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ORIGINAL PAPER

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$\boldsymbol{J}$-, M- and $\boldsymbol{L}$-integrals of line charges and line forces

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Abstract In this work, the $\boldsymbol{J}$-, M- and $\boldsymbol{L}$-integrals of two line charges and two line forces in generalized plane strain are derived in the framework of three-dimensional (3D) electrostatics and three-dimensional compatible linear elasticity, respectively, in order to study the interaction between them. The key point in this derivation is achieved by expressing the $\boldsymbol{J}$-, M- and $\boldsymbol{L}$-integrals of point charges and point forces in three dimensions in terms of the corresponding three-dimensional Green functions, that is, the three-dimensional Green function of the Laplace operator and the three-dimensional Green tensor of the Navier operator, respectively. The major mathematical tool used in deriving the $\boldsymbol{J}$-, M- and L_3-integrals of line charges and line forces from the corresponding $\boldsymbol{J}$-, M- and $\boldsymbol{L}$-integrals of point charges and point forces is the method of embedding or method of descent of Green functions to two dimensions (2D) from the corresponding Green functions in 3D. The analytical expressions of $\boldsymbol{J}$-, M- and L_3-integrals of line charges and line forces in antiplane and plane strain are derived and discussed. The $\boldsymbol{J}$-integral is the electrostatic part of the Lorentz force (electrostatic interaction force) between two line charges in electrostatics and the Cherepanov force between two line forces in elasticity. The M-integral of two line sources (charges or forces) equals half the electrostatic interaction energy in electrostatics and half the elastic interaction energy in elasticity of these two line sources, respectively. The L_3-integral of two line sources (charges or forces) is the z-component of the configurational vector moment or the rotational moment representing the total torque about the z-axis caused by the interaction of the two line sources. An important outcome is that the $\boldsymbol{J}$-integral is twice the negative gradient of the M-integral, leading to the result that the $\boldsymbol{J}$-integral for line charges and line forces is a conservative force (with the corresponding interaction energy playing the role of the potential energy) and consequently an irrotational vector field. Finally, the obtained $\boldsymbol{J}$-, M- and L_3-integrals being functions of the distance and of the angle, are able to describe the physical behaviour of the interaction of two line sources (charges and forces), since they represent the fundamental and necessary quantities, that is, the interaction force, interaction energy and total torque produced by the interaction of these two line sources.

1 Introduction

The $\boldsymbol{J}$-, M- and $\boldsymbol{L}$-integrals are fundamental concepts in continuum mechanics theories like elasticity, electroelasticity, thermoelasticity as well as in defect theories like fracture mechanics, dislocation theory and inclusion theory (see, e.g., [12,14,22,29,32,33]). Seven conservation laws of elasticity, which are related to translational, rotational, and scaling symmetries, were originally derived by Günther [18], and Knowles and Sternberg [24].

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The corresponding conservation integrals are the $\boldsymbol{J}$-, $\boldsymbol{L}$- and M-integrals introduced by Budiansky and Rice [11]. Budiansky and Rice [11] were the first to give a physical interpretation to the $\boldsymbol{J}$-, $\boldsymbol{L}$- and M-integrals as the energy-release rates per unit cavity translation, rotation and expansion, respectively. The $\boldsymbol{J}$-integral for cracks is widely used and known as the $\boldsymbol{J}$-integral of Rice [39], which is the driving force acting on the crack tip and it is a material or configurational force [33]. The $\boldsymbol{J}$-integral of dislocations is equivalent to the well-known Peach-Koehler force that is the interaction force between two dislocations [23,40,45] and it is also a configurational force. In the framework of configurational or Eshelbian mechanics, Lazar and Kirchner [29] showed that, in the general case of anisotropic linear first strain gradient elasticity for non-homogeneous and incompatible media with dislocations and body forces present, the $\boldsymbol{J}$-integral gives the sum of configurational forces of different origin, namely the Peach-Koehler force for dislocations, the Cherepanov force due to body forces and the Eshelby force due to inhomogeneities, the M-integral is equivalent to the total configurational or material work and the $\boldsymbol{L}$-integral is equivalent to the sum of configurational or material vector moments also of different origin (see also [3]). An extensive introduction about the $\boldsymbol{J}$-, M- and $\boldsymbol{L}$-integrals can be found in Agiasofitou and Lazar [2], Lazar and Agiasofitou [28] and the references therein. A recent monograph devoted in the field of configurational mechanics, presenting among others, various vistas of the material or configurational forces and highlighting its advantages to tackle problems in continuum defect mechanics, providing a dissipation-consistent approach, has been given by Steinmann [42].

Special attention should be paid to the M-integral and its physical interpretation. Rice [40] was the first to raise the question "What is M for a dislocation?" and studying the M-integral when centered on a dislocation line in the framework of two-dimensional, compatible linear elasticity, he found that the M-integral equals to the "dislocation energy factor". In fracture mechanics, first Chen [12] wrote explicitly that the M-integral equals twice the change of the total potential energy owed to single cracking of a central crack in a plane elastic body. In the framework of three-dimensional incompatible elasticity of dislocations, the physical interpretation of the M- and $\boldsymbol{L}$-integrals for dislocations was given for the first time in Agiasofitou and Lazar [2]. Namely, the M-integral between two straight dislocations (per unit dislocation length) is equal to the half of the interaction energy of the two dislocations (per unit dislocation length) depending on the distance and on the angle, plus twice the corresponding pre-logarithmic energy factor and the L_3-integral between two straight dislocations is the z-component of the configurational vector moment or the rotational moment about the z-axis caused by the interaction of the two dislocations [2]. Moreover, the $\boldsymbol{J}$-, M- and $\boldsymbol{L}$-integrals of body charges and point charges in electrostatics, and the $\boldsymbol{J}$-, M- and $\boldsymbol{L}$-integrals of body forces and point forces in elasticity have been given by Lazar and Agiasofitou [28]. In particular, it is proven that the $\boldsymbol{J}$-integral of body charges in electrostatics represents the electrostatic part of the Lorentz force, and the $\boldsymbol{J}$-integral of body forces in elasticity represents the Cherepanov force, both of them are configurational forces. The M-integral of two point sources (charges or forces) equals half the electrostatic interaction energy in electrostatics and half the elastic interaction energy in elasticity of these two point sources. The $\boldsymbol{L}$-integral represents the configurational vector moment or total torque between two body or point sources (charges or forces). The physical interpretation of the M-integral given above in [2,28] is in full agreement with the physical interpretation of the M-integral in fracture mechanics as we have discussed above given in [12].

It becomes clear that, in general, the $\boldsymbol{J}$-integral is equivalent to the interaction force (electrostatic part of the Lorentz force in electrostatics, Peach-Koehler force of dislocations, Cherepanov force of point forces) and it is a configurational force. The M-integral is interaction energy (electrostatic interaction energy in electrostatics, interaction energy between two dislocations, elastic interaction energy between two point forces in elasticity) and it is not representing a "force" as misleadingly stated in the literature as "self-similar expansion force" [41] or "generalized force" [17]. Finally, the $\boldsymbol{L}$-integral represents the total torque and it is a configurational vector moment [2,27,28]. This physical interpretation of the examined integrals has been also shown to be true in more complicated cases such as for electro-elastic dislocations in piezoelectric materials [3] and even for the complex structure of quasicrystals [26].

A crucial point that is noteworthy to be highlighted and which can shed light to the misunderstandings around the physical interpretation of the M-integral is that the analytical expression of the M-integral for a two-dimensional body is different than that for a three-dimensional body. This happens because the M-integral depends on the space dimension (see related discussion in [2]), since it is inherently related to the group of scaling transformations and the scaling or canonical dimension of the field variable of elasticity, that is of the displacement field, is different in 2D than that in 3D (see, e.g., [2,28]). For physical phenomena such as straight dislocations and line forces that have to be treated in three dimensions (so $n = 3$) due to their physics, if one reduces the study of these problems to two dimensions (so $n = 2$), then the M-integral is just a constant (see, e.g., [4,5,10,20,40,41]) loosing the important contribution which offers the functional dependence on the distance and on the angle between the examined defects or sources like dislocations, charges, point forces, etc. (see [2,28]).

A line force is a line singularity in a three-dimensional medium. Lazar and Maugin [31] found closed analytical solutions for the displacement fields of line forces, free of singularities, in the framework of simplified first strain gradient elasticity. In addition, generalized plane strain is a two-dimensional deformation in a three-dimensional body (see, e.g., [34]). Therefore, the problem of the interaction of two line forces has to be studied in the framework of three-dimensional compatible linear elasticity, considering for the calculation of the M-integral the space dimension equals three ($n = 3$). Regarding previous results in the literature as far as the $\boldsymbol{J}$-, M- and L_3-integrals of line forces are concerned, the $\boldsymbol{J}$-, M- and L_3-integrals of two-dimensional elastic line singularities (among them also line forces) in the framework of plane strain linear elasticity have been derived by Seo et al. [41] but only in a marginal case without providing the general analytical expressions of the examined integrals. Moreover, the M-integral in [41] being a constant (since it has been considered $n = 2$) cannot capture the interaction between the two line forces in contrast to the M-integral derived in this work (for $n = 3$), which is a function of the distance and of the angle and in this way the physics of the physical phenomenon can be appropriately described (see related discussion in the Appendix A). It is important to pay attention to this matter, since the results and statements in [41] about the M-integral of line forces have been taken over in a recent review article [20] about the M-integral.

This article is organized as follows. In Sect. 2, the $\boldsymbol{J}^{(AB)}$-, $M^{(AB)}$- and $\boldsymbol{L}^{(AB)}$-integrals of point charges and point forces in 3D are given for the first time in terms of the three-dimensional Green function $G^{(3)}(R)$ of the Laplace operator and the three-dimensional Green tensor $\boldsymbol{G}^{(3)}(\boldsymbol{R})$ of the Navier operator, respectively, giving the opportunity to use the method of embedding or method of descent of Green functions to 2D from 3D in Sect. 3 and to obtain the desired $\boldsymbol{J}^{(AB)}$-, $M^{(AB)}$- and $L_3^{(AB)}$-integrals of line charges and line forces. The explicit expressions of the $\boldsymbol{J}^{(AB)}$-, $M^{(AB)}$- and $L_3^{(AB)}$-integrals for two parallel line forces in antiplane and plane strain are derived in Sects. 4 and 5 examining all possible directions of strengths. In addition, the $J_r^{(AB)}$-, $J_\varphi^{(AB)}$-, $M^{(AB)}$- and $L_3^{(AB)}$-integrals are given in terms of the corresponding pre-logarithmic energy factor of line forces which is introduced here for the first time in analogy to the pre-logarithmic energy factor of dislocations. The result that the $\boldsymbol{J}^{(AB)}$-integral is twice the negative gradient of the $M^{(AB)}$-integral gives rise to prove that the $\boldsymbol{J}^{(AB)}$-integral for line charges and line forces is a conservative force in Sect. 6. The results are summarized in the conclusions in Sect. 7. The Appendix A serves to make the necessary comparisons with the already given results in the literature and to highlight important aspects concerning the $M^{(AB)}$-integral.

2 $\boldsymbol{J}$-, $\boldsymbol{M}$- and $\boldsymbol{L}$-integrals of point charges and point forces in 3D

In this Section, we present in brief the $\boldsymbol{J}^{(AB)}$-, $M^{(AB)}$- and $\boldsymbol{L}^{(AB)}$-integrals of point charges in electrostatics and point forces in elasticity in 3D that have been derived in Lazar and Agiasofitou [28]. We recall only the results that are necessary for the purpose of this work. The reader is addressed to [28] for an extensive presentation and detailed derivations. In particular, the $\boldsymbol{J}^{(AB)}$-, $M^{(AB)}$- and $\boldsymbol{L}^{(AB)}$-integrals of point charges and point forces in 3D with the "source field" applied at the origin derived in [28], are used and further expressed in terms of the corresponding three-dimensional Green functions providing the appropriate expressions for the forthcoming calculations.

2.1 Electrostatics: point charges in 3D

Let us consider an unbounded, linear, homogeneous and isotropic medium. In electrostatics, the field variable is the *electrostatic potential* $\varphi = \varphi(\boldsymbol{x})$. The *electric field strength vector* $E_k = E_k(\boldsymbol{x})$ is defined as the negative gradient of the electrostatic potential φ:

$$E_k = -\partial_k \varphi\,, \quad (\boldsymbol{E} = -\mathrm{grad}\,\varphi)\,, \tag{1}$$

if the *electrostatic Bianchi identity*, which has the meaning of electrostatic compatibility condition, is fulfilled

$$\epsilon_{ijk}\partial_j E_k = 0\,, \quad (\mathrm{curl}\,\boldsymbol{E} = 0)\,, \tag{2}$$

where ϵ_{ijk} is the Levi-Civita tensor and ∂_j denotes partial differentiation with respect to the spatial coordinates x_j, $j = 1, 2, 3$.

Considering the interaction between an electric field $E_i^{(B)} = E_i^{(B)}(\boldsymbol{x})$ or the corresponding electrostatic potential $\varphi^{(B)} = \varphi^{(B)}(\boldsymbol{x})$ as source field and a body charge density $q^{(A)} = q^{(A)}(\boldsymbol{x})$ as receiver field, then the $J_k^{(AB)}$-, $M^{(AB)}$- *and* $L_k^{(AB)}$*-integrals for continuous body charge density distributions* are given by [28]

$$J_k^{(AB)} = \int_V q^{(A)} E_k^{(B)} \,\mathrm{d}V\,, \tag{3}$$

$$M^{(AB)} = \int_V \Big(x_k q^{(A)} E_k^{(B)} - \frac{1}{2}\, q^{(A)} \varphi^{(B)}\Big) \,\mathrm{d}V\,, \tag{4}$$

$$L_k^{(AB)} = \int_V \epsilon_{kji}\, x_j q^{(A)} E_i^{(B)} \,\mathrm{d}V\,, \tag{5}$$

where $i, j, k = 1, 2, 3$. We recall here that the $\boldsymbol{J}$*-integral of electrostatics* represents the *electrostatic force*, which is the electrostatic part of the Lorentz force

$$\mathcal{F}_k^{\mathrm{L}} = \int_V f_k^{\mathrm{L}} \,\mathrm{d}V\,, \tag{6}$$

since

$$f_k^{\mathrm{L}} = q E_k \tag{7}$$

is the *electrostatic part of the Lorentz force density*. The latter is the electrostatic force density on a body charge density q in presence of an electric field E_k. Therefore, it can be seen in Eq. (3) that the $J_k^{(AB)}$-integral is the electrostatic force on the body charge density $q^{(A)}$ in presence of the electric field $E_k^{(B)}$:

$$J_k^{(AB)} \equiv \mathcal{F}_k^{\mathrm{L}(AB)}\,. \tag{8}$$

It should be noted that for the derivation of the $M^{(AB)}$-integral given by Eq. (4), the space dimension has been taken equal to three, $n = 3$ (see [28]).

We consider now that the body contains two point charge densities, $q^{(A)}$ located at point A with position vector $\boldsymbol{x}^{(A)}$ and $q^{(B)}$ located at point B with position vector $\boldsymbol{x}^{(B)}$. The point B is the "source-point" (point of sender) where the charge density $q^{(B)}$, playing the role of a "source charge", is located; and A is the "field-point" (point of receiver) where the charge density $q^{(A)}$, playing the role of a test charge measuring the electrostatic force, energy and torque, is located. The two body charge densities can be modelled in terms of the point charges $\hat{q}^{(A)}$ and $\hat{q}^{(B)}$ and the three-dimensional Dirac delta function $\delta(\boldsymbol{x})$ as follows

$$q^{(A)}(\boldsymbol{x}) = \hat{q}^{(A)}\, \delta(\boldsymbol{x} - \boldsymbol{x}^{(A)})\,, \tag{9}$$

$$q^{(B)}(\boldsymbol{x}) = \hat{q}^{(B)}\, \delta(\boldsymbol{x} - \boldsymbol{x}^{(B)})\,. \tag{10}$$

In the case that the interaction is between a point charge (source charge) $\hat{q}^{(B)}$ located at the origin of the coordinate system O, that is $\boldsymbol{x}^{(B)} = 0$, and a point charge (test charge) $\hat{q}^{(A)}$ located at the position $\boldsymbol{R} = \boldsymbol{x}^{(A)} = (\bar{x}, \bar{y}, \bar{z})$, then the $\boldsymbol{J}^{(AB)}$-, $M^{(AB)}$- *and* $\boldsymbol{L}^{(AB)}$*-integrals in 3D for point charges with the "source charge" at the origin* read as [28]

$$J_k^{(AB)} = \frac{\hat{q}^{(A)}\hat{q}^{(B)}}{4\pi\varepsilon}\, \frac{R_k}{R^3} = \mathcal{F}_k^{\mathrm{L}(AB)}\,, \tag{11}$$

$$M^{(AB)} = \frac{\hat{q}^{(A)}\hat{q}^{(B)}}{8\pi\varepsilon}\, \frac{1}{R} = \frac{1}{2}\, U^{(AB)}\,, \tag{12}$$

$$L_k^{(AB)} = 0\,, \tag{13}$$

where $R = |\boldsymbol{x}^{(A)}| = \sqrt{\bar{x}^2 + \bar{y}^2 + \bar{z}^2}$ and ε denotes the *permittivity of material* or *dielectric constant of matter*.

