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ORIGINAL PAPER

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The crust shell and the edge beams of third-gradient continua in current and referential description

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Abstract A former publication has shown that some of the free boundary conditions of a third-gradient continuum can be interpreted as the equilibrium conditions of forces and moments of a crust shell and of edge beams. This was elaborated in the current (Eulerian) description. But it is obvious that a referential (Lagrangian) description of these physical phenomena must look very similar. This is, indeed, demonstrated in the paper at hand and disproves contrary claims in the literature. Moreover, the inductivist idea is omnipresent in the literature that a fictitious cut and a free surface show identical behaviour. But this is only correct with simple materials and not valid with higher gradient ones. We study a cube as a simple example where this wrong approach predicts the following absurd interaction between a subcube and the remainder of the cube: No actions at all appear on the fictitious cuts but there is one single force at the vertex of the subcube. Our findings do not depend on any specific constitutive assumption.

1 The actual status of gradient theories

During the last two centuries, the theory of materials of continuum mechanics was to a large extent a synonym of the theory of first-gradient materials, called simple materials [1,2]. In the last decades, however, theories of materials with higher gradients, already studied by Piola [3] in the nineteenth century, received an increasing attention, starting with Toupin [4] and Mindlin [5]. The basics of these theories can, e.g., be found in [6–11] among many others and in the references therein. The paper at hand restricts the discussion to third-gradient materials. Constitutive properties are only attributed to the bulk and not to surfaces or lines.

Unfortunately, even the most recent publications reveal the impression that the behaviour of gradient materials is not yet understood. This is apparent from the following four facts.

1. A free surface behaves like a crust shell with cutting forces and cutting moments. They may be regarded as enhanced surface tensions. This was shown in [12] but is not yet grasped.
2. The papers [13,14] compare the current (Eulerian) and the referential (Lagrangian) description. An action in one description corresponds to an enormous cascade of actions in the other.
3. The papers [13,15,16] emphasize that fictitious cuts of a body show the same behaviour as free surfaces. They assert that the interactions on a fictitious cut depend on the curvature of the cut and that additional interactions appear at edges and vertices of the cut.
4. Three higher-order surface actions that are present on a fixed boundary as reactions can, of course, be prescribed on free surfaces but do not appear anywhere in the literature.

In what follows, we denote the dimension of force by [F] and that of length by [L].

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To 1: In contrast to the approach of Gurtin and Murdoch [17] and its extension in [18], we do not attribute constitutive laws to surfaces or edges of a body and do not introduce a surface energy. Nevertheless, it was shown in [12] that a free surface of a third gradient material behaves like a shell with cutting forces and moments that therefore have the character of reactions. This is an unavoidable consequence of the bulk behaviour. Such a phenomenon is well known from the simplest gradient theory, namely the Kirchhoff plate.

To 2: The crust shell possesses a force tensor $\mathbf{S}$ and a moment tensor $\mathbf{M}$ of dimension [F/L] and [FL/L], respectively, in the current description. If we switch to the referential description, we must find similar tensors $\mathbf{S}_R$ and $\mathbf{M}_R$. Moreover, we have surface load vectors $\mathbf{t}^*$, $\check{\mathbf{t}}^*$, and $\hat{\mathbf{t}}^*$ of dimensions [F/L^2], [FL/L^2], and [FL^2/L^2] in the current description and similar vectors $\mathbf{t}^*_R$, $\check{\mathbf{t}}^*_R$, and $\hat{\mathbf{t}}^*_R$ in the referential description. The two descriptions treat the same physical behaviour so that we expect the following simple connections.

$$\mathbf{S}\cdot\mathbf{e}\,ds = \mathbf{S}_R\cdot\mathbf{e}_R\,ds_R, \qquad \mathbf{M}\cdot\mathbf{e}\,ds = \mathbf{M}_R\cdot\mathbf{e}_R\,ds_R \tag{1}$$

$$\mathbf{t}^*\,dA = \mathbf{t}^*_R\,dA_R, \qquad \check{\mathbf{t}}^*\,dA = \check{\mathbf{t}}^*_R\,dA_R, \qquad \hat{\mathbf{t}}^*\,dA = \hat{\mathbf{t}}^*_R\,dA_R \tag{2}$$

Here dA and dA_R denote the current and referential surface element and ds and ds_R are the current and referential line element of a cut through the shell, while $\mathbf{e}$ and $\mathbf{e}_R$ are the unit vectors normal to the cut in the tangential plane. It is easy to understand why the cited papers present monstrous connections between their current and referential actions. If we want to derive the boundary conditions of a free surface, we must split the current derivative ∇ into a tangential part ∇_T and a normal part d_n in order to be able to apply the divergence theorem of surfaces. The same split of the referential derivative ∇_R, however, causes the mentioned difficulties since the normal derivatives of the two descriptions are not simply connected. In the paper at hand, the solution of the problem is based on the introduction of an alternate split of ∇_R that, indeed, produces the simple connections (1), (2). Nevertheless, the reader will see that the proofs of these connections are by no means trivial. The same procedure is applied to edge beams where patches of the surface meet. Here again, an alternate split of ∇_R allows a simple connection between the current and referential loads.

The mentioned cascades have inadmissible consequences. For example, a part of a current moment appears to be represented by a referential double force. But forces, moments and hence also double forces have a physical meaning. A moment acts on a rotation and a double force on a stretching. This cannot depend on the description.

To 3: Three experts in continuum mechanics published an impressive analysis of the genesis of scientific ideas in mechanics [19]. They point out that the inductivist approach that simply extrapolates the previous knowledge is not helpful to promote the understanding of new phenomena. They substantiate it with the famous example of Bertrand Russell [20]: *"Domestic animals expect food when they see the person who usually feeds them. We know that all these rather crude expectations of uniformity are liable to be misleading. The man who has fed the chicken every day throughout its life at last wrings its neck instead, showing that more refined views as to the uniformity of nature would have been useful to the chicken."*

Now, people have grown up with the belief that a fictitious cut and a free surface show the same behaviour because they have in mind the theory of simple materials where the interaction is indeed described by Cauchy's stress vector in both cases. The idea that this must also be the case with higher gradient materials is obviously inductivist. This could have been known since 150 years as even the simplest gradient theory, the Kirchhoff plate, shows this different behaviour. What happens really on a free surface? One of its boundary conditions will be shown to read in the current description

$$\mathbf{t}^* + \mathbf{S}\cdot\nabla_T = \mathbf{0} \tag{3}$$

It results from the application of the divergence theorem of surfaces and can be interpreted as the equilibrium condition of force of a crust shell. The dimension of $\mathbf{t}^*$ is [F/L^2], that of $\mathbf{S}$ is [F/L], and that of ∇_T is [1/L]. The crust tensor $\mathbf{S}$ is only defined on the surface. Its tangential divergence is needed to carry aside the difference $\mathbf{t}^* = \mathbf{t}_{\text{ext}} - \mathbf{t}$ of the external and internal surface forces. Where a patch of the surface ends, a force per unit length $\mathbf{S}\cdot\mathbf{e}$ is applied to an edge beam or to a fixed boundary.

It is obvious that no such shells with cutting forces of dimension [F/L] can be found in the interior of a body. They would only exist and apply their line force $\mathbf{S}\cdot\mathbf{e}$ to a free or fixed boundary as long as someone imagines such a fictitious cut. The appearance of the force tensor $\mathbf{S}$ and its tangential derivative $\mathbf{S}\cdot\nabla_T$ is an unavoidable result of the application of the divergence theorem of surfaces. Since such a force tensor cannot be present on a fictitious cut, it is obvious that the application of the divergence theorem of surfaces on such a fictitious cut makes no sense, although it is widely practised in the literature. On the other hand, the application on a free surface is a consequence of the fact that the tangential derivatives of a surface function are not independent but are determined by the function itself and require an integration by parts. This has been pointed out by Mindlin [5].

To 4: The missing prescribed actions will be specified in Sect. 7.

2 Outline of the investigation

Geometrical tools are provided in Sect. 3. The most important topic is the construction of alternate splittings of the referential derivative on crust shells and edge beams.

Section 4 introduces the true stress tensors $\mathbf{T}$, T, and $\mathcal{T}$ of second, third, and fourth order that are dual to the first three current derivatives of a virtual velocity field. There is no one-to-one correspondence between the n-th derivatives in the current and referential description (if $n > 1$) and so there is also no such simple correspondence between the mentioned tensors and their referential counterparts, the nominal stresses $\mathbf{T}_R$, T_R, and $\mathcal{T}_R$.

The application of the divergence theorem in Sect. 5 transforms the volumetric representation of the internal virtual power and reveals the role of the effective true stress tensors $\bar{\mathbf{T}}$ and $\bar{\mathsf{T}}$. There is again no one-to-one correspondence between $\bar{\mathsf{T}}$ and its nominal counterpart $\bar{\mathsf{T}}_R$. The surface terms reveal that the interaction on any fictitious cut is performed by the local normal components of the three tensors $\mathcal{T}$, $\bar{\mathsf{T}}$, and $\bar{\mathbf{T}}$.

The treatment of real boundaries is only possible if the current or referential derivative is decomposed so that the tangential derivatives are made explicit. This is done in Sect. 6.

Section 7 clarifies the external actions that are admissible on free surfaces. Then, the boundary conditions are derived, and some of them can only be obtained if the divergence theorem of surfaces is applied.

Section 8 compares the current and referential field and boundary quantities and reveals very simple connections.

The subject of Sect. 9 is the crust shell. Some of the boundary conditions on a free surface can be interpreted as the equilibrium conditions of forces and moments of a shell.

Section 10 shows that an edge where two patches of the surface meet has the behaviour of a beam.

Section 11 presents a simple example: A cube loaded by a single force at one vertex and by line forces, moments, and double forces on three edges. A state of stress is presented that satisfies the field equation and all boundary conditions. Finally, a subcube is considered, and the interaction with its surroundings along the fictitious cuts is evaluated. It is shown that this interaction is erroneously claimed to happen by a single force at the vertex alone if the divergence theorem of surfaces is applied to these fictitious cuts without any physical reason.

A conclusion is found in Sect. 12.

The appendix contains proofs that belong to Sect. 8 but were to extensive to be included there.

3 Geometrical tools

3.1 Notation

A dot denotes a contraction. Multiple contractions are written $\mathbf{a}\otimes\mathbf{b} : \mathbf{e}\otimes\mathbf{f} = \mathbf{a}\cdot\mathbf{e}\,\mathbf{b}\cdot\mathbf{f}$, $\mathbf{a}\otimes\mathbf{b}\otimes\mathbf{c} :\cdot\, \mathbf{e}\otimes\mathbf{f}\otimes\mathbf{g} = \mathbf{a}\cdot\mathbf{e}\,\mathbf{b}\cdot\mathbf{f}\,\mathbf{c}\cdot\mathbf{g}$, $\mathbf{a}\otimes\mathbf{b}\otimes\mathbf{c}\otimes\mathbf{d} :: \mathbf{e}\otimes\mathbf{f}\otimes\mathbf{g}\otimes\mathbf{h} = \mathbf{a}\cdot\mathbf{e}\,\mathbf{b}\cdot\mathbf{f}\,\mathbf{c}\cdot\mathbf{g}\,\mathbf{d}\cdot\mathbf{h}$. We define the Gibbsian vector of a tensor by $\mathbf{a}\otimes\mathbf{b}|_\times = \mathbf{a}\times\mathbf{b} = -2\,\mathrm{axi}\,(\mathbf{a}\otimes\mathbf{b})$ and extend this definition to $\mathbf{a}\otimes\mathbf{b}\otimes\mathbf{c}|_\times = \mathbf{a}\times\mathbf{b}\otimes\mathbf{c}$. The transpose of a tensor shall be defined by $\mathbf{a}\otimes\mathbf{b}^T = \mathbf{b}\otimes\mathbf{a}$ and $\mathbf{a}\otimes\mathbf{b}\otimes\mathbf{c}^T = \mathbf{b}\otimes\mathbf{a}\otimes\mathbf{c}$. If direct notation becomes too cumbersome, we switch to Cartesian components.

3.2 Current and reference placement

The position of a particle in the current and the reference placement is denoted by the position vectors $\mathbf{r}$ and $\mathbf{r}_R$, respectively. The derivatives with respect to these positions are ∇ and ∇_R. The local transplacement $\mathbf{F}$ and its inverse $\mathbf{G}$ are defined by:

$$\mathbf{F} = \mathbf{r}\otimes\nabla_R \equiv \mathbf{G}^{-1} \tag{4}$$

and we have the connection

$$\nabla_R = \nabla \cdot \mathbf{F}, \quad \nabla = \nabla_R \cdot \mathbf{G} \qquad \nabla_{Rk} = \nabla_p f_{pk}, \quad \nabla_l = \nabla_{Rq} g_{ql} \tag{5}$$

We need the gradient of **F** that is symmetric in the last two entries.

$$\mathbf{F} \otimes \nabla_R = \mathbf{r} \otimes \nabla_R \otimes \nabla_R, \qquad f_{abc} = f_{acb} \equiv f_{ab} \nabla_{Rc} \tag{6}$$

The ratio J of the volume elements and its derivative are

$$J = \frac{dV}{dV_R} = \det \mathbf{F}, \quad dJ = J\, \mathbf{F}^{-T} : d\mathbf{F} = J\, \mathbf{G}^T : d\mathbf{F}, \quad \nabla_{Rc} J = J g_{vu} f_{uvc}, \quad \nabla_{Rc}\left(\frac{1}{J}\right) = -\frac{1}{J} g_{vu} f_{uvc} \tag{7}$$

Moreover,

$$d\mathbf{G} = -\mathbf{G} \cdot d\mathbf{F} \cdot \mathbf{G}, \qquad g_{ab} \nabla_{Rc} = -g_{ax} f_{xyc} g_{yb} \tag{8}$$

3.3 Surfaces

Let $\mathbf{n}$ and $\mathbf{n}_R$ denote the unit normal vectors and dA and dA_R the areas of corresponding surface elements at a point. We will need the projection tensors onto the tangential planes.

$$\mathbf{P} = \mathbf{1} - \mathbf{n} \otimes \mathbf{n}, \qquad \mathbf{P}_R = \mathbf{1} - \mathbf{n}_R \otimes \mathbf{n}_R \tag{9}$$

where $\mathbf{1}$ is the identity tensor. We make use of the well-known formula

$$\mathbf{n}\, dA = J\, \mathbf{G}^T \cdot \mathbf{n}_R\, dA_R \tag{10}$$

and introduce the following auxiliary quantity N

$$N \equiv J \frac{dA_R}{dA} = \sqrt{\mathbf{n} \cdot \mathbf{F} \cdot \mathbf{F}^T \cdot \mathbf{n}} = \frac{1}{\sqrt{\mathbf{n}_R \cdot \mathbf{G} \cdot \mathbf{G}^T \cdot \mathbf{n}_R}} \tag{11}$$

So we obtain the connections between the normal vectors and the areas of the surface elements:

$$\mathbf{n} = N \mathbf{G}^T \cdot \mathbf{n}_R, \qquad Q \equiv \frac{dA_R}{dA} = \frac{N}{J} \tag{12}$$

The introduction of the following modified normal vector

$$\tilde{\mathbf{n}} = \mathbf{G} \cdot \mathbf{n} = N \mathbf{G} \cdot \mathbf{G}^T \cdot \mathbf{n}_R \tag{13}$$

and the following modified projection tensor

$$\tilde{\mathbf{P}} = \mathbf{G} \cdot \mathbf{P} \cdot \mathbf{F} = \mathbf{1} - \mathbf{G} \cdot \mathbf{n} \otimes \mathbf{n} \cdot \mathbf{F} = \mathbf{1} - N \tilde{\mathbf{n}} \otimes \mathbf{n}_R \tag{14}$$

will turn out to be of utmost importance. Note that $\tilde{\mathbf{n}}$ is not a unit vector and that the tensor $\tilde{\mathbf{P}}$ is not symmetric.

The following identities will also be needed.

$$N \tilde{\mathbf{n}} \cdot \mathbf{n}_R = 1, \qquad \mathbf{n}_R \cdot \tilde{\mathbf{P}} = \mathbf{0}, \qquad \tilde{\mathbf{P}} \cdot \tilde{\mathbf{n}} = \mathbf{0} \tag{15}$$

$$\tilde{\mathbf{P}} \cdot \mathbf{P}_R = \mathbf{P}_R \qquad \mathbf{P}_R \cdot \tilde{\mathbf{P}} = \tilde{\mathbf{P}}, \qquad \tilde{\mathbf{P}} \cdot \tilde{\mathbf{P}} = (\mathbf{1} - N \tilde{\mathbf{n}} \otimes \mathbf{n}_R) \cdot (\mathbf{1} - N \tilde{\mathbf{n}} \otimes \mathbf{n}_R) = \mathbf{1} - N \tilde{\mathbf{n}} \otimes \mathbf{n}_R = \tilde{\mathbf{P}} \tag{16}$$

$$\mathbf{G} \cdot \mathbf{G}^T - \tilde{\mathbf{n}} \otimes \tilde{\mathbf{n}} = \mathbf{G} \cdot \mathbf{G}^T - \tilde{\mathbf{n}} \otimes N \mathbf{n}_R \cdot \mathbf{G} \cdot \mathbf{G}^T = \tilde{\mathbf{P}} \cdot \mathbf{G} \cdot \mathbf{G}^T \tag{17}$$

3.4 Tangential derivatives

When treating surfaces, it is useful to split the spatial derivative into a tangential and a normal part.

$$\nabla = \nabla_T + \mathbf{n}\, d_n, \qquad \nabla_R = \nabla_{RT} + \mathbf{n}_R\, d_{Rn} \tag{18}$$

We note $\mathbf{P}_R \cdot \mathbf{F}^T \cdot \mathbf{n} = N\, \mathbf{P}_R \cdot \mathbf{n}_R = \mathbf{0}$ and obtain

$$\nabla_T = \mathbf{P} \cdot \nabla, \qquad \nabla_{RT} = \mathbf{P}_R \cdot \nabla_R = \mathbf{P}_R \cdot \mathbf{F}^T \cdot \nabla = \mathbf{P}_R \cdot \mathbf{F}^T \cdot \nabla_T \tag{19}$$

and also

$$d_{Rn} = \mathbf{n}_R \cdot \nabla_R, \qquad d_n = \mathbf{n} \cdot \nabla = \mathbf{n} \cdot \mathbf{G}^T \cdot \mathbf{F}^T \cdot \nabla = \tilde{\mathbf{n}} \cdot \nabla_R \tag{20}$$

The last identity reveals the benefit of the vector $\tilde{\mathbf{n}}$. Note that $\tilde{\mathbf{n}}$ is not normalized and hence d_n is not a directional derivative in the reference placement in contrast to d_{Rn}.

The key to a successful definition of referential quantities is the introduction of an oblique decomposition of the referential derivative. We note (15) and (14) and find

$$\nabla_{RT} \cdot \tilde{\mathbf{P}} = \nabla_R \cdot \tilde{\mathbf{P}} = \nabla_R \cdot (\mathbf{1} - N\, \tilde{\mathbf{n}} \otimes \mathbf{n}_R) = \nabla_R - N(\tilde{\mathbf{n}} \cdot \nabla_R)\, \mathbf{n}_R = \nabla_R - N\, \mathbf{n}_R\, d_n \tag{21}$$

and hence

$$\nabla_R = \nabla_{RT} \cdot \tilde{\mathbf{P}} + N\, \mathbf{n}_R\, d_n \tag{22}$$

Let $\delta\mathbf{v} = \delta\dot{\mathbf{r}}$ be a virtual velocity field. We represent its first and second derivative with the help of this decomposition.

$$\delta\mathbf{v} \otimes \nabla_R = \delta\mathbf{v} \otimes \nabla_{RT} \cdot \tilde{\mathbf{P}} + N\big(d_n\, \delta\mathbf{v}\big) \otimes \mathbf{n}_R \tag{23}$$

$$\delta\mathbf{v} \otimes \nabla_R \otimes \nabla_R = \delta\mathbf{v} \otimes \nabla_R \otimes \nabla_{RT} \cdot \tilde{\mathbf{P}} + N d_n\big(\delta\mathbf{v} \otimes \nabla_R\big) \otimes \mathbf{n}_R \tag{24}$$

with

$$\begin{aligned} d_n\big(\delta\mathbf{v} \otimes \nabla_R\big) &= (\tilde{\mathbf{n}} \cdot \nabla_R)(\delta\mathbf{v} \otimes \nabla_R) = (\tilde{\mathbf{n}} \cdot \nabla_R \delta\mathbf{v}) \otimes \nabla_R - (\delta\mathbf{v} \otimes \nabla_R) \cdot (\tilde{\mathbf{n}} \otimes \nabla_R) \\ &= \big(d_n \delta\mathbf{v}\big) \otimes \nabla_R - (\delta\mathbf{v} \otimes \nabla_R) \cdot \big(\tilde{\mathbf{n}} \otimes \nabla_{RT} \cdot \tilde{\mathbf{P}} + N d_n \tilde{\mathbf{n}} \otimes \mathbf{n}_R\big) \\ &= \big(d_n \delta\mathbf{v}\big) \otimes \nabla_{RT} \cdot \tilde{\mathbf{P}} + N\big(d_n^2 \delta\mathbf{v}\big) \otimes \mathbf{n}_R - \delta\mathbf{v} \otimes \nabla_{RT} \cdot \tilde{\mathbf{P}} \cdot \big(\tilde{\mathbf{n}} \otimes \nabla_{RT} \cdot \tilde{\mathbf{P}} + N d_n \tilde{\mathbf{n}} \otimes \mathbf{n}_R\big) \\ &\quad - N\big(d_n \delta\mathbf{v}\big) \otimes \mathbf{n}_R \cdot \big(\tilde{\mathbf{n}} \otimes \nabla_{RT} \cdot \tilde{\mathbf{P}} + N d_n \tilde{\mathbf{n}} \otimes \mathbf{n}_R\big) \end{aligned} \tag{25}$$

So we arrive at

$$\begin{aligned} \delta\mathbf{v} \otimes \nabla_R \otimes \nabla_R = {} & \Big(\delta\mathbf{v} \otimes \nabla_{RT} \cdot \tilde{\mathbf{P}} + N\big(d_n\, \delta\mathbf{v}\big) \otimes \mathbf{n}_R\Big) \otimes \nabla_{RT} \cdot \tilde{\mathbf{P}} \\ & + N\Big(\big(d_n \delta\mathbf{v}\big) \otimes \nabla_{RT} \cdot \tilde{\mathbf{P}} + N\big(d_n^2 \delta\mathbf{v}\big) \otimes \mathbf{n}_R - \delta\mathbf{v} \otimes \nabla_{RT} \cdot \tilde{\mathbf{P}} \cdot \big(\tilde{\mathbf{n}} \otimes \nabla_{RT} \cdot \tilde{\mathbf{P}} + N d_n \tilde{\mathbf{n}} \otimes \mathbf{n}_R\big) \\ & - N\big(d_n \delta\mathbf{v}\big) \otimes \mathbf{n}_R \cdot \big(\tilde{\mathbf{n}} \otimes \nabla_{RT} \cdot \tilde{\mathbf{P}} + N d_n \tilde{\mathbf{n}} \otimes \mathbf{n}_R\big)\Big) \otimes \mathbf{n}_R \end{aligned} \tag{26}$$

The following identity will later be needed.

$$\begin{aligned} \big((\delta\mathbf{v} \otimes \nabla_T) \cdot \mathbf{X}\big) \otimes \nabla_T &= \overline{(\delta\mathbf{v} \,\acute{\otimes}\, \nabla_T)} \cdot \mathbf{X} \otimes \nabla_T + (\delta\mathbf{v} \otimes \nabla_T) \cdot (\acute{\mathbf{X}} \otimes \nabla_T) \\ &= \Big(\mathbf{X}^T \cdot ((\delta\mathbf{v} \otimes \nabla_T) \otimes \nabla_T)^T\Big)^T + (\delta\mathbf{v} \otimes \nabla_T) \cdot \big(\mathbf{X} \otimes \nabla_T\big) \end{aligned} \tag{27}$$

Here $\mathbf{X}$ is any second-order tensor. An accent indicates on which field the derivative is to be applied. We may note $\nabla_T \cdot \mathbf{n} = 0$ and obtain an interesting special case.

$$\begin{aligned} \mathbf{0} &= \Big((\delta\mathbf{v} \otimes \nabla_T) \cdot (\mathbf{n} \otimes \mathbf{n})\Big) \otimes \nabla_T \\ &= \Big(\mathbf{n} \otimes \mathbf{n}) \cdot \big((\delta\mathbf{v} \otimes \nabla_T) \otimes \nabla_T\big)^T\Big)^T + (\delta\mathbf{v} \otimes \nabla_T) \cdot \big((\mathbf{n} \otimes \mathbf{n}) \otimes \nabla_T\big) \end{aligned} \tag{28}$$

The sum is zero, but the two single terms do not vanish separately. One should also keep in mind that $(\delta\mathbf{v} \otimes \nabla_T) \otimes \nabla_T$ is in general not symmetric in the last two entries. Therefore, the notation $\delta\mathbf{v} \otimes \nabla_T \otimes \nabla_T$ is avoided.

