussion that for $\frac{1}{2} < x < 1$ and $\tilde{u}'(x) = y$, the singular measure $\hat{Q}^s_y$ is supported by A, while for $1 < x < \infty$ and $\tilde{u}'(x) = y$, the singular measure $\hat{Q}^s_y$ is supported by B. One may also pass to the limits $x \nearrow 1$ and $x \searrow 1$ to show that $\hat{Q}^s_{y_1}$ is supported by A, while $\hat{Q}^s_{y_2}$ is supported by B. It turns out that for general $y \in [y_1, y_2]$ of the form $y = \mu y_1 + (1-\mu)y_2$, we have the affine relations

$$y\hat{Q}^s_y[A] = \mu y_1 \hat{Q}^s_{y_1}[A] \tag{1.9}$$

and

$$y\hat{Q}^s_y[B] = (1-\mu) y_2 \hat{Q}^s_{y_2}[B]. \tag{1.10}$$

Indeed, because $\mathbb{E}_Q[\tilde{S}_1 - \tilde{S}_0] = 0$ for any equivalent martingale measure Q for $\tilde{S}$, we also obtain $\langle \hat{Q}_y, \tilde{S}_1 - \tilde{S}_0\rangle = 0$ by weak-star continuity. As $y\hat{Q}^r_y$ does not depend on y, we obtain that $\langle y\hat{Q}^s_y, \tilde{S}_1 - \tilde{S}_0\rangle$ does not depend on y either, for $y \in [y_1, y_2]$. On the set $C_N \cap A$ (resp. $C_N \cap B$), the random variable $\tilde{S}_1 - \tilde{S}_0$ equals $-\frac{1}{2}$ (resp. -1), up to an error of at most $\frac{1}{N}$ which disappears in the limit $N \mapsto \infty$. This implies that $-\frac{1}{2}y\hat{Q}^s_y[A] - y\hat{Q}^s_y[B]$ is constant when y varies in $[y_1, y_2]$ and readily yields the affine relations (1.9) and (1.10).

Finally, let us have a closer look where the mistake in Remark 4.2 of [1] occurred. In that argument, we fixed numbers $0 < y_1 < y_2$ (which may or may not coincide with the y_1, y_2 considered above) and considered the value function $\tilde{v}(y)$ as in (1.8). For $y_1 \neq y_2$, we have that $y_1\hat{Q}_{y_1}$ is different from $y_2\hat{Q}_{y_2}$, as shown in [1]. Up to this point, the reasoning was correct. We then tacitly (and incorrectly) assumed that this implies that their regular parts $y_1 \hat{Q}^r_{y_1}$ and $y_2 \hat{Q}^r_{y_2}$ must be different too! This would imply the strict inequality claimed in Remark 4.2 of [1]. But as we just have seen, it may happen that these two measures coincide. In addition, the singular parts $y \hat{Q}^s_y$ satisfy the affine relations (1.9) and (1.10), which also prevent the inequality in Remark 4.2 of [1] becoming strict.

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