It can be seen in Eq. (11) that the electrostatic $\boldsymbol{J}^{(AB)}$-integral is the electrostatic force $\boldsymbol{\mathcal{F}}^{\mathrm{L}(AB)}$ acting on the point charge $\hat{q}^{(A)}$ that is placed at the position $\boldsymbol{x}^{(A)}$ due to the point charge $\hat{q}^{(B)}$ placed at the position $\boldsymbol{x}^{(B)} \equiv O$ and it is provided by the famous *Coulomb law*, which is the experimentally established law for the electrostatic part of the Lorentz force between two point charges in isotropic media. The electrostatic force along the line from B to A is repulsive if $\hat{q}^{(B)}$ and $\hat{q}^{(A)}$ have the same sign, and attractive if their signs are opposite. Equation (12) shows that the $M^{(AB)}$-integral is half the electrostatic interaction energy $U^{(AB)}$ of the two point charges (see, e.g., [21,28]). Equation (13) gives a zero value for the $L^{(AB)}$-integral, which is reasonable since there is no torque between two scalar fields.

The important observation that can be made at this point is that the non-vanishing $\boldsymbol{J}^{(AB)}$- *and* $M^{(AB)}$-*integrals of point charges in 3D*, Eqs. (11) and (12), can be written *in terms of the three-dimensional Green function* $G^{(3)}(R)$ *of the Laplace operator* as follows

$$J_k^{(AB)} = -\hat{q}^{(A)}\hat{q}^{(B)}\,\partial_k G^{(3)}(R)\,, \tag{14}$$

$$M^{(AB)} = \frac{1}{2}\,\hat{q}^{(A)}\hat{q}^{(B)}G^{(3)}(R)\,, \tag{15}$$

where the *three-dimensional Green function of the Laplace operator* is given by (see, e.g., [9,28])

$$G^{(3)}(R) = \frac{1}{4\pi\varepsilon}\,\frac{1}{R}\,. \tag{16}$$

2.2 Elasticity: point forces in 3D

Let us consider an unbounded, linear, homogeneous and isotropic elastic body. In elasticity, the field variable is the *displacement vector* $\boldsymbol{u} = \boldsymbol{u}(\boldsymbol{x})$. The *elastic distortion tensor* $\beta_{ik} = \beta_{ik}(\boldsymbol{x})$ can be written as gradient of a displacement vector field u_i:

$$\beta_{ik} = \partial_k u_i\,, \tag{17}$$

if the *compatibility condition of elasticity*

$$\epsilon_{jlk}\partial_l\beta_{ik} = 0$$

is fulfilled.

Considering the interaction between an elastic distortion $\beta_{ik}^{(B)}(\boldsymbol{x})$ or the corresponding *displacement field* $u_i^{(B)}(\boldsymbol{x})$ as source field and a body force density $f_i^{(A)}(\boldsymbol{x})$ as receiver field, then the $\boldsymbol{J}^{(AB)}$-, $M^{(AB)}$- *and* $\boldsymbol{L}^{(AB)}$-*integrals in 3D for continuous body force density distributions* are given by [28]

$$J_k^{(AB)} = \int_V f_i^{(A)}\beta_{ik}^{(B)}\,\mathrm{d}V\,, \tag{18}$$

$$M^{(AB)} = \int_V \Big(x_k f_i^{(A)}\beta_{ik}^{(B)} + \frac{1}{2}\,f_i^{(A)}u_i^{(B)}\Big)\,\mathrm{d}V\,, \tag{19}$$

$$L_k^{(AB)} = \int_V \epsilon_{kji}\Big(x_j f_m^{(A)}\beta_{mi}^{(B)} + f_j^{(A)}u_i^{(B)}\Big)\,\mathrm{d}V\,. \tag{20}$$

In Eq. (18), it can be seen that the $J_k^{(AB)}$-integral is the *Cherepanov force* $\mathcal{F}_k^{\mathrm{C}(AB)}$ on the body force density $f_i^{(A)}$ in presence of the elastic distortion $\beta_{ik}^{(B)}$:

$$J_k^{(AB)} \equiv \mathcal{F}_k^{\mathrm{C}(AB)}\,. \tag{21}$$

To recall, the *Cherepanov force* is given by

$$\mathcal{F}_k^{\mathrm{C}} = \int_V f_k^{\mathrm{C}}\,\mathrm{d}V\,, \tag{22}$$

where

$$f_k^{\mathrm{C}} = f_i\beta_{ik} \tag{23}$$

is the so-called *Cherepanov force density* [13], which is the configurational force density acting on a body force density f_i in presence of an elastic distortion β_{ik}.

The body force densities $f_i^{(A)}$ and $f_j^{(B)}$ can be modelled with the point forces $\hat{f}_i^{(A)}$ and $\hat{f}_j^{(B)}$ as follows

$$f_i^{(A)}(\boldsymbol{x}) = \hat{f}_i^{(A)}\,\delta(\boldsymbol{x} - \boldsymbol{x}^{(A)})\,, \tag{24}$$

$$f_j^{(B)}(\boldsymbol{x}') = \hat{f}_j^{(B)}\,\delta(\boldsymbol{x}' - \boldsymbol{x}^{(B)})\,, \tag{25}$$

where $\boldsymbol{x}^{(A)}$ and $\boldsymbol{x}^{(B)}$ are the position vectors of the points A and B and $\hat{f}_i^{(A)}$ and $\hat{f}_j^{(B)}$ are the strengths of the point forces, respectively. For the interaction between a point force (source force) of strength, $\hat{f}_j^{(B)}$, located at the point B which coincides with the origin of the coordinate system O, so that the position vector $\boldsymbol{x}^{(B)} = 0$ and a point force (test force) of strength, $\hat{f}_i^{(A)}$, located at the point A with position vector $\boldsymbol{R} = \boldsymbol{x}^{(A)}$, the *$\boldsymbol{J}^{(AB)}$-, $M^{(AB)}$- and $\boldsymbol{L}^{(AB)}$-integrals in 3D for point forces with the "source force" applied at the origin* are given by [28]

$$J_k^{(AB)} = -\frac{\hat{f}_i^{(A)}\hat{f}_j^{(B)}}{16\pi\mu(1-\nu)R^2}\left[(3-4\nu)\delta_{ij}\frac{R_k}{R} - \delta_{ik}\frac{R_j}{R} - \delta_{jk}\frac{R_i}{R} + 3\,\frac{R_iR_jR_k}{R^3}\right] = \mathcal{F}_k^{\mathrm{C}(AB)}\,, \tag{26}$$

$$M^{(AB)} = -\frac{\hat{f}_i^{(A)}\hat{f}_j^{(B)}}{32\pi\mu(1-\nu)R}\left[(3-4\nu)\delta_{ij} + \frac{R_iR_j}{R^2}\right] = \frac{1}{2}\,U^{(AB)}\,, \tag{27}$$

$$L_k^{(AB)} = \frac{\epsilon_{kji}\hat{f}_l^{(A)}\hat{f}_i^{(B)}}{16\pi\mu(1-\nu)R}\left[(3-4\nu)\delta_{jl} + \frac{R_jR_l}{R^2}\right] = \mathcal{T}_k^{(AB)}\,, \tag{28}$$

where μ is the *shear modulus* and ν is the *Poisson ratio*. Here, $U^{(AB)}$ denotes the elastic interaction energy and $\mathcal{T}_k^{(AB)}$ denotes the total torque vector of the two point forces. It has to be noticed that for the derivation of the $M^{(AB)}$-integral in Eqs. (19) and (27), the space dimension has been taken equal to three, $n = 3$ (see [28]).

The crucial step here is to observe that in the above expressions of the $J_k^{(AB)}$-, $M^{(AB)}$- and $L_k^{(AB)}$-integrals, Eqs. (26)–(28), the three-dimensional Green tensor $G_{ij}^{(3)}(\boldsymbol{R})$ of the Navier operator and its gradient appear. Hence, the *$J_k^{(AB)}$-, $M^{(AB)}$- and $L_k^{(AB)}$-integrals of point forces in 3D can be given in terms of the three-dimensional Green tensor function $G_{ij}^{(3)}(\boldsymbol{R})$ of the Navier operator* by the following simple formulas:

$$J_k^{(AB)} = \hat{f}_i^{(A)}\hat{f}_j^{(B)}\,\partial_k G_{ij}^{(3)}(\boldsymbol{R})\,, \tag{29}$$

$$M^{(AB)} = -\frac{1}{2}\,\hat{f}_i^{(A)}\hat{f}_j^{(B)}G_{ij}^{(3)}(\boldsymbol{R})\,, \tag{30}$$

$$L_k^{(AB)} = \epsilon_{kji}\hat{f}_i^{(B)}\hat{f}_l^{(A)}G_{jl}^{(3)}(\boldsymbol{R})\,, \tag{31}$$

where the *three-dimensional Green tensor of the Navier operator* is given by (see, e.g., [28,32,35])

$$G_{ij}^{(3)}(\boldsymbol{R}) = \frac{1}{16\pi\mu(1-\nu)R}\left[(3-4\nu)\delta_{ij} + \frac{R_iR_j}{R^2}\right]. \tag{32}$$

3 Method of embedding or method of descent to 2D from 3D

Line charges and line forces can be viewed as infinitely long line sources along the entire z-axis in 3D. By symmetry, the problem of line sources is independent of $\bar{z}$ and is a 2D problem which is embedded in the 3D space (see, e.g., [9]). Therefore, the solution of line sources depending on $(\bar{x}, \bar{y})$ can be obtained from the solution of point sources depending on $(\bar{x}, \bar{y}, \bar{z})$ using the projection from 3D to 2D.

In this manner, the $\boldsymbol{J}$-, M- and $\boldsymbol{L}$-integrals of line charges and line forces can be constructed from the $\boldsymbol{J}$-, M- and $\boldsymbol{L}$-integrals of point charges and point forces using the *method of embedding* [9] or the so-called *method of descent* following Hadamard [19], because one is descending to 2D from 3D via the following relations [9,43]

$$J_k(\bar{x}, \bar{y}) = \int_{-\infty}^{\infty} J_k(\bar{x}, \bar{y}, \bar{z})\,\mathrm{d}\bar{z}\,, \tag{33}$$

$$M(\bar{x},\bar{y}) = \int_{-\infty}^{\infty} M(\bar{x},\bar{y},\bar{z})\,\mathrm{d}\bar{z}\,, \tag{34}$$

$$L_k(\bar{x},\bar{y}) = \int_{-\infty}^{\infty} L_k(\bar{x},\bar{y},\bar{z})\,\mathrm{d}\bar{z}\,. \tag{35}$$

Eqs. (33)–(35) give the *projection* of the three-dimensional $J_k(\bar{x},\bar{y},\bar{z})$-, $M(\bar{x},\bar{y},\bar{z})$- and $L_k(\bar{x},\bar{y},\bar{z})$-integrals onto the $\bar{x}\bar{y}$-plane and these are the $J_k(\bar{x},\bar{y})$-, $M(\bar{x},\bar{y})$- and $L_k(\bar{x},\bar{y})$-integrals per unit length l_z embedded in the 3D space. For reasons of brevity, per unit length l_z is omitted to be written explicitly in the formulas of the $J_k(\bar{x},\bar{y})$-, $M(\bar{x},\bar{y})$- and $L_k(\bar{x},\bar{y})$-integrals. However, it is implied that it is per unit length l_z. It has to be noticed that the second term in the M-integral in Eqs. (4) and (19) arises by considering the space dimension[1] $n = 3$ and survives without being influenced by the projection, which is very reasonable since line sources "live" in 3D.

A great advantage of expressing the $J_k^{(AB)}$-, $M^{(AB)}$- and $L_k^{(AB)}$-integrals of point charges in 3D, Eqs. (14), (15) and (13), in terms of the three-dimensional Green function $G^{(3)}(R)$, Eq. (16), and the $J_k^{(AB)}$-, $M^{(AB)}$- and $L_k^{(AB)}$-integrals of point forces in 3D, Eqs. (29), (30) and (31), in terms of the three-dimensional Green tensor function $G_{ij}^{(3)}(\boldsymbol{R})$, Eq. (32), is that the projection of the three-dimensional $J_k^{(AB)}(\bar{x},\bar{y},\bar{z})$-, $M^{(AB)}(\bar{x},\bar{y},\bar{z})$- and $L_k^{(AB)}(\bar{x},\bar{y},\bar{z})$-integrals onto the $\bar{x}\bar{y}$-plane via Eqs. (33)–(35) is reduced to the projection or descending of the corresponding Green functions from 3D to 2D, which is a well-known technique in the theory of partial differential equations (see, e.g., [9,43]) and it will be applied in the forthcoming Sects. 4 and 5.

The method of projection from 3D to 2D has also been used in dislocation theory for the calculation of solutions of straight dislocations in isotropic materials (see, e.g, [6,35]), whereas the method of the projection-slice theorem has been used for dislocations in anisotropic materials in [30].

4 $\boldsymbol{J}$-, M- and L_3-integrals of line charges

In this Section, we consider two infinitely long straight line charges parallel to z-direction in an unbounded, homogeneous and isotropic body.

Substituting the $J_k^{(AB)}$-, $M^{(AB)}$- and $L_k^{(AB)}$-integrals of two point charges, Eqs. (14), (15) and (13) into Eqs. (33), (34) and (35), respectively, we obtain the $J_k^{(AB)}$-, $M^{(AB)}$- *and* $L_k^{(AB)}$-*integrals of two parallel line charges*

$$J_k^{(AB)}(\bar{x},\bar{y}) = -\hat{q}^{(A)}\hat{q}^{(B)}\,\partial_k G^{(2)}(\bar{x},\bar{y})\,, \qquad k = 1,2\,, \tag{36}$$

$$M^{(AB)}(\bar{x},\bar{y}) = \frac{1}{2}\,\hat{q}^{(A)}\hat{q}^{(B)}G^{(2)}(\bar{x},\bar{y})\,, \tag{37}$$

$$L_3^{(AB)}(\bar{x},\bar{y}) = 0\,, \tag{38}$$

where

$$G^{(2)}(\bar{x},\bar{y}) = \int_{-\infty}^{\infty} G^{(3)}(\bar{x},\bar{y},\bar{z})\,\mathrm{d}\bar{z} \tag{39}$$

is the *two-dimensional Green function of the Laplace operator*. Using Eq. (16), Eq. (39) is explicitly written (see, e.g., [9])

$$G^{(2)}(\bar{x},\bar{y}) = -\frac{1}{2\pi\varepsilon}\,\ln\frac{\bar{r}}{L}\,, \tag{40}$$

where $\bar{r} = \sqrt{\bar{x}^2+\bar{y}^2}$ and L denotes the length of the line charge in z-direction. Note that the argument of a logarithm must be dimensionless (see, e.g., [9]) and this is guaranteed by dividing $\bar{r}$ with the length L in the two-dimensional Green function (40) due to the projection from 3D to 2D. In fact, the limits of the integral from $-\infty$ to ∞ in Eq. (39) are replaced by finite numbers from $-L/2$ to $L/2$ and the final expression (40) is valid for large $L/\bar{r}$ (see [9]). From a mathematical point of view, L is a *cut-off parameter*.

[1] The scaling or canonical dimension of the scalar field φ is the same as the scaling dimension of the vector field $\boldsymbol{u}$: $d_\varphi = d_{\boldsymbol{u}} = -1/2$ (see [15,28]).