3.5 The boundary curve of a surface patch

The unit tangent vector **s** and the unit normal vector **e** in the tangential plane of such a curve are connected by

$$\mathbf{n} = \mathbf{e} \times \mathbf{s}, \qquad \mathbf{e} = -\mathbf{n} \times \mathbf{s}, \qquad \mathbf{s} = \mathbf{n} \times \mathbf{e} \tag{29}$$

and a line element is $d\mathbf{r} = \mathbf{s}\,ds$ or $d\mathbf{r}_R = \mathbf{s}_R\,ds_R$ in the current and reference placement, respectively. If we interpret the surface as a shell and want to discuss cutting loads, then we need the oriented cutting line element

$$\mathbf{e}_R\,ds_R = -\mathbf{n}_R \times\, d\mathbf{r}_R = -\frac{1}{N}\mathbf{n}\cdot\mathbf{F}\times\mathbf{G}\cdot d\mathbf{r} = -\Big(\frac{1}{N}\mathbf{n}\cdot\mathbf{F}\times\mathbf{G}\times\mathbf{n}\Big)\cdot\mathbf{e}\,ds \equiv \mathbf{N}\cdot\mathbf{e}\,ds \tag{30}$$

This can be rewritten to give the simple representation

$$\mathbf{e}_R\,ds_R = \mathbf{N}\cdot\mathbf{e}\,ds \qquad \text{with} \qquad \mathbf{N} = Q\,\mathbf{P}_R\cdot\mathbf{F}^T \tag{31}$$

The proof is easy if we choose an orthonormal basis such that $\mathbf{n}_R = \mathbf{e}_3$. Then,

$$\begin{aligned}
\mathbf{N} &= -\frac{1}{N}\mathbf{n}\cdot\mathbf{F}\times\mathbf{G}\times\mathbf{n} = -N\mathbf{n}_R\times\mathbf{G}\times\mathbf{G}^T\cdot\mathbf{n}_R = -\,N\,(\mathbf{e}_1\times\mathbf{e}_2)\times\mathbf{G}\times\mathbf{G}^T\cdot\mathbf{e}_3\\
&= N\big(\mathbf{e}_1\otimes\mathbf{e}_2\cdot\mathbf{G} - \mathbf{e}_2\otimes\mathbf{e}_1\cdot\mathbf{G}\big)\times\mathbf{G}^T\cdot\mathbf{e}_3\\
&= N\,\Big(\mathbf{e}_1\otimes\big((\mathbf{e}_2\cdot\mathbf{G})\times(\mathbf{e}_3\cdot\mathbf{G})\big) - \mathbf{e}_2\otimes\big((\mathbf{e}_1\cdot\mathbf{G})\times(\mathbf{e}_3\cdot\mathbf{G})\big)\Big)\\
&= \frac{N}{J}\big(\mathbf{e}_1\otimes\mathbf{F}\cdot\mathbf{e}_1 + \mathbf{e}_2\otimes\mathbf{F}\cdot\mathbf{e}_2\big) = Q\,\mathbf{P}_R\cdot\mathbf{F}^T
\end{aligned} \tag{32}$$

The field **N** has an interesting property: The planar part of its tangential divergence vanishes. To show this, we recall the divergence theorem of a surface. If a field **A** satisfies $\mathbf{A}\cdot\mathbf{n} = \mathbf{0}$ everywhere on the surface, then

$$\oint \mathbf{A}\cdot\mathbf{e}\,ds = \int \mathbf{A}\cdot\nabla_T\,dA \tag{33}$$

is valid for any part of the surface. Therefore,

$$\begin{aligned}
\oint \mathbf{N}\cdot\mathbf{e}\,ds &= \int \mathbf{N}\cdot\nabla_T\,dA = \oint \mathbf{e}_R ds_R\\
&= \oint \mathbf{P}_R\cdot\mathbf{e}_R ds_R = \int \mathbf{P}_R\cdot\nabla_{RT}\,dA_R = \int Q(\mathbf{P}_R\cdot\nabla_{RT})\,dA
\end{aligned} \tag{34}$$

which implies

$$\mathbf{N}\cdot\nabla_T = Q\,(\mathbf{P}_R\cdot\nabla_{RT}) = -Q\,\mathbf{n}_R\,(\mathbf{n}_R\cdot\nabla_{RT}) \tag{35}$$

The projection into the tangential plane of the reference placement yields indeed:

$$\mathbf{P}_R\cdot\acute{\mathbf{N}}\cdot\nabla_T = \mathbf{0} \tag{36}$$

The connection

$$d\mathbf{r} = \mathbf{s}\,ds = \mathbf{F}\cdot d\mathbf{r}_R = \mathbf{F}\cdot\mathbf{s}_R\,ds_R \tag{37}$$

implies

$$L \equiv \frac{ds}{ds_R} = \sqrt{\mathbf{s}_R\cdot\mathbf{F}^T\cdot\mathbf{F}\cdot\mathbf{s}_R} = \frac{1}{\sqrt{\mathbf{s}\cdot\mathbf{G}^T\cdot\mathbf{G}\cdot\mathbf{s}}}, \qquad \mathbf{s}_R = L\,\mathbf{G}\cdot\mathbf{s}, \qquad \mathbf{s} = \frac{1}{L}\,\mathbf{F}\cdot\mathbf{s}_R \tag{38}$$

and hence

$$\frac{\partial}{\partial s_R} = \mathbf{s}_R\cdot\nabla_R = L\,\mathbf{s}\cdot\mathbf{G}^T\cdot\mathbf{F}^T\cdot\nabla = L\,\mathbf{s}\cdot\nabla = L\frac{\partial}{\partial s} \tag{39}$$

The introduction of the following modified tangent vector

$$\tilde{\mathbf{s}} = \frac{1}{L}\mathbf{F}^T \cdot \mathbf{s} = \frac{1}{L^2}\mathbf{F}^T \cdot \mathbf{F} \cdot \mathbf{s}_R \tag{40}$$

will turn out to be of importance. We find

$$\tilde{\mathbf{s}} \cdot \mathbf{s}_R = 1, \qquad \tilde{\mathbf{s}} \cdot \tilde{\mathbf{n}} = \frac{1}{L}\mathbf{s} \cdot \mathbf{F} \cdot \mathbf{G} \cdot \mathbf{n} = \frac{1}{L}\mathbf{s} \cdot \mathbf{n} = 0, \qquad \tilde{\mathbf{P}}^T \cdot \tilde{\mathbf{s}} = \left(\mathbf{1} - N\mathbf{n}_R \otimes \tilde{\mathbf{n}}\right) \cdot \tilde{\mathbf{s}} = \tilde{\mathbf{s}} \tag{41}$$

We introduce the projector onto the plane normal to $\mathbf{s}$

$$\mathbf{P}_s = \mathbf{1} - \mathbf{s} \otimes \mathbf{s} \tag{42}$$

and may split the current derivative into the derivative in the direction of the curve and the derivative in the plane normal to the curve.

$$\nabla = \mathbf{s}\frac{\partial}{\partial s} + \nabla_P \qquad \text{with} \qquad \nabla_P = \mathbf{P}_s \cdot \nabla \tag{43}$$

Here again, it is useful to introduce a modified split of the referential derivative.

$$\nabla_R = \mathbf{F}^T \cdot \nabla = \mathbf{F}^T \cdot \left(\mathbf{s}\frac{\partial}{\partial s} + \nabla_P\right) = \tilde{\mathbf{s}}\frac{\partial}{\partial s_R} + \mathbf{F}^T \cdot \nabla_P \tag{44}$$

4 True and nominal stresses

The internal virtual power of a whole body or of a subbody made of a third-gradient material is given by:

$$\delta\Pi_{\text{int}} = \int \left(\mathbf{T} : \delta\mathbf{v} \otimes \nabla + \mathsf{T} \vdots\ \delta\mathbf{v} \otimes \nabla \otimes \nabla + \mathcal{T} :: \delta\mathbf{v} \otimes \nabla \otimes \nabla \otimes \nabla\right) dV \tag{45}$$

We identify a second-order true stress tensor $\mathbf{T}$ (dimension [F/L^2]), a third-order true stress tensor T ([FL/L^2]), and a fourth-order true stress tensor $\mathcal{T}$ ([FL^2/L^2]). $\mathbf{T}$ is symmetric due to the principle of invariance under superimposed rigid body motions. We notice that only those parts of the tensors T and $\mathcal{T}$ that are symmetric in the last two or three entries, respectively, enter the expression of the virtual power. So this expression cannot give information on any remaining part. Therefore, it is reasonable to lay down the assumption that these remaining parts do not exist and that T and $\mathcal{T}$ enjoy the same symmetries in the last entries as their dual virtual velocity gradients.

The expression above is our only constitutive restriction. The topic of the paper at hand is nothing else but the mathematical exploration of the behaviour of this constitutive model. No further assumption (like elasticity, plasticity, fluidity, etc.) is laid down. Moreover, we will detect the existence of a crust shell on a free surface, but this is not the consequence of any additional constitutive assumption on surfaces.

We want to describe the internal virtual power also in the reference placement where nominal instead of true stresses appear, characterized by the index R.

$$\delta\Pi_{\text{int}} = \int \left(\mathbf{T}_R : \delta\mathbf{v} \otimes \nabla_R + \mathsf{T}_R \vdots \delta\mathbf{v} \otimes \nabla_R \otimes \nabla_R + \mathcal{T}_R :: \delta\mathbf{v} \otimes \nabla_R \otimes \nabla_R \otimes \nabla_R\right) dV_R \tag{46}$$

We are interested in the connection between the true and nominal stresses. We notice (5) and start with

$$\begin{aligned} \mathbf{T}_R : \delta\mathbf{v} \otimes \nabla_R \, dV_R &= t_{R\,ik}(\delta v_i \nabla_{Rk})\, dV_R = t_{R\,ik}(\delta v_i \nabla_p) f_{pk}\, dV_R \\ &= \mathbf{T}_2 : \delta\mathbf{v} \otimes \nabla\, dV = t_{2\,ip}(\delta v_i \nabla_p)\, dV = J\, t_{2\,ip}(\delta v_i \nabla_p)\, dV_R \end{aligned} \tag{47}$$

$$\Longrightarrow \qquad J t_{2\,ip} = t_{R\,ik} f_{pk}, \qquad J\,\mathbf{T}_2 = \mathbf{T}_R \cdot \mathbf{F}^T \tag{48}$$

The index 2 indicates that this is not yet the full tensor $\mathbf{T}$ but only one contribution. Next we notice (5) and (6) and study

$$\begin{aligned}
\mathsf{T}_R : \cdot\, \delta\mathbf{v} \otimes \nabla_R \otimes \nabla_R \, dV_R &= t_{R\,ikl}(\delta v_i \nabla_{Rk} \nabla_{Rl})\, dV_R = t_{R\,ikl}\big((\delta v_i \nabla_p) f_{pk}\big)\nabla_{Rl}\, dV_R \\
&= \Big(t_{R\,ikl}(\delta v_i \nabla_p \nabla_q) f_{pk} f_{ql} + t_{R\,ikl}(\delta v_i \nabla_p) f_{pkl}\Big)\, dV_R \\
&= \mathsf{T}_3 : \cdot\, \delta\mathbf{v} \otimes \nabla \otimes \nabla \, dV + \mathbf{T}_3 : \delta\mathbf{v} \otimes \nabla \, dV \\
&= \Big(t_{3\,ipq}(\delta v_i \nabla_p \nabla_q) + t_{3\,ip}(\delta v_i \nabla_p)\Big)\, J\, dV_R \qquad (49)\\
\Longrightarrow \quad & J\, t_{3\,ipq} = t_{R\,ikl} f_{pk} f_{ql}, \qquad J\, t_{3\,ip} = t_{R\,ikl} f_{pkl} \qquad (50)
\end{aligned}$$

Finally, we discuss

$$\begin{aligned}
&\mathcal{T}_R :: \delta\mathbf{v} \otimes \nabla_R \otimes \nabla_R \otimes \nabla_R \, dV_R = t_{R\,iklm}(\delta v_i \nabla_{Rk} \nabla_{Rl} \nabla_{Rm})\, dV_R \\
&\quad = t_{R\,iklm}\Big((\delta v_i \nabla_p \nabla_q) f_{pk} f_{ql} + (\delta v_i \nabla_p) f_{pkl}\Big)\nabla_{Rm}\, dV_R \\
&\quad = t_{R\,iklm}\Big((\delta v_i \nabla_p \nabla_q \nabla_r) f_{pk} f_{ql} f_{rm} + (\delta v_i \nabla_p \nabla_q) f_{pkm} f_{ql} + (\delta v_i \nabla_p \nabla_q) f_{pk} f_{qlm} \\
&\qquad + (\delta v_i \nabla_p \nabla_q) f_{pkl} f_{qm} + (\delta v_i \nabla_p) f_{pklm}\Big)\, dV_R \\
&\quad = \Big(\mathcal{T}_4 :: \delta\mathbf{v} \otimes \nabla \otimes \nabla \otimes \nabla + \mathsf{T}_4 : \cdot\, \delta\mathbf{v} \otimes \nabla \otimes \nabla + \mathbf{T}_4 : \delta\mathbf{v} \otimes \nabla\Big)\, dV \\
&\quad = \Big(t_{4ipqr}(\delta v_i \nabla_p \nabla_q \nabla_r) + t_{4\,ipq}(\delta v_i \nabla_p \nabla_q) + t_{4\,ip} \delta v_i \nabla_p\Big)\, J dV_R \qquad (51)\\
&\Longrightarrow J\, t_{4\,ipqr} = t_{R\,iklm} f_{pk} f_{ql} f_{rm}, \\
&\qquad J\, t_{4\,ipq} = t_{R\,iklm}\big(f_{pkm} f_{ql} + f_{qlm} f_{pk} + f_{pkl} f_{qm}\big), \quad J\, t_{4\,ip} = t_{R\,iklm} f_{pklm} \qquad (52)
\end{aligned}$$

After all, we have the following connections between the true and the nominal stresses.

$$J\, t_{ipqr} = J\, t_{4\,ipqr} = t_{R\,iklm} f_{pk} f_{ql} f_{rm} \qquad (53)$$

$$J\, t_{ipq} = J\, t_{3\,ipq} + J\, t_{4\,ipq} = t_{R\,ikl} f_{pk} f_{ql} + t_{R\,iklm}\big(f_{pkm} f_{ql} + f_{qlm} f_{pk} + f_{pkl} f_{qm}\big) \qquad (54)$$

$$J\, t_{ip} = J\, t_{2\,ip} + J\, t_{3\,ip} + J\, t_{4\,ip} = t_{R\,ik} f_{pk} + t_{R\,ikl} f_{pkl} + t_{R\,iklm} f_{pklm} \qquad (55)$$

First we learn that there is no one-to-one correspondence between the current and referential stress tensors of order two and three.

Second we note that t_{ipq} on the left-hand side of (54) is symmetric in p and q. But the second term on the right-hand side does not enjoy this symmetry. Therefore, the first term cannot be symmetric, too. Instead, we find the difference

$$t_{Riuv} - t_{Rivu} = (t_{Riklu} g_{vw} - t_{Riklv} g_{uw}) f_{wkl} \qquad (56)$$

The fact that T_R is not symmetric in the last two entries, in general, seems confusing at first sight. The properties of the skew part cannot be inferred from the virtual power (46) since it does not enter this expression. But, fortunately, this is not necessary at all since the skew part is fully determined by $\mathcal{T}_R$, $\mathbf{F}$ and $\mathbf{F} \otimes \nabla_R$ according to (56). If we restrict our attention to second-gradient materials, then $\mathcal{T}_R$ does not exist and T_R is symmetric anyhow.

5 True and nominal effective stresses

The internal virtual power may be reformulated by multiple application of the divergence theorem.

$$\delta\Pi_{\text{int}} = \int \Big((\mathcal{T} \cdot \mathbf{n}) : \cdot\, \delta\mathbf{v} \otimes \nabla \otimes \nabla + (\bar{\mathsf{T}} \cdot \mathbf{n}) : \delta\mathbf{v} \otimes \nabla + \delta\mathbf{v} \cdot \bar{\mathbf{T}} \cdot \mathbf{n}\Big)\, dA - \int \delta\mathbf{v} \cdot \bar{\mathbf{T}} \cdot \nabla \, dV \qquad (57)$$

$$\begin{aligned}
&= \int \Big((\mathcal{T}_R \cdot \mathbf{n}_R) : \cdot\, \delta\mathbf{v} \otimes \nabla_R \otimes \nabla_R + (\bar{\mathsf{T}}_R \cdot \mathbf{n}_R) : \delta\mathbf{v} \otimes \nabla_R + \delta\mathbf{v} \cdot \bar{\mathbf{T}}_R \cdot \mathbf{n}_R\Big)\, dA_R \\
&\quad - \int \delta\mathbf{v} \cdot \bar{\mathbf{T}}_R \cdot \nabla_R \, dV_R \qquad (58)
\end{aligned}$$

We introduced the true and nominal effective stress tensors.

$$\bar{\mathsf{T}} = \mathsf{T} - \mathcal{T} \cdot \nabla, \qquad \bar{\mathbf{T}} = \mathbf{T} - \bar{\mathsf{T}} \cdot \nabla \tag{59}$$

$$\bar{\mathsf{T}}_R = \mathsf{T}_R - \mathcal{T}_R \cdot \nabla_R, \qquad \bar{\mathbf{T}}_R = \mathbf{T}_R - \bar{\mathsf{T}}_R \cdot \nabla_R$$
$$\bar{t}_{R\,ikl} = t_{R\,ikl} - t_{R\,iklm} \nabla_{Rm}, \qquad \bar{t}_{R\,ik} = t_{R\,ik} - \bar{t}_{R\,ikl} \nabla_{Rl} \tag{60}$$

We are interested in the connection between the true and nominal effective stresses. We find with (6), (7) and (53)

$$\begin{aligned} t_{ipqr} \nabla_r &= \Big(J^{-1} t_{R\,iklm} f_{pk} f_{ql} f_{rm} \Big) \nabla_{Rs} g_{sr} \\ &= J^{-1} \Big(\underline{-g_{vu} f_{uvm} t_{R\,iklm} f_{pk} f_{ql}} + (t_{R\,iklm} \nabla_{Rm}) f_{pk} f_{ql} + t_{R\,iklm} f_{pkm} f_{ql} \\ &\quad + t_{R\,iklm} f_{pk} f_{qlm} + \underline{t_{R\,iklm} f_{pk} f_{ql} f_{rms} g_{sr}} \Big) \end{aligned} \tag{61}$$

where the underlined terms cancel each other and

$$\bar{t}_{ipq} = t_{ipq} - t_{ipqr} \nabla_r = -J^{-1} \Big((t_{R\,iklm} \nabla_{Rm} - t_{Rikl}) f_{pk} f_{ql} - t_{R\,iklm} f_{pkl} f_{qm} \Big) \tag{62}$$

and hence the connection

$$J\, \bar{t}_{ipq} = \bar{t}_{R\,ikl} f_{pk} f_{ql} + t_{R\,iklm} f_{pkl} f_{qm} \tag{63}$$

The divergence yields

$$\begin{aligned} \bar{t}_{ipq} \nabla_q = J^{-1} \Big(& \underline{- g_{vu} f_{uvl} \bar{t}_{R\,ikl} f_{pk}} + (\bar{t}_{R\,ikl} \nabla_{Rl}) f_{pk} + \bar{t}_{R\,ikl} f_{pkl} + \underline{\bar{t}_{R\,ikl} f_{pk} f_{qls} g_{sq}} \\ & \underline{- g_{vu} f_{uvm} t_{R\,iklm} f_{pkl}} + (t_{R\,iklm} \nabla_{Rm}) f_{pkl} + t_{R\,iklm} f_{pklm} + \underline{t_{R\,iklm} f_{pkl} f_{qms} g_{sq}} \Big) \end{aligned} \tag{64}$$

$$\begin{aligned} \bar{t}_{ip} = t_{ip} - \bar{t}_{ipq} \nabla_q = J^{-1} \Big(& t_{R\,iklm} f_{pklm} + (\bar{t}_{R\,ikl} + t_{R\,iklm} \nabla_{Rm}) f_{pkl} + (\bar{t}_{R\,ik} + \bar{t}_{R\,ikl} \nabla_{Rl}) f_{pk} \\ & - (\bar{t}_{R\,ikl} \nabla_{Rl}) f_{pk} - \bar{t}_{R\,ikl} f_{pkl} - (t_{R\,iklm} \nabla_{Rm}) f_{pkl} - t_{R\,iklm} f_{pklm} \Big) \end{aligned} \tag{65}$$

This expression reduces to a very simple one-to-one connection between the current and referential second-order effective stress tensors, while we found no such connection with the third-order tensors (see (63)).

$$J\, \bar{t}_{ip} = \bar{t}_{R\,ik} f_{pk}, \qquad J \bar{\mathbf{T}} = \bar{\mathbf{T}}_R \cdot \mathbf{F}^T \tag{66}$$

Noting (10) we obtain the force on a surface element in current and referential description.

$$\bar{\mathbf{T}} \cdot \mathbf{n}\, dA = \bar{\mathbf{T}}_R \cdot J^{-1} \mathbf{F}^T \cdot \mathbf{n}\, dA = \bar{\mathbf{T}}_R \cdot \mathbf{n}_R\, dA_R \tag{67}$$

If we restrict our attention to first-gradient (i.e. simple) materials, then $\bar{\mathbf{T}} = \mathbf{T}$, and we find the well-known connection between the true and nominal Cauchy stress vectors. Now consider a second-gradient material. We expect the coincidence of the additional terms.

$$(\bar{\mathsf{T}} \cdot \mathbf{n}) : \delta \mathbf{v} \otimes \nabla\, dA = (\bar{\mathsf{T}}_R \cdot \mathbf{n}_R) : \delta \mathbf{v} \otimes \nabla_R\, dA_R \tag{68}$$

This can be proved as follows—note (63) with $t_{Riklm} \equiv 0$

$$\bar{t}_{ipq} n_q \delta v_i \nabla_p dA = \frac{1}{J} \bar{t}_{R\,ikl} f_{pk} f_{ql}\, N\, g_{xq} n_{Rx} \delta v_i \nabla_{Ry} g_{yp} \frac{1}{Q} dA_R = \bar{t}_{R\,ikl}\, n_{Rl} \delta v_i \nabla_{Rk} dA_R \tag{69}$$

The interaction on a cut was restricted to the normal component of a second-order tensor in the context of a simple material. Now we have in addition the normal component $\bar{\mathsf{T}} \cdot \mathbf{n}$ of a third-order tensor, which itself is a second-order tensor. Its symmetric part acts on the symmetric part of the virtual velocity gradient $\delta \mathbf{v} \otimes \nabla$ that describes a stretching. So it represents double forces. The skew part acts on the skew part of the virtual velocity gradient that describes a spinning. So it represents single moments. It makes no sense to apply the same argument to the symmetric and skew part of $\delta\mathbf{v} \otimes \nabla_R$ and to invent "referential double forces and moments" without a one-to-one correspondence with the current ones since double forces and moments are physical concepts.