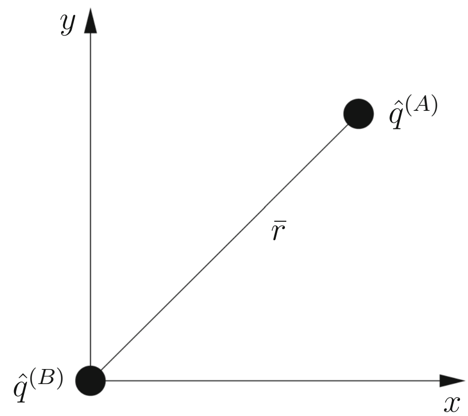

Fig. 1 Interaction between two parallel line charges $\hat{q}^{(A)}$ and $\hat{q}^{(B)}$ located at positions $(\bar{x}, \bar{y})$ and $(0, 0)$, respectively

Comparing the $J_k^{(AB)}$-, $M^{(AB)}$- and $L_k^{(AB)}$-integrals of point charges in 3D, Eqs. (14), (15) and (13), with the corresponding ones of line charges, Eqs. (36), (37) and (38), it is important to make the following observation. In electrostatics, the projection of the $J_k^{(AB)}$-, $M^{(AB)}$- and $L_k^{(AB)}$-integrals from 3D to 2D is reduced to the projection of the Green function of the Laplace operator from 3D to 2D: $G^{(3)}(\bar{x}, \bar{y}, \bar{z}) \longrightarrow G^{(2)}(\bar{x}, \bar{y})$ leading in this way to a known and standard quantity in the theory of partial differential equations, $G^{(2)}(\bar{x}, \bar{y})$, (see, e.g., [9,43]). Indeed, substituting the two-dimensional Green function (40) into Eqs. (36)–(38), we obtain the explicit expressions for the $J_k^{(AB)}$-, $M^{(AB)}$- *and* $L_k^{(AB)}$*-integrals of two parallel line charges:*

$$J_1^{(AB)}(\bar{x}, \bar{y}) = \frac{\hat{q}^{(A)}\hat{q}^{(B)}}{2\pi\varepsilon}\, \frac{\bar{x}}{\bar{x}^2 + \bar{y}^2} = \mathcal{F}_1^{\mathrm{L}(AB)}\,, \tag{41}$$

$$J_2^{(AB)}(\bar{x}, \bar{y}) = \frac{\hat{q}^{(A)}\hat{q}^{(B)}}{2\pi\varepsilon}\, \frac{\bar{y}}{\bar{x}^2 + \bar{y}^2} = \mathcal{F}_2^{\mathrm{L}(AB)}\,, \tag{42}$$

$$M^{(AB)}(\bar{x}, \bar{y}) = -\frac{\hat{q}^{(A)}\hat{q}^{(B)}}{4\pi\varepsilon}\, \ln\frac{\bar{r}}{L} = \frac{1}{2}\, U^{(AB)}\,, \tag{43}$$

$$L_3^{(AB)}(\bar{x}, \bar{y}) = 0\,. \tag{44}$$

Here, $\bar{r} = \sqrt{\bar{x}^2 + \bar{y}^2}$ is the distance between the two line charges (see Fig. 1). Equations (41) and (42) give the non-vanishing components of the electrostatic part of the Lorentz force $\boldsymbol{\mathcal{F}}^{\mathrm{L}(AB)}$, which is the electrostatic interaction force between the two line charges and is plotted in Fig. 2. It can be seen that the electrostatic force $\boldsymbol{\mathcal{F}}^{\mathrm{L}(AB)}$ is repulsive if $\hat{q}^{(A)}\hat{q}^{(B)} > 0$ (Fig. 2(a)) and is attractive if $\hat{q}^{(A)}\hat{q}^{(B)} < 0$ (Fig. 2(b)). The $M^{(AB)}$-integral, Eq. (43), equals half the electrostatic interaction energy (per unit length) of the two line charges

$$M^{(AB)} = \frac{1}{2}\, U^{(AB)}\,, \tag{45}$$

where the *electrostatic interaction energy* is given by (see [28])

$$U^{(AB)} = \hat{q}^{(A)}\varphi^{(B)} = -\frac{\hat{q}^{(A)}\hat{q}^{(B)}}{2\pi\varepsilon}\, \ln\frac{\bar{r}}{L} \tag{46}$$

with the electrostatic potential of a line charge

$$\varphi^{(B)} = G^{(2)}\hat{q}^{(B)}\,. \tag{47}$$

In a cylindrical coordinate system with basis $(\boldsymbol{e}_r, \boldsymbol{e}_\varphi, \boldsymbol{e}_z)$, the two components $J_r^{(AB)}$ and $J_\varphi^{(AB)}$ of the $J_k^{(AB)}$-integral of two line charges simplify to

$$J_r^{(AB)} = J_1^{(AB)}\cos\varphi + J_2^{(AB)}\sin\varphi = \frac{\hat{q}^{(A)}\hat{q}^{(B)}}{2\pi\varepsilon}\, \frac{1}{\bar{r}} = \mathcal{F}_r^{\mathrm{L}(AB)}\,, \tag{48}$$

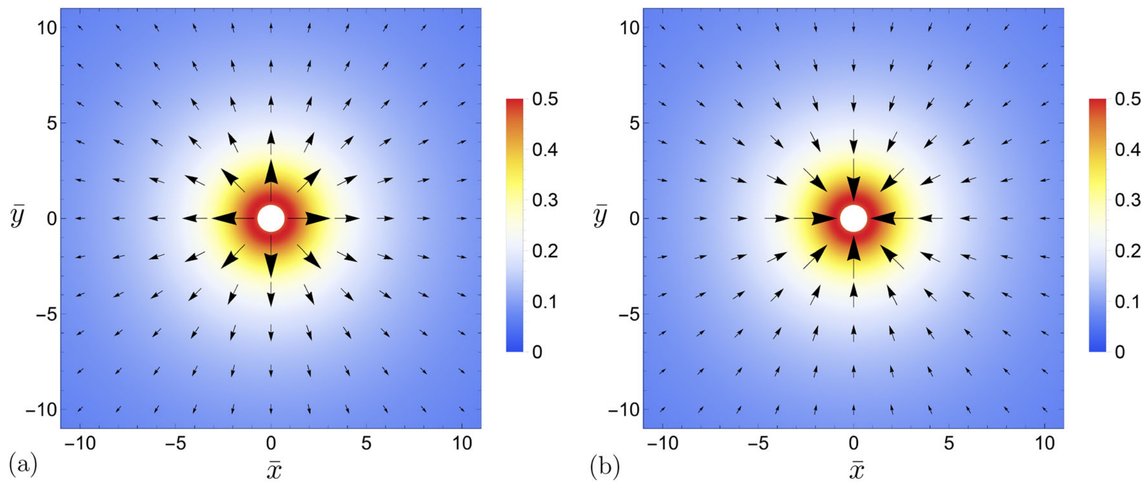

Fig. 2 Electrostatic force $\boldsymbol{\mathcal{F}}^{\mathrm{L}(AB)}$ given by the $\boldsymbol{J}^{(AB)}$-integral ($J_1^{(AB)}(\bar{x},\bar{y})$, $J_2^{(AB)}(\bar{x},\bar{y})$) of two parallel line charges with: (a) $\hat{q}^{(A)}\hat{q}^{(B)} > 0$, and (b) $\hat{q}^{(A)}\hat{q}^{(B)} < 0$ plotted in units of $\frac{|\hat{q}^{(A)}\hat{q}^{(B)}|}{2\pi\varepsilon}$

$$J_\varphi^{(AB)} = J_2^{(AB)} \cos\varphi - J_1^{(AB)} \sin\varphi = 0 = \mathcal{F}_\varphi^{\mathrm{L}(AB)} \,. \tag{49}$$

It can be seen that the electrostatic force between two parallel line charges is a purely radial force, since $\mathcal{F}_\varphi^{\mathrm{L}(AB)} = 0$. Hence,

$$\boldsymbol{\mathcal{F}}^{\mathrm{L}(AB)} = \mathcal{F}_r^{\mathrm{L}(AB)}(\bar{r})\, \boldsymbol{e}_r \,,$$

which means that the *electrostatic force is a central force* (in 2D). If $\mathcal{F}_r^{\mathrm{L}(AB)}$ is positive (or negative), then the electrostatic force is repulsive (or attractive). In other words, $\boldsymbol{\mathcal{F}}^{\mathrm{L}(AB)}$ is repulsive if $\mathcal{F}_r^{\mathrm{L}(AB)} > 0$, that is $\hat{q}^{(A)}\hat{q}^{(B)} > 0$ and is attractive if $\mathcal{F}_r^{\mathrm{L}(AB)} < 0$, that is $\hat{q}^{(A)}\hat{q}^{(B)} < 0$. Therefore, the Coulomb law (see, e.g., [21,38]) stating that charges with opposite sign attract each other (Coulomb attraction) and charges with same sign repel each other (Coulomb repulsion) is reproduced also for line charges using the concept of the $\boldsymbol{J}^{(AB)}$-integral.

To sum up, the $\boldsymbol{J}^{(AB)}$-, $M^{(AB)}$- and $L_3^{(AB)}$-integrals of two parallel line charges $\hat{q}^{(A)}$ and $\hat{q}^{(B)}$ are given by:

$$J_r^{(AB)} = \frac{\hat{q}^{(A)}\hat{q}^{(B)}}{2\pi\varepsilon}\frac{1}{\bar{r}} = \mathcal{F}_r^{\mathrm{L}(AB)} \,, \tag{50}$$

$$J_\varphi^{(AB)} = 0 = \mathcal{F}_\varphi^{\mathrm{L}(AB)} \,, \tag{51}$$

$$M^{(AB)} = -\frac{\hat{q}^{(A)}\hat{q}^{(B)}}{4\pi\varepsilon} \ln\frac{\bar{r}}{L} = \frac{1}{2}\, U^{(AB)} \,, \tag{52}$$

$$L_3^{(AB)} = 0 \,. \tag{53}$$

5 $\boldsymbol{J}$-, M- and L_3-integrals of line forces

In this Section, we consider two infinitely long straight line forces (parallel to z-direction) of uniform strength per unit length in an unbounded, homogeneous and isotropic elastic body.

Substituting the $J_k^{(AB)}$-, $M^{(AB)}$- and $L_k^{(AB)}$-integrals of two point forces, Eqs. (29), (30) and (31) into Eqs. (33), (34) and (35), respectively, we obtain *the basic formulas of the* $J_k^{(AB)}$-, $M^{(AB)}$*- and* $L_3^{(AB)}$*-integrals of two parallel line forces in generalized plane strain*:

$$J_k^{(AB)}(\bar{x},\bar{y}) = \hat{f}_i^{(A)}\hat{f}_j^{(B)}\, \partial_k G_{ij}^{(2)}(\bar{x},\bar{y}) \,, \qquad i,j = 1,2,3 \,, \quad k = 1,2 \tag{54}$$

$$M^{(AB)}(\bar{x},\bar{y}) = -\frac{1}{2}\,\hat{f}_i^{(A)}\hat{f}_j^{(B)}G_{ij}^{(2)}(\bar{x},\bar{y})\,, \qquad i,j=1,2,3 \tag{55}$$

$$L_3^{(AB)}(\bar{x},\bar{y}) = \epsilon_{3ji}\,\hat{f}_i^{(B)}\hat{f}_l^{(A)}G_{jl}^{(2)}(\bar{x},\bar{y})\,, \qquad i,j,l=1,2,3\,, \tag{56}$$

where

$$G_{ij}^{(2)}(\bar{x},\bar{y}) = \int_{-\infty}^{\infty} G_{ij}^{(3)}(\bar{x},\bar{y},\bar{z})\,\mathrm{d}\bar{z}\,, \qquad i,j=1,2,3\,, \tag{57}$$

is the *two-dimensional Green tensor of generalized plane strain*.

Using Eq. (32), we can obtain from Eq. (57) the *two-dimensional Green tensor function of plane strain* (see, e.g., [35])

$$G_{ij}^{(2)}(\bar{x},\bar{y}) = -\frac{1}{8\pi\mu(1-\nu)}\left[(3-4\nu)\delta_{ij}\ln\frac{\bar{r}}{L} - \frac{\bar{x}_i\bar{x}_j}{\bar{r}^2}\right], \qquad i,j=1,2 \tag{58}$$

and the *two-dimensional Green function of antiplane strain* ($i=j=3$)

$$G_{33}^{(2)}(\bar{x},\bar{y}) = -\frac{1}{2\pi\mu}\,\ln\frac{\bar{r}}{L}\,, \tag{59}$$

where $(\bar{x}_1,\bar{x}_2)=(\bar{x},\bar{y})$ and L denotes the length of the line force in z-direction.

Comparing the $J_k^{(AB)}$-, $M^{(AB)}$- and $L_k^{(AB)}$-integrals of point forces in 3D, Eqs. (29), (30) and (31), with the corresponding ones of line forces, Eqs. (54), (55) and (56), we conclude to an analogous observation like in electrostatics. In particular in elasticity, the projection of the $J_k^{(AB)}$-, $M^{(AB)}$- and $L_k^{(AB)}$-integrals from 3D to 2D is reduced to the projection of the three-dimensional Green tensor function to the two-dimensional Green tensor function of generalized plane strain: $G_{ij}^{(3)}(\bar{x},\bar{y},\bar{z}) \longrightarrow G_{ij}^{(2)}(\bar{x},\bar{y})$, $i,j=1,2,3$ leading to known and standard quantities in the mathematical theory of elasticity: $G_{ij}^{(2)}(\bar{x},\bar{y})$, $i,j=1,2$ and $G_{33}^{(2)}(\bar{x},\bar{y})$ (see, e.g., [35]). Note that in the projection: $G_{33}^{(3)}(\bar{x},\bar{y},\bar{z}) \longrightarrow G_{33}^{(2)}(\bar{x},\bar{y})$, an unimportant constant term has been neglected which may be disregarded because it satisfies the homogeneous Laplace equation (see [43]).

In the following subsections, we examine all possible cases of parallel line forces in z-direction considering all possible directions of strengths and applying the basic formulas of $J_k^{(AB)}$-, $M^{(AB)}$- and $L_3^{(AB)}$-integrals of parallel line forces given by Eqs. (54), (55) and (56). It should be clarified here that the line direction is the z-direction but the strength can be in x-, y- or z-direction.

5.1 Line forces in antiplane strain

We examine first the case of two parallel line forces with strengths $\hat{f}_3^{(A)}$ and $\hat{f}_3^{(B)}$ in z-direction.

If we substitute the two-dimensional Green function of antiplane strain, Eq. (59), into Eqs. (54), (55) and (56), we obtain the $J_k^{(AB)}$-, $M^{(AB)}$- *and* $L_3^{(AB)}$-*integrals of parallel line forces in* z-*direction with strengths* $\hat{f}_3^{(A)}$ *and* $\hat{f}_3^{(B)}$ *for antiplane strain:*

$$J_1^{(AB)}(\bar{x},\bar{y}) = -\frac{\hat{f}_3^{(A)}\hat{f}_3^{(B)}}{2\pi\mu}\,\frac{\bar{x}}{\bar{x}^2+\bar{y}^2} = \mathcal{F}_1^{\mathrm{C}(AB)}\,, \tag{60}$$

$$J_2^{(AB)}(\bar{x},\bar{y}) = -\frac{\hat{f}_3^{(A)}\hat{f}_3^{(B)}}{2\pi\mu}\,\frac{\bar{y}}{\bar{x}^2+\bar{y}^2} = \mathcal{F}_2^{\mathrm{C}(AB)}\,, \tag{61}$$

$$M^{(AB)}(\bar{x},\bar{y}) = \frac{\hat{f}_3^{(A)}\hat{f}_3^{(B)}}{4\pi\mu}\,\ln\frac{\bar{r}}{L} = \frac{1}{2}\,U^{(AB)}\,, \tag{62}$$

$$L_3^{(AB)}(\bar{x},\bar{y}) = 0 = \mathcal{T}_3^{(AB)}\,. \tag{63}$$

Eqs. (60) and (61) give the non-vanishing components of the Cherepanov force $\boldsymbol{\mathcal{F}}^{\mathrm{C}(AB)}$ (per unit length l_z) of the two line forces, which is plotted in Figs. 3(a) and 4(a). It can be seen that the Cherepanov force $\boldsymbol{\mathcal{F}}^{\mathrm{C}(AB)}$

is attractive if $\hat{f}_3^{(A)}\hat{f}_3^{(B)} > 0$ (Fig. 3(a)) and is repulsive if $\hat{f}_3^{(A)}\hat{f}_3^{(B)} < 0$ (Fig. 4(a)). The $M^{(AB)}$-integral, Eq. (62), equals half the elastic interaction energy $U^{(AB)}$ per unit length l_z of the two line forces

$$M^{(AB)} = \frac{1}{2}\, U^{(AB)}\,, \tag{64}$$

where the *elastic interaction energy (per unit length) for line forces in antiplane strain* is given by (see [28])

$$U^{(AB)} = -\hat{f}_3^{(A)} u_3^{(B)} = \frac{\hat{f}_3^{(A)}\hat{f}_3^{(B)}}{2\pi\mu}\, \ln\frac{\bar{r}}{L}\,, \tag{65}$$

with the corresponding displacement field

$$u_3^{(B)} = G_{33}^{(2)}\, \hat{f}_3^{(B)}\,. \tag{66}$$

In Eq. (63), we find that the $L_3^{(AB)}$-integral is zero, which is an expected result since for two parallel line forces with strengths in z-direction no torque $\mathcal{T}_3^{(AB)}$ can be produced.

In analogy to dislocation theory where the pre-logarithmic energy factor is a well-known quantity (see, e.g., [7,8,40]), we introduce here the prefactor of the logarithmic term in the $M^{(AB)}$-integral, Eq. (62), as the *pre-logarithmic energy factor of line forces with strengths in z-direction*:

$$K_{33}^{(AB)} = -\frac{Q_{33}}{4\pi}\, \hat{f}_3^{(A)}\hat{f}_3^{(B)} \qquad \text{with} \qquad Q_{33} = -\frac{1}{\mu}\,. \tag{67}$$

The matrix

$$Q_{jk} = -\frac{1}{4\mu(1-\nu)}\big[(3-4\nu)\delta_{jk} + \tau_j\tau_k\big]\,, \tag{68}$$

in an orthogonal coordinate system basis $(\boldsymbol{m}, \boldsymbol{n}, \boldsymbol{\tau})$ so that the z-axis (and therefore the line of the force) is parallel to $\boldsymbol{\tau} = \boldsymbol{m}\times\boldsymbol{n}$, is well-known in the integral formalism of anisotropic elasticity (see, e.g., [7,8]). In isotropic elasticity, Q_{ij} is a negative definite diagonal matrix and reads as

$$Q_{ij} = \begin{pmatrix} -\dfrac{3-4\nu}{4\mu(1-\nu)} & 0 & 0 \\ 0 & -\dfrac{3-4\nu}{4\mu(1-\nu)} & 0 \\ 0 & 0 & -\dfrac{1}{\mu} \end{pmatrix}. \tag{69}$$

Therefore, Eq. (62) can be rewritten as

$$M^{(AB)} = K_{33}^{(AB)}\, \ln\frac{\bar{r}}{L}\,. \tag{70}$$

In cylindrical coordinates, the two components $J_r^{(AB)}$ and $J_\varphi^{(AB)}$ of the $\boldsymbol{J}^{(AB)}$*-integral for two parallel line forces in z-direction in antiplane strain* read as

$$J_r^{(AB)} = -\frac{\hat{f}_3^{(A)}\hat{f}_3^{(B)}}{2\pi\mu}\,\frac{1}{\bar{r}} = \mathcal{F}_r^{\mathrm{C}(AB)}\,, \tag{71}$$

$$J_\varphi^{(AB)} = 0 = \mathcal{F}_\varphi^{\mathrm{C}(AB)}\,, \tag{72}$$

or in terms of the pre-logarithmic energy factor $K_{33}^{(AB)}$, Eq. (67),

$$J_r^{(AB)} = -2K_{33}^{(AB)}\,\frac{1}{\bar{r}}\,. \tag{73}$$

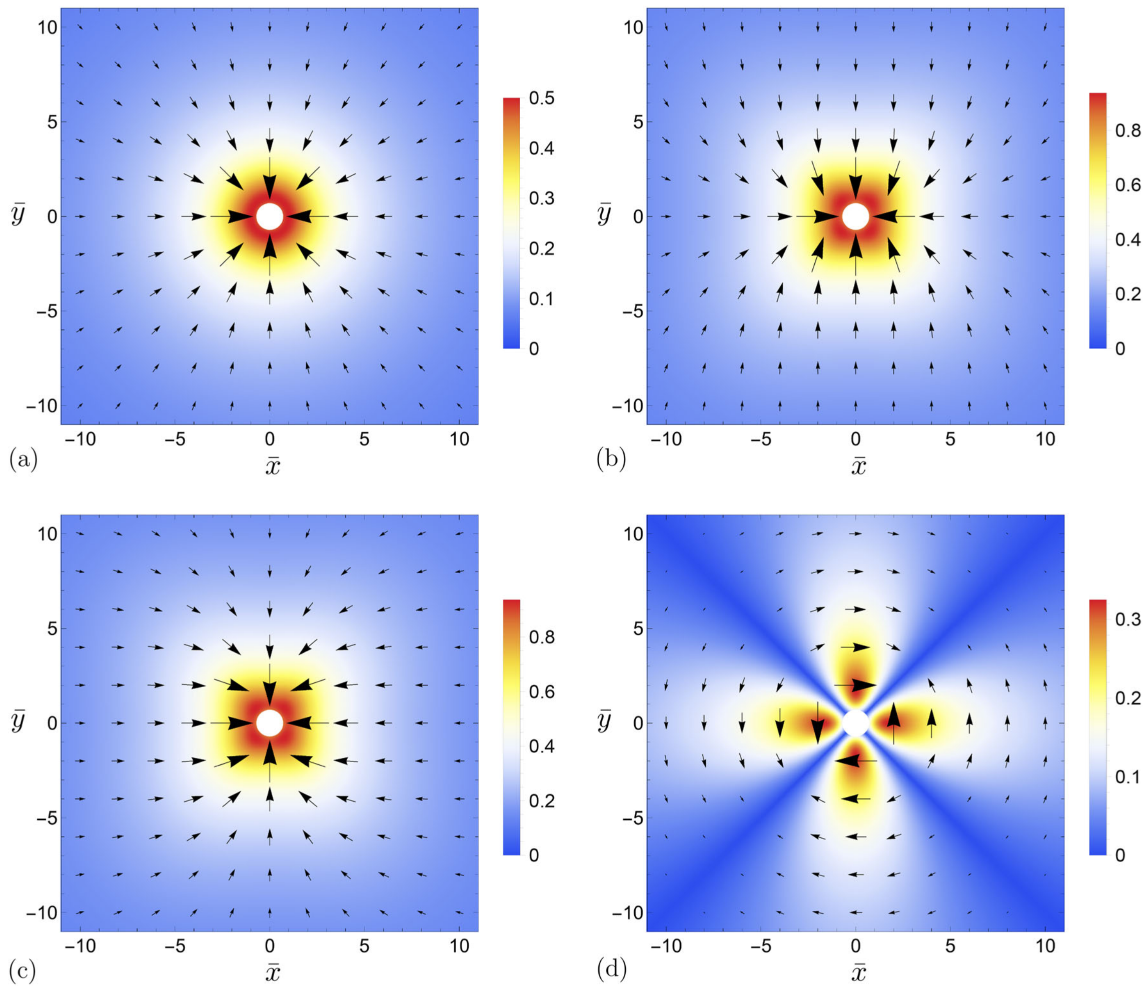

Fig. 3 The Cherepanov force $\boldsymbol{\mathcal{F}}^{\mathrm{C}(AB)}$ calculated by the $\boldsymbol{J}^{(AB)}$-integral $(J_1^{(AB)}(\bar{x}, \bar{y}),\, J_2^{(AB)}(\bar{x}, \bar{y}))$ of two parallel line forces with: (a) strengths in z-direction and $\hat{f}_3^{(A)} \hat{f}_3^{(B)} > 0$ (plotted in units of $\frac{\hat{f}_3^{(A)} \hat{f}_3^{(B)}}{2\pi\mu}$), (b) strengths in x-direction and $\hat{f}_1^{(A)} \hat{f}_1^{(B)} > 0$, (c) strengths in y-direction and $\hat{f}_2^{(A)} \hat{f}_2^{(B)} > 0$, and (d) perpendicular directions of strengths and $\hat{f}_1^{(A)} \hat{f}_2^{(B)} > 0$, plotted in units of $\frac{\hat{f}_i^{(A)} \hat{f}_j^{(B)}}{8\pi\mu(1-\nu)}$ and $\nu = 0.3$

It can be seen in Eqs. (71) and (72) that the Cherepanov force between two parallel line forces in z-direction in antiplane strain is a purely radial force:

$$\boldsymbol{\mathcal{F}}^{\mathrm{C}(AB)} = \mathcal{F}_r^{\mathrm{C}(AB)}(\bar{r})\, \boldsymbol{e}_r\,.$$

This means that in the examined case, the *Cherepanov force is a central force* (in 2D). $\boldsymbol{\mathcal{F}}^{\mathrm{C}(AB)}$ is repulsive if $\mathcal{F}_r^{\mathrm{C}(AB)} > 0$ or $\hat{f}_3^{(A)} \hat{f}_3^{(B)} < 0$, that is, if the forces are antiparallel and $\boldsymbol{\mathcal{F}}^{\mathrm{C}(AB)}$ is attractive if $\mathcal{F}_r^{\mathrm{C}(AB)} < 0$ or $\hat{f}_3^{(A)} \hat{f}_3^{(B)} > 0$, that is, if the forces are parallel. This behaviour is similar to magnetostatics, where the magnetostatic part of the Lorentz force between two parallel wires carrying the currents I_1 and I_2 is attractive if the currents flow in the same direction and repulsive if the currents flow in opposite directions (see, e.g., [21]).

To sum up, the $J_k^{(AB)}$-, $M^{(AB)}$- and $L_3^{(AB)}$-integrals for parallel line forces in antiplane strain are given by

$$J_r^{(AB)} = -\frac{\hat{f}_3^{(A)} \hat{f}_3^{(B)}}{2\pi\mu}\, \frac{1}{\bar{r}} = \mathcal{F}_r^{\mathrm{C}(AB)}\,, \tag{74}$$

$$J_\varphi^{(AB)} = 0 = \mathcal{F}_\varphi^{\mathrm{C}(AB)}\,, \tag{75}$$

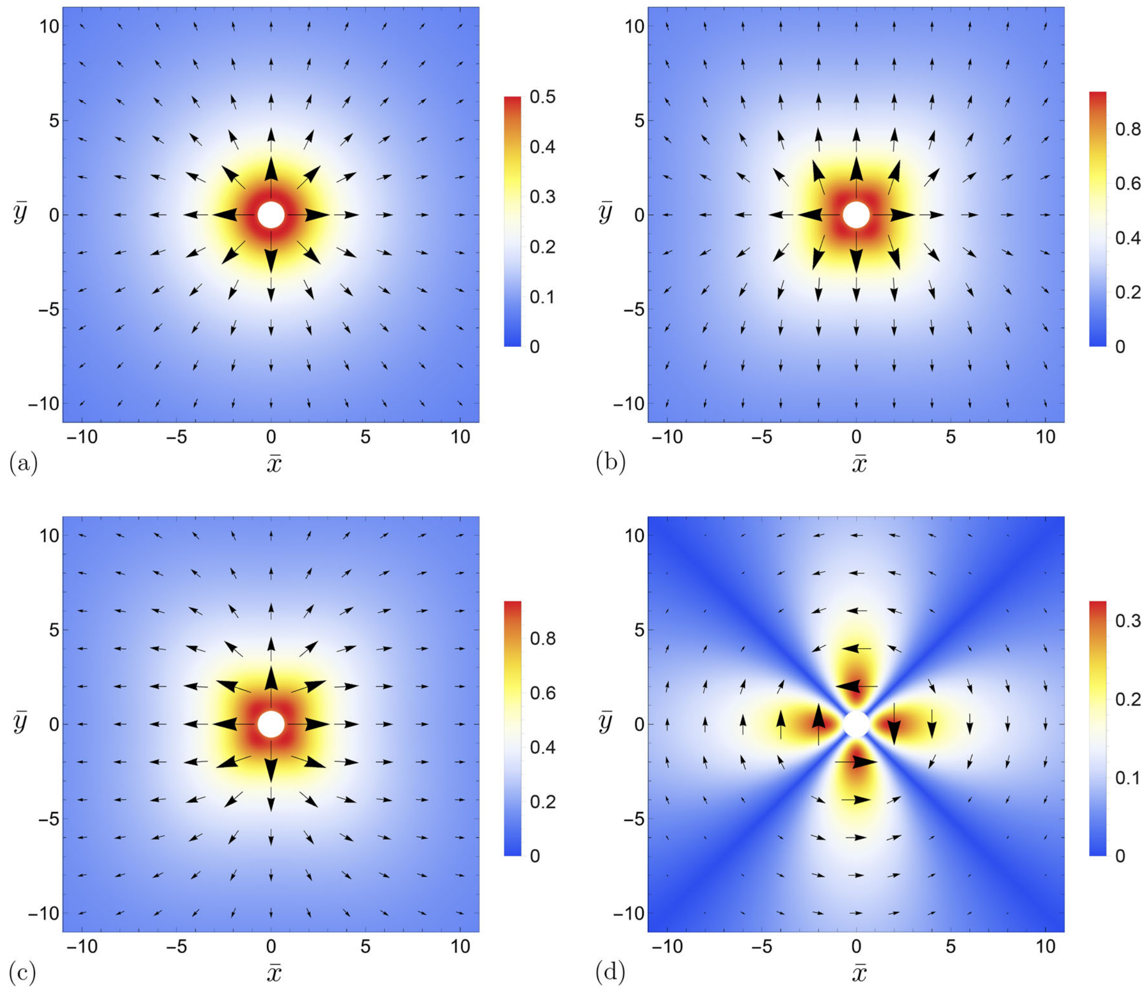

Fig. 4 The Cherepanov force $\boldsymbol{\mathcal{F}}^{C(AB)}$ calculated by the $\boldsymbol{J}^{(AB)}$-integral $(J_1^{(AB)}(\bar{x},\bar{y}),\ J_2^{(AB)}(\bar{x},\bar{y}))$ of two parallel line forces with: (a) strengths in z-direction and $\hat{f}_3^{(A)}\hat{f}_3^{(B)} < 0$ (plotted in units of $\frac{|\hat{f}_3^{(A)}\hat{f}_3^{(B)}|}{2\pi\mu}$), (b) strengths in x-direction and $\hat{f}_1^{(A)}\hat{f}_1^{(B)} < 0$, (c) strengths in y-direction and $\hat{f}_2^{(A)}\hat{f}_2^{(B)} < 0$, and (d) perpendicular directions of strengths and $\hat{f}_1^{(A)}\hat{f}_2^{(B)} < 0$, plotted in units of $\frac{|\hat{f}_i^{(A)}\hat{f}_j^{(B)}|}{8\pi\mu(1-\nu)}$ and $\nu = 0.3$

$$M^{(AB)} = \frac{\hat{f}_3^{(A)}\hat{f}_3^{(B)}}{4\pi\mu}\,\ln\frac{\bar{r}}{L} = \frac{1}{2}\,U^{(AB)}\,, \tag{76}$$

$$L_3^{(AB)} = 0 = \mathcal{T}_3^{(AB)}\,, \tag{77}$$

or in terms of the corresponding pre-logarithmic energy factor $K_{33}^{(AB)}$ read as

$$J_r^{(AB)} = -2K_{33}^{(AB)}\,\frac{1}{\bar{r}}\,, \tag{78}$$

$$J_\varphi^{(AB)} = 0\,, \tag{79}$$

$$M^{(AB)} = K_{33}^{(AB)}\,\ln\frac{\bar{r}}{L}\,, \tag{80}$$

$$L_3^{(AB)} = 0\,. \tag{81}$$

5.2 Line forces in plane strain

We continue with the case of two parallel line forces whose strengths $\hat{f}_i^{(A)}$ and $\hat{f}_j^{(B)}$, $i, j = 1, 2$ acting in the $\bar{x}\bar{y}$-plane have arbitrary directions.

If we substitute the two-dimensional Green tensor function of plane strain, Eq. (58), into Eqs. (54), (55) and (56), we obtain the $J_k^{(AB)}$-, $M^{(AB)}$- *and* $L_3^{(AB)}$*-integrals of parallel line forces in z-direction with strengths* $\hat{f}_i^{(A)}$ *and* $\hat{f}_j^{(B)}$ *in plane strain:*

$$J_k^{(AB)}(\bar{x}, \bar{y}) = -\frac{\hat{f}_i^{(A)}\hat{f}_j^{(B)}}{8\pi\mu(1-\nu)\bar{r}^2}\left[(3-4\nu)\delta_{ij}\bar{x}_k - \delta_{ik}\bar{x}_j - \delta_{jk}\bar{x}_i + 2\,\frac{\bar{x}_i\bar{x}_j\bar{x}_k}{\bar{r}^2}\right] = \mathcal{F}_k^{\mathrm{C}(AB)}\,, \tag{82}$$

$$M^{(AB)}(\bar{x}, \bar{y}) = \frac{\hat{f}_i^{(A)}\hat{f}_j^{(B)}}{16\pi\mu(1-\nu)}\left[(3-4\nu)\delta_{ij}\ln\frac{\bar{r}}{L} - \frac{\bar{x}_i\bar{x}_j}{\bar{r}^2}\right] = \frac{1}{2}\,U^{(AB)}\,, \tag{83}$$

$$L_3^{(AB)}(\bar{x}, \bar{y}) = -\frac{\epsilon_{3ji}\hat{f}_k^{(A)}\hat{f}_i^{(B)}}{8\pi\mu(1-\nu)}\left[(3-4\nu)\delta_{jk}\ln\frac{\bar{r}}{L} - \frac{\bar{x}_j\bar{x}_k}{\bar{r}^2}\right] = \mathcal{T}_3^{(AB)}\,, \tag{84}$$

where $i, j, k = 1, 2$. It is interesting to observe that both $M^{(AB)}$- and $L_3^{(AB)}$-integrals include a logarithmic part $\ln(\bar{r}/L)$, which originates from the two-dimensional Green tensor function, Eq. (58). The $J_k^{(AB)}$-integral represents the Cherepanov force $\boldsymbol{\mathcal{F}}^{\mathrm{C}(AB)}$ between the line forces for plane strain. The *elastic interaction energy (per unit length) of two line forces with strengths* $\hat{f}_i^{(A)}$ *and* $\hat{f}_j^{(B)}$ is given by (see, e.g., [28])

$$U^{(AB)} = -\hat{f}_i^{(A)}u_i^{(B)} = \frac{\hat{f}_i^{(A)}\hat{f}_j^{(B)}}{8\pi\mu(1-\nu)}\left[(3-4\nu)\delta_{ij}\ln\frac{\bar{r}}{L} - \frac{\bar{x}_i\bar{x}_j}{\bar{r}^2}\right], \qquad i, j = 1, 2\,, \tag{85}$$

where the displacement vector of the line force $\hat{f}_j^{(B)}$ reads

$$u_i^{(B)} = G_{ij}^{(2)}\hat{f}_j^{(B)}\,. \tag{86}$$

Therefore, the $M^{(AB)}$-integral in Eq. (83) equals half of the elastic interaction energy (per unit length) of the two line forces

$$M^{(AB)}(\bar{x}, \bar{y}) = \frac{1}{2}\,U^{(AB)}\,. \tag{87}$$

It is interesting to note that the elastic interaction energy of the two line forces, Eq. (85), has a similar form to the elastic interaction energy (per unit length) of two parallel edge dislocations with general Burgers vectors given in [25,36].