Things become more complicated in the case of a third-gradient material. Then, we expect the following coincidence.

$$\Big((\mathcal{T} \cdot \mathbf{n}) :\cdot \delta\mathbf{v} \otimes \nabla \otimes \nabla + (\bar{\mathsf{T}} \cdot \mathbf{n}) : \delta\mathbf{v} \otimes \nabla\Big)\, dA \tag{70}$$

$$= \Big((\mathcal{T}_R \cdot \mathbf{n}_R) :\cdot \delta\mathbf{v} \otimes \nabla_R \otimes \nabla_R + (\bar{\mathsf{T}}_R \cdot \mathbf{n}_R) : \delta\mathbf{v} \otimes \nabla_R\Big)\, dA_R \tag{71}$$

It is obvious from (63) that the terms with the first and second virtual velocity gradient cannot be identified separately in the current and the reference placement.

This time, we encounter also the normal component of a fourth-order tensor. This third-order tensor represents triple forces and various kinds of double moments.

Let us now assume that the equation (57) is applied to a subbody that is surrounded by fictitious cuts. The interactions across these cuts are seen to be $\bar{\mathbf{T}} \cdot \mathbf{n}$, $\bar{\mathsf{T}} \cdot \mathbf{n}$, and $\mathcal{T} \cdot \mathbf{n}$. It is not surprising that the interactions become more and more intricate if higher gradients enter the constitutive description. Two facts should, however, be noticed:

- The interactions depend only on the local orientation of the cut and not on its curvature.
- No additional contributions appear at edges or vertices of the boundary of a subbody if its surface is not smooth.

Contrary claims are based on an inappropriate application of the divergence theorem of surfaces on fictitious cuts. This point will be clarified in Sects. 7 and 11.

6 Decomposition of the derivatives on a boundary

We want to derive boundary conditions on a free surface. Beforehand, it is necessary to apply the split of the first and second derivative of the virtual velocity provided in (23) and (26) to the expression (71).

The second integrand becomes

$$(\bar{\mathsf{T}}_R \cdot \mathbf{n}_R) : \delta\mathbf{v} \otimes \nabla_R = (\bar{\mathsf{T}}_R \cdot \mathbf{n}_R) : \delta\mathbf{v} \otimes \nabla_{RT} \cdot \tilde{\mathbf{P}} + N(\bar{\mathsf{T}}_R : \mathbf{n}_R \otimes \mathbf{n}_R) \cdot \big(d_n \delta\mathbf{v}\big) \tag{72}$$

The first integrand is:

$$\begin{aligned}
\big(\mathcal{T}_R \cdot \mathbf{n}_R\big) :\cdot \delta\mathbf{v} \otimes \nabla_R \otimes \nabla_R &= \big(\mathcal{T}_R \cdot \mathbf{n}_R\big) :\cdot \big(\delta\mathbf{v} \otimes \nabla_{RT} \cdot \tilde{\mathbf{P}}\big) \otimes \nabla_{RT} \cdot \tilde{\mathbf{P}} \\
&+ \big(\mathcal{T}_R \cdot \mathbf{n}_R\big) :\cdot \Big(N\big(d_n\, \delta\mathbf{v}\big) \otimes \mathbf{n}_R\Big) \otimes \nabla_{RT} \cdot \tilde{\mathbf{P}} \\
&+ N\big(\mathcal{T}_R \cdot \mathbf{n}_R\big) :\cdot \big(d_n \delta\mathbf{v}\big) \otimes \nabla_{RT} \cdot \tilde{\mathbf{P}} \otimes \mathbf{n}_R \\
&- N^2\big(\mathcal{T}_R \cdot \mathbf{n}_R\big) :\cdot \big(d_n \delta\mathbf{v}\big) \otimes \mathbf{n}_R \cdot \big(\tilde{\mathbf{n}} \otimes \nabla_{RT} \cdot \tilde{\mathbf{P}} + N d_n \tilde{\mathbf{n}} \otimes \mathbf{n}_R\big) \otimes \mathbf{n}_R \\
&- N\big(\mathcal{T}_R \cdot \mathbf{n}_R\big) :\cdot (\delta\mathbf{v} \otimes \nabla_{RT}) \cdot \tilde{\mathbf{P}} \cdot (\tilde{\mathbf{n}} \otimes \nabla_{RT} \cdot \tilde{\mathbf{P}} + N d_n \tilde{\mathbf{n}} \otimes \mathbf{n}_R) \otimes \mathbf{n}_R \\
&+ N^2\big(\mathcal{T}_R \cdot \mathbf{n}_R\big) :\cdot \big(d_n^2 \delta\mathbf{v}\big) \otimes \mathbf{n}_R \otimes \mathbf{n}_R
\end{aligned} \tag{73}$$

We rearrange that and note the symmetry of $\mathcal{T}$:

$$\begin{aligned}
\big(\mathcal{T}_R \cdot \mathbf{n}_R\big) :\cdot \delta\mathbf{v} \otimes \nabla_R \otimes \nabla_R &= \big(\mathcal{T}_R \cdot \mathbf{n}_R\big) \cdot \tilde{\mathbf{P}}^T :\cdot \big(\delta\mathbf{v} \otimes \nabla_{RT} \cdot \tilde{\mathbf{P}}\big) \otimes \nabla_{RT} \\
&+ 2N\big(\mathcal{T}_R : \mathbf{n}_R \otimes \mathbf{n}_R\big) \cdot \tilde{\mathbf{P}}^T : \big(d_n\, \delta\mathbf{v}\big) \otimes \nabla_{RT} \\
&+ N\big(\mathcal{T}_R \cdot \mathbf{n}_R\big) \cdot \tilde{\mathbf{P}}^T :\cdot \big(d_n\, \delta\mathbf{v}\big) \otimes \big(\mathbf{n}_R \otimes \nabla_{RT}\big) \\
&+ \big(\mathcal{T}_R : \mathbf{n}_R \otimes \mathbf{n}_R\big) \cdot \tilde{\mathbf{P}}^T : \big(d_n\, \delta\mathbf{v}\big) \otimes (\nabla_{RT} N) \\
&- N^2\big(\mathcal{T}_R : \mathbf{n}_R \otimes \mathbf{n}_R\big) : \big(d_n \delta\mathbf{v}\big) \otimes \mathbf{n}_R \cdot \big(\tilde{\mathbf{n}} \otimes \nabla_{RT} \cdot \tilde{\mathbf{P}} + N d_n \tilde{\mathbf{n}} \otimes \mathbf{n}_R\big) \\
&- N\big(\mathcal{T}_R : \mathbf{n}_R \otimes \mathbf{n}_R\big) : (\delta\mathbf{v} \otimes \nabla_{RT}) \cdot \tilde{\mathbf{P}} \cdot (\tilde{\mathbf{n}} \otimes \nabla_{RT} \cdot \tilde{\mathbf{P}} + N d_n \tilde{\mathbf{n}} \otimes \mathbf{n}_R) \\
&+ N^2\big(\mathcal{T}_R :\cdot \mathbf{n}_R \otimes \mathbf{n}_R \otimes \mathbf{n}_R\big) \cdot \big(d_n^2 \delta\mathbf{v}\big)
\end{aligned} \tag{74}$$

The first term must be evaluated. We make use of (27) and replace the current notions by referential ones.

$$
\begin{aligned}
&(\mathcal{T}_R \cdot \mathbf{n}_R) \cdot \tilde{\mathbf{P}}^T \,\vdots\, \big((\delta\mathbf{v} \otimes \nabla_{RT}) \cdot \tilde{\mathbf{P}}\big) \otimes \nabla_{RT} \\
&= (\mathcal{T}_R \cdot \mathbf{n}_R) \cdot \tilde{\mathbf{P}}^T \,\vdots\, \Big(\Big(\tilde{\mathbf{P}}^T \cdot ((\delta\mathbf{v} \otimes \nabla_{RT}) \otimes \nabla_{RT})^T\Big)^T + (\delta\mathbf{v} \otimes \nabla_{RT}) \cdot \big(\tilde{\mathbf{P}} \otimes \nabla_{RT}\big)\Big) \\
&= \Big(\tilde{\mathbf{P}} \cdot (\mathcal{T}_R \cdot \mathbf{n}_R)^T\Big)^T \cdot \tilde{\mathbf{P}}^T \,\vdots\, (\delta\mathbf{v} \otimes \nabla_{RT}) \otimes \nabla_{RT} + \Big((\mathcal{T}_R \cdot \mathbf{n}_R) \cdot \tilde{\mathbf{P}}^T : (\nabla_{RT} \otimes \tilde{\mathbf{P}}^T)^T\Big) : \delta\mathbf{v} \otimes \nabla_{RT}
\end{aligned}
\tag{75}
$$

We are now able to reformulate the internal virtual power (58) and obtain

$$
\begin{aligned}
\delta\Pi_{\text{int}} = &\int \Big(\mathbf{t}_R \cdot \delta\mathbf{v} + \mathbf{Z}_R : \delta\mathbf{v} \otimes \nabla_{RT} + \mathsf{Y}_R \,\vdots\, (\delta\mathbf{v} \otimes \nabla_{RT}) \otimes \nabla_{RT}\Big)\, dA_R \\
&+ \int \Big(\check{\mathbf{t}}_R \cdot d_n\delta\mathbf{v} + \check{\mathbf{Z}}_R : d_n\delta\mathbf{v} \otimes \nabla_{RT}\Big)\, dA_R + \int \hat{\mathbf{t}}_R \cdot d_n^2\delta\mathbf{v}\, dA_R - \int \delta\mathbf{v} \cdot \bar{\mathbf{T}}_R \cdot \nabla_R\, dV_R
\end{aligned}
\tag{76}
$$

with the following referential dynamic quantities.

$$
\mathbf{t}_R = \bar{\mathbf{T}}_R \cdot \mathbf{n}_R \tag{77}
$$

$$
\begin{aligned}
\check{\mathbf{t}}_R = &N\bar{\mathsf{T}}_R : \mathbf{n}_R \otimes \mathbf{n}_R + N\big(\mathcal{T}_R \cdot \mathbf{n}_R\big) \cdot \tilde{\mathbf{P}}^T : \big(\mathbf{n}_R \otimes \nabla_{RT}\big) + \big(\mathcal{T}_R : \mathbf{n}_R \otimes \mathbf{n}_R\big) \cdot \tilde{\mathbf{P}}^T \cdot (\nabla_{RT} N) \\
&- N^2\big(\mathcal{T}_R : \mathbf{n}_R \otimes \mathbf{n}_R\big) \cdot \tilde{\mathbf{P}}^T \cdot (\nabla_{RT} \otimes \tilde{\mathbf{n}}) \cdot \mathbf{n}_R - N^3\big(\mathcal{T}_R \,\vdots\, \mathbf{n}_R \otimes \mathbf{n}_R \otimes \mathbf{n}_R\big)(\mathbf{n}_R \cdot d_n\tilde{\mathbf{n}})
\end{aligned}
\tag{78}
$$

$$
\begin{aligned}
\mathbf{Z}_R = &\Big(\bar{\mathsf{T}}_R \cdot \mathbf{n}_R - N\big(\mathcal{T}_R : \mathbf{n}_R \otimes \mathbf{n}_R\big) \cdot \tilde{\mathbf{P}}^T \cdot (\nabla_{RT} \otimes \tilde{\mathbf{n}}) \\
&- N^2\big(\mathcal{T}_R \,\vdots\, \mathbf{n}_R \otimes \mathbf{n}_R \otimes \mathbf{n}_R\big) \otimes d_n\tilde{\mathbf{n}}\Big) \cdot \tilde{\mathbf{P}}^T + \Big((\mathcal{T}_R \cdot \mathbf{n}_R) \cdot \tilde{\mathbf{P}}^T : \big(\nabla_{RT} \otimes \tilde{\mathbf{P}}^T\big)^T\Big) \cdot \mathbf{P}_R
\end{aligned}
\tag{79}
$$

$$
\hat{\mathbf{t}}_R = N^2\mathcal{T}_R \,\vdots\, \mathbf{n}_R \otimes \mathbf{n}_R \otimes \mathbf{n}_R, \qquad \check{\mathbf{Z}}_R = 2N\big(\mathcal{T}_R : \mathbf{n}_R \otimes \mathbf{n}_R\big) \cdot \tilde{\mathbf{P}}^T \tag{80}
$$

$$
\mathsf{Y}_R = \Big(\tilde{\mathbf{P}} \cdot \big(\mathcal{T}_R \cdot \mathbf{n}_R\big)^T\Big)^T \cdot \tilde{\mathbf{P}}^T, \qquad y_{Ripq} = t_{Riklm} n_{Rm} \tilde{p}_{pk} \tilde{p}_{ql} \tag{81}
$$

If the reference placement coincides with the current placement, then we have $\tilde{\mathbf{n}} = \mathbf{n}$, $\tilde{\mathbf{P}} = \mathbf{P}$, $N \equiv 1$, $\mathbf{n} \cdot d_n\mathbf{n} = 0$, $(\nabla_T \otimes \mathbf{n}) \cdot \mathbf{n} = \mathbf{0}$, and the internal virtual power (57) becomes

$$
\begin{aligned}
\delta\Pi_{\text{int}} = &\int \Big(\mathbf{t} \cdot \delta\mathbf{v} + \mathbf{Z} : \delta\mathbf{v} \otimes \nabla_T + \mathsf{Y} \,\vdots\, (\delta\mathbf{v} \otimes \nabla_T) \otimes \nabla_T\Big)\, dA \\
&+ \int \Big(\check{\mathbf{t}} \cdot d_n\delta\mathbf{v} + \check{\mathbf{Z}} : d_n\delta\mathbf{v} \otimes \nabla_T\Big)\, dA + \int \hat{\mathbf{t}} \cdot d_n^2\delta\mathbf{v}\, dA - \int \delta\mathbf{v} \cdot \bar{\mathbf{T}} \cdot \nabla\, dV
\end{aligned}
\tag{82}
$$

with the following current dynamic quantities.

$$
\mathbf{t} = \bar{\mathbf{T}} \cdot \mathbf{n}, \qquad \check{\mathbf{t}} = \bar{\mathsf{T}} : \mathbf{n} \otimes \mathbf{n} + \big(\mathcal{T} \cdot \mathbf{n}\big) : \big(\mathbf{n} \otimes \nabla_T\big) \tag{83}
$$

$$
\mathbf{Z} = \Big(\bar{\mathsf{T}} \cdot \mathbf{n} - \big(\mathcal{T} : \mathbf{n} \otimes \mathbf{n}\big) \cdot \big(\nabla_T \otimes \mathbf{n}\big) - \big(\mathcal{T} \,\vdots\, \mathbf{n} \otimes \mathbf{n} \otimes \mathbf{n}\big) \otimes d_n\mathbf{n} + \big(\mathcal{T} \cdot \mathbf{n}\big) : (\nabla_T \otimes \mathbf{P})^T\Big) \cdot \mathbf{P} \tag{84}
$$

$$
\hat{\mathbf{t}} = \mathcal{T} \,\vdots\, \mathbf{n} \otimes \mathbf{n} \otimes \mathbf{n}, \qquad \check{\mathbf{Z}} = 2\big(\mathcal{T} : \mathbf{n} \otimes \mathbf{n}\big) \cdot \mathbf{P}, \qquad \mathsf{Y} = \Big(\mathbf{P} \cdot \big(\mathcal{T} \cdot \mathbf{n}\big)^T\Big)^T \cdot \mathbf{P} \tag{85}
$$

The normal components $\bar{\mathbf{T}} \cdot \mathbf{n}$, $\bar{\mathsf{T}} \cdot \mathbf{n}$, and $\mathcal{T} \cdot \mathbf{n}$ depend only on the local value of $\mathbf{n}$. The splitting, however, creates the quantities $\check{\mathbf{t}}$ and $\mathbf{Z}$ that depend also on the tangential derivative $-\mathbf{C} = \mathbf{n} \otimes \nabla_T$ of $\mathbf{n}$, i.e. on the symmetric local curvature tensor $\mathbf{C}$.

The following properties will later be needed.

$$
\mathbf{0} = \mathbf{Z}_R \cdot \mathbf{n}_R, \qquad \mathbf{0} = \mathbf{Z} \cdot \mathbf{n}, \qquad \mathbf{0} = \mathsf{Y}_R \cdot \mathbf{n}_R = \mathbf{n}_R \cdot \mathsf{Y}_R^T, \qquad \mathbf{0} = \mathsf{Y} \cdot \mathbf{n} = \mathbf{n} \cdot \mathsf{Y}^T \tag{86}
$$

The definition of $\mathbf{Z}$ and Y and hence also of $\mathbf{Z}_R$ and Y_R is not unique. The following statement is a consequence of the connection (28) of the first and second tangential derivative.

$$\begin{aligned}0 &= (\mathcal{T}\cdot\mathbf{n})\cdot\mathbf{P} :\cdot \Big(\Big((\mathbf{n}\otimes\mathbf{n})\cdot\big((\delta\mathbf{v}\otimes\nabla_T)\otimes\nabla_T\big)^T\Big)^T + (\delta\mathbf{v}\otimes\nabla_T)\cdot\big((\mathbf{n}\otimes\mathbf{n})\otimes\nabla_T\big)\Big)\\ &= \Big(\mathbf{n}\otimes\mathbf{n}\cdot(\mathcal{T}\cdot\mathbf{n})^T\Big)^T\cdot\mathbf{P} :\cdot(\delta\mathbf{v}\otimes\nabla_T)\otimes\nabla_T + \Big((\mathcal{T}\cdot\mathbf{n})\cdot\mathbf{P} : \big(\nabla_T\otimes(\mathbf{n}\otimes\mathbf{n})\big)^T\Big)\cdot\mathbf{P} : \delta\mathbf{v}\otimes\nabla_T\\ &\equiv \mathsf{Y}_{\text{add}} :\cdot(\delta\mathbf{v}\otimes\nabla_T)\otimes\nabla_T + \mathbf{Z}_{\text{add}} : \delta\mathbf{v}\otimes\nabla_T\end{aligned} \tag{87}$$

with

$$\mathsf{Y}_{\text{add}} = \Big(\mathbf{n}\otimes\mathbf{n}\cdot(\mathcal{T}\cdot\mathbf{n})^T\Big)^T\cdot\mathbf{P} \tag{88}$$

$$\mathbf{Z}_{\text{add}} = \Big((\mathcal{T}\cdot\mathbf{n})\cdot\mathbf{P} : \big(\nabla_T\otimes(\mathbf{n}\otimes\mathbf{n})\big)^T\Big)\cdot\mathbf{P} = \mathcal{T}\cdot\mathbf{n} : \mathbf{n}\otimes(\nabla_T\otimes\mathbf{n}) \tag{89}$$

Any multiple of these additional contributions of $\mathbf{Z}$ and Y does not influence the current version (82) of the internal virtual power. Special modified values are

$$\mathbf{Z}_{\text{mod}} = \mathbf{Z} + \mathbf{Z}_{\text{add}} = \Big(\bar{\mathsf{T}}\cdot\mathbf{n} - (\mathcal{T} : \mathbf{n}\otimes\mathbf{n})\cdot(\nabla_T\otimes\mathbf{n}) - (\mathcal{T} :\cdot \mathbf{n}\otimes\mathbf{n}\otimes\mathbf{n})\otimes d_n\mathbf{n}\Big)\cdot\mathbf{P} \tag{90}$$

$$\mathsf{Y}_{\text{mod}} = \mathsf{Y} + \mathsf{Y}_{\text{add}} = (\mathcal{T}\cdot\mathbf{n})\cdot\mathbf{P} \tag{91}$$

These modified values were derived in [12] under the additional choice $d_n\mathbf{n} = \mathbf{0}$. In general, we will have $\mathbf{n}\cdot\mathsf{Y}_{\text{mod}}^T \neq \mathbf{0}$.