Regarding the physical interpretation of the $\boldsymbol{L}$-integral, Agiasofitou and Lazar [3] studying electro-elastic dislocations in piezoelectric materials considering additionally inhomogeneities in presence of body forces and body charges showed that the $\boldsymbol{L}$-integral is the *total configurational* or *material vector moment* representing the *total torque*, since it is able to capture every produced configurational vector moment density: the configurational vector moment density produced by the total configurational force density plus an intrinsic or field vector moment density due to body force density vector and the eigendistortion tensor plus the configurational vector moment density due to the inherent material anisotropy of the piezoelectric materials. Hence, the $L_3^{(AB)}$-integral, Eq. (84), represents the *total torque* $\mathcal{T}_3^{(AB)}$ of two parallel line forces in plane strain and it can be decomposed into its orbital and intrinsic parts as follows[2]

$$L_3^{(AB)}(\bar{x}, \bar{y}) = \mathcal{M}_3^{(AB)}(\bar{x}, \bar{y}) + \mathcal{S}_3^{(AB)}(\bar{x}, \bar{y}) \equiv \mathcal{T}_3^{(AB)}\,. \tag{88}$$

The orbital part, which we call the *orbital torque*, is given as torque produced by the Cherepanov force

$$\mathcal{M}_3^{(AB)}(\bar{x}, \bar{y}) = \epsilon_{3ji}\bar{x}_j J_i^{(AB)}(\bar{x}, \bar{y}) = \epsilon_{3ji}\bar{x}_j\mathcal{F}_i^{\mathrm{C}(AB)}(\bar{x}, \bar{y}) \tag{89}$$

[2] Such a terminology is also used for the torque on an electric dipole in electrostatics (see [16]).

and the intrinsic (or spin) part, which we call the *intrinsic torque*, is due to the displacement vector field $u_i^{(B)}$ (which is a spin 1 field) and the line force $\hat{f}_j^{(A)}$

$$\mathcal{S}_3^{(AB)}(\bar{x},\bar{y}) = \epsilon_{3ji}\hat{f}_j^{(A)}u_i^{(B)}(\bar{x},\bar{y}) = \epsilon_{3ji}\hat{f}_j^{(A)}\hat{f}_k^{(B)}G_{ik}^{(2)}(\bar{x},\bar{y})\,. \tag{90}$$

If we substitute Eq. (82) into Eq. (89), the orbital torque reads as

$$\mathcal{M}_3^{(AB)}(\bar{x},\bar{y}) = \frac{\epsilon_{3ji}\hat{f}_k^{(A)}\hat{f}_l^{(B)}}{8\pi\mu(1-\nu)}\,\frac{\bar{x}_j\big[\delta_{ik}\bar{x}_l + \delta_{il}\bar{x}_k\big]}{\bar{r}^2} \tag{91}$$

and if we substitute Eq. (58) into Eq. (90), the intrinsic torque reads as

$$\mathcal{S}_3^{(AB)}(\bar{x},\bar{y}) = -\frac{\epsilon_{3ji}\hat{f}_j^{(A)}\hat{f}_k^{(B)}}{8\pi\mu(1-\nu)}\left[(3-4\nu)\delta_{ik}\ln\frac{\bar{r}}{L} - \frac{\bar{x}_i\bar{x}_k}{\bar{r}^2}\right]. \tag{92}$$

Remark 1 In order to gain more insight into the ***L***-integral and the above representation, it is interesting to recall its derivation based on Noether's theorem as it has been done in Agiasofitou and Lazar [1] using the prolongation method of Olver [37], where it can be seen that the ***L***-integral arises from the symmetry group of rotations of both, independent and dependent variables in space, giving rise to two kinds of torque, the orbital torque and the intrinsic one, respectively. The intrinsic torque is related to the vectorial nature of the displacement field.

In the following, we examine all possible combinations of the directions of strengths, applying Eqs. (82), (83) and (84) in order to find the explicit expressions of the $J_k^{(AB)}$-, $M^{(AB)}$- and $L_3^{(AB)}$-integrals for plane strain.

5.2.1 Line forces with strengths in x-direction

We consider first the case of two parallel line forces with strengths $\hat{f}_1^{(A)}$ and $\hat{f}_1^{(B)}$ acting in x-axis.

Then the $J_k^{(AB)}$-, $M^{(AB)}$- and $L_3^{(AB)}$-integrals, Eqs. (82), (83) and (84), reduce to

$$J_1^{(AB)}(\bar{x},\bar{y}) = -\frac{\hat{f}_1^{(A)}\hat{f}_1^{(B)}}{8\pi\mu(1-\nu)}\,\frac{\bar{x}}{\bar{x}^2+\bar{y}^2}\left[(3-4\nu) - \frac{2\bar{y}^2}{\bar{x}^2+\bar{y}^2}\right] = \mathcal{F}_1^{\mathrm{C}(AB)}\,, \tag{93}$$

$$J_2^{(AB)}(\bar{x},\bar{y}) = -\frac{\hat{f}_1^{(A)}\hat{f}_1^{(B)}}{8\pi\mu(1-\nu)}\,\frac{\bar{y}}{\bar{x}^2+\bar{y}^2}\left[(3-4\nu) + \frac{2\bar{x}^2}{\bar{x}^2+\bar{y}^2}\right] = \mathcal{F}_2^{\mathrm{C}(AB)}\,, \tag{94}$$

$$M^{(AB)}(\bar{x},\bar{y}) = \frac{\hat{f}_1^{(A)}\hat{f}_1^{(B)}}{16\pi\mu(1-\nu)}\left[(3-4\nu)\ln\frac{\bar{r}}{L} - \frac{\bar{x}^2}{\bar{x}^2+\bar{y}^2}\right] = \frac{1}{2}\,U^{(AB)}\,, \tag{95}$$

$$L_3^{(AB)}(\bar{x},\bar{y}) = -\frac{\hat{f}_1^{(A)}\hat{f}_1^{(B)}}{8\pi\mu(1-\nu)}\,\frac{\bar{x}\bar{y}}{\bar{x}^2+\bar{y}^2} = \mathcal{T}_3^{(AB)}\,. \tag{96}$$

The $J_1^{(AB)}$- and $J_2^{(AB)}$-integrals, Eqs. (93) and (94), give the non-vanishing components of the Cherepanov force $\boldsymbol{\mathcal{F}}^{\mathrm{C}(AB)}$ (per unit length l_z) of the two line forces with strengths in x-direction, which is plotted in Figs. 3(b) and 4(b). It can be seen that the Cherepanov force $\boldsymbol{\mathcal{F}}^{\mathrm{C}(AB)}$ is attractive if $\hat{f}_1^{(A)}\hat{f}_1^{(B)} > 0$ (Fig. 3(b)) and repulsive if $\hat{f}_1^{(A)}\hat{f}_1^{(B)} < 0$ (Fig. 4(b)). The $M^{(AB)}$-integral (95) gives half the interaction energy (per unit length l_z) of the two line forces with strengths in x-direction and is plotted in Figs. 5(a) and 6(a). The $L_3^{(AB)}$-integral (96) gives the total torque about the z-axis, caused by the interaction of the two line forces with strengths in x-direction. The total torque (96) is zero if $\bar{x} = 0$ or $\bar{y} = 0$, which means that if the line force $\hat{f}_1^{(A)}$ is applied in one of the coordinate axis then no torque is produced. Using the decomposition of the total torque (88) into the orbital part (91) and the intrinsic part (92), the orbital torque for the case under study reduces to

$$\mathcal{M}_3^{(AB)}(\bar{x},\bar{y}) = -\frac{\hat{f}_1^{(A)}\hat{f}_1^{(B)}}{4\pi\mu(1-\nu)}\,\frac{\bar{x}\bar{y}}{\bar{x}^2+\bar{y}^2} \tag{97}$$

and the intrinsic torque becomes

$$\mathcal{S}_3^{(AB)}(\bar{x}, \bar{y}) = \frac{\hat{f}_1^{(A)}\hat{f}_1^{(B)}}{8\pi\mu(1-\nu)}\,\frac{\bar{x}\,\bar{y}}{\bar{x}^2+\bar{y}^2}\,. \tag{98}$$

In cylindrical coordinates, the $J_k^{(AB)}$-, $M^{(AB)}$- and $L_3^{(AB)}$-integrals, Eqs. (93)–(96), take the form

$$J_r^{(AB)} = -\frac{\hat{f}_1^{(A)}\hat{f}_1^{(B)}(3-4\nu)}{8\pi\mu(1-\nu)}\,\frac{1}{\bar{r}} = \mathcal{F}_r^{\mathrm{C}(AB)}\,, \tag{99}$$

$$J_\varphi^{(AB)} = -\frac{\hat{f}_1^{(A)}\hat{f}_1^{(B)}}{8\pi\mu(1-\nu)}\,\frac{\sin 2\varphi}{\bar{r}} = \mathcal{F}_\varphi^{\mathrm{C}(AB)}\,, \tag{100}$$

$$M^{(AB)} = \frac{\hat{f}_1^{(A)}\hat{f}_1^{(B)}}{16\pi\mu(1-\nu)}\left[(3-4\nu)\ln\frac{\bar{r}}{L} - \cos^2\varphi\right] = \frac{1}{2}\,U^{(AB)}\,, \tag{101}$$

$$L_3^{(AB)} = -\frac{\hat{f}_1^{(A)}\hat{f}_1^{(B)}}{16\pi\mu(1-\nu)}\,\sin 2\varphi = \mathcal{T}_3^{(AB)}\,. \tag{102}$$

In Eq. (99), it can be seen that the radial force component $\mathcal{F}_r^{\mathrm{C}(AB)}$ is repulsive if $\hat{f}_1^{(A)}\hat{f}_1^{(B)} < 0$, that is $\hat{f}_1^{(A)}$ and $\hat{f}_1^{(B)}$ are antiparallel and is attractive if $\hat{f}_1^{(A)}\hat{f}_1^{(B)} > 0$, that is $\hat{f}_1^{(A)}$ and $\hat{f}_1^{(B)}$ are parallel. Moreover, it is easy to see that due to the non-vanishing tangential force component $\mathcal{F}_\varphi^{\mathrm{C}(AB)}$, Eq. (100), the Cherepanov force between two parallel line forces with strengths in x-direction is not a purely radial force in contrast to the case of antiplane strain. In addition, $\mathcal{F}_\varphi^{\mathrm{C}(AB)} = 0$ if $\varphi = \{0, \pi/2, \pi, 3\pi/2\}$.

The $L_3^{(AB)}$-integral, Eq. (102), measures the total torque about the z-axis produced by the interaction of the line forces $\hat{f}_1^{(A)}$ and $\hat{f}_1^{(B)}$. So as to achieve more insight into the $L_3^{(AB)}$-integral, using its decomposition to orbital and spin parts, it can be seen that

$$\begin{aligned} L_3^{(AB)} &= \mathcal{M}_3^{(AB)} + \mathcal{S}_3^{(AB)} \\ &= \bar{r}\,J_\varphi^{(AB)} - \frac{1}{2}\,\bar{r}\,J_\varphi^{(AB)} = \frac{1}{2}\,\bar{r}\,J_\varphi^{(AB)} = \frac{1}{2}\,\bar{r}\mathcal{F}_\varphi^{\mathrm{C}(AB)}\,. \end{aligned} \tag{103}$$

Therefore, it turns out that the $L_3^{(AB)}$-integral of two parallel line forces with strengths in x-direction provides the total torque, which is produced finally only by the tangential Cherepanov force $\mathcal{F}_\varphi^{\mathrm{C}(AB)}$, and the rotational direction of the line force $\hat{f}_1^{(A)}$ with respect to $\hat{f}_1^{(B)}$ is counter-clockwise if $L_3^{(AB)} > 0$ and clockwise if $L_3^{(AB)} < 0$. This means that the tangential component $\mathcal{F}_\varphi^{\mathrm{C}(AB)}$ of the Cherepanov force tends to rotate the line force $\hat{f}_1^{(A)}$ with respect to the line force $\hat{f}_1^{(B)}$ at distance $\bar{r}$ counter-clockwise if $\mathcal{F}_\varphi^{\mathrm{C}(AB)} > 0$ and clockwise if $\mathcal{F}_\varphi^{\mathrm{C}(AB)} < 0$. Note that $\mathcal{M}_3^{(AB)} = \mathcal{M}_3^{(AB)}(\varphi)$ and $\mathcal{S}_3^{(AB)} = \mathcal{S}_3^{(AB)}(\varphi)$ are functions depending only on the angle φ. Due to the relation (103), $L_3^{(AB)} = 0$ has the same roots as $J_\varphi^{(AB)} = 0$ with respect to φ. That is, $L_3^{(AB)} = 0$ if $\varphi = \{0, \pi/2, \pi, 3\pi/2\}$, which means that there is no torque at these positions as one can also see in Figs. 5(b) and 6(b).

The prefactor of the logarithmic term in the $M^{(AB)}$-integral, Eq. (101), is the *pre-logarithmic energy factor of line forces with strengths in x-direction*:

$$K_{11}^{(AB)} = -\frac{Q_{11}}{4\pi}\,\hat{f}_1^{(A)}\hat{f}_1^{(B)} \qquad \text{with} \qquad Q_{11} = -\frac{3-4\nu}{4\mu(1-\nu)}\,. \tag{104}$$

Eqs. (99)–(102) can be rewritten in terms of the pre-logarithmic energy factor $K_{11}^{(AB)}$, Eq. (104), as follows

$$J_r^{(AB)} = -2K_{11}^{(AB)}\,\frac{1}{\bar{r}}\,, \tag{105}$$

$$J_\varphi^{(AB)} = -\frac{2K_{11}^{(AB)}}{3-4\nu}\,\frac{\sin 2\varphi}{\bar{r}}\,, \tag{106}$$

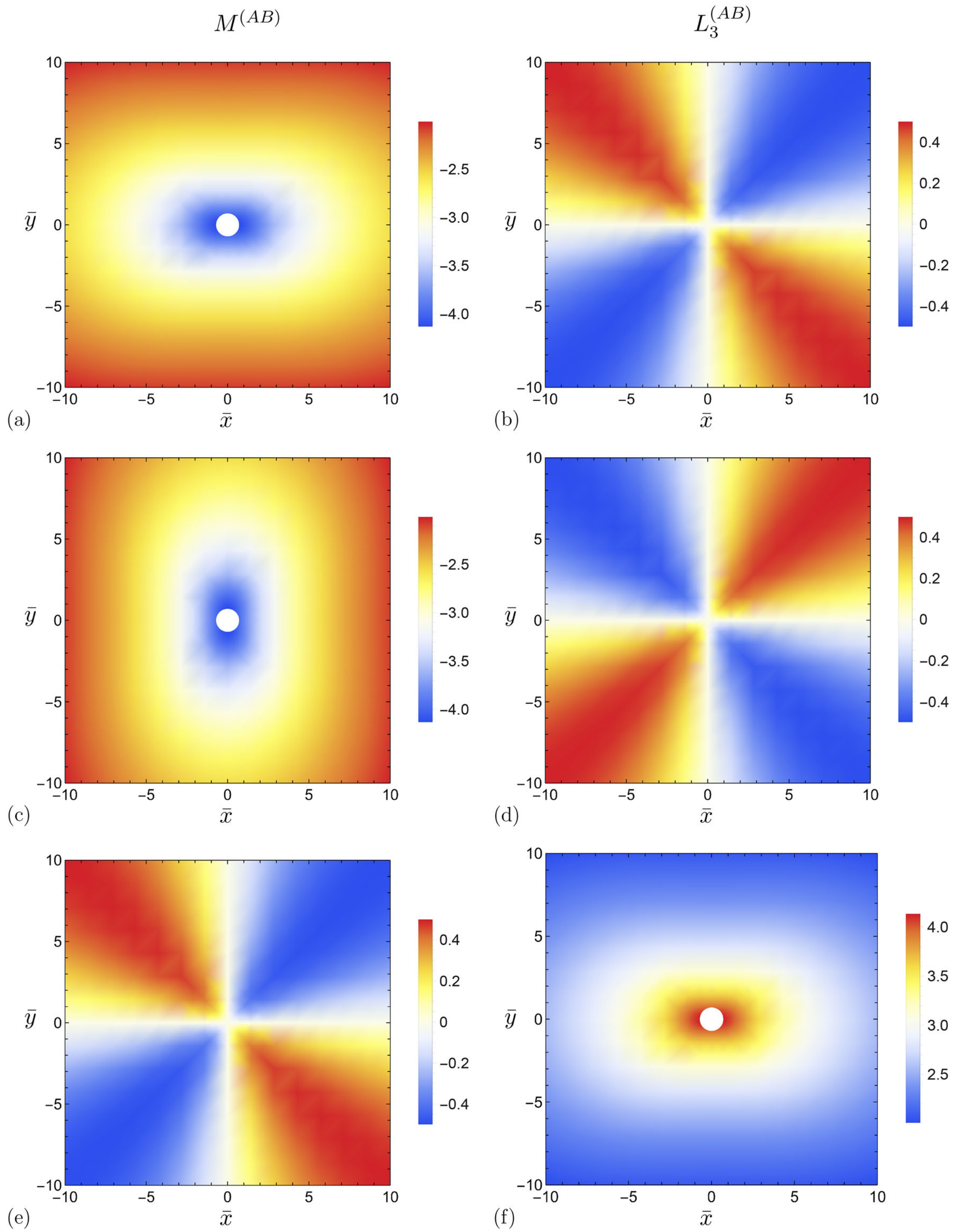

Fig. 5 The $M^{(AB)}$ and $L_3^{(AB)}$-integrals of two parallel line forces with: (a) and (b) strengths in x-direction and $\hat{f}_1^{(A)}\hat{f}_1^{(B)} > 0$; (c) and (d) strengths in y-direction and $\hat{f}_2^{(A)}\hat{f}_2^{(B)} > 0$; (e) and (f) perpendicular directions of strengths and $\hat{f}_1^{(A)}\hat{f}_2^{(B)} > 0$, respectively, plotted in units of $\frac{\hat{f}_i^{(A)}\hat{f}_j^{(B)}}{8\pi\mu(1-\nu)}$ and $\nu = 0.3$, $L = 100$

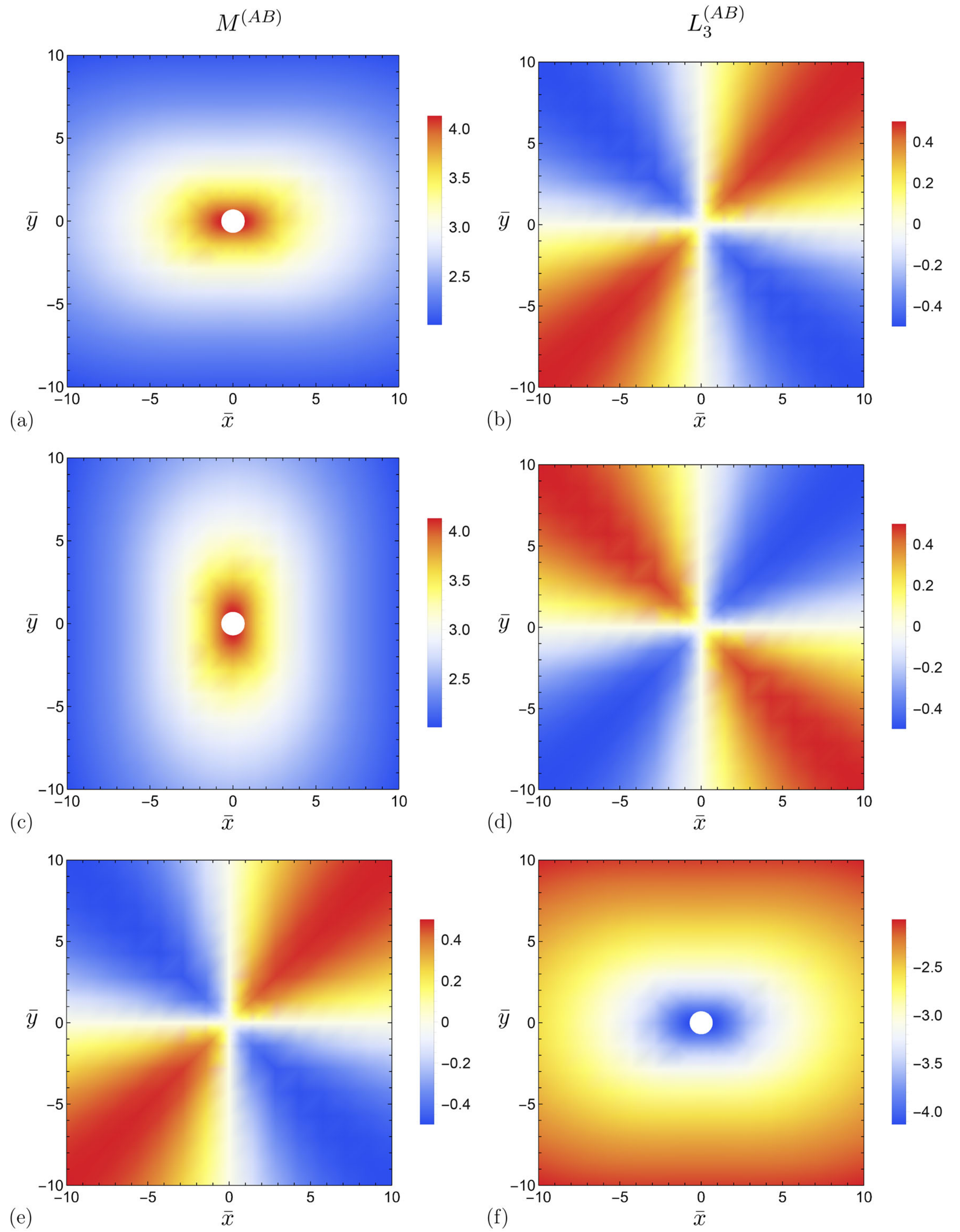

Fig. 6 The $M^{(AB)}$- and $L_3^{(AB)}$-integrals of two parallel line forces with: (a) and (b) strengths in x-direction and $\hat{f}_1^{(A)}\hat{f}_1^{(B)} < 0$; (c) and (d) strengths in y-direction and $\hat{f}_2^{(A)}\hat{f}_2^{(B)} < 0$; (e) and (f) perpendicular directions of strengths and $\hat{f}_1^{(A)}\hat{f}_2^{(B)} < 0$, respectively, plotted in units of $\frac{|\hat{f}_i^{(A)}\hat{f}_j^{(B)}|}{8\pi\mu(1-\nu)}$ and $\nu = 0.3$, $L = 100$

$$M^{(AB)} = K_{11}^{(AB)}\left[\ln\frac{\bar r}{L} - \frac{1}{3-4\nu}\cos^2\varphi\right], \tag{107}$$

$$L_3^{(AB)} = -\frac{K_{11}^{(AB)}}{3-4\nu}\,\sin 2\varphi\,. \tag{108}$$

5.2.2 Line forces with strengths in y-direction

The next case that has to be examined is the case of two parallel line forces with strengths $\hat f_2^{(A)}$ and $\hat f_2^{(B)}$ acting in y-direction.

The $J_k^{(AB)}$-, $M^{(AB)}$- and $L_3^{(AB)}$-integrals, Eqs. (82), (83) and (84), reduce to

$$J_1^{(AB)}(\bar x,\bar y) = -\frac{\hat f_2^{(A)}\hat f_2^{(B)}}{8\pi\mu(1-\nu)}\,\frac{\bar x}{\bar x^2+\bar y^2}\left[(3-4\nu)+\frac{2\bar y^2}{\bar x^2+\bar y^2}\right] = \mathcal{F}_1^{\mathrm{C}(AB)}\,, \tag{109}$$

$$J_2^{(AB)}(\bar x,\bar y) = -\frac{\hat f_2^{(A)}\hat f_2^{(B)}}{8\pi\mu(1-\nu)}\,\frac{\bar y}{\bar x^2+\bar y^2}\left[(3-4\nu)-\frac{2\bar x^2}{\bar x^2+\bar y^2}\right] = \mathcal{F}_2^{\mathrm{C}(AB)}\,, \tag{110}$$

$$M^{(AB)}(\bar x,\bar y) = \frac{\hat f_2^{(A)}\hat f_2^{(B)}}{16\pi\mu(1-\nu)}\left[(3-4\nu)\ln\frac{\bar r}{L}-\frac{\bar y^2}{\bar x^2+\bar y^2}\right] = \frac{1}{2}\,U^{(AB)}\,, \tag{111}$$

$$L_3^{(AB)}(\bar x,\bar y) = \frac{\hat f_2^{(A)}\hat f_2^{(B)}}{8\pi\mu(1-\nu)}\,\frac{\bar x\bar y}{\bar x^2+\bar y^2} = \mathcal{T}_3^{(AB)}\,. \tag{112}$$

The $J_1^{(AB)}$- and $J_2^{(AB)}$-integrals, Eqs. (109) and (110), give the non-vanishing components of the Cherepanov force $\boldsymbol{\mathcal{F}}^{\mathrm{C}(AB)}$ per unit length l_z of the two line forces with strengths in y-direction, which is plotted in Figs. 3(c) and 4(c). It can be seen that the Cherepanov force $\boldsymbol{\mathcal{F}}^{\mathrm{C}(AB)}$ is attractive if $\hat f_2^{(A)}\hat f_2^{(B)} > 0$ (Fig. 3(c)) and is repulsive if $\hat f_2^{(A)}\hat f_2^{(B)} < 0$ (Fig. 4(c)). The $M^{(AB)}$-integral (111) gives half the elastic interaction energy per unit length l_z of the two line forces with strengths in y-direction and is plotted in Figs. 5(c) and 6(c). The $L_3^{(AB)}$-integral, Eq. (112), represents the total torque about the z-axis caused by the interaction of the two line forces with strengths in y-direction. The total torque (112) is zero if $\bar x = 0$ or $\bar y = 0$, that is if the line force $\hat f_2^{(A)}$ is applied in one of the coordinate axis. Using the decomposition of the total torque (88) into the orbital part (91) and the intrinsic part (92), the orbital torque reduces to

$$\mathcal{M}_3^{(AB)}(\bar x,\bar y) = \frac{\hat f_2^{(A)}\hat f_2^{(B)}}{4\pi\mu(1-\nu)}\,\frac{\bar x\bar y}{\bar x^2+\bar y^2} \tag{113}$$

and the intrinsic torque becomes

$$\mathcal{S}_3^{(AB)}(\bar x,\bar y) = -\frac{\hat f_2^{(A)}\hat f_2^{(B)}}{8\pi\mu(1-\nu)}\,\frac{\bar x\bar y}{\bar x^2+\bar y^2}\,. \tag{114}$$

In cylindrical coordinates, the $J_k^{(AB)}$-, $M^{(AB)}$- and $L_3^{(AB)}$-integrals, Eqs. (109)–(112), take the form

$$J_r^{(AB)} = -\frac{\hat f_2^{(A)}\hat f_2^{(B)}(3-4\nu)}{8\pi\mu(1-\nu)}\,\frac{1}{\bar r} = \mathcal{F}_r^{\mathrm{C}(AB)}\,, \tag{115}$$

$$J_\varphi^{(AB)} = \frac{\hat f_2^{(A)}\hat f_2^{(B)}}{8\pi\mu(1-\nu)}\,\frac{\sin 2\varphi}{\bar r} = \mathcal{F}_\varphi^{\mathrm{C}(AB)}\,, \tag{116}$$

$$M^{(AB)} = \frac{\hat f_2^{(A)}\hat f_2^{(B)}}{16\pi\mu(1-\nu)}\left[(3-4\nu)\ln\frac{\bar r}{L}-\sin^2\varphi\right] = \frac{1}{2}\,U^{(AB)}\,, \tag{117}$$

$$L_3^{(AB)} = \frac{\hat f_2^{(A)}\hat f_2^{(B)}}{16\pi\mu(1-\nu)}\,\sin 2\varphi = \mathcal{T}_3^{(AB)}\,. \tag{118}$$

It is easy to see that the Cherepanov force between two line forces with strengths in y-direction is not a purely radial force due to the non-vanishing component $\mathcal{F}_\varphi^{C(AB)}$, Eq. (116). It can be seen in Eq. (115) that the radial force component $\mathcal{F}_r^{C(AB)}$ is repulsive if $\hat{f}_2^{(A)}\hat{f}_2^{(B)} < 0$ and is attractive if $\hat{f}_2^{(A)}\hat{f}_2^{(B)} > 0$. It holds, $\mathcal{F}_\varphi^{C(AB)} = 0$ if $\varphi = \{0, \pi/2, \pi, 3\pi/2\}$.

The $L_3^{(AB)}$-integral measures the total torque produced by the interaction of the line forces $\hat{f}_2^{(A)}$ and $\hat{f}_2^{(B)}$. It can be easily checked that the relation (103) also holds for parallel line forces with strengths in y-direction. Indeed,

$$\mathcal{M}_3^{(AB)} = \bar{r}\, J_\varphi^{(AB)}\,, \qquad \mathcal{S}_3^{(AB)} = -\frac{1}{2}\,\bar{r}\, J_\varphi^{(AB)} \tag{119}$$

and finally

$$L_3^{(AB)} = \frac{1}{2}\,\bar{r}\, J_\varphi^{(AB)} = \frac{1}{2}\,\bar{r}\,\mathcal{F}_\varphi^{C(AB)}\,. \tag{120}$$

Therefore, the $L_3^{(AB)}$-integral for two parallel line forces with strengths in y-direction provides the total torque and it is produced only by the tangential Cherepanov force $\mathcal{F}_\varphi^{C(AB)}$. The rotational direction of the line force $\hat{f}_2^{(A)}$ with respect to $\hat{f}_2^{(B)}$ is counter-clockwise if $L_3^{(AB)} > 0$ and clockwise if $L_3^{(AB)} < 0$. Hence, the tangential component $\mathcal{F}_\varphi^{C(AB)}$ of the Cherepanov force, Eq. (116), tends to rotate the line force $\hat{f}_2^{(A)}$ with respect to the line force $\hat{f}_2^{(B)}$ at distance $\bar{r}$ counter-clockwise if $\mathcal{F}_\varphi^{C(AB)} > 0$ and clockwise if $\mathcal{F}_\varphi^{C(AB)} < 0$. The positions with vanishing torque $L_3^{(AB)} = 0$ are $\varphi = \{0, \pi/2, \pi, 3\pi/2\}$ as one can see in the Figs. 5(d) and 6(d).

The prefactor of the logarithmic term in the $M^{(AB)}$-integral, Eq. (117), is the *pre-logarithmic energy factor of line forces with strengths in y-direction*:

$$K_{22}^{(AB)} = -\frac{Q_{22}}{4\pi}\,\hat{f}_2^{(A)}\hat{f}_2^{(B)} \qquad \text{with} \qquad Q_{22} = -\frac{3-4\nu}{4\mu(1-\nu)}\,. \tag{121}$$

Equations (115)–(118) in terms of the pre-logarithmic energy factor $K_{22}^{(AB)}$ are rewritten as

$$J_r^{(AB)} = -2K_{22}^{(AB)}\,\frac{1}{\bar{r}}\,, \tag{122}$$

$$J_\varphi^{(AB)} = \frac{2K_{22}^{(AB)}}{3-4\nu}\,\frac{\sin 2\varphi}{\bar{r}}\,, \tag{123}$$

$$M^{(AB)} = K_{22}^{(AB)}\left[\ln\frac{\bar{r}}{L} - \frac{1}{3-4\nu}\sin^2\varphi\right], \tag{124}$$

$$L_3^{(AB)} = \frac{K_{22}^{(AB)}}{3-4\nu}\,\sin 2\varphi\,. \tag{125}$$

5.2.3 Line forces with perpendicular strengths

The last case is that of two parallel line forces with strengths $\hat{f}_1^{(A)}$ acting in x-direction and $\hat{f}_2^{(B)}$ acting in y-direction, which presents special features due to the particular geometry.