7 The boundary conditions of a free surface

Now imagine an extremely thin layer at the surface. It is loaded from the interior of the body by these dynamic quantities, endowed with a negative sign. In case of a simple material, they reduce to a force per unit reference area $-\mathbf{t}_R = -\mathbf{T}_R\cdot\mathbf{n}_R$, and this must be compensated by an external force $\mathbf{t}_{R\,\text{ext}}$ of equal amount. If we treat gradient materials then we encounter, of course, also tensors of higher order as external loads. The virtual power of all these external loads can be read from (76).

$$\begin{aligned}\delta\Pi_{\text{ext}} = &\int\Big(\mathbf{t}_{R\text{ext}}\cdot\delta\mathbf{v} + \underbrace{\mathbf{Z}_{R\text{ext}}} : \delta\mathbf{v}\otimes\nabla_{RT} + \underbrace{\mathsf{Y}_{R\text{ext}}} :\cdot (\delta\mathbf{v}\otimes\nabla_{RT})\otimes\nabla_{RT}\Big)\,dA_R\\ &+\int\Big(\check{\mathbf{t}}_{R\text{ext}}\cdot d_n\delta\mathbf{v} + \underbrace{\check{\mathbf{Z}}_{R\text{ext}}} : d_n\delta\mathbf{v}\otimes\nabla_{RT}\Big)\,dA_R\\ &+\int\hat{\mathbf{t}}_{R\text{ext}}\cdot d_n^2\delta\mathbf{v}\,dA_R + \int\delta\mathbf{v}\cdot\varrho_R(\mathbf{b}-\mathbf{a})\,dV_R + \delta\Pi_{\text{add}}\end{aligned} \tag{92}$$

It has been supplemented by an additional contribution $\delta\Pi_{\text{add}}$ of line tractions and point forces that will be specified later. The external action within the volume is given by the well-known contributions of the body force $\mathbf{b}$ and the inertial force $-\mathbf{a}$ per unit mass.

The three underbraced external actions are those that are ignored in the literature as has been mentioned under point 4 of the introduction. We meet them again at the end of this section.

The principle of virtual power postulates the equality of the internal and external virtual power of a body. (An application to a subbody is vacuous since no external surface actions are prescribed for a subbody.) So we arrive at the following statement.

$$\begin{aligned}0 = \delta\Pi_{\text{ext}} - \delta\Pi_{\text{int}} = &\int\Big(\mathbf{t}_R^*\cdot\delta\mathbf{v} + \mathbf{Z}_R^* : \delta\mathbf{v}\otimes\nabla_{RT} + \mathsf{Y}_R^* :\cdot (\delta\mathbf{v}\otimes\nabla_{RT})\otimes\nabla_{RT}\Big)\,dA_R\\ &+\int\Big(\check{\mathbf{t}}_R^*\cdot d_n\delta\mathbf{v} + \check{\mathbf{Z}}_R^* : d_n\delta\mathbf{v}\otimes\nabla_{RT}\Big)\,dA_R\\ &+\int\hat{\mathbf{t}}_R^*\cdot d_n^2\delta\mathbf{v}\,dA_R + \int\delta\mathbf{v}\cdot\Big(\bar{\mathbf{T}}_R\cdot\nabla_R + \varrho_R(\mathbf{b}-\mathbf{a})\Big)\,dV_R + \delta\Pi_{\text{add}}\end{aligned} \tag{93}$$

with the differences

$$\mathbf{t}_R^* = \mathbf{t}_{R\text{ext}} - \mathbf{t}_R, \quad \check{\mathbf{t}}_R^* = \check{\mathbf{t}}_{R\text{ext}} - \check{\mathbf{t}}_R, \quad \hat{\mathbf{t}}_R^* = \hat{\mathbf{t}}_{R\text{ext}} - \hat{\mathbf{t}}_R,$$
$$\mathbf{Z}_R^* = \mathbf{Z}_{R\text{ext}} - \mathbf{Z}_R, \quad \check{\mathbf{Z}}_R^* = \check{\mathbf{Z}}_{R\text{ext}} - \check{\mathbf{Z}}_R, \quad \mathsf{Y}_R^* = \mathsf{Y}_{R\text{ext}} - \mathsf{Y}_R \tag{94}$$

By the way, if we consider fixed boundaries, then the external agents are reactions and all the differences vanish.

The postulate (93) must be valid for any virtual velocity field $\delta\mathbf{v}$ that satisfies possible displacement constraints. So we obtain the field equation

$$\bar{\mathbf{T}}_R \cdot \nabla_R + \varrho_R(\mathbf{b} - \mathbf{a}) = \mathbf{0} \tag{95}$$

and the boundary condition—note (80)—

$$\mathbf{0} = \hat{\mathbf{t}}_R^* = \hat{\mathbf{t}}_{R\text{ext}} - N^2 \mathcal{T}_R \,\because\, \mathbf{n}_R \otimes \mathbf{n}_R \otimes \mathbf{n}_R \tag{96}$$

on a free surface. Moreover, the vanishing of the differences $\mathbf{t}_R^*, \check{\mathbf{t}}_R^*, \mathbf{Z}_R^*, \check{\mathbf{Z}}_R^*, \mathsf{Y}_R^*$ of the actions on both sides of the surface would obviously be sufficient but, unfortunately, is not necessary. The reason is that the tangential derivatives of $\delta\mathbf{v}$ and $d_n\delta\mathbf{v}$ are not arbitrary but are determined by the fields $\delta\mathbf{v}$ and $d_n\delta\mathbf{v}$ on the free surface. This difficulty was first detected 150 years ago in the context of the simplest second-gradient theory, namely the Kirchhoff plate, and was treated by Thomson and Tait [21].

The remedy is the divergence theorem of the surface. It is applied twice to the first line and once to the second line of (93).

$$\int \Big(\mathbf{t}_R^* \cdot \delta\mathbf{v} + \mathbf{Z}_R^* : \delta\mathbf{v} \otimes \nabla_{RT} + \mathsf{Y}_R^* \,\because\, (\delta\mathbf{v} \otimes \nabla_{RT}) \otimes \nabla_{RT}\Big)\, dA_R$$
$$= \int \delta\mathbf{v} \cdot \Big(\mathbf{t}_R^* - \mathbf{Z}_R^* \cdot \nabla_{RT} + \Big(\big(\mathsf{Y}_R^* \cdot \nabla_{RT}\big) \cdot \mathbf{P}_R\Big) \cdot \nabla_{RT}\Big)\, dA_R$$
$$+ \oint \Big(\delta\mathbf{v} \cdot \Big(\mathbf{Z}_R^* - \big(\mathsf{Y}_R^* \cdot \nabla_{RT}\big) \cdot \mathbf{P}_R\Big) + \delta\mathbf{v} \otimes \nabla_{RT} : \mathsf{Y}_R^*\Big) \cdot \mathbf{e}_R\, ds_R \tag{97}$$

$$\int \Big(d_n\delta\mathbf{v} \cdot \check{\mathbf{t}}_R^* + \check{\mathbf{Z}}_R^* : d_n\delta\mathbf{v} \otimes \nabla_{RT}\Big)\, dA_R$$
$$= \int d_n\delta\mathbf{v} \cdot \Big(\check{\mathbf{t}}_R^* - \check{\mathbf{Z}}_R^* \cdot \nabla_{RT}\Big)\, dA_R + \oint d_n\delta\mathbf{v} \cdot \check{\mathbf{Z}}_R^* \cdot \mathbf{e}_R\, ds_R \tag{98}$$

In this section, we restrict the discussion to the surface terms. The line terms are discussed in Sect. 10.

Since $\delta\mathbf{v}$ and $d_n\delta\mathbf{v}$ are arbitrary on the surface, we infer the boundary conditions

$$\mathbf{t}_R^* + \mathbf{S}_R \cdot \nabla_{RT} = \mathbf{0} \tag{99}$$

and

$$\check{\mathbf{t}}_R^* + \check{\mathbf{S}}_R \cdot \nabla_{RT} = \mathbf{0} \tag{100}$$

with the following two tensors

$$\mathbf{S}_R = -\mathbf{Z}_R^* + \Big(\mathsf{Y}_R^* \cdot \nabla_{RT}\Big) \cdot \mathbf{P}_R \tag{101}$$

and

$$\check{\mathbf{S}}_R = -\check{\mathbf{Z}}_R^* \tag{102}$$

We call them crust tensors since they are only defined on free surfaces. These crust tensors satisfy $\mathbf{S}_R \cdot \mathbf{n}_R = \mathbf{0}$ and $\check{\mathbf{S}}_R \cdot \mathbf{n}_R = \mathbf{0}$ and hence

$$\mathbf{S}_R = \mathbf{S}_R \cdot \mathbf{P}_R, \qquad \check{\mathbf{S}}_R = \check{\mathbf{S}}_R \cdot \mathbf{P}_R \tag{103}$$

Noting (80), we find

$$\check{\mathbf{S}}_R = 2N\big(\mathcal{T}_R : \mathbf{n}_R \otimes \mathbf{n}_R\big) \cdot \tilde{\mathbf{P}}^T - \check{\mathbf{Z}}_{R\,\mathrm{ext}} \tag{104}$$

Next we want to evaluate $\mathbf{S}_R$ and make use of (81), (79), (16) and (96).

$$\begin{aligned} y_{Ripq} \nabla_{Rs} p_{Rsq} p_{Rpz} &= \big(t_{Riklm} n_{Rm} \tilde{p}_{pk} \tilde{p}_{ql}\big) \nabla_{Rs} p_{Rsq} p_{Rpz} \\ &= \tilde{p}_{pk}(t_{Riklm} n_{Rm} \tilde{p}_{ql}) \nabla_{Rs} p_{Rsq} p_{Rpz} + t_{Riklm} n_{Rm} (\tilde{p}_{pk} \nabla_{Rs}) p_{Rsq} \tilde{p}_{ql} p_{Rpz} \\ &= \tilde{p}_{zk}(t_{Riklm} n_{Rm} \tilde{p}_{ql}) \nabla_{Rs} p_{Rsq} + t_{Riklm} n_{Rm} \tilde{p}_{sl} (\tilde{p}_{pk} \nabla_{Rs}) p_{Rpz} \end{aligned} \tag{105}$$

$$\begin{aligned} \mathbf{S}_R = &\Big(\bar{\mathsf{T}}_R \cdot \mathbf{n}_R - N\big(\mathcal{T}_R : \mathbf{n}_R \otimes \mathbf{n}_R\big) \cdot \tilde{\mathbf{P}}^T \cdot \big(\nabla_{RT} \otimes \tilde{\mathbf{n}}\big) - \hat{\mathbf{t}}_{R\mathrm{ext}} \otimes d_n \tilde{\mathbf{n}}\Big) \cdot \tilde{\mathbf{P}}^T \\ &-\Big(\big((\mathcal{T}_R \cdot \mathbf{n}_R) \cdot \tilde{\mathbf{P}}^T\big) \cdot \nabla_{RT}\Big) \cdot \tilde{\mathbf{P}}^T - \mathbf{Z}_{R\,\mathrm{ext}} + (\mathsf{Y}_{R\,\mathrm{ext}} \cdot \nabla_{RT}) \cdot \mathbf{P}_R \end{aligned} \tag{106}$$

If the reference placement coincides with the current placement, then we have

$$\check{\mathbf{S}} = 2\big(\mathcal{T} : \mathbf{n} \otimes \mathbf{n}\big) \cdot \mathbf{P} - \check{\mathbf{Z}}_{\mathrm{ext}} \tag{107}$$

$$\mathbf{S} = \Big(\bar{\mathsf{T}} \cdot \mathbf{n} - \big(\mathcal{T} : \mathbf{n} \otimes \mathbf{n}\big) \cdot \big(\nabla_T \otimes \mathbf{n}\big) - \hat{\mathbf{t}}_{\mathrm{ext}} \otimes d_n \mathbf{n} - \big((\mathcal{T} \cdot \mathbf{n}) \cdot \mathbf{P}\big) \cdot \nabla_T\Big) \cdot \mathbf{P} - \mathbf{Z}_{\mathrm{ext}} + (\mathsf{Y}_{\mathrm{ext}} \cdot \nabla_T) \cdot \mathbf{P} \tag{108}$$

The following observation is very important: The dimension of $\mathbf{t}_R^*$ is [F/L^2] and that of $\mathbf{S}_R$ is [F/L]. The dimension of $\check{\mathbf{t}}_R^*$ is [FL/L^2] and that of $\check{\mathbf{S}}_R$ is [FL/L]. So $\mathbf{t}_R^*$ and $\check{\mathbf{t}}_R^*$ are the differences of forces and of double forces or moments, respectively, per unit area of both sides of the surface. On the other hand, $\mathbf{S}_R$ and $\check{\mathbf{S}}_R$ are forces and double forces or moments, respectively, per unit length. We will later compare some of these quantities with the cutting loads of a shell and hence introduce the notion of a crust shell. $\mathbf{S}_R$ and $\check{\mathbf{S}}_R$ are only defined on the free boundary. If we study fictitious cuts of the body, then we do not encounter such quantities per unit length. They only appeared because we postulated the fulfilment of boundary conditions that only make sense on a free boundary surface. This distinction would have been lost if we had mistakenly applied the divergence theorem of surfaces not only to free surfaces but also to fictitious cuts. Then, we had the following strange behaviour: Wherever a fictitious cut reaches the real boundary of the body, components of $\mathbf{S}_R$ and $\check{\mathbf{S}}_R$ are set free but they only exist as long as we imagine this cutting surface. The problem is discussed again in Sect. 11.2. Our remarks on fictitious cuts do not concern singular surfaces or lines in the interior of a body where external loads are applied. The treatment of such surfaces is similar to that of a free boundary and shall not be detailed here.

Finally we want to compare our representation of the boundary conditions with those that can be found in the literature. We rewrite the equations (99) and (100) with the help of (94), (101), (102).

$$\mathbf{t}_R - \Big(\mathbf{Z}_R - (\mathsf{Y}_R \cdot \nabla_{RT}) \cdot \mathbf{P}_R - \underbrace{\mathbf{Z}_{R\,\mathrm{ext}}} + (\underbrace{\mathsf{Y}_{R\,\mathrm{ext}}} \cdot \nabla_{RT}) \cdot \mathbf{P}_R\Big) \cdot \nabla_{RT} = \mathbf{t}_{R\,\mathrm{ext}} \tag{109}$$

$$\check{\mathbf{t}}_R - \Big(\check{\mathbf{Z}}_R - \underbrace{\check{\mathbf{Z}}_{R\,\mathrm{ext}}}\Big) \cdot \nabla_{RT} = \check{\mathbf{t}}_{R\,\mathrm{ext}} \tag{110}$$

A representation of this kind is indeed common in the literature, e.g. [5] and, with a different split of ∇_R, [18], page 28. We notice two aspects:

- The contributions of the underbraced terms are absent in the literature. But they allow a more complex physical situation.
- The second-order tensors in the brackets are usually not recognized as surface quantities that are of different dimension than the vectors on the right-hand sides and can be linked with the cutting loads of a shell.

8 The connection between the field and boundary quantities in current and referential description

If we choose the current placement as the reference placement, then we have to put $\mathbf{F} \equiv \mathbf{1}$, $N \equiv 1$, $\tilde{\mathbf{n}} \equiv \mathbf{n} \equiv \mathbf{n}_R$, $\tilde{\mathbf{P}} \equiv \mathbf{P} \equiv \mathbf{P}_R$, $\nabla \equiv \nabla_R$, $dA \equiv dA_R$.

First we compare the versions of the field equation.

$$\big(\bar{\mathbf{T}}_R \cdot \nabla_R + \varrho_R(\mathbf{b}-\mathbf{a})\big)\, dV_R = \big(\bar{\mathbf{T}} \cdot \nabla + \varrho(\mathbf{b}-\mathbf{a})\big)\, dV = \mathbf{0} \tag{111}$$

We apply the well-known identity

$$\mathbf{0} = \int \mathbf{n}\, dA = \int N\, \mathbf{G}^T \cdot \mathbf{n}_R \frac{1}{Q}\, dA_R = \int (J\, \mathbf{G}^T) \cdot \nabla_R\, dV_R \qquad \Longrightarrow \qquad (J\, \mathbf{G}^T) \cdot \nabla_R = \mathbf{0} \tag{112}$$

and find with (66)

$$\bar{\mathbf{T}}_R \cdot \nabla_R\, dV_R = (\bar{\mathbf{T}} \cdot J\, \mathbf{G}^T) \cdot \nabla_R\, dV_R = \acute{\bar{\mathbf{T}}} \cdot J\, \mathbf{G}^T \cdot \nabla_R \frac{1}{J}\, dV = \bar{\mathbf{T}} \cdot \nabla\, dV \tag{113}$$

Moreover, $\varrho_R = J\varrho$ so that the current and the referential version of the field equation are indeed equivalent.

Next we compare the boundary conditions of the free surface.

$$\hat{\mathbf{t}}^*_R\, dA_R = \hat{\mathbf{t}}^*\, dA = \mathbf{0} \tag{114}$$

$$\big(\mathbf{t}^*_R + \mathbf{S}_R \cdot \nabla_{RT}\big)\, dA_R = \big(\mathbf{t}^* + \mathbf{S} \cdot \nabla_T\big)\, dA = \mathbf{0} \tag{115}$$

$$\big(\check{\mathbf{t}}^*_R + \check{\mathbf{S}}_R \cdot \nabla_{RT}\big)\, dA_R = \big(\check{\mathbf{t}}^* + \check{\mathbf{S}} \cdot \nabla_T\big)\, dA = \mathbf{0} \tag{116}$$

with the crust tensors

$$\mathbf{S}_R = -\mathbf{Z}^*_R + \Big(\mathsf{Y}^*_R \cdot \nabla_{RT}\Big) \cdot \mathbf{P}_R, \qquad \mathbf{S} = -\mathbf{Z}^* + \Big(\mathsf{Y}^* \cdot \nabla_T\Big) \cdot \mathbf{P} \tag{117}$$

and

$$\check{\mathbf{S}}_R = -\check{\mathbf{Z}}^*_R, \qquad \check{\mathbf{S}} = -\check{\mathbf{Z}}^* \tag{118}$$

The fact that $\mathbf{Z}$ and Y are not unique has no influence on the uniqueness of $\mathbf{S}$ since $-\mathbf{Z}_{\text{add}} + \Big(\mathsf{Y}_{\text{add}} \cdot \nabla_T\Big) \cdot \mathbf{P} = \mathbf{0}$ is valid.

We are interested in the connection of the crust tensors. Therefore, we integrate (115) with (31)

$$\begin{aligned}\mathbf{0} &= \int \big(\mathbf{t}^*_R + \mathbf{S}_R \cdot \nabla_{RT}\big)\, dA_R = \int \mathbf{t}^*_R\, dA_R + \oint \mathbf{S}_R \cdot \mathbf{e}_R\, ds_R = \int \mathbf{t}^*_R Q\, dA + \oint \mathbf{S}_R \cdot \mathbf{N} \cdot \mathbf{e}\, ds \\ &= \int \big(\mathbf{t}^* + \mathbf{S} \cdot \nabla_T\big)\, dA = \int \mathbf{t}^*\, dA + \oint \mathbf{S} \cdot \mathbf{e}\, ds \end{aligned} \tag{119}$$

Since $\mathbf{e}$ is an arbitrary vector in the tangential plane and (103) is valid, we can identify the tensors $\mathbf{S}_R \cdot \mathbf{N}$ and $\mathbf{S}$. Equation (116) yields an analogous result. So we have

$$\mathbf{S}_R \cdot \mathbf{N} = \mathbf{S}, \qquad \check{\mathbf{S}}_R \cdot \mathbf{N} = \check{\mathbf{S}} \tag{120}$$

The connection of the tangential derivatives of the crust tensors can then be obtained from (103), (19), (36), (31).

$$\begin{aligned}\mathbf{S} \cdot \nabla_T &= (\mathbf{S}_R \cdot \mathbf{N}) \cdot \nabla_T = \acute{\mathbf{S}}_R \cdot \mathbf{N} \cdot \nabla_T + \mathbf{S}_R \cdot \mathbf{P}_R \cdot (\acute{\mathbf{N}} \cdot \nabla_T) \\ &= Q\acute{\mathbf{S}}_R \cdot \mathbf{P}_R \cdot \mathbf{F}^T \cdot \nabla_T = Q\mathbf{S}_R \cdot \nabla_{RT}, \qquad \check{\mathbf{S}} \cdot \nabla_T = Q\check{\mathbf{S}}_R \cdot \nabla_{RT}\end{aligned} \tag{121}$$

So three more connections appear.

$$Q\mathbf{t}^*_R = \mathbf{t}^*, \qquad Q\check{\mathbf{t}}^*_R = \check{\mathbf{t}}^* \tag{122}$$

$$Q\hat{\mathbf{t}}^*_R = \hat{\mathbf{t}}^* = \mathbf{0} \tag{123}$$

We arrive at the following important observation: There exist simple one-to-one connections between the versions of these five quantities that characterize the dynamic behaviour of a free surface.

If we compare, e.g., the representations (106) and (108) of $\mathbf{S}_R$ and $\mathbf{S}$, then the validity of such simple connections is by no means obvious. So we might wish to prove all these connections in an independent manner. Consider, for example,

$$Q\mathbf{t}_R^* = Q\mathbf{t}_{R\,\text{ext}} - Q\mathbf{t}_R = \mathbf{t}^* = \mathbf{t}_{\text{ext}} - \mathbf{t} \qquad \Longleftrightarrow \qquad Q\mathbf{t}_{R\,\text{ext}} = \mathbf{t}_{\text{ext}}, \qquad Q\mathbf{t}_R = \mathbf{t} \tag{124}$$

The last statement can be proved with the help of (66), (83), and (77).

$$\mathbf{t} = \bar{\mathbf{T}} \cdot \mathbf{n} = \frac{1}{J}\bar{\mathbf{T}}_R \cdot \mathbf{F}^T \cdot N\mathbf{G}^T \cdot \mathbf{n}_R = \frac{N}{J}\bar{\mathbf{T}}_R \cdot \mathbf{n}_R = Q\mathbf{t}_R \tag{125}$$

In what follows, it is sufficient to ignore the external actions since they must, of course, satisfy the same transformations as the internal ones.