The $J_k^{(AB)}$-, $M^{(AB)}$- and $L_3^{(AB)}$-integrals, Eqs. (82), (83) and (84), become

$$J_1^{(AB)}(\bar{x},\bar{y}) = \frac{\hat{f}_1^{(A)}\hat{f}_2^{(B)}}{8\pi\mu(1-\nu)}\,\frac{\bar{y}(\bar{y}^2-\bar{x}^2)}{(\bar{x}^2+\bar{y}^2)^2} = \mathcal{F}_1^{C(AB)}\,, \tag{126}$$

$$J_2^{(AB)}(\bar{x},\bar{y}) = \frac{\hat{f}_1^{(A)}\hat{f}_2^{(B)}}{8\pi\mu(1-\nu)}\,\frac{\bar{x}(\bar{x}^2-\bar{y}^2)}{(\bar{x}^2+\bar{y}^2)^2} = \mathcal{F}_2^{C(AB)}\,, \tag{127}$$

$$M^{(AB)}(\bar{x},\bar{y}) = -\frac{\hat{f}_1^{(A)}\hat{f}_2^{(B)}}{16\pi\mu(1-\nu)}\,\frac{\bar{x}\bar{y}}{\bar{x}^2+\bar{y}^2} = \frac{1}{2}\,U^{(AB)}\,, \tag{128}$$

$$L_3^{(AB)}(\bar{x},\bar{y}) = -\frac{\hat{f}_1^{(A)}\hat{f}_2^{(B)}}{8\pi\mu(1-\nu)}\left[(3-4\nu)\ln\frac{\bar{r}}{L} - \frac{\bar{x}^2}{\bar{x}^2+\bar{y}^2}\right] = \mathcal{T}_3^{(AB)}. \tag{129}$$

The $J_1^{(AB)}$- and $J_2^{(AB)}$-integrals, Eqs. (126) and (127), give the non-vanishing components of the Cherepanov force per unit length l_z of the two line forces, which is plotted in Figs. 3(d) and 4(d). The Cherepanov force $\boldsymbol{\mathcal{F}}^{C(AB)}$ is zero at $\bar{x} = \pm|\bar{y}|$. The $M^{(AB)}$-integral, Eq. (128), is half the elastic interaction energy (per unit length l_z) of the two line forces with perpendicular strengths. $M^{(AB)}$ is zero if $\bar{x} = 0$ or $\bar{y} = 0$, that is the $M^{(AB)}$-integral or the elastic interaction energy vanishes if the line force with strength $\hat{f}_1^{(A)}$ applies on the x- or y-axis, as one can also see in Figs. 5(e) and 6(e). The $L_3^{(AB)}$-integral, Eq. (129), represents the total torque about the z-axis caused by the interaction of the two line forces. Using the decomposition of the total torque (88) into its orbital part (91) and its intrinsic part (92), the orbital torque reduces to

$$\mathcal{M}_3^{(AB)}(\bar{x},\bar{y}) = \frac{\hat{f}_1^{(A)}\hat{f}_2^{(B)}}{8\pi\mu(1-\nu)}\,\frac{\bar{x}^2-\bar{y}^2}{\bar{x}^2+\bar{y}^2} \tag{130}$$

and the intrinsic torque becomes

$$\mathcal{S}_3^{(AB)}(\bar{x},\bar{y}) = -\frac{\hat{f}_1^{(A)}\hat{f}_2^{(B)}}{8\pi\mu(1-\nu)}\left[(3-4\nu)\ln\frac{\bar{r}}{L} - \frac{\bar{y}^2}{\bar{x}^2+\bar{y}^2}\right]. \tag{131}$$

In cylindrical coordinates, the $J_k^{(AB)}$-, $M^{(AB)}$- and $L_3^{(AB)}$-integrals, Eqs. (126)–(129), reduce to

$$J_r^{(AB)} = 0 = \mathcal{F}_r^{C(AB)}, \tag{132}$$

$$J_\varphi^{(AB)} = \frac{\hat{f}_1^{(A)}\hat{f}_2^{(B)}}{8\pi\mu(1-\nu)}\,\frac{\cos 2\varphi}{\bar{r}} = \mathcal{F}_\varphi^{C(AB)}, \tag{133}$$

$$M^{(AB)} = -\frac{\hat{f}_1^{(A)}\hat{f}_2^{(B)}}{32\pi\mu(1-\nu)}\,\sin 2\varphi = \frac{1}{2}\,U^{(AB)}, \tag{134}$$

$$L_3^{(AB)} = -\frac{\hat{f}_1^{(A)}\hat{f}_2^{(B)}}{8\pi\mu(1-\nu)}\left[(3-4\nu)\ln\frac{\bar{r}}{L} - \cos^2\varphi\right] = \mathcal{T}_3^{(AB)}. \tag{135}$$

It can be seen in Eq. (132), that this is the only case where the radial Cherepanov force component $\mathcal{F}_r^{C(AB)}$ vanishes and this is a consequence of the fact that the $M^{(AB)}$-integral depends only on the angle φ and not on the distance $\bar{r}$. This connection will become clear in the next Section, where the fundamental relations between the $\boldsymbol{J}^{(AB)}$- and $M^{(AB)}$-integrals are given. The force equilibrium positions can be obtained by taking $\mathcal{F}_\varphi^{C(AB)} = 0$, which holds for $\varphi = \{\pi/4, 3\pi/4, 5\pi/4, 7\pi/4\}$. The equilibrium is considered as stable when the elastic interaction energy $U^{(AB)}$ or equivalently the $M^{(AB)}$-integral possesses a minimum at the force equilibrium positions and unstable when it possesses a maximum. It is easy to check that for $\hat{f}_1^{(A)}\hat{f}_2^{(B)} > 0$, the $M^{(AB)}$-integral has a minimum at $\varphi = \{\pi/4, 5\pi/4\}$ and a maximum at $\varphi = \{3\pi/4, 7\pi/4\}$ (see Fig. 5(e)) and for $\hat{f}_1^{(A)}\hat{f}_2^{(B)} < 0$, the $M^{(AB)}$-integral has a minimum at $\varphi = \{3\pi/4, 7\pi/4\}$ and a maximum at $\varphi = \{\pi/4, 5\pi/4\}$ (see Fig. 6(e)) corresponding to the stable and unstable equilibrium positions, respectively. In other words, the positions $\varphi = \{\pi/4, 5\pi/4\}$ are the stable force equilibrium positions for $\hat{f}_1^{(A)}\hat{f}_2^{(B)} > 0$ (see also Fig. 3(d)) and the positions $\varphi = \{3\pi/4, 7\pi/4\}$ are the stable force equilibrium positions for $\hat{f}_1^{(A)}\hat{f}_2^{(B)} < 0$ (see also Fig. 4(d)).[3]

On the other hand, the $M^{(AB)}$-integral is just a function of $\sin 2\varphi$ and it no longer contains a logarithmic part as one can see in Eq. (134). Moreover, it is remarkable that the case of line forces with perpendicular strengths is the only case where the $M^{(AB)}$-integral representing the elastic interaction energy does not include a logarithmic part and for that reason there is no pre-logarithmic energy factor, $K_{12}^{(AB)} = 0$. The latter is in accordance with the formula (137) that we will see in the next subsection, due to the fact that $Q_{12} = 0$. Finally, in the case of line forces with perpendicular strengths the biggest contribution is given by the $L_3^{(AB)}$-integral,

[3] Note that in these positions the orbital torque $\mathcal{M}_3^{(AB)} = \bar{r}\mathcal{F}_\varphi^{C(AB)}$ produced by the tangential Cherepanov force $\mathcal{F}_\varphi^{C(AB)}$ is also zero (see also Eq. (136)).

Eq. (135), and this is reasonable since due to the geometry of the line forces, the produced total torque is significant.

Furthermore, it is interesting to decompose the $L_3^{(AB)}$-integral in its orbital and intrinsic parts as follows

$$\begin{aligned} L_3^{(AB)} &= \mathcal{M}_3^{(AB)} + \mathcal{S}_3^{(AB)} \\ &= \bar{r}\, J_\varphi^{(AB)} + \Big(-\frac{\hat{f}_1^{(A)}\, \hat{f}_2^{(B)}}{8\pi\mu(1-\nu)}\left[(3-4\nu)\ln\frac{\bar{r}}{L} - \frac{1}{2}\right] - \frac{1}{2}\,\bar{r}\, J_\varphi^{(AB)}\Big) \\ &= \frac{1}{2}\,\bar{r}\, J_\varphi^{(AB)} - \frac{\hat{f}_1^{(A)}\, \hat{f}_2^{(B)}}{8\pi\mu(1-\nu)}\left[(3-4\nu)\ln\frac{\bar{r}}{L} - \frac{1}{2}\right] \\ &= \frac{1}{2}\,\bar{r}\, \mathcal{F}_\varphi^{\mathrm{C}(AB)} - \frac{\hat{f}_1^{(A)}\, \hat{f}_2^{(B)}}{8\pi\mu(1-\nu)}\left[(3-4\nu)\ln\frac{\bar{r}}{L} - \frac{1}{2}\right]. \end{aligned} \tag{136}$$

In Eq. (136), it can be seen that the $L_3^{(AB)}$-integral of two parallel line forces with perpendicular strengths provides the total torque, which is not only produced by the tangential Cherepanov force $\mathcal{F}_\varphi^{\mathrm{C}(AB)}$ as in the previous cases, since additional terms even of logarithmic type appear. Note that in this case, $\mathcal{M}_3^{(AB)} = \mathcal{M}_3^{(AB)}(\varphi)$ but $\mathcal{S}_3^{(AB)} = \mathcal{S}_3^{(AB)}(\bar{r}, \varphi)$ depends not only on φ but also on $\bar{r}$.

Remark 2 It can be seen that the $J_r^{(AB)}$-, $J_\varphi^{(AB)}$-, $M^{(AB)}$- and $L_3^{(AB)}$-integrals of line forces in antiplane and in all cases of plane strain are functions of the radius $\bar{r}$ and angle φ capturing the interaction between the two line forces. It is significant to emphasize this result concerning the M-integral, since it is in contrast to the results and statements of Seo et al. [41], where the M-integral is a constant and not a function of $\bar{r}$ and φ and consequently "do not recognize any elastic field produced by other interacting defects", which is not true as it is shown in this paper (see also Appendix A).

Remark 3 It should be emphasized once again that the physical interpretation of the M-integral is that of the electrostatic interaction energy in electrostatics and the elastic interaction energy in elasticity, therefore the M-integral is in general (interaction) energy and it is misleading to attach the physical interpretation of the "configurational force" or "self-similar expansion force" to the M-integral (see, e.g., [41]).

5.3 Line forces with parallel/antiparallel strengths in x-, y- and z-direction in terms of the pre-logarithmic energy factors

In this Section, we write the $J_k^{(AB)}$-, $M^{(AB)}$- and $L_3^{(AB)}$-integrals of line forces with parallel or antiparallel strengths in terms of the corresponding pre-logarithmic energy factor in order to see the functional dependence on $\bar{r}$ and φ, in which the examined integrals obey.

Comparing the pre-logarithmic energy factors, Eqs. (67), (104) and (121), we see that they can be written as[4]

$$K_{ij}^{(AB)} = -\frac{Q_{ij}}{4\pi}\,\hat{f}_i^{(A)}\,\hat{f}_j^{(B)}\,, \quad i = j = 1, 2, 3 \tag{137}$$

with

$$Q_{11} = Q_{22} = -\frac{3-4\nu}{4\mu(1-\nu)}\quad,\quad Q_{33} = -\frac{1}{\mu}\,,\quad \text{other}\quad Q_{ij} = 0\,.$$

Using the above definition and comparing the expressions of $J_r^{(AB)}$-, $J_\varphi^{(AB)}$-, $M^{(AB)}$- and $L_3^{(AB)}$-integrals for parallel line forces in antiplane and plane strain with strengths directed in the same axis, that is with parallel or antiparallel strengths, written in terms of the pre-logarithmic energy factors $K_{ij}^{(AB)}$, $i, j = 1, 2, 3$, which have been collected in Table 1, we observe that the examined integrals present the following functional dependence on $\bar{r}$ and φ:

$$J_r^{(AB)} = K_{ii}^{(AB)}\, f\Big(\frac{1}{\bar{r}}\Big)\,, \qquad\qquad i = 1, 2, 3 \tag{138}$$

[4] No summation convention is assumed in Eqs. (137)–(142).

Table 1 Functional dependence of the $J_k^{(AB)}$-, $M^{(AB)}$- and $L_3^{(AB)}$-integrals of line forces with parallel/antiparallel strengths on the distance $\bar{r}$ and angle φ

Line forces	$J_r^{(AB)}$	$J_\varphi^{(AB)}$	$M^{(AB)}$	$L_3^{(AB)}$
Antiplane strain				
$\hat{f}_3^{(A)},\ \hat{f}_3^{(B)}$	$-2K_{33}^{(AB)}\frac{1}{\bar{r}}$	0	$K_{33}^{(AB)}\ \ln\frac{\bar{r}}{L}$	0
Plane strain				
$\hat{f}_1^{(A)},\ \hat{f}_1^{(B)}$	$-2K_{11}^{(AB)}\frac{1}{\bar{r}}$	$-\frac{2K_{11}^{(AB)}}{3-4\nu}\frac{\sin 2\varphi}{\bar{r}}$	$K_{11}^{(AB)}\left[\ln\frac{\bar{r}}{L}-\frac{1}{3-4\nu}\cos^2\varphi\right]$	$-\frac{K_{11}^{(AB)}}{3-4\nu}\sin 2\varphi$
$\hat{f}_2^{(A)},\ \hat{f}_2^{(B)}$	$-2K_{22}^{(AB)}\frac{1}{\bar{r}}$	$\frac{2K_{22}^{(AB)}}{3-4\nu}\frac{\sin 2\varphi}{\bar{r}}$	$K_{22}^{(AB)}\left[\ln\frac{\bar{r}}{L}-\frac{1}{3-4\nu}\sin^2\varphi\right]$	$\frac{K_{22}^{(AB)}}{3-4\nu}\sin 2\varphi$
$\hat{f}_i^{(A)},\ \hat{f}_i^{(B)}$	$K_{ii}^{(AB)}\ f\left(\frac{1}{\bar{r}}\right)$	$K_{ii}^{(AB)}\ f\left(\frac{\sin 2\varphi}{\bar{r}}\right)$	$K_{ii}^{(AB)}\ f\left(\ln\frac{\bar{r}}{L},\ \sin^2\varphi\right)$	$K_{ii}^{(AB)}\ f\left(\sin 2\varphi\right)$

$$J_\varphi^{(AB)} = K_{ii}^{(AB)}\ f\left(\frac{\sin 2\varphi}{\bar{r}}\right), \qquad i = 1, 2 \tag{139}$$

$$M^{(AB)} = K_{33}^{(AB)}\ \ln\frac{\bar{r}}{L}, \tag{140}$$

$$M^{(AB)} = K_{ii}^{(AB)}\ f\left(\ln\frac{\bar{r}}{L},\ \sin^2\varphi\right), \qquad i = 1, 2 \tag{141}$$

$$L_3^{(AB)} = K_{ii}^{(AB)}\ f\left(\sin 2\varphi\right), \qquad i = 1, 2\,. \tag{142}$$

Remark 4 An exception from the above scheme is the case of parallel line forces with perpendicular strengths where the dependence of the integrals as logarithmic or trigonometric functions changes and it does not follow the above scheme of functional dependence. In any case: $K_{12}^{(AB)} = 0$.

6 The $\boldsymbol{J}$-integral is a conservative force

In this Section, important relations between the $\boldsymbol{J}^{(AB)}$-, $M^{(AB)}$- and $L_3^{(AB)}$-integrals of line charges and line forces are given and discussed showing that they are inherently connected. Moreover, it is shown that the $\boldsymbol{J}$-integral for line charges and line forces is a conservative force.

In cylindrical coordinates, it can be seen after some calculations that the $J_r^{(AB)}$-, $J_\varphi^{(AB)}$- and $M^{(AB)}$-integrals given in this work for line charges and line forces satisfy the following relations:

$$J_r^{(AB)} = -2\,\frac{\partial M^{(AB)}}{\partial \bar{r}}\,, \tag{143}$$

$$J_\varphi^{(AB)} = -\frac{2}{\bar{r}}\,\frac{\partial M^{(AB)}}{\partial \varphi}\,. \tag{144}$$

Also, in Cartesian coordinates, the following relations between the $\boldsymbol{J}^{(AB)}$- and $M^{(AB)}$-integrals for line charges and line forces hold

$$J_1^{(AB)} = -2\,\frac{\partial M^{(AB)}}{\partial \bar{x}}\,, \tag{145}$$

$$J_2^{(AB)} = -2\,\frac{\partial M^{(AB)}}{\partial \bar{y}}\,. \tag{146}$$

Eqs. (145) and (146) can be written in the following compact form

$$J_k^{(AB)} = -2\,\frac{\partial M^{(AB)}}{\partial \bar{x}_k}\,, \quad k = 1, 2\,, \tag{147}$$

which shows that the $\boldsymbol{J}^{(AB)}$-integral is nothing but twice the negative gradient of the $M^{(AB)}$-integral

$$\boldsymbol{J}^{(AB)} = -2\,\mathrm{grad} M^{(AB)}\,. \tag{148}$$

On the other hand, it has been shown throughout the paper that for line charges and for all cases of line forces it holds also the relation

$$M^{(AB)} = \frac{1}{2}\,U^{(AB)}\,, \tag{149}$$

which in combination with Eq. (148) yields

$$\boldsymbol{J}^{(AB)} \equiv \boldsymbol{\mathcal{F}}^{\mathrm{L}(AB)} = -\,\mathrm{grad} U^{(AB)}\,, \quad \boldsymbol{J}^{(AB)} \equiv \boldsymbol{\mathcal{F}}^{\mathrm{C}(AB)} = -\,\mathrm{grad} U^{(AB)}\,. \tag{150}$$

Eq. (150) provides the important result that the $\boldsymbol{J}^{(AB)}$*-integral representing the electrostatic part of the Lorentz force* $\boldsymbol{\mathcal{F}}^{\mathrm{L}(AB)}$ *of two line charges in electrostatics and the Cherepanov force* $\boldsymbol{\mathcal{F}}^{\mathrm{C}(AB)}$ *of two line forces in elasticity is a conservative force* and the corresponding (electrostatic or elastic) interaction energy $U^{(AB)}$ plays the role of the potential energy for the corresponding force in electrostatics and elasticity, respectively. As a consequence, the curl of $\boldsymbol{J}^{(AB)}$-integral is zero, that is,

$$\mathrm{curl}\,\boldsymbol{J}^{(AB)} \equiv \mathrm{curl}\,\boldsymbol{\mathcal{F}}^{\mathrm{L}(AB)} = 0\,, \quad \mathrm{curl}\,\boldsymbol{J}^{(AB)} \equiv \mathrm{curl}\,\boldsymbol{\mathcal{F}}^{\mathrm{C}(AB)} = 0\,, \tag{151}$$

which means that the $\boldsymbol{J}^{(AB)}$*-integral is an irrotational* (curl-less or curl-free) *vector field*.

Equations (143)–(151) are valid for line charges and for all cases of line forces and they are fundamental relations. It is of great significance to notice that the fundamental relations between the $\boldsymbol{J}^{(AB)}$- and $M^{(AB)}$-integrals given in this work for line charges and line forces in isotropic case are also valid for straight screw and edge dislocations with parallel or antiparallel Burgers vectors in isotropic materials as found in [2]; for parallel electro-elastic screw dislocations in hexagonal piezoelectric materials as showed in [3]; and for parallel screw dislocations in one-dimensional hexagonal quasicrystals, even if they have a complicated structure, as found in [26].

Remark 5 It is noted that for line forces in plane strain with parallel or antiparallel strengths, the following relations also hold between the $L_3^{(AB)}$-, $J_\varphi^{(AB)}$- and $M^{(AB)}$-integrals

$$L_3^{(AB)} = -\frac{\partial M^{(AB)}}{\partial \varphi}\,, \tag{152}$$

$$L_3^{(AB)} = \frac{1}{2}\,\bar{r}\,J_\varphi^{(AB)}\,. \tag{153}$$

It should be emphasized that for line forces with perpendicular strengths, the relations (152) and (153) are not valid.

7 Conclusions

In this work, we have determined the $\boldsymbol{J}^{(AB)}$-, $M^{(AB)}$- and $L_3^{(AB)}$-integrals of line charges and line forces using the method of embedding or method of descent of the corresponding Green functions to 2D from 3D. The main results can be summarized as follows:

- The $\boldsymbol{J}^{(AB)}$-integral of two line charges $\hat{q}^{(A)}$ and $\hat{q}^{(B)}$ is the electrostatic part of the Lorentz force $\boldsymbol{\mathcal{F}}^{\mathrm{L}(AB)}$ (electrostatic interaction force) between the two line charges. It is a purely radial force

$$\boldsymbol{\mathcal{F}}^{\mathrm{L}(AB)} = (\mathcal{F}_r^{\mathrm{L}(AB)}, 0) \tag{154}$$

  and is repulsive if the charges are of the same sign and attractive if they are of the opposite sign. In other words, the $\boldsymbol{J}^{(AB)}$-integral of two line charges provides nothing but the Coulomb law

$$J_r^{(AB)} \equiv \mathcal{F}_r^{\mathrm{L}(AB)} = \begin{cases} > 0, & \text{repulsive}, \quad \hat{q}^{(A)}\hat{q}^{(B)} > 0, \\ < 0, & \text{attractive}, \quad \hat{q}^{(A)}\hat{q}^{(B)} < 0. \end{cases} \tag{155}$$

- The $\boldsymbol{J}^{(AB)}$-integral of two line forces $\hat{f}_i^{(A)}$ and $\hat{f}_j^{(B)}$ is the Cherepanov force $\boldsymbol{\mathcal{F}}^{\mathrm{C}(AB)}$ (interaction force) between the two line forces.
- The Cherepanov force of two parallel line forces in antiplane strain is a purely radial force

$$\boldsymbol{\mathcal{F}}^{\mathrm{C}(AB)} = (\mathcal{F}_r^{\mathrm{C}(AB)}, 0) \tag{156}$$

and is attractive if the line forces are parallel and repulsive if they are antiparallel. That is,

$$J_r^{(AB)} \equiv \mathcal{F}_r^{\mathrm{C}(AB)} = \begin{cases} < 0, & \text{attractive}, \quad \hat{f}_3^{(A)} \hat{f}_3^{(B)} > 0, \\ > 0, & \text{repulsive}, \quad \hat{f}_3^{(A)} \hat{f}_3^{(B)} < 0. \end{cases} \tag{157}$$

- There is an analogy between the electrostatic part of the Lorentz force $\boldsymbol{\mathcal{F}}^{\mathrm{L}(AB)}$ for two parallel line charges (in z-direction) and the Cherepanov force $\boldsymbol{\mathcal{F}}^{\mathrm{C}(AB)}$ for two parallel line forces (in z-direction) in antiplane strain. Both forces, $\boldsymbol{\mathcal{F}}^{\mathrm{L}(AB)}$ and $\boldsymbol{\mathcal{F}}^{\mathrm{C}(AB)}$, are central forces in two-dimensions (see Eqs. (154) and (156)). However, the two forces have "opposite behaviour". In other words, $\boldsymbol{\mathcal{F}}^{\mathrm{L}(AB)}$ is attractive (or repulsive) if the charges have opposite sign (or same sign), whereas $\boldsymbol{\mathcal{F}}^{\mathrm{C}(AB)}$ is attractive (or repulsive) if the line forces are parallel (or antiparallel) (see Eqs. (155) and (157)).
- The Cherepanov force in antiplane or plane strain with parallel or antiparallel strengths shows the same behaviour for all cases and it is attractive if the line forces are parallel and repulsive if they are antiparallel. That is,[5]

$$\boldsymbol{J}^{(AB)} \equiv \boldsymbol{\mathcal{F}}^{\mathrm{C}(AB)} = \begin{cases} \text{attractive}, \quad \hat{f}_i^{(A)} \hat{f}_i^{(B)} > 0, \quad i = 1, 2, 3, \\ \text{repulsive}, \quad \hat{f}_i^{(A)} \hat{f}_i^{(B)} < 0, \quad i = 1, 2, 3. \end{cases} \tag{158}$$

- The Cherepanov force for line forces in plane strain with parallel or antiparallel strengths is not a purely radial force and its tangential component $J_\varphi^{(AB)} \equiv \mathcal{F}_\varphi^{\mathrm{C}(AB)}$ shows in which rotational direction one line force will be rotated with respect to the other line force due to Eqs. (103) and (120).
- The $L_3^{(AB)}$-integral for two parallel line forces with parallel or antiparallel strengths in x- or y-direction is the total torque produced only by the tangential Cherepanov force $\mathcal{F}_\varphi^{\mathrm{C}(AB)}$. The rotational direction of the line force $\hat{f}_i^{(A)}$, $i = 1, 2$ with respect to $\hat{f}_i^{(B)}$, $i = 1, 2$ is counter-clockwise if $L_3^{(AB)} > 0$ and clockwise if $L_3^{(AB)} < 0$. Hence, the tangential component $\mathcal{F}_\varphi^{\mathrm{C}(AB)}$ of the Cherepanov force tends to rotate the line force $\hat{f}_i^{(A)}$ with respect to the line force $\hat{f}_i^{(B)}$ at distance $\bar{r}$ counter-clockwise if $\mathcal{F}_\varphi^{\mathrm{C}(AB)} > 0$ and clockwise if $\mathcal{F}_\varphi^{\mathrm{C}(AB)} < 0$.
- The $M^{(AB)}$-integral of two line sources (charges or forces) equals half the electrostatic interaction energy in electrostatics and half the elastic interaction energy in elasticity between these two line sources, respectively.
- The $L_3^{(AB)}$-integral of two line sources (charges or forces) is the z-component of the configurational vector moment or the rotational moment representing the total torque about the z-axis caused by the interaction between the two line sources.
- The $\boldsymbol{J}^{(AB)}$-integral, representing the electrostatic part of the Lorentz force $\boldsymbol{\mathcal{F}}^{\mathrm{L}(AB)}$ of two line charges in electrostatics and the Cherepanov force $\boldsymbol{\mathcal{F}}^{\mathrm{C}(AB)}$ of two line forces in elasticity, is a conservative force and the corresponding (electrostatic or elastic) interaction energy $U^{(AB)}$ plays the role of the potential energy for the corresponding force in electrostatics and elasticity, respectively.
- The $\boldsymbol{J}^{(AB)}$-integral for line charges and line forces is an irrotational vector field.

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[5] No summation convention is used here.

A Appendix

In this Appendix, the $J_1^{(AB)}$-, $J_2^{(AB)}$-, $M^{(AB)}$- and $L_3^{(AB)}$-integrals of line forces in plane strain are given for $(\bar{x}, \bar{y}) = (a, 0)$ with $a > 0$ in order to make the necessary comparisons with already given results in the literature concerning line forces (see Seo et al. [41]). Obviously, this example is marginal and restricted, since in this case the angle φ is zero and one looses the influence of the angle dependence, which as it is shown here is important. However, only this case has been considered in [41]. The obtained results are listed in Table 2.

Comparing the obtained results in Table 2 and the corresponding results in Table 2 in Seo et al. [41], the following noteworthy observations can be made:

- Concerning the $J_1^{(AB)}$- and $J_2^{(AB)}$-integrals, it can be seen that there is a difference in the sign. Despite the fact that Seo et al. [41] have found (without giving any derivation) the $J_1^{(AB)}$- and $J_2^{(AB)}$-integrals with different sign, making a wrong physical interpretation afterwards, they concluded to the same physical interpretation as in this work.
  That means, the Cherepanov force $J_1^{(AB)} \equiv \mathcal{F}_1^{\mathrm{C}(AB)}$ is attractive if the line forces $\hat{f}_1^{(A)}$ and $\hat{f}_1^{(B)}$, $\hat{f}_2^{(A)}$ and $\hat{f}_2^{(B)}$ are parallel, and repulsive if they are antiparallel. That is,[6]

$$\mathcal{F}_1^{\mathrm{C}(AB)} = \begin{cases} < 0, & \text{attractive}, \quad \hat{f}_i^{(A)} \hat{f}_i^{(B)} > 0, \quad i = 1, 2 \\ > 0, & \text{repulsive}, \quad \hat{f}_i^{(A)} \hat{f}_i^{(B)} < 0, \quad i = 1, 2. \end{cases} \tag{A.1}$$

  On the other hand, the Cherepanov force $J_2^{(AB)} \equiv \mathcal{F}_2^{\mathrm{C}(AB)}$ is not vanishing only for parallel line forces with perpendicular strengths and it is repulsive if $\hat{f}_1^{(A)}$ and $\hat{f}_2^{(B)}$ have the same sign, and attractive if their signs are opposite. That is,

$$\mathcal{F}_2^{\mathrm{C}(AB)} = \begin{cases} > 0, & \text{repulsive}, \quad \hat{f}_1^{(A)} \hat{f}_2^{(B)} > 0, \\ < 0, & \text{attractive}, \quad \hat{f}_1^{(A)} \hat{f}_2^{(B)} < 0. \end{cases} \tag{A.2}$$

- Concerning the M-integral, there is a substantial discrepancy between the results. The M-integral in [41] is calculated for "two-dimensional elastic line singularities in the framework of plane strain linear elasticity" and for line forces it is constant in all cases. That means on the one hand, it cannot capture the interaction between the two line forces and on the other hand that it is an unphysical result, since as it has been shown in this paper the M-integral represents the elastic interaction energy between the two line forces. Moreover, the M-integral, in the last two cases in Table 2 in [41] for line forces with perpendicular strengths and with parallel strengths in y-direction, is given by the exact same formula, which is a rather strange result. A crucial point that has to be emphasized as we have already mentioned in the Introduction is that the M-integral in [41] is calculated taking the space dimension equal to two ($n = 2$) and not three as it should be, since a line force is a line singularity in a three-dimensional medium and plane strain is a two-dimensional deformation in a three-dimensional body, therefore the space dimension should be taken equal to three. As it can be seen here in Table 2, the M-integral for parallel line forces with parallel/antiparallel strengths in x- or y-direction which has been calculated taking the space dimension equal to three ($n = 3$), is a logarithmic function depending on the distance a, describing in this way the interaction of the two line forces.
- Concerning the L_3-integral, the result in Table 2 is in agreement with the result given in [41], only if we consider that there is a typing mistake (wrong placement of the parenthesis) in [41].

Finally, it should be noted that in Seo et al. [41] a comparison between the $\boldsymbol{J}^{(AB)}$-, $M^{(AB)}$- and $L_3^{(AB)}$-integrals of line forces in plane strain and the $\boldsymbol{J}$-, M- and L_3-integrals of edge dislocations has been made. However, such a comparison could lead to misleading conclusions, since these two problems have different physics. A dislocation is a topological defect and in particular is a line defect in a three-dimensional crystal and has to be treated in the framework of three-dimensional incompatible elasticity taking into account plastic fields, whereas a line force is nothing but the source term in the force equilibrium condition and has to be treated in the framework of three-dimensional compatible elasticity. Even if there are similarities between the two problems, there are still important differences from a physical point of view. The $\boldsymbol{J}$-, M- and L_3-integrals

[6] No summation convention is used here.

Table 2 The $J_1^{(AB)}$-, $J_2^{(AB)}$-, $M^{(AB)}$- and $L_3^{(AB)}$-integrals of two interacting line forces in plane strain for $(\bar{x}, \bar{y}) = (a, 0)$ with $a > 0$

Line forces	$J_1^{(AB)}$	$J_2^{(AB)}$	$M^{(AB)}$	$L_3^{(AB)}$
$\hat{f}_1^{(A)},\ \hat{f}_1^{(B)}$	$-\frac{\hat{f}_1^{(A)}\hat{f}_1^{(B)}(3-4\nu)}{8\pi\mu(1-\nu)}\frac{1}{a}$	0	$\frac{\hat{f}_1^{(A)}\hat{f}_1^{(B)}}{16\pi\mu(1-\nu)}\left[(3-4\nu)\ln\frac{a}{L}-1\right]$	0
$\hat{f}_1^{(A)},\ \hat{f}_2^{(B)}$	0	$\frac{\hat{f}_1^{(A)}\hat{f}_2^{(B)}}{8\pi\mu(1-\nu)}\frac{1}{a}$	0	$-\frac{\hat{f}_1^{(A)}\hat{f}_2^{(B)}}{8\pi\mu(1-\nu)}\left[(3-4\nu)\ln\frac{a}{L}-1\right]$
$\hat{f}_2^{(A)},\ \hat{f}_2^{(B)}$	$-\frac{\hat{f}_2^{(A)}\hat{f}_2^{(B)}(3-4\nu)}{8\pi\mu(1-\nu)}\frac{1}{a}$	0	$\frac{\hat{f}_2^{(A)}\hat{f}_2^{(B)}(3-4\nu)}{16\pi\mu(1-\nu)}\ln\frac{a}{L}$	0

of straight screw and edge dislocations have been derived in Agiasofitou and Lazar [2] in the framework of three-dimensional incompatible elasticity of dislocations and there are discrepancies with the results reported in [41] in the framework of plane strain elasticity. Comparisons of the related results are out of the aim of this paper.

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