Next we check (123) with (80)

$$Q\hat{\mathbf{t}}_R = QN^2\mathcal{T}_R : \mathbf{n}_R \otimes \mathbf{n}_R \otimes \mathbf{n}_R = \hat{\mathbf{t}} = \mathcal{T} : \mathbf{n} \otimes \mathbf{n} \otimes \mathbf{n} = \mathbf{0} \tag{126}$$

$$\begin{aligned} Q\hat{t}_{Ri} &= QN^2 t_{Riklm} n_{Rk} n_{Rl} n_{Rm} = \hat{t}_i = t_{ipqr} n_p n_q n_r \\ &= \frac{1}{J} N^3 t_{Riklm} f_{pk} f_{ql} f_{rm} g_{up} n_{Ru} g_{vq} n_{Rv} g_{wr} n_{Rw} = QN^2 t_{Riklm} n_{Rk} n_{Rl} n_{Rm} = 0 \end{aligned} \tag{127}$$

The following identity is a consequence of (14) and will now be needed.

$$\mathbf{P} = \mathbf{F} \cdot \tilde{\mathbf{P}} \cdot \mathbf{G} = \mathbf{G}^T \cdot \tilde{\mathbf{P}}^T \cdot \mathbf{F}^T, \qquad p_{sp} = f_{sx}\tilde{p}_{xy}g_{yp} = p_{ps} = g_{up}\tilde{p}_{vu}f_{sv} \tag{128}$$

The second crust tensor (118) together with (80) was

$$\check{\mathbf{S}}_R = \check{\mathbf{Z}}_R = 2N(\mathcal{T}_R : \mathbf{n}_R \otimes \mathbf{n}_R) \cdot \tilde{\mathbf{P}}^T, \qquad \check{\mathbf{S}} = 2(\mathcal{T} : \mathbf{n} \otimes \mathbf{n}) \cdot \mathbf{P} \tag{129}$$

We ignore $\check{\mathbf{Z}}_{R\,\text{ext}}$ as has been announced. The correctness of the connection (120) $\check{\mathbf{S}} = \check{\mathbf{S}}_R \cdot \mathbf{N} = Q\check{\mathbf{S}}_R \cdot \mathbf{P}_R \cdot \mathbf{F}^T$ can now easily be seen.

$$\begin{aligned} \check{s}_{is} &= 2\,t_{ipqr} n_q n_r p_{ps} = 2\frac{1}{J} N^2 t_{R\,iklm} f_{pk} f_{ql} f_{rm} g_{aq} n_{Ra} g_{br} n_{Rb} g_{up} \tilde{p}_{vu} f_{sv} \\ &= 2QN t_{R\,iklm} n_{Rl} n_{Rm} \tilde{p}_{vk} f_{sv} = Q\check{s}_{Riv} p_{Rvw} f_{sw} \end{aligned} \tag{130}$$

Finally, we want to find a connection between $\mathbf{Z}_R$ and $\mathbf{Z}$ and between Y_R and Y. We identify some corresponding terms of (76) and (82) and make use of (19) and (27).

$$\begin{aligned} &\Big(\mathbf{Z} : \delta\mathbf{v} \otimes \nabla_T + \mathsf{Y} : \cdot (\delta\mathbf{v} \otimes \nabla_T) \otimes \nabla_T\Big)\, dA \\ &= \Big(\mathbf{Z}_R : \delta\mathbf{v} \otimes \nabla_{RT} + \mathsf{Y}_R : \cdot\, (\delta\mathbf{v} \otimes \nabla_{RT}) \otimes \nabla_{RT}\Big)\, dA_R \\ &= Q\Big(\mathbf{Z}_R : (\delta\mathbf{v} \otimes \nabla_T) \cdot \mathbf{F} \cdot \mathbf{P}_R + \mathsf{Y}_R : \cdot\, \Big((\delta\mathbf{v} \otimes \nabla_T) \cdot \mathbf{F} \cdot \mathbf{P}_R\Big) \otimes \nabla_T \cdot \mathbf{F} \cdot \mathbf{P}_R\Big)\, dA \\ &= Q\Big(\mathbf{Z}_R \cdot \mathbf{P}_R \cdot \mathbf{F}^T : (\delta\mathbf{v} \otimes \nabla_T) + \mathsf{Y}_R \cdot \mathbf{P}_R \cdot \mathbf{F}^T : \cdot\, \Big((\delta\mathbf{v} \otimes \nabla_T) \cdot \mathbf{F} \cdot \mathbf{P}_R\Big) \otimes \nabla_T\Big)\, dA \\ &= Q\Big(\mathbf{Z}_R \cdot \mathbf{P}_R \cdot \mathbf{F}^T : (\delta\mathbf{v} \otimes \nabla_T) + \Big(\mathbf{F} \cdot \mathbf{P}_R \cdot \mathsf{Y}_R^T\Big)^T \cdot \mathbf{P}_R \cdot \mathbf{F}^T : \cdot (\delta\mathbf{v} \otimes \nabla_T) \otimes \nabla_T \\ &\quad + \Big(\mathsf{Y}_R \cdot \mathbf{P}_R \cdot \mathbf{F}^T : (\nabla_T \otimes (\mathbf{P}_R \cdot \mathbf{F}^T))^T\Big) : \delta\mathbf{v} \otimes \nabla_T\Big)\, dA \\ &= Q\Big(\Big(\mathbf{Z}_R \cdot \mathbf{P}_R \cdot \mathbf{F}^T + \mathsf{Y}_R \cdot \mathbf{P}_R \cdot \mathbf{F}^T : (\nabla_T \otimes (\mathbf{P}_R \cdot \mathbf{F}^T))^T\Big) : \delta\mathbf{v} \otimes \nabla_T \\ &\quad + \Big(\mathbf{F} \cdot \mathbf{P}_R \cdot \mathsf{Y}_R^T\Big)^T \cdot \mathbf{P}_R \cdot \mathbf{F}^T : \cdot\, (\delta\mathbf{v} \otimes \nabla_T) \otimes \nabla_T\Big)\, dA \end{aligned} \tag{131}$$

A comparison shows the connections

$$\mathbf{Z} = Q\Big(\mathbf{Z}_R \cdot \mathbf{P}_R \cdot \mathbf{F}^T + \mathsf{Y}_R \cdot \mathbf{P}_R \cdot \mathbf{F}^T : (\nabla_T \otimes (\mathbf{P}_R \cdot \mathbf{F}^T))^T\Big)$$
$$= Q\Big(\mathbf{Z}_R \cdot \mathbf{F}^T + \mathsf{Y}_R \cdot \mathbf{F}^T : (\nabla_T \otimes (\mathbf{P}_R \cdot \mathbf{F}^T))^T\Big) = \mathbf{Z}_R \cdot \mathbf{N} + \mathsf{Y}_R \cdot \mathbf{N} : (\nabla_T \otimes (\mathbf{P}_R \cdot \mathbf{F}^T))^T \quad (132)$$

$$\mathsf{Y} = Q\Big(\mathbf{F} \cdot \mathbf{P}_R \cdot \mathsf{Y}_R^T\Big)^T \cdot \mathbf{P}_R \cdot \mathbf{F}^T = Q\Big(\mathbf{F} \cdot \mathsf{Y}_R^T\Big)^T \cdot \mathbf{F}^T = \frac{1}{Q}\Big(\mathbf{N}^T \cdot \mathsf{Y}_R^T \cdot \mathbf{N}\Big)^T \quad (133)$$

The projector $\mathbf{P}_R$ can be omitted because of (86).

After all, the proofs of the connections $\mathbf{S} = \mathbf{S}_R \cdot \mathbf{N}$ and $\check{\mathbf{t}} = Q\check{\mathbf{t}}_R$ are due. They are intricate and are therefore postponed to the appendix.

9 The crust shell

9.1 The equilibrium of forces

We want to demonstrate that some of the boundary conditions of a free surface may be interpreted as the equilibrium conditions of a shell. The differential equations (115)

$$\mathbf{0} = \mathbf{t}^* + \mathbf{S} \cdot \nabla_T = Q\left(\mathbf{t}_R^* + \mathbf{S}_R \cdot \nabla_{RT}\right) \quad (134)$$

had the consequence (119).

$$\mathbf{0} = \int \mathbf{t}_R^* \, dA_R + \oint \mathbf{S}_R \cdot \mathbf{e}_R \, ds_R = \int \mathbf{t}^* \, dA + \oint \mathbf{S} \cdot \mathbf{e} \, ds \quad (135)$$

This can immediately be interpreted as an equilibrium condition of forces acting on a surface patch and on its boundary. Here $\mathbf{t}_R^* \, dA_R = \mathbf{t}^* \, dA$ is the resultant force acting on an area element from the inside and the outside, while $\mathbf{S}_R \cdot \mathbf{e}_R \, ds_R = \mathbf{S} \cdot \mathbf{e} \, ds$ represents the force on a line element of the boundary. So the first crust tensors $\mathbf{S}$ and $\mathbf{S}_R$ have to be interpreted as the current and referential force tensor of the shell, respectively. If we study second-gradient materials, then this force tensor is the only cutting quantity of the crust shell. It has been discussed in [22].

A third-gradient material body possesses in addition the boundary condition (116) that contains the second crust tensor $\check{\mathbf{S}}$ or $\check{\mathbf{S}}_R$. Its interpretation is more complicated. Therefore, we first remember further aspects of the behaviour of shells.

9.2 The equilibrium of moments

The moments acting on an element of a surface patch or of its boundary may be represented by:

$$\mathbf{m}^* \, dA = \mathbf{m}_R^* \, dA_R = Q\mathbf{m}_R^* \, dA \quad (136)$$

and—note (31)—

$$\mathbf{M} \cdot \mathbf{e} \, ds = \mathbf{M}_R \cdot \mathbf{e}_R \, ds_R = \mathbf{M}_R \cdot \mathbf{N} \cdot \mathbf{e} \, ds \quad (137)$$

where $\mathbf{M}$ and $\mathbf{M}_R$ denote the current and referential moment tensor of the shell, respectively. So we must have

$$\mathbf{m}^* = Q\mathbf{m}_R^*, \qquad \mathbf{M} = \mathbf{M}_R \cdot \mathbf{N} \quad (138)$$

The equilibrium condition of moments reads

$$\mathbf{0} = \int \left(\mathbf{m}^* + \mathbf{r} \times \mathbf{t}^*\right) dA + \oint \left(\mathbf{M} \cdot \mathbf{e} + \mathbf{r} \times \mathbf{S} \cdot \mathbf{e}\right) ds = \int \Big(\mathbf{m}^* + \mathbf{r} \times \mathbf{t}^* + \left(\mathbf{M} + \mathbf{r} \times \mathbf{S}\right) \cdot \nabla_T\Big) \, dA$$
$$= \int \left(\mathbf{m}_R^* + \mathbf{r} \times \mathbf{t}_R^*\right) dA_R + \oint \left(\mathbf{M}_R \cdot \mathbf{e}_R + \mathbf{r} \times \mathbf{S}_R \cdot \mathbf{e}_R\right) ds_R$$
$$= \int \Big(\mathbf{m}_R^* + \mathbf{r} \times \mathbf{t}_R^* + \left(\mathbf{M}_R + \mathbf{r} \times \mathbf{S}_R\right) \cdot \nabla_{RT}\Big) \, dA_R \quad (139)$$

We have with (103)

$$(\mathbf{r} \times \mathbf{S}) \cdot \nabla_T = \mathbf{r} \times \acute{\mathbf{S}} \cdot \nabla_T + \acute{\mathbf{r}} \times \mathbf{S} \cdot \nabla_T \tag{140}$$

$$\acute{\mathbf{r}} \times \mathbf{S} \cdot \nabla_T = (\mathbf{r} \otimes \nabla_T) \cdot \mathbf{S}^T\big|_\times = (\mathbf{P} \cdot \mathbf{S}^T)\big|_\times = -\mathbf{S}\big|_\times \tag{141}$$

$$\acute{\mathbf{r}} \times \mathbf{S}_R \cdot \nabla_{RT} = (\mathbf{r} \otimes \nabla_{RT}) \cdot \mathbf{S}_R^T\big|_\times = (\mathbf{F} \cdot \mathbf{P}_R \cdot \mathbf{S}_R^T)\big|_\times = -(\mathbf{S}_R \cdot \mathbf{F}^T)\big|_\times \tag{142}$$

We introduce the equilibrium condition of forces (134) and arrive at the equilibrium condition of moments.

$$\mathbf{0} = \mathbf{m}^* + \mathbf{M} \cdot \nabla_T - \mathbf{S}\big|_\times = Q\left(\mathbf{m}_R^* + \mathbf{M}_R \cdot \nabla_{RT} - (\mathbf{S}_R \cdot \mathbf{F}^T)\big|_\times\right) \tag{143}$$

We note the representation (101) of the tensor $\mathbf{S}_R$ and apply integration by parts.

$$\begin{aligned} \mathbf{S}_R \cdot \mathbf{F}^T &= -\mathbf{Z}_R^* \cdot \mathbf{F}^T + (\mathsf{Y}_R^* \cdot \nabla_{RT}) \cdot \mathbf{P}_R \cdot \mathbf{F}^T \\ &= -\mathbf{Z}_R^* \cdot \mathbf{F}^T - \mathsf{Y}_R^* : \left(\nabla_{RT} \otimes (\mathbf{P}_R \cdot \mathbf{F}^T)\right)^T + \left(\mathbf{F} \cdot \mathbf{P}_R \cdot \mathsf{Y}_R^{*T}\right)^T \cdot \nabla_{RT} \end{aligned} \tag{144}$$

$$\begin{aligned} s_{Rit} f_{zt} &= -z_{Rit}^* f_{zt} + (y_{Ritq}^* \nabla_{Rs}) p_{Rsq} p_{Rtp} f_{zp} \\ &= -z_{Rit}^* f_{zt} - y_{Ritq}^* p_{Rqs} \nabla_{Rs}(p_{Rtp} f_{zp}) + \left(f_{zp} p_{Rpt} y_{Ritq}^*\right) \nabla_{Rs} p_{Rsq} \end{aligned} \tag{145}$$

Then, the equilibrium condition (143) with (86) becomes

$$\mathbf{0} = \mathbf{m}_R^* + \left(\mathbf{Z}_R^* \cdot \mathbf{F}^T + \mathsf{Y}_R^* : \left(\nabla_{RT} \otimes (\mathbf{P}_R \cdot \mathbf{F}^T)\right)^T\right)\Big|_\times + \left(\mathbf{M}_R + \left(\mathbf{F} \cdot \mathsf{Y}_R^{*T}\right)\Big|_\times\right) \cdot \nabla_{RT} \tag{146}$$

9.3 The role of the second crust tensor

The boundary condition (100) with (102) was derived from (98):

$$0 = d_n \delta \mathbf{v} \cdot (\check{\mathbf{t}}_R^* + \check{\mathbf{S}}_R \cdot \nabla_{RT}) dA_R = d_n \delta \mathbf{v} \cdot (\check{\mathbf{t}}^* + \check{\mathbf{S}} \cdot \nabla_T) dA \tag{147}$$

We introduce a virtual angular velocity

$$\psi = \mathbf{n} \times d_n \delta \mathbf{v} \tag{148}$$

and obtain a split of $d_n \delta \mathbf{v}$ into stretching and spinning.

$$d_n \delta \mathbf{v} = d_n \delta \mathbf{v} \cdot \mathbf{n}\, \mathbf{n} - \mathbf{n} \times \psi \tag{149}$$

This also induces a split of the boundary condition.

$$\begin{aligned} 0 &= Q d_n \delta \mathbf{v} \cdot \mathbf{n}\left(\mathbf{n} \cdot \check{\mathbf{t}}_R^* + \mathbf{n} \cdot (\check{\mathbf{S}}_R \cdot \nabla_{RT})\right) + Q\psi \cdot \left(\mathbf{n} \times (\check{\mathbf{t}}_R^* + \check{\mathbf{S}}_R \cdot \nabla_{RT})\right) \\ &= d_n \delta \mathbf{v} \cdot \mathbf{n}\left(\mathbf{n} \cdot \check{\mathbf{t}}^* + \mathbf{n} \cdot (\check{\mathbf{S}} \cdot \nabla_T)\right) + \psi \cdot \left(\mathbf{n} \times (\check{\mathbf{t}}^* + \check{\mathbf{S}} \cdot \nabla_T)\right) \end{aligned} \tag{150}$$

The first bracket works on a stretching and therefore represents a double force t_{total} that must vanish.

$$\begin{aligned} 0 = Q t_{R\,\text{total}} &= Q\mathbf{n} \cdot \check{\mathbf{t}}_R^* - Q\check{\mathbf{S}}_R : (\mathbf{n} \otimes \nabla_{RT}) + Q(\mathbf{n} \cdot \check{\mathbf{S}}_R) \cdot \nabla_{RT} \\ = t_{\text{total}} &= \mathbf{n} \cdot \check{\mathbf{t}}^* - \check{\mathbf{S}} : (\mathbf{n} \otimes \nabla_T) + (\mathbf{n} \cdot \check{\mathbf{S}}) \cdot \nabla_T \end{aligned} \tag{151}$$

The vector field $\mathbf{n} \cdot \check{\mathbf{S}}$ of dimension [FL/L] represents a non-classical cutting quantity of the crust shell, namely a double force per unit length. The scalar field $\mathbf{n} \cdot \check{\mathbf{t}}^* - \check{\mathbf{S}} : (\mathbf{n} \otimes \nabla_T)$ is a double force per unit area acting in the direction of $\mathbf{n}$.

The second bracket represents a moment vector $\mathbf{m}_{\text{total}}$ in the tangential plane normal to $\mathbf{n}$ that must vanish when multiplied with any ψ.

$$\begin{aligned}\mathbf{0} &= Q\mathbf{m}_{R\,\text{total}} = Q\mathbf{n}\times\left(\check{\mathbf{t}}_R^* + \check{\mathbf{S}}_R\cdot\nabla_{RT}\right) = Q\mathbf{n}\times\check{\mathbf{t}}_R^* + Q\mathbf{P}\cdot\left(\check{\mathbf{S}}_R\cdot(\nabla_{RT}\otimes\mathbf{n})\right)\big|_\times + Q\mathbf{P}\cdot(\mathbf{n}\times\check{\mathbf{S}}_R)\cdot\nabla_{RT}\\ &= \mathbf{m}_{\text{total}} = \mathbf{n}\times\left(\check{\mathbf{t}}^* + \check{\mathbf{S}}\cdot\nabla_T\right) = \mathbf{n}\times\check{\mathbf{t}}^* + \mathbf{P}\cdot\left(\check{\mathbf{S}}\cdot(\nabla_T\otimes\mathbf{n})\right)\big|_\times + \mathbf{P}\cdot(\mathbf{n}\times\check{\mathbf{S}})\cdot\nabla_T \end{aligned} \tag{152}$$

We identify this statement with the tangential part of the equilibrium condition of moments (146) of a shell.

$$\begin{aligned}\mathbf{0} &= \mathbf{P}\cdot\mathbf{m}_R^* + \mathbf{P}\cdot\left(\mathbf{Z}_R^*\cdot\mathbf{F}^T + \mathsf{Y}_R^* : \left(\nabla_{RT}\otimes(\mathbf{P}_R\cdot\mathbf{F}^T)\right)^T\right)\Big|_\times + \mathbf{P}\cdot\left(\mathbf{M}_R + \left(\mathbf{F}\cdot\mathsf{Y}_R^{*T}\right)\Big|_\times\right)\cdot\nabla_{RT}\\ &= \mathbf{n}\times\check{\mathbf{t}}_R^* + \mathbf{P}\cdot\left(\check{\mathbf{S}}_R\cdot(\nabla_{RT}\otimes\mathbf{n})\right)\big|_\times + \mathbf{P}\cdot(\mathbf{n}\times\check{\mathbf{S}}_R)\cdot\nabla_{RT}\end{aligned} \tag{153}$$

If we define the moment tensor of the crust shell by

$$\mathbf{M}_R = \mathbf{n}\times\check{\mathbf{S}}_R - \left(\mathbf{F}\cdot\mathsf{Y}_R^{*T}\right)\big|_\times \tag{154}$$

then we interpret the tangential part of the boundary condition that contains the second crust tensor as the tangential part of the equilibrium condition of moments of the crust shell.

The current representation is:

$$\mathbf{M} = \mathbf{n}\times\check{\mathbf{S}} + \mathsf{Y}^*\big|_\times \tag{155}$$

The second term should originally read $(\mathbf{P}\cdot\mathsf{Y}^T)^T|_\times$ and could be simplified to $\mathsf{Y}|_\times$ because of (86). An addition of Y_{add} does not influence the tensor $\mathbf{M}$ since $\mathbf{P}\cdot\mathsf{Y}_{\text{add}}^T = \mathbf{0}$ according to (88). So $\mathbf{M}$ does not depend on the choice of Y.

The tangential part of the moment per unit area is seen to be:

$$\mathbf{P}\cdot\mathbf{m}_R^* = \mathbf{n}\times\check{\mathbf{t}}_R^* - \mathbf{P}\cdot\left(\mathbf{Z}_R^*\cdot\mathbf{F}^T - \check{\mathbf{S}}_R\cdot(\nabla_{RT}\otimes\mathbf{n}) + \mathsf{Y}_R^* : \left(\nabla_{RT}\otimes(\mathbf{P}_R\cdot\mathbf{F}^T)\right)^T\right)\Big|_\times \tag{156}$$

We are also interested in the normal part of $\mathbf{m}_R^*$. We need

$$\mathbf{n}\cdot\left(\mathbf{M}_R + \left(\mathbf{F}\cdot\mathsf{Y}_R^{*T}\right)\Big|_\times\right)\cdot\nabla_{RT} = \mathbf{n}\cdot\left(\mathbf{n}\times\check{\mathbf{S}}_R\right)\cdot\nabla_{RT} = (\underline{\mathbf{n}\times\mathbf{n}})\cdot(\check{\mathbf{S}}_R\cdot\nabla_{RT}) - \mathbf{n}\cdot\left(\check{\mathbf{S}}_R\cdot(\nabla_{RT}\otimes\mathbf{n})\right)\Big|_\times \tag{157}$$

The underlined term vanishes. So we obtain from (146)

$$\mathbf{n}\cdot\mathbf{m}_R^* = -\mathbf{n}\cdot\left(\mathbf{Z}_R^*\cdot\mathbf{F}^T - \check{\mathbf{S}}_R\cdot(\nabla_{RT}\otimes\mathbf{n}) + \mathsf{Y}_R^* : \left(\nabla_{RT}\otimes(\mathbf{P}_R\cdot\mathbf{F}^T)\right)^T\right)\Big|_\times \tag{158}$$

and finally the total moment per unit area.

$$\mathbf{m}_R^* = \mathbf{n}\times\check{\mathbf{t}}_R^* - \left(\mathbf{Z}_R^*\cdot\mathbf{F}^T - \check{\mathbf{S}}_R\cdot(\nabla_{RT}\otimes\mathbf{n}) + \mathsf{Y}_R^* : \left(\nabla_{RT}\otimes(\mathbf{P}_R\cdot\mathbf{F}^T)\right)^T\right)\Big|_\times \tag{159}$$

We note

$$\left(\nabla_T\otimes\mathbf{P}\right)^T = -\acute{\mathbf{n}}\otimes\nabla_T\otimes\mathbf{n} - \mathbf{n}\otimes\nabla_T\otimes\acute{\mathbf{n}} \tag{160}$$

and find the current representation

$$\begin{aligned}\mathbf{m}^* &= \mathbf{n}\times\check{\mathbf{t}}^* - \left(\mathbf{Z}^* - \check{\mathbf{S}}\cdot(\nabla_T\otimes\mathbf{n}) + \mathsf{Y}^* : \left(\nabla_T\otimes\mathbf{P}\right)^T\right)\Big|_\times\\ &= \mathbf{n}\times\left(\check{\mathbf{t}}^* - \mathsf{Y}^* : (\nabla_T\otimes\mathbf{n})\right) - \left(\mathbf{Z}^* - \check{\mathbf{S}}\cdot(\nabla_T\otimes\mathbf{n})\right)\Big|_\times\end{aligned} \tag{161}$$

10 The edge beams

We want to show that the edges where two patches of the crust meet behave like beams and are therefore called edge beams. We recall the edge terms of (97) and (98) with (101) and (102) and augment it by the virtual power of a vector function $\mathbf{h}_{R\text{ext}}$ of dimension [F/L] and a tensor function $\mathbf{H}_{R\text{ext}}$ of dimension [FL/L] that represent external loads on the edge beam per unit length. (This is part of the formerly announced additional power $\delta\Pi_{\text{add}}$.) The principle of virtual power yields the following postulate for one edge line.

$$\begin{aligned}
0 &= \sum \int \Big(\delta\mathbf{v}\cdot\big(\mathbf{Z}^*_R - (\mathsf{Y}^*_R\cdot\nabla_{RT})\cdot\mathbf{P}_R\big) + \delta\mathbf{v}\otimes\nabla_{RT} : \mathsf{Y}^*_R + d_n\delta\mathbf{v}\cdot\check{\mathbf{Z}}^*_R\Big)\cdot\mathbf{e}_R\, ds_R \\
&\quad + \int \big(\mathbf{h}_{R\text{ext}}\cdot\delta\mathbf{v} + \mathbf{H}_{R\text{ext}} : \delta\mathbf{v}\otimes\nabla_R\big)\, ds_R \\
&= \sum \int \Big(-\delta\mathbf{v}\cdot\mathbf{S}_R + \delta\mathbf{v}\otimes\nabla_{RT} : \mathsf{Y}^*_R - d_n\delta\mathbf{v}\cdot\check{\mathbf{S}}_R\Big)\cdot\mathbf{e}_R\, ds_R + \int \big(\mathbf{h}_{R\text{ext}}\cdot\delta\mathbf{v} + \mathbf{H}_{R\text{ext}} : \delta\mathbf{v}\otimes\nabla_R\big)\, ds_R \\
&= \int \Big(\delta\mathbf{v}\cdot\Big(-\sum\mathbf{S}_R\cdot\mathbf{e}_R + \mathbf{h}_{R\text{ext}}\Big) + \Big(-\sum\check{\mathbf{S}}_R\cdot\mathbf{e}_R\otimes\tilde{\mathbf{n}} + \sum\hat{\mathbf{S}}_R + \mathbf{H}_{R\text{ext}}\Big) : \delta\mathbf{v}\otimes\nabla_R\Big)\, ds_R
\end{aligned} \tag{162}$$

We introduced the second-order tensor—note (86)—

$$\hat{\mathbf{S}}_R \equiv \mathsf{Y}^*_R\cdot\mathbf{e}_R = \big(\mathsf{Y}^*_R\cdot\mathbf{e}_R\big)\cdot\mathbf{P}_R \tag{163}$$

of dimension [FL/L]. The sums are to be extended over the two adjacent patches. (Indices are omitted.) The equation can be reformulated with (41) and (44) by partial integration along the arc length s_R.

$$\begin{aligned}
0 &= \int \delta\mathbf{v}\cdot\Big(-\sum\mathbf{S}_R\cdot\mathbf{e}_R + \mathbf{h}_{R\text{ext}}\Big)\, ds_R + \int \Big(-\sum\check{\mathbf{S}}_R\cdot\mathbf{e}_R\otimes\tilde{\mathbf{n}} + \sum\hat{\mathbf{S}}_R + \mathbf{H}_{R\text{ext}}\Big) : \frac{\partial\delta\mathbf{v}}{\partial s_R}\otimes\tilde{\mathbf{s}}\, ds_R \\
&\quad + \int \Big(-\sum\check{\mathbf{S}}_R\cdot\mathbf{e}_R\otimes\tilde{\mathbf{n}} + \sum\hat{\mathbf{S}}_R + \mathbf{H}_{R\text{ext}}\Big) : \delta\mathbf{v}\otimes\nabla_P\cdot\mathbf{F}\, ds_R \\
&= \int \delta\mathbf{v}\cdot\Big(-\sum\mathbf{S}_R\cdot\mathbf{e}_R + \mathbf{h}_{R\text{ext}} - \frac{\partial}{\partial s_R}\Big(\sum\hat{\mathbf{S}}_R\cdot\tilde{\mathbf{s}} + \mathbf{H}_{R\text{ext}}\cdot\tilde{\mathbf{s}}\Big)\, ds_R \\
&\quad + \Big[\Big(\sum\hat{\mathbf{S}}_R\cdot\tilde{\mathbf{s}} + \mathbf{H}_{R\text{ext}}\cdot\tilde{\mathbf{s}}\Big)\cdot\delta\mathbf{v}\Big]_{s_{R0}}^{s_{R1}} \\
&\quad + \int \Big(-\sum\check{\mathbf{S}}_R\cdot\mathbf{e}_R\otimes\tilde{\mathbf{n}}\cdot\mathbf{F}^T + \sum(\hat{\mathbf{S}}_R\cdot\mathbf{F}^T + \mathbf{H}_{R\text{ext}}\cdot\mathbf{F}^T\Big)\cdot\mathbf{P}_s : \delta\mathbf{v}\otimes\nabla_P\, ds_R
\end{aligned} \tag{164}$$

The last line made use of $\nabla_P = \mathbf{P}_s\cdot\nabla_P$ according to (43).

If the current placement is chosen as the reference placement, then the result reduces to

$$\begin{aligned}
0 &= \int \delta\mathbf{v}\cdot\Big(-\sum\mathbf{S}\cdot\mathbf{e} + \mathbf{h}_{\text{ext}} - \frac{\partial}{\partial s}\Big(\sum\hat{\mathbf{S}}\cdot\mathbf{s} + \mathbf{H}_{\text{ext}}\cdot\mathbf{s}\Big)\Big)\, ds \\
&\quad + \Big[\Big(\sum\hat{\mathbf{S}}\cdot\mathbf{s} + \mathbf{H}_{\text{ext}}\cdot\mathbf{s}\Big)\cdot\delta\mathbf{v}\Big]_{s_0}^{s_1} \\
&\quad + \int \Big(-\sum\check{\mathbf{S}}\cdot\mathbf{e}\otimes\mathbf{n} + \sum\hat{\mathbf{S}}\cdot\mathbf{e}\otimes\mathbf{e} + \mathbf{H}_{\text{ext}}\cdot\mathbf{P}_s\Big) : \delta\mathbf{v}\otimes\nabla_P\, ds
\end{aligned} \tag{165}$$

We made use of $\mathbf{P}\cdot\mathbf{P}_s = (\mathbf{e}\otimes\mathbf{e} + \mathbf{s}\otimes\mathbf{s})\cdot(\mathbf{e}\otimes\mathbf{e} + \mathbf{n}\otimes\mathbf{n}) = \mathbf{e}\otimes\mathbf{e}$ and introduced

$$\hat{\mathbf{S}} \equiv \mathsf{Y}^*\cdot\mathbf{e} = \big(\mathsf{Y}^*\cdot\mathbf{e}\big)\cdot\mathbf{P} \tag{166}$$

One might raise the objection that the original orientation of the arc length of the two adjacent patches is opposite. But this orientation has actually no influence on the above results since a reversal changes the sign of d/ds_R, $\mathbf{s}$, and $\tilde{\mathbf{s}}$ simultaneously.

Now we recall the equilibrium of forces of a beam element. It reads in current and referential description

$$\mathbf{0} = \Big(\frac{d\mathbf{p}}{ds} + \mathbf{t}_L\Big)\, ds = \Big(\frac{d\mathbf{p}}{ds_R} + \mathbf{t}_{RL}\Big)\, ds_R \tag{167}$$

where $\mathbf{p}$ is the cutting force of the beam (of dimension [F]) and $\mathbf{t}_L$ or $\mathbf{t}_{RL}$ is the force per unit length (of dimension [F/L]) in current or referential description. We compare this with the conditions that emerge from (164) or (165) when $\delta\mathbf{v}$ is arbitrary along the edge beam and find

$$\begin{aligned}\mathbf{t}_L\, ds &= \Big(-\sum \mathbf{S}\cdot\mathbf{e} + \mathbf{h}_{\text{ext}} - \frac{\partial}{\partial s}\big(\mathbf{H}_{\text{ext}}\cdot\mathbf{s}\big)\Big)ds\\ &= \mathbf{t}_{RL} ds_R = \Big(-\sum \mathbf{S}_R\cdot\mathbf{e}_R + \mathbf{h}_{R\text{ext}} - \frac{\partial}{\partial s_R}\big(\mathbf{H}_{R\text{ext}}\cdot\tilde{\mathbf{s}}\big)\Big)\, ds_R\\ &= \Big(-\sum \mathbf{S}_R\cdot\mathbf{e}_R + \mathbf{h}_{R\text{ext}} - \frac{\partial}{\partial s_R}\Big(\frac{1}{L}\mathbf{H}_{R\text{ext}}\cdot\mathbf{F}^T\cdot\mathbf{s}\Big)\Big)\, ds_R \end{aligned} \tag{168}$$

We know already $\mathbf{S}\cdot\mathbf{e}\, ds = \mathbf{S}_R\cdot\mathbf{e}_R\, ds_R = \mathbf{S}_R\cdot\mathbf{N}\cdot\mathbf{e}\, ds$. So we may choose the connections

$$L\,\mathbf{h}_{\text{ext}} = \mathbf{h}_{R\text{ext}}, \qquad L\mathbf{H}_{\text{ext}} = \mathbf{H}_{R\text{ext}}\cdot\mathbf{F}^T \tag{169}$$

We observe that the distributed force on the edge beam results from the force tensors in the two patches and from an external force and the derivative of an external moment. The cutting force of the edge beam is seen to be:

$$\mathbf{p} = -\sum \hat{\mathbf{S}}\cdot\mathbf{s} = -\sum \hat{\mathbf{S}}_R\cdot\tilde{\mathbf{s}} = -\sum \frac{1}{L}\hat{\mathbf{S}}_R\cdot\mathbf{F}^T\cdot\mathbf{s} \tag{170}$$

The equality of the two descriptions is derived from (133) and (31) as follows.

$$\hat{\mathbf{S}}^T\, ds = \mathsf{Y}^{*T}\cdot\mathbf{e}\, ds = Q\mathbf{F}\cdot\mathsf{Y}_R^{*T}\cdot\mathbf{P}_R\cdot\mathbf{F}^T\cdot\mathbf{e}\, ds = \mathbf{F}\cdot\mathsf{Y}_R^{*T}\cdot\mathbf{N}\cdot\mathbf{e}\, ds = \mathbf{F}\cdot\mathsf{Y}_R^{*T}\cdot\mathbf{e}_R\, ds_R = \frac{1}{L}\mathbf{F}\cdot\hat{\mathbf{S}}_R^T\, ds \tag{171}$$

By the way, the interpretation of $\mathbf{p}$ and $\mathbf{t}_L$ in [12], eqs. (67), (68) is not correct.

Now we note $\tilde{\mathbf{n}}\cdot\mathbf{F}^T = \mathbf{n}$ and $\mathbf{P}_R\cdot\mathbf{F}^T\cdot\mathbf{P}_s = \mathbf{P}_R\cdot\mathbf{F}^T\cdot\mathbf{e}\otimes\mathbf{e} + \mathbf{P}_R\cdot\mathbf{F}^T\cdot\mathbf{n}\otimes\mathbf{n} = \mathbf{P}_R\cdot\mathbf{F}^T\cdot\mathbf{e}\otimes\mathbf{e} + N\underline{\mathbf{P}_R\cdot\mathbf{n}_R}\otimes\mathbf{n}$. The underlined vector vanishes. When $\delta\mathbf{v}\otimes\nabla_P$ is arbitrary along the edge beam, then we obtain from (164) and (165)

$$\begin{aligned}\mathbf{0} &= \Big(-\sum \check{\mathbf{S}}_R\cdot\mathbf{e}_R\otimes\mathbf{n} + \sum \hat{\mathbf{S}}_R\cdot\mathbf{F}^T\cdot\mathbf{e}\otimes\mathbf{e} + \mathbf{H}_{R\text{ext}}\cdot\mathbf{F}^T\cdot\mathbf{P}_s\Big)\, ds_R\\ &= \Big(-\sum \check{\mathbf{S}}\cdot\mathbf{e}\otimes\mathbf{n} + \sum \hat{\mathbf{S}}\cdot\mathbf{e}\otimes\mathbf{e} + \mathbf{H}_{\text{ext}}\cdot\mathbf{P}_s\Big)\, ds \end{aligned} \tag{172}$$

The first contributions are equal because of (120) and (31), the second ones because of (171) and the third one because of (169).

The skew part of $\delta\mathbf{v}\otimes\nabla_P$ describes an angular velocity and the vanishing of the skew part—or equivalently of the negative vector—of the expression (172) is therefore an equilibrium condition of moments. It reads

$$\mathbf{0} = -\sum \mathbf{n}\times\check{\mathbf{S}}\cdot\mathbf{e} + \sum \mathbf{e}\times\hat{\mathbf{S}}\cdot\mathbf{e} + \mathbf{k}_{\text{ext}} \tag{173}$$

where we abbreviated the external line moment by

$$\mathbf{k}_{\text{ext}} \equiv -\mathbf{H}_{\text{ext}}\cdot\mathbf{P}_s\big|_\times \tag{174}$$

Now we have with (166)

$$-\mathsf{Y}^*\big|_\times\cdot\mathbf{e} = -\hat{\mathbf{S}}\big|_\times = -\big(\hat{\mathbf{S}}\cdot\mathbf{P}\big)\big|_\times = -\big(\hat{\mathbf{S}}\cdot(\mathbf{e}\otimes\mathbf{e}+\mathbf{s}\otimes\mathbf{s})\big)\big|_\times = \mathbf{e}\times\hat{\mathbf{S}}\cdot\mathbf{e} + \mathbf{s}\times\hat{\mathbf{S}}\cdot\mathbf{s} \tag{175}$$

and hence according to (155)

$$-\mathbf{n}\times\check{\mathbf{S}}\cdot\mathbf{e} + \mathbf{e}\times\hat{\mathbf{S}}\cdot\mathbf{e} = -\mathbf{M}\cdot\mathbf{e} + \mathsf{Y}^*\Big|_\times\cdot\mathbf{e} + \mathbf{e}\times\hat{\mathbf{S}}\cdot\mathbf{e} = -\mathbf{M}\cdot\mathbf{e} - \mathbf{s}\times\hat{\mathbf{S}}\cdot\mathbf{s} \tag{176}$$

So we arrive at the condition

$$\mathbf{0} = -\sum \mathbf{M}\cdot\mathbf{e} - \mathbf{s}\times\sum \hat{\mathbf{S}}\cdot\mathbf{s} + \mathbf{k}_{\text{ext}} \tag{177}$$

If no cutting moments (bending and torsional moments) are present in a beam, then the equilibrium of moments of a beam element reads

$$\mathbf{0} = \mathbf{m}_L ds + \mathbf{s}\, ds \times \mathbf{p} = \mathbf{m}_{RL} ds_R + (\mathbf{F} \cdot \mathbf{s}_R\, ds_R) \times \mathbf{p} \tag{178}$$

Here $\mathbf{m}_L$ and $\mathbf{m}_{RL}$ denote the moment per unit length (of dimension [FL/L]). If we note (170), then we see that (177) is nothing but the equilibrium condition of moments of the edge bar if we put

$$\begin{aligned} \mathbf{m}_L ds &= -\sum \mathbf{M} \cdot \mathbf{e} ds + \mathbf{k}_{\text{ext}} ds \\ &= \mathbf{m}_{RL} ds_R = -\sum \mathbf{M}_R \cdot \mathbf{e}_R ds_R - \left(\mathbf{H}_{R\text{ext}} \cdot \mathbf{F}^T \cdot \mathbf{P}_s\right)\Big|_\times ds_R \end{aligned} \tag{179}$$

We see that the distributed moment on the edge beam results from the moment tensors in the two patches and from an external moment.

We must also discuss the equilibrium at a vertex where several edge beams meet. The second line of (165) with (170) yields

$$\sum_{\text{edge beams}} \left(-\mathbf{p} + \mathbf{H}_{\text{ext}} \cdot \mathbf{s}\right) + \mathbf{p}_{\text{vertex}} = \mathbf{0} \tag{180}$$

since $\delta\mathbf{v}$ is arbitrary at the vertex. The external force $\mathbf{p}_{\text{vertex}}$ that may be applied at the vertex has the dimension [F]. Its virtual power $\mathbf{p}_{\text{vertex}} \cdot \delta\mathbf{v}$ is another part of $\delta\Pi_{\text{add}}$.

Finally, we have to interpret the symmetric part of the condition (172).

$$\begin{aligned} \mathbf{0} = &\Big(-\sum\left(\check{\mathbf{S}}_R \cdot \mathbf{e}_R \otimes \mathbf{n} + \mathbf{n} \otimes \check{\mathbf{S}}_R \cdot \mathbf{e}_R\right) + \sum\left(\hat{\mathbf{S}}_R \cdot \mathbf{F}^T \cdot \mathbf{e} \otimes \mathbf{e} + \mathbf{e} \otimes \hat{\mathbf{S}}_R \cdot \mathbf{F}^T \cdot \mathbf{e}\right) \\ &+\mathbf{H}_{R\text{ext}} \cdot \mathbf{F}^T \cdot \mathbf{P}_s + \mathbf{P}_s \cdot \mathbf{F} \cdot \mathbf{H}^T_{R\text{ext}}\Big) ds_R \\ = &\Big(-\sum\left(\check{\mathbf{S}} \cdot \mathbf{e} \otimes \mathbf{n} + \mathbf{n} \otimes \check{\mathbf{S}} \cdot \mathbf{e}\right) + \sum\left(\hat{\mathbf{S}} \cdot \mathbf{e} \otimes \mathbf{e} + \mathbf{e} \otimes \hat{\mathbf{S}} \cdot \mathbf{e}\right) + \mathbf{H}_{\text{ext}} \cdot \mathbf{P}_s + \mathbf{P}_s \cdot \mathbf{H}^T_{\text{ext}}\Big) ds \end{aligned} \tag{181}$$

The component in the direction of $\mathbf{s}$ yields

$$\begin{aligned} \mathbf{0} &= \Big(-\sum(\mathbf{s} \cdot \check{\mathbf{S}}_R \cdot \mathbf{e}_R)\,\mathbf{n} + \sum(\mathbf{s} \cdot \hat{\mathbf{S}}_R \cdot \mathbf{F}^T \cdot \mathbf{e})\,\mathbf{e} + \mathbf{s} \cdot \mathbf{H}_{R\text{ext}} \cdot \mathbf{F}^T \cdot \mathbf{P}_s\Big) ds_R \\ &= \Big(-\sum(\mathbf{s} \cdot \check{\mathbf{S}} \cdot \mathbf{e})\,\mathbf{n} + \sum(\mathbf{s} \cdot \hat{\mathbf{S}} \cdot \mathbf{e})\,\mathbf{e} + \mathbf{s} \cdot \mathbf{H}_{\text{ext}} \cdot \mathbf{P}_s\Big) ds \end{aligned} \tag{182}$$

But this condition is already satisfied since it is identical with the vector product of $\mathbf{s}$ with the condition (173). So only the projection of the equation (181) onto the plane normal to $\mathbf{s}$ yields an additional restriction. This remaining tensor equation is equivalent to three scalar equations that describe the balance of double forces on a line element of the edge beam.

11 An example: a cube under vertex and edge loads

11.1 Stresses in the cube and cutting loads in the crust shells and edge beams

We use the current description and Cartesian co-ordinates. The surfaces $x = 0$, $y = 0$, $z = 0$ of the cube are fixed and the boundary conditions are $\mathbf{v} = d_n\mathbf{v} = d_n^2\mathbf{v} = \mathbf{0}$. The surfaces $x = l$, $y = l$, $z = l$ are free, and there are no external agents:

$$\mathbf{t}_{\text{ext}} = \mathbf{0}, \quad \check{\mathbf{t}}_{\text{ext}} = \mathbf{0}, \quad \hat{\mathbf{t}}_{\text{ext}} = \mathbf{0} \tag{183}$$

$$\mathbf{Z}_{\text{ext}} = \mathbf{0}, \quad \check{\mathbf{Z}}_{\text{ext}} = \mathbf{0}, \quad \Upsilon_{\text{ext}} = \mathbf{0} \tag{184}$$

External loads are only applied at the vertex $x = y = z = l$ and at the three edge beams that meet at the vertex and are specified later.

We will see that the following state of stress satisfies the field equation and all the boundary conditions.

$$t_{xx} = 0, \quad t_{xy} = 2\tau_{11} + 2\tau_{12} \tag{185}$$
$$t_{xxx} = 0, \quad t_{xyy} = 3\tau_{33} + 3\tau_{21} \tag{186}$$
$$t_{xxy} = 3\tau_{42} + \tau_{11}(x-a), \quad t_{xyz} = 3\tau_{41} + 3\tau_{33} + \tau_{12}(y+z-2a) \tag{187}$$
$$t_{xxxx} = 0, \quad t_{xyyy} = 0, \quad t_{xxxy} = 0 \tag{188}$$
$$t_{xxyy} = \tau_{21}(x-a) \tag{189}$$
$$t_{xyyz} = \tau_{33}(y+z-2a) \tag{190}$$
$$t_{xxyz} = \tau_{41}x + \tau_{42}(y+z) \tag{191}$$

All the other components of the stress tensors $\mathbf{T}$, T, and $\mathcal{T}$ are obtained from one of the following symmetry transforms.

$$(x \to y, \quad y \to x), \qquad (x \to z, \quad z \to x), \qquad (y \to z, \quad z \to y) \tag{192}$$
$$(x \to y, \quad y \to z, \quad z \to x), \qquad (x \to z, \quad z \to y, \quad y \to x) \tag{193}$$

So the state of stress is fully characterized by only seven parameters. The effective stresses become according to (59)

$$\bar{t}_{xxx} = 0, \quad \bar{t}_{xyy} = 2\tau_{33} + 2\tau_{21} \tag{194}$$
$$\bar{t}_{xxy} = 2\tau_{42} + \tau_{11}(x-a), \quad \bar{t}_{xyz} = 2\tau_{41} + \tau_{33} + \tau_{12}(y+z-2a) \tag{195}$$
$$\bar{t}_{xx} = 0, \quad \bar{t}_{xy} = \tau_{11} + \tau_{12} \tag{196}$$

The homogeneous field equation $\bar{\mathbf{T}} \cdot \nabla = \mathbf{0}$ is satisfied since $\bar{\mathbf{T}}$ is constant.

We obtain the following dynamic quantities on any surface $x = const$. Greek indices α, β take the values y and z.

$$\mathbf{t}: \quad t_i = \bar{t}_{ix}: \quad t_x = 0, \quad t_y = t_z = \tau_{11} + \tau_{12} \tag{197}$$
$$\check{\mathbf{t}}: \quad \check{t}_i = \bar{t}_{ixx}: \quad \check{t}_x = 0, \quad \check{t}_y = \check{t}_z = 2\tau_{33} + 2\tau_{21} \tag{198}$$
$$\mathbf{Z}: \quad z_{i\alpha} = \bar{t}_{i\alpha x}: \quad z_{xy} = z_{xz} = 2\tau_{42} + \tau_{11}(x-a) \tag{199}$$
$$z_{zy} = 2\tau_{41} + \tau_{33} + \tau_{12}(x+y-2a) \tag{200}$$
$$z_{yz} = 2\tau_{41} + \tau_{33} + \tau_{12}(x+z-2a) \tag{201}$$
$$z_{yy} = 2\tau_{42} + \tau_{11}(y-a), \quad z_{zz} = 2\tau_{42} + \tau_{11}(z-a) \tag{202}$$
$$\hat{\mathbf{t}}: \quad \hat{t}_i = t_{ixxx} = 0 \tag{203}$$
$$\check{\mathbf{Z}}: \quad \check{z}_{i\alpha} = 2t_{i\alpha xx}: \quad \check{z}_{xy}/2 = \check{z}_{xz}/2 = 0 \tag{204}$$
$$\check{z}_{zy}/2 = \tau_{33}(x+y-2a), \quad \check{z}_{yz}/2 = \tau_{33}(x+z-2a) \tag{205}$$
$$\check{z}_{yy}/2 = \tau_{21}(y-a), \quad \check{z}_{zz}/2 = \tau_{21}(z-a) \tag{206}$$
$$\mathsf{Y}: \quad y_{i\alpha\beta} = y_{i\beta\alpha} = t_{i\alpha\beta x}: \quad y_{xyy} = y_{xzz} = \tau_{21}(x-a) \tag{207}$$
$$y_{xyz} = y_{xzy} = \tau_{41}x + \tau_{42}(y+z) \tag{208}$$
$$y_{yyy} = 0, \quad y_{yyz} = y_{yzy} = \tau_{41}y + \tau_{42}(x+z) \tag{209}$$
$$y_{yzz} = \tau_{33}(x+z-2a), \quad y_{zyy} = \tau_{33}(x+y-2a) \tag{210}$$
$$y_{zzz} = 0, \quad y_{zyz} = y_{zzy} = \tau_{41}z + \tau_{42}(x+y) \tag{211}$$

If the considered surface is a fictitious cut through the cube, then we identify the following interactions across this surface. The vector $\mathbf{t}$ ([F/L^2]) applies shear stresses. The normal component $\check{t}_x$ of the vector $\check{\mathbf{t}}$ ([FL/L^2]) represents a double force in the direction of x but does not appear here while the tangential components give rise to moments. The symmetric part of the tensor $\mathbf{Z}$ ([FL/L^2]) represents double forces and the vector of the skew part moments. A double force in the direction of a vector $\mathbf{g} = g_x\mathbf{e}_x + g_y\mathbf{e}_y + g_z\mathbf{e}_z$ is given by:

$$\mathbf{Z} : \mathbf{g} \otimes \mathbf{g} = z_{yy}g_y^2 + z_{zz}g_z^2 + (z_{yz} + z_{zy})g_y g_z + z_{xy}g_x g_y + z_{xz}g_x g_z \tag{212}$$

So the components z_{yy}, z_{zz}, $(z_{yz} + z_{zy})/2$, $z_{xy}/2$, $z_{xz}/2$ of the symmetric part are responsible for double forces, the components of the skew part $(z_{yz} - z_{zy})/2$, $z_{xy}/2$, $z_{xz}/2$ describe moments. Finally, the tensor Y

($[\mathrm{FL}^2/\mathrm{L}^2]$) describes the action of triple forces or double moments. A triple force in the direction of a vector $\mathbf{g} = g_x\mathbf{e}_x + g_y\mathbf{e}_y + g_z\mathbf{e}_z$ is given by:

$$\mathsf{Y} : \cdot\, \mathbf{g} \otimes \mathbf{g} \otimes \mathbf{g} = y_{xyy} g_x g_y^2 + y_{xzz} g_x g_z^2 + 2y_{xyz} g_x g_y g_z \\ +(y_{zyy} + 2y_{yyz}) g_y^2 g_z + (y_{yzz} + 2y_{zyz}) g_y g_z^2 \tag{213}$$

So the components $y_{xyy}/2$, $y_{xzz}/2$, y_{xyz}, $(y_{zyy} + 2y_{yyz})/2$, $(y_{yzz} + 2y_{zyz})/2$ are responsible for triple forces, while the remaining components $y_{xyy}/2$, $y_{xzz}/2$, y_{xyz}, $(y_{zyy} - 2y_{yyz})/2$, $(y_{yzz} - 2y_{zyz})/2$ describe various kinds of double moments.

If we put $x = 0$, then the surface under consideration is a fixed boundary and the mentioned actions are reactions that are applied to the cube.

If the considered surface is a free boundary with the prescriptions (184), then we need the components of the crust tensor $\check{\mathbf{S}}$ that is here identical with $\check{\mathbf{Z}}$ according to (118) and of the crust tensor $\mathbf{S}$ that is, according to (117),

$$\mathbf{S} = \mathbf{Z} - \mathbf{A} \quad \text{with} \quad \mathbf{A} \equiv (\mathsf{Y} \cdot \nabla_T) \cdot \mathbf{P} \tag{214}$$

We find

$$a_{i\alpha} = y_{i\alpha\beta},_{\beta} = y_{i\alpha y},_y + y_{i\alpha z},_z : \tag{215}$$

$$a_{xy} = a_{xz} = a_{yy} = a_{zz} = \tau_{42}, \quad a_{zy} = a_{yz} = \tau_{33} + \tau_{41} \tag{216}$$

and therefore

$$s_{xy} = s_{xz} = \tau_{42} + \tau_{11}(x - a) \tag{217}$$

$$s_{zy} = \tau_{41} + \tau_{12}(x + y - 2a), \quad s_{yz} = \tau_{41} + \tau_{12}(x + z - 2a) \tag{218}$$

$$s_{yy} = \tau_{42} + \tau_{11}(y - a), \quad s_{zz} = \tau_{42} + \tau_{11}(z - a) \tag{219}$$

Next we compute the divergence of these tensors $\mathbf{b} \equiv \mathbf{S} \cdot \nabla_T$ and $\mathbf{c} \equiv \check{\mathbf{S}} \cdot \nabla_T$.

$$b_x = s_{xy},_y + s_{xz},_z = 0, \qquad b_y = s_{yy},_y + s_{yz},_z = b_z = s_{zy},_y + s_{zz},_z = \tau_{11} + \tau_{12} \tag{220}$$

$$c_x = \check{s}_{xy},_y + \check{s}_{xz},_z = 0, \quad c_y = \check{s}_{yy},_y + \check{s}_{yz},_z = c_z = \check{s}_{zy},_y + \check{s}_{zz},_z = 2\tau_{33} + 2\tau_{21} \tag{221}$$

We find $\mathbf{b} = \mathbf{t}$ and $\mathbf{c} = \check{\mathbf{t}}$, and we also have $\hat{\mathbf{t}} = \mathbf{0}$. So the boundary conditions (96), (99), and (100) with (183) of a free surface are also satisfied. Note that these results are independent of the position x of the surface. The tensor $\mathbf{S}$ is the force tensor of the crust shell. We are also interested in the moment tensor according to (155) with (184).

$$\mathbf{M} = \mathbf{e}_x \times \check{\mathbf{S}} - \mathbf{U} \quad \text{with} \quad \mathbf{U} \equiv \mathsf{Y}\big|_{\times} \tag{222}$$

$$u_{x\alpha} = y_{yz\alpha} - y_{zy\alpha}, \quad u_{y\alpha} = -y_{xz\alpha}, \quad u_{z\alpha} = y_{xy\alpha} \tag{223}$$

So we obtain

$$-m_{xy} = u_{xy} = \tau_{42}(x + z) - \tau_{33}(x + y - 2a) + \tau_{41} y \tag{224}$$

$$-m_{xz} = u_{xz} = -\tau_{42}(x + y) + \tau_{33}(x + z - 2a) - \tau_{41} z \tag{225}$$

$$m_{yy} = -\check{s}_{zy} - u_{yy} = -2\tau_{33}(x + y - 2a) + \tau_{41} x + \tau_{42}(y + z) \tag{226}$$

$$m_{zz} = \check{s}_{yz} - u_{zz} = 2\tau_{33}(x + z - 2a) - \tau_{41} x - \tau_{42}(y + z) \tag{227}$$

$$m_{zy} = \check{s}_{yy} - u_{zy} = \tau_{21}(2y - x - a) \tag{228}$$

$$m_{yz} = -\check{s}_{zz} - u_{yz} = -\tau_{21}(2z - x - a) \tag{229}$$

Next we compute the divergence $\mathbf{d} \equiv \mathbf{M} \cdot \nabla_T$.

$$d_x = m_{xy},_y + m_{xz},_z = 0 \tag{230}$$

$$d_y = m_{yy},_y + m_{yz},_z = -d_z = -m_{zy},_y - m_{zz},_z = -2\tau_{21} - 2\tau_{33} + \tau_{42} \tag{231}$$

Furthermore, we need the vectors $\mathbf{v} \equiv \mathbf{S}\big|_\times$ and $\mathbf{w} \equiv \mathbf{Z}\big|_\times$.

$$v_x = s_{yz} - s_{zy} = \tau_{12}(z - y), \quad v_y = -s_{xz} = -v_z = -s_{xy} = -\tau_{42} - \tau_{11}(x - a) \tag{232}$$

$$w_x = z_{yz} - z_{zy} = \tau_{12}(z - y), \quad w_y = -z_{xz} = -w_z = -z_{xy} = -2\tau_{42} - \tau_{11}(x - a) \tag{233}$$

The equilibrium condition of moments of the crust shell reads according to (143) with (161):

$$\mathbf{M} \cdot \nabla_T = \mathbf{S}\big|_\times + \mathbf{e}_x \times \check{\mathbf{t}} - \mathbf{Z}\big|_\times \tag{234}$$

It is easily checked that it is satisfied.

We must also check the balance of the double forces in the direction of the normal $\mathbf{n} = \mathbf{e}_x$ according to (151).

$$0 = -\mathbf{n} \cdot \check{\mathbf{t}} + (\mathbf{n} \cdot \check{\mathbf{S}}) \cdot \nabla_T = -\check{t}_x + \check{s}_{xy},_y + \check{s}_{xz},_z \tag{235}$$

This is trivially satisfied here.

Next we consider the edge beam with $x = y = l$. We characterize the contribution of the free surface $x = l$ of our cube to the cutting force of the beam by the index X and find from (170)

$$\mathbf{p}_X = \mathsf{Y}_X : \mathbf{e}_z \otimes \mathbf{e}_y, \qquad p_{X\,i} = y_{X\,izy} \tag{236}$$

$$p_{X\,x} = y_{X\,xzy} = p_{X\,y} = y_{X\,yzy} = \tau_{41} l + \tau_{42}(l + z), \quad p_{X\,z} = y_{X\,zzy} = 2\tau_{42} l + \tau_{41} z \tag{237}$$

The equilibrium conditions of forces of the beam yield according to (167) and (168) with $\mathbf{H}_{\text{ext}} \cdot \mathbf{e}_z = \mathbf{0}$

$$\frac{d\mathbf{p}_X}{ds} = -\mathbf{t}_{L\,X} = \mathbf{S}_X \cdot \mathbf{e}_y - \mathbf{h}_{\text{ext}\,X} \tag{238}$$

$$\tau_{42} = \frac{dp_{X\,x}}{dz} = \frac{dp_{X\,y}}{dz} = s_{X\,xy} - h_{\text{ext}\,X\,x} = s_{X\,yy} - h_{\text{ext}\,X\,y} = \tau_{42} + \tau_{11}(l - a) - h_{\text{ext}\,X\,y} \tag{239}$$

$$\tau_{41} = \frac{dp_{X\,z}}{dz} = s_{X\,zy} - h_{\text{ext}\,X\,z} = \tau_{41} + 2\tau_{12}(l - a) - h_{\text{ext}\,X\,z} \tag{240}$$

The contribution of the surface $y = l$ with index Y is obtained by interchanging x and y. Equilibrium holds if the following conditions are satisfied.

$$\begin{aligned} h_{\text{ext}\,x} &= h_{\text{ext}\,X\,x} + h_{\text{ext}\,Y\,x} = h_{\text{ext}\,y} = h_{\text{ext}\,X\,y} + h_{\text{ext}\,Y\,y} = 2\tau_{11}(l - a), \\ h_{\text{ext}\,z} &= h_{\text{ext}\,X\,z} + h_{\text{ext}\,Y\,z} = 4\tau_{12}(l - a) \end{aligned} \tag{241}$$

So the prescription of these constant line forces on the edge beam determines two of the parameters of the state of stress.

$$\tau_{11} = \frac{h_{\text{ext}\,x}}{2(l - a)} = \frac{h_{\text{ext}\,y}}{2(l - a)}, \qquad \tau_{12} = \frac{h_{\text{ext}\,z}}{4(l - a)} \tag{242}$$

The total cutting force of the beam is given by:

$$\mathbf{p} = \mathbf{p}_X + \mathbf{p}_Y, \qquad p_x = p_y = 2\big(\tau_{41} l + \tau_{42} l + \tau_{42} z\big), \qquad p_z = 4\tau_{42} l + 2\tau_{41} z \tag{243}$$

and assumes the value

$$p_x = p_y = p_z = 2\big(2\tau_{42} + \tau_{41}\big) l \tag{244}$$

at the vertex ($z = l$). Each of the three beams that meet at the vertex provides the same force, and so the single force that has to be applied at the vertex is three times this value. This implies one more restriction on the stress parameters.

$$6\big(2\tau_{42} + \tau_{41}\big)\, l = p_{\text{vertex}\,x} = p_{\text{vertex}\,y} = p_{\text{vertex}\,z} \tag{245}$$

Next we study the equilibrium conditions of moments of the beam. The contributions of the two adjacent surfaces yield according to (178) and (179)

$$\mathbf{s} \times \mathbf{p}_X = -\mathbf{m}_{X\,L} = \mathbf{M}_X \cdot \mathbf{e}_y - \mathbf{k}_{\text{ext}\,X} \tag{246}$$
$$\tau_{41} l + \tau_{42} l + \tau_{42} z = p_{X\,x} = m_{X\,yy} - k_{\text{ext}\,X\,y} = \tau_{41} l + \tau_{42} l + \tau_{42} z - 4\tau_{33}(l-a) - k_{\text{ext}\,X\,y} \tag{247}$$
$$\tau_{41} l + \tau_{42} l + \tau_{42} z = p_{X\,y} = -m_{X\,xy} + k_{\text{ext}\,X\,x} = \tau_{41} l + \tau_{42} l + \tau_{42} z - 2\tau_{33}(l-a) + k_{\text{ext}\,X\,x} \tag{248}$$
$$\mathbf{s} \times \mathbf{p}_Y = -\mathbf{m}_{Y\,L} = \mathbf{M}_Y \cdot \mathbf{e}_x - \mathbf{k}_{\text{ext}\,Y} \tag{249}$$
$$\tau_{41} l + \tau_{42} l + \tau_{42} z = p_{Y\,x} = m_{Y\,yx} - k_{\text{ext}\,Y\,y} = \tau_{41} l + \tau_{42} l + \tau_{42} z - 2\tau_{33}(l-a) - k_{\text{ext}\,Y\,y} \tag{250}$$
$$\tau_{41} l + \tau_{42} l + \tau_{42} z = p_{Y\,y} = -m_{Y\,xx} + k_{\text{ext}\,Y\,x} = \tau_{41} l + \tau_{42} l + \tau_{42} z - 4\tau_{33}(l-a) + k_{\text{ext}\,Y\,x} \tag{251}$$

Equilibrium requires

$$-k_{\text{ext}\,X\,y} = k_{\text{ext}\,Y\,x} = 2k_{\text{ext}\,X\,x} = -2k_{\text{ext}\,Y\,y} = 4\tau_{33}\,(l-a) \tag{252}$$

This implies

$$k_{\text{ext}\,x} = k_{\text{ext}\,X\,x} + k_{\text{ext}\,Y\,x} = -k_{\text{ext}\,y} = -k_{\text{ext}\,X\,y} - k_{\text{ext}\,Y\,y} = 6\tau_{33}\,(l-a) \tag{253}$$

and the prescription of these constant line moments on the edge beam provides another restriction of the stress parameters.

$$\tau_{33} = \frac{k_{\text{ext}\,x}}{6(l-a)} = -\frac{k_{\text{ext}\,y}}{6(l-a)} \tag{254}$$

No external line moment around the z-axis has to be applied since the contributions of the two adjacent surfaces cancel each other:

$$m_{X\,zy} = -m_{Y\,zx} = \tau_{21}\,(l-a) \tag{255}$$

Finally we have to check the balance of the double forces that act on the beam at $x = y = l$ in the plane normal to $\mathbf{s} = \mathbf{e}_z$ according to (181). The three components are:

$$-\check{s}_{Xxy} - y_{Yxxx} + h_{\text{ext}\,xx} = 0 \tag{256}$$
$$-\check{s}_{Yyx} - y_{Xyyy} + h_{\text{ext}\,yy} = 0 \tag{257}$$
$$-\check{s}_{Xyy} - \check{s}_{Yxx} - y_{Xxyy} - y_{Yyxx} + h_{\text{ext}\,xy} + h_{\text{ext}\,yx} = 0 \tag{258}$$

and yield

$$h_{\text{ext}\,xx} = h_{\text{ext}\,yy} = 0 \tag{259}$$
$$h_{\text{ext}\,xy} + h_{\text{ext}\,yx} = 6\tau_{21}\,(l-a) \tag{260}$$

and hence

$$\tau_{21} = \frac{h_{\text{ext}\,xy} + h_{\text{ext}\,yx}}{6(l-a)} \tag{261}$$

After all we are able to understand the complex statics of the cube. First a single force is applied to the vertex and is taken by three edge beams that are additionally loaded by constant line forces, line moments, and line double forces. The magnitude of the cutting force of the beam with $x = y = l$, e.g., decays with decreasing z, and the remainder at $z = 0$ is balanced by a reaction force at the fixed surface. In between the beam surrenders line forces, line moments, and line double forces to the two adjacent crust shells. These possess cutting forces and moments. A part of them is applied to the fixed boundaries, while their divergences render loads to the bulk body of the cube. These are finally balanced by reaction forces at the three fixed boundary surfaces.

11.2 The illegal treatment of fictitious cuts

It is a widespread bad habit to apply the divergence theorem of surfaces to fictitious cuts. This means that these cuts are treated like free surfaces with the prescriptions (184). Let us consider a subcube with side length $a < l$. If we consider the three cuts around it as free surfaces, then we notice the following behaviour:

- There are no actions on the three surfaces since (183) is satisfied.
- There are no line forces, line moments, and line double forces on the three edges. (Replace l by a in (241), (253), and (260).)
- There is a single force $p_{\mathrm{vertex}\,x} = p_{\mathrm{vertex}\,y} = p_{\mathrm{vertex}\,z} = 6(2\tau_{42} + \tau_{41})a$ at the vertex $x = y = z = a$. (Replace l by a in (245)).

We are confronted by the absurd claim that the interaction between the subcube and the remainder of the cube is realized by one single force alone at the vertex of the subcube. Actually, however, there is a wealth of interactions across the three cuts as we have seen in equations (197) to (211), but there are never any interactions on the edges and on the vertices of a subbody.

12 Conclusion

The paper presents an enhancement of a former one [12]. It had been shown there that some of the free boundary conditions of a third-gradient continuum can be interpreted as the equilibrium conditions of force and moment of a shell and of beams. So the body seems endowed with a crust shell and edge beams. Now, forces and moments are fundamental concepts of mechanics since the time of Archimedes. Therefore, the physical behaviour of the crust shell cannot depend on the description, either current (Eulerian) or referential (Lagrangian). Instead, these two kinds of description must be closely related. Although there is no one-to-one correspondence between the true and nominal stress tensors and effective stress tensors of the same order within the body, we find, indeed, such a correspondence between the cutting loads (forces, moments, and double forces) of the crust shell. This can easily be inferred from the comparison of the current and referential version of the boundary conditions. But someone might question these simple connections, since some of the surface loads and the cutting loads of the crust shell look extremely different in the two descriptions. Therefore, an independent check of these connections is performed.

The introduced concepts are illustrated by the case of a cube that is loaded at a vertex and three edges. The various interactions on fictitious cuts and the cutting loads in the crust shells and edge beams are evaluated. It is demonstrated that the bad habit to apply the divergence theorem of surfaces to fictitious cuts produces an absurd result.

No specific constitutive assumptions have been laid down in the investigation. An application to second-gradient fluids can be found in [22]. Elasticity with a stress-free reference placement will surely be the next candidate for future research.

A Detailed proofs

A.1 Tangential derivatives of some geometric quantities

We start with (11)

$$N^{-2} = \mathbf{n}_R \cdot \mathbf{G} \cdot \mathbf{G}^T \cdot \mathbf{n}_R, \qquad -2N^{-3}(\nabla_{RT} N) = 2\mathbf{n}_R \cdot \mathbf{G} \cdot (\mathbf{G}^T \cdot \mathbf{n}_R) \otimes \nabla_{RT} \tag{262}$$

and obtain the tangential derivative of N

$$\begin{aligned} \nabla_{RT} N &= -N^3 \mathbf{n}_R \cdot \mathbf{G} \cdot (\mathbf{G}^T \cdot \mathbf{n}_R) \otimes \nabla_{RT} = -N^2 \tilde{\mathbf{n}} \cdot \mathbf{F}^T \cdot (\mathbf{G}^T \cdot \mathbf{n}_R) \otimes \nabla_{RT} \\ &= -N^2 \tilde{\mathbf{n}} \cdot (\mathbf{n}_R \otimes \nabla_{RT}) - N^2 \tilde{\mathbf{n}} \cdot \mathbf{F}^T \cdot \acute{\mathbf{G}}^T \cdot \mathbf{n}_R \otimes \nabla_{RT} \end{aligned} \tag{263}$$

We take into account that the curvature tensor $\mathbf{C}_R = -\mathbf{n}_R \otimes \nabla_{RT}$ of the reference surface is planar and symmetric and obtain with (8)

$$(\nabla_{Rb} N) p_{Rba} = -N^2 \tilde{n}_y \Big(p_{Ryz} \big(n_{Rz} \nabla_{Rb} + n_{Rb} \nabla_{Rz} \big)/2 - n_{Ru} g_{uw} f_{wyb} \Big) p_{Rba} \tag{264}$$

We also find the tangential derivative of $\tilde{\mathbf{n}} = N\mathbf{G} \cdot \mathbf{G}^T \cdot \mathbf{n}_R$.

$$\tilde{\mathbf{n}} \otimes \nabla_{RT} = \mathbf{G} \cdot \mathbf{G}^T \cdot \mathbf{n}_R \otimes (\nabla_{RT} N) + N\mathbf{G} \cdot \mathbf{G}^T \cdot (\mathbf{n}_R \otimes \nabla_{RT}) + N\mathbf{n}_R \cdot (\mathbf{G} \cdot \mathbf{G}^T \otimes \nabla_{RT}) \tag{265}$$

$$\begin{aligned}(\tilde{n}_x \nabla_{Rb}) p_{Rba} &= -N\tilde{n}_x \tilde{n}_y \Big(p_{Ryz} \big(n_{Rz} \nabla_{Rb} + n_{Rb} \nabla_{Rz} \big)/2 - n_{Ru} g_{uw} f_{wyb} \Big) p_{Rba} \\ &\quad + N g_{xu} g_{yu} p_{Ryz} \big(n_{Rz} \nabla_{Rb} + n_{Rb} \nabla_{Rz} \big)/2 \; p_{Rba} \\ &\quad - N n_{Ru} g_{uw} g_{xv} g_{yv} f_{wyb} p_{Rba} - N n_{Ru} g_{uv} g_{xw} g_{yv} f_{wyb} p_{Rba} \\ &= N \tilde{p}_{xw} g_{wu} g_{yu} p_{Ryz} \big(n_{Rz} \nabla_{Rb} + n_{Rb} \nabla_{Rz} \big)/2 \; p_{Rba} \\ &\quad - N n_{Ru} \big(g_{uw} \tilde{p}_{xz} g_{zv} + g_{xw} g_{uv} \big) g_{yv} f_{wyb} p_{Rba}\end{aligned} \tag{266}$$

Equation (17) is used to simplify the expression.
Next we need the tangential derivative of Q and make use of (7).

$$\begin{aligned}(\nabla_{Rc} Q) p_{Rca} &= \nabla_{Rc} \Big(\frac{N}{J} \Big) p_{Rca} = -QN\tilde{n}_y \Big(p_{Ryz} \big(n_{Rz} \nabla_{Rc} + n_{Rc} \nabla_{Rz} \big)/2 \\ &\quad - n_{Rz} g_{zw} f_{wyc} \Big) p_{Rca} - Q\delta_{yz} g_{zw} f_{wyc} p_{Rca} \\ &= -QN\tilde{n}_y p_{Ryz} \big(n_{Rz} \nabla_{Rc} + n_{Rc} \nabla_{Rz} \big)/2 p_{Rca} - Q\tilde{p}_{yz} g_{zw} f_{wyc} p_{Rca}\end{aligned} \tag{267}$$

After all the modified projector is

$$\tilde{\mathbf{P}} = \mathbf{1} - N\tilde{\mathbf{n}} \otimes \mathbf{n}_R, \qquad \tilde{p}_{el} = \delta_{el} - N\tilde{n}_e n_{Rl} \tag{268}$$

and its tangential derivative

$$\begin{aligned}\tilde{p}_{el} \nabla_{Rc} p_{Rca} &= -N\tilde{n}_e (n_{Rl} \nabla_{Rc}) p_{Rca} - N n_{Rl} (\tilde{n}_e \nabla_{Rc}) p_{Rca} - \tilde{n}_e n_{Rl} (\nabla_{Rc} N) p_{Rca} \\ &= -N\tilde{n}_e \delta_{ly} p_{Ryz} \big(n_{Rz} \nabla_{Rc} + n_{Rc} \nabla_{Rz} \big)/2 p_{Rca} \\ &\quad - N^2 n_{Rl} \Big(\tilde{p}_{ew} g_{wu} g_{yu} p_{Ryz} \big(n_{Rz} \nabla_{Rc} + n_{Rc} \nabla_{Rz} \big)/2 - n_{Ru} \big(g_{uw} \tilde{p}_{ez} g_{zv} + g_{ew} g_{uv} \big) g_{yv} f_{wyc} \Big) p_{Rca} \\ &\quad + N^2 \tilde{n}_e n_{Rl} \tilde{n}_y \Big(p_{Ryz} \big(n_{Rz} \nabla_{Rc} + n_{Rc} \nabla_{Rz} \big)/2 - n_{Ru} g_{uw} f_{wyc} \Big) p_{Rca}\end{aligned} \tag{269}$$

A.2 Comparison of current and referential quantities

We want to prove the connections $\mathbf{S} = \mathbf{S}_R \cdot \mathbf{N}$ and $\check{\mathbf{t}} = Q\check{\mathbf{t}}_R$ and remember that external actions had been separated and hence can be ignored.
We consider first the component equation of $\mathbf{S}_R \cdot \mathbf{N} = Q\mathbf{S}_R \cdot \mathbf{P}_R \cdot \mathbf{F}^T = Q\mathbf{S}_R \cdot \mathbf{F}^T$ where $\mathbf{S}_R$ and $\mathbf{S}$ are given by (106) and (108), respectively.

$$\begin{aligned}Q s_{Rit} f_{zt} &= Q \Big(\bar{t}_{Rikl} n_{Rl} - N t_{Riplm} n_{Rl} n_{Rm} \tilde{p}_{cp} (\nabla_{Rc} \tilde{n}_k) - (t_{Riklm} n_{Rm} \tilde{p}_{el}) \nabla_{R_c} {}_P Rce \Big) \tilde{p}_{tk} f_{zt} \\ &= \underline{Q \bar{t}_{Rikl} n_{Rl} \tilde{p}_{tk} f_{zt}} - QN t_{Riplm} n_{Rl} n_{Rm} \tilde{p}_{cp} (\nabla_{Rc} \tilde{n}_k) \tilde{p}_{tk} f_{zt} \underline{- Q (t_{Riklm} n_{Rm}) \nabla_{R_c} \tilde{p}_{cl} \tilde{p}_{tk} f_{zt}} \\ &\quad - Q t_{Riklm} n_{Rm} (\tilde{p}_{el} \nabla_{R_c})_P Rce \tilde{p}_{tk} f_{zt}\end{aligned} \tag{270}$$

This should be identical with the component representation of **S**:

$$
\begin{aligned}
s_{iz} &= \bar{t}_{ipq} n_q p_{pz} - t_{ipqr} n_q n_r p_{ps} \frac{1}{2} (\nabla_s n_w + \nabla_w n_s) p_{wz} - (t_{ipqr} n_r p_{qa}) \nabla_b p_{ba} p_{pz} \\
&= Q \bar{t}_{R\,ikl} f_{pk} f_{ql} g_{aq} n_{Ra} g_{xp} \tilde{p}_{yx} f_{zy} + Q t_{R\,iklm} f_{pkl} f_{qm} g_{aq} n_{Ra} g_{xp} \tilde{p}_{yx} f_{zy} \\
&\quad - Q N t_{R\,iklm} f_{pk} f_{ql} f_{rm} g_{aq} n_{Ra} g_{br} n_{Rb} g_{xp} \tilde{p}_{yx} f_{sy} \frac{1}{2} \Big(g_{cs} \nabla_{Rc} (N g_{ew} n_{Re}) + g_{cw} \nabla_{Rc} (N g_{es} n_{Re}) \Big) f_{wu} \tilde{p}_{uv} g_{vz} \\
&\quad - \Big(\frac{1}{J} t_{R\,iklm} f_{pk} f_{ql} f_{rm} N g_{xr} n_{Rx} g_{uq} \tilde{p}_{vu} f_{av} \Big) \nabla_{Rw} g_{wb} f_{bc} \tilde{p}_{ce} g_{ea} g_{sp} \tilde{p}_{ts} f_{zt} \\
&= Q \bar{t}_{R\,ikl} n_{Rl} \tilde{p}_{yk} f_{zy} + Q t_{R\,iklm} f_{pkl} n_{Rm} g_{xp} \tilde{p}_{yx} f_{zy} \\
&\quad - Q N t_{R\,iklm} n_{Rl} n_{Rm} \tilde{p}_{yk} f_{sy} \frac{1}{2} \Big(g_{cs} \nabla_{Rc} (N g_{ew} n_{Re}) + g_{cw} \nabla_{Rc} (N g_{es} n_{Re}) \Big) f_{wu} \tilde{p}_{uv} g_{vz} \\
&\quad - \Big(Q t_{R\,iklm} n_{Rm} f_{pk} f_{av} \tilde{p}_{vl} \Big) \nabla_{Rc} \tilde{p}_{ce} g_{ea} g_{sp} \tilde{p}_{ts} f_{zt} \\
&= \underline{Q \bar{t}_{R\,ikl} n_{Rl} \tilde{p}_{yk} f_{zy}} + Q t_{R\,iklm} f_{pkl} n_{Rm} g_{xp} \tilde{p}_{yx} f_{zy} \\
&\quad \underbrace{- Q N t_{R\,iklm} n_{Rl} n_{Rm} (\nabla_{Rc} N) \frac{1}{2} (\tilde{p}_{ck} \tilde{p}_{ev} + \tilde{p}_{cv} \tilde{p}_{ek}) n_{Re} g_{vz}} \\
&\quad - Q N^2 t_{R\,iklm} n_{Rl} n_{Rm} \tilde{p}_{ek} \frac{1}{2} \Big(\nabla_{Rc} n_{Re} + \nabla_{Re} n_{Rc} \Big) \tilde{p}_{cv} g_{vz} \\
&\quad + Q N^2 t_{R\,iklm} n_{Rl} n_{Rm} n_{Re} g_{ex} f_{xyc} \frac{1}{2} (\tilde{p}_{ck} \tilde{p}_{yv} + \tilde{p}_{cv} \tilde{p}_{yk}) g_{vz} \\
&\quad - (Q \nabla_{Rc}) t_{R\,iklm} n_{Rm} \tilde{p}_{cl} \tilde{p}_{tk} f_{zt} \underline{- Q (t_{R\,iklm} n_{Rm}) \nabla_{Rc} \tilde{p}_{cl} \tilde{p}_{tk} f_{zt}} \\
&\quad - Q t_{R\,iklm} n_{Rm} f_{pkc} \tilde{p}_{cl} g_{sp} \tilde{p}_{ts} f_{zt} - Q t_{R\,iklm} n_{Rm} f_{avc} \tilde{p}_{vl} \tilde{p}_{ce} g_{ea} \tilde{p}_{tk} f_{zt} - Q t_{R\,iklm} n_{Rm} (\tilde{p}_{el} \nabla_{Rc}) \tilde{p}_{ce} \tilde{p}_{tk} f_{zt}
\end{aligned}
\tag{271}
$$

The underbraced term vanishes because of $\mathbf{n}_R \cdot \tilde{\mathbf{P}} = \mathbf{0}$. Moreover, $\tilde{\mathbf{P}} \cdot \tilde{\mathbf{P}} = \tilde{\mathbf{P}}$ according to (16) was used. The equality of the corresponding underlined terms is evident. We split the remainder into a part that is linear in $\mathbf{F} \otimes \nabla_R$ and one that is linear in $\mathbf{n}_R \otimes \nabla_R$.
First we compare the factors with $Q t_{R\,iklm} n_{Rm} f_{abc}$ and call them R_{klabcz} and A_{klabcz} in the referential and current description, respectively.

$$
\begin{aligned}
R_{klabcz} &= N^2 n_{Rl} n_{Ru} \Big(\tilde{p}_{ck} g_{ua} g_{xv} g_{bv} \tilde{p}_{tx} + \tilde{p}_{ck} g_{ea} g_{uv} g_{bv} \tilde{p}_{te} - g_{ua} \tilde{p}_{cx} g_{xv} g_{bv} \tilde{p}_{tk} - g_{ea} g_{uv} g_{bv\,P} Rce \tilde{p}_{tk} \\
&\quad + \tilde{n}_e \tilde{n}_b g_{ua\,P} Rce \tilde{p}_{tk} \Big) f_{zt}
\end{aligned}
\tag{272}
$$

$$
\begin{aligned}
A_{klabcz} &= g_{xa} \tilde{p}_{yx} f_{zy} \delta_{kb} \delta_{lc} + N^2 n_{Rl} n_{Re} g_{ea} \frac{1}{2} (\tilde{p}_{ck} \tilde{p}_{bv} + \tilde{p}_{cv} \tilde{p}_{bk}) g_{vz} + \tilde{p}_{bx} g_{xa} \tilde{p}_{cl} \tilde{p}_{tk} f_{zt} - \tilde{p}_{cl} g_{sa} \tilde{p}_{ts} f_{zt} \delta_{kb} \\
&\quad - \tilde{p}_{bl} \tilde{p}_{ce} g_{ea} \tilde{p}_{tk} f_{zt} - N^2 n_{Rl} n_{Ru} g_{ua} \tilde{p}_{cx} g_{xv} g_{bv} \tilde{p}_{tk} f_{zt} \\
&\quad - N^2 n_{Rl} n_{Ru} g_{ea} g_{uv} g_{bv} \tilde{p}_{ce} \tilde{p}_{tk} f_{zt} + N^2 n_{Rl} n_{Ru} \underbrace{\tilde{p}_{ce} \tilde{n}_e} \tilde{n}_b g_{ua} \tilde{p}_{tk} f_{zt}
\end{aligned}
\tag{273}
$$

The underbraced term vanishes because of $\tilde{\mathbf{P}} \cdot \tilde{\mathbf{n}} = \mathbf{0}$. The two expressions must be symmetrized in (k, l) and (b, c) because of $t_{Riklm} = t_{Rilkm}$ und $f_{abc} = f_{acb}$. Nevertheless it is by no means obvious that they are identical. Since they depend on six parameters we have to secure the identity of $3^6 = 729$ single expressions. This task can easily be performed with the help of a computer algebra system. Without loss of generality, we may choose a Cartesian system such that $\mathbf{n}_R = \mathbf{e}_3$. Moreover we may apply a rigid rotation of the body such that the local transplacement becomes

$$
\mathbf{F} = f_{11} \mathbf{e}_1 \otimes \mathbf{e}_1 + f_{12} \mathbf{e}_1 \otimes \mathbf{e}_2 + f_{13} \mathbf{e}_1 \otimes \mathbf{e}_3 + f_{22} \mathbf{e}_2 \otimes \mathbf{e}_2 + f_{23} \mathbf{e}_2 \otimes \mathbf{e}_3 + f_{33} \mathbf{e}_3 \otimes \mathbf{e}_3 \tag{274}
$$

and hence $\mathbf{n} = \mathbf{n}_R = \mathbf{e}_3$. The summation over indices that occur twice can be done with a matrix multiplication routine. Initially there is no coincidence in 18 cases with $k = l = 3$. However, the boundary condition (126) reduces to $t_{Ri333} = 0$ so that these differing expressions are both multiplied by zero.

Next we compare the factors with $Qt_{R\,iklm} n_{Rm} \big(n_{Rx} \nabla_{Rc} + n_{Rc} \nabla_{Rx}\big)/2$ and call them R_{klxcz} and A_{klxcz} in the referential and current description, respectively.

$$\begin{aligned} R_{klxcz} &= -N^2 n_{Rl} \tilde{p}_{tw} g_{wu} g_{yu} p_{Ryx} \tilde{p}_{ck} f_{zt} + N \tilde{n}_e p_{Rlx\,p} R_{ce} \tilde{p}_{tk} f_{zt} \\ &\quad + N^2 n_{Rl} \tilde{p}_{cw} g_{wu} g_{yu} p_{Ryx} \tilde{p}_{tk} f_{zt} - N^2 n_{Rl} \tilde{n}_e \tilde{n}_y p_{Ryx\,p} R_{ce} \tilde{p}_{tk} f_{zt} \end{aligned} \tag{275}$$

$$\begin{aligned} A_{klxcz} &= -N^2 n_{Rl} \tilde{p}_{xk} \tilde{p}_{cv} g_{vz} + N \tilde{n}_y p_{Ryx} \tilde{p}_{cl} \tilde{p}_{tk} f_{zt} + N \underbrace{\tilde{p}_{ce} \tilde{n}_e} \delta_{ly} p_{Ryx} \tilde{p}_{tk} f_{zt} \\ &\quad + N^2 n_{Rl} \tilde{p}_{cw} g_{wu} g_{yu} p_{Ryx} \tilde{p}_{tk} f_{zt} - N^2 n_{Rl} \underbrace{\tilde{p}_{ce} \tilde{n}_e} \tilde{n}_y p_{Ryx} \tilde{p}_{tk} f_{zt} \end{aligned} \tag{276}$$

The underbraced terms vanish because of $\tilde{\mathbf{P}} \cdot \tilde{\mathbf{n}} = \mathbf{0}$. The two expressions must be symmetrized in (k, l) and (x, c) because of $t_{Riklm} = t_{Rilkm}$ und $n_{Rx} \nabla_{Rc} = n_{Rc} \nabla_{Rx}$. Since they depend on five parameters, we have to compare $3^5 = 243$ single expressions, and indeed, they turn out to be identical.
Finally we have to check the statement (122) $Q\check{\mathbf{t}}_R = \check{\mathbf{t}}$ where $\check{\mathbf{t}}_R$ is given by (78) — the last term vanishes due to (126) — and $\check{\mathbf{t}}$ is given by (83). The component of $Q\check{\mathbf{t}}_R$ is

$$\begin{aligned} Q\check{t}_{Ri} &= QN\bar{t}_{Rikl} n_{Rk} n_{Rl} + QNt_{Riklm} n_{Rm} \tilde{p}_{el} \frac{1}{2}\Big((\nabla_{Rc} n_{Re}) + (\nabla_{Re} n_{Rc}) \Big) p_{Rck} \\ &\quad + Qt_{Riklm} n_{Rl} n_{Rm} \tilde{p}_{ck} (\nabla_{Rc} N) - QN^2 t_{Riklm} n_{Rl} n_{Rm} \tilde{p}_{ck} (\nabla_{Rc} \tilde{n}_e) n_{Re} \\ &= \underline{QN\bar{t}_{Rikl} n_{Rk} n_{Rl}} + QNt_{Riklm} n_{Rm} \tilde{p}_{el} \frac{1}{2}\Big((\nabla_{Rc} n_{Re}) + (\nabla_{Re} n_{Rc}) \Big) p_{Rck} \\ &\quad - QN^2 t_{Riklm} n_{Rl} n_{Rm} \tilde{p}_{ck} \tilde{n}_y \Big(p_{Rye} \big(n_{Re} \nabla_{Rc} + n_{Rc} \nabla_{Re}\big)/2 - n_{Re} g_{ew} f_{wyc} \Big) \\ &\quad - QN^3 t_{Riklm} n_{Rl} n_{Rm} \tilde{p}_{ck} g_{wu} g_{yu} p_{Ryz} \big(n_{Rz} \nabla_{Rc} + n_{Rc} \nabla_{Rz}\big)/2 \underbrace{n_{Re} \tilde{p}_{ew}} \\ &\quad + QN^3 t_{Riklm} n_{Rl} n_{Rm} \tilde{p}_{ck} n_{Ru} \big(g_{uw} \underbrace{n_{Re} \tilde{p}_{ez}} g_{zv} + g_{ew} g_{uv} n_{Re} \big) g_{yv} f_{wyc} \end{aligned} \tag{277}$$

The underbraced terms vanish. This has to be compared with the component of $\check{\mathbf{t}}$.

$$\begin{aligned} \check{t}_i &= \bar{t}_{ipq} n_p n_q + t_{ipqr} n_r p_{qs} (n_s \nabla_w + n_w \nabla_s)/2 p_{wp} \\ &= QN\bar{t}_{R\,ikl} f_{pk} f_{ql} g_{bp} n_{Rb} g_{aq} n_{Ra} + QNt_{R\,iklm} f_{pkl} f_{qm} g_{bp} n_{Rb} g_{aq} n_{Ra} \\ &\quad + Qt_{R\,iklm} f_{pk} f_{ql} f_{rm} g_{ar} n_{Ra} g_{xq} \tilde{p}_{yx} f_{sy} \frac{1}{2}\Big(g_{cs} \nabla_{Rc} (N g_{ew} n_{Re}) + g_{cw} \nabla_{Rc} (N g_{es} n_{Re}) \Big) f_{wu} \tilde{p}_{uv} g_{vp} \\ &= QN\bar{t}_{R\,ikl} n_{Rk} n_{Rl} + QNt_{R\,iklm} n_{Rm} f_{pkl} g_{bp} n_{Rb} \\ &\quad + Qt_{R\,iklm} n_{Rm} \tilde{p}_{yl} f_{sy} \frac{1}{2}\Big(g_{cs} \nabla_{Rc} (N g_{ew} n_{Re}) + g_{cw} \nabla_{Rc} (N g_{es} n_{Re}) \Big) f_{wu} \tilde{p}_{uk} \\ &= \underline{QN\bar{t}_{R\,ikl} n_{Rk} n_{Rl}} + QNt_{R\,iklm} n_{Rm} f_{pkl} g_{bp} n_{Rb} \\ &\quad - QNt_{R\,iklm} n_{Rm} n_{Re} g_{ea} f_{abc} \frac{1}{2}\Big(\tilde{p}_{cl} \tilde{p}_{bk} + \tilde{p}_{bl} \tilde{p}_{ck} \Big) \\ &\quad + Qt_{R\,iklm} n_{Rm} \tilde{p}_{cl} N \frac{1}{2}\Big((\nabla_{Rc} n_{Re}) + (\nabla_{Re} n_{Rc}) \Big) \tilde{p}_{ek} \\ &\quad + Qt_{R\,iklm} n_{Rm} \frac{1}{2}\Big(\underbrace{n_{Re} \tilde{p}_{ek}} \tilde{p}_{cl} + \underbrace{n_{Re} \tilde{p}_{el}} \tilde{p}_{ck} \Big) (\nabla_{Rc} N) \end{aligned} \tag{278}$$

The underbraced terms vanish again.
The equivalence of the underlined terms is obvious. Next we compare the factors with $Qt_{R\,iklm} n_{Rm} f_{abc}$ and call them R_{klabc} and A_{klabc} in the referential and current description, respectively.

$$R_{klabc} = N g_{ea} n_{Re} \Big(N n_{Rl} \tilde{p}_{ck} \tilde{n}_b + N^2 n_{Rl} \tilde{p}_{ck} n_{Ru} g_{uv} g_{bv} \Big) \tag{279}$$

$$A_{klabc} = N g_{ea} n_{Re} \Big(\delta_{bk} \delta_{cl} - (\tilde{p}_{cl} \tilde{p}_{bk} + \tilde{p}_{bl} \tilde{p}_{ck})/2 \Big) \tag{280}$$

The two expressions must be symmetrized in (k, l) and (b, c). Since they depend on five parameters, we have to compare $3^5 = 243$ single expressions. Initially there is no coincidence in 9 cases with $k = l = 3$. However, the boundary condition (126) reduces to $t_{Ri333} = 0$ so that these differing expressions are both multiplied by zero.
Finally we compare the factors with $Qt_{R\,iklm} n_{Rm} \left(n_{Rx} \nabla_{Rc} + n_{Rc} \nabla_{Rx}\right)/2$ and call them R_{klxc} and A_{klxc} in the referential and current description, respectively.

$$R_{klxc} = N \tilde{p}_{xl} p_{Rck} - N^2 n_{Rl} \tilde{p}_{ck} \tilde{n}_y p_{Ryx} \tag{281}$$

$$A_{klxc} = N \tilde{p}_{cl} \tilde{p}_{xk} \tag{282}$$

The two expressions must be symmetrized in (k, l) and (x, c). Since they depend on four parameters, we have to compare $3^4 = 81$ single expressions and, indeed, they turn out to be identical.

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