Acta Mech 234, 3241–3257 (2023)
https://doi.org/10.1007/s00707-023-03537-y

ORIGINAL PAPER

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Continuous dependence and convergence for Moore–Gibson–Thompson heat equation

Received: 3 August 2022 / Revised: 15 February 2023 / Accepted: 4 March 2023 / Published online: 7 April 2023

Abstract In this paper, we investigate how the solutions vary when the relaxation parameter, the conductivity rate parameter, or the thermal conductivity parameter change in the case of the Moore-Gibson-Thompson heat equation. In fact, we prove that they can be controlled by a term depending upon the square of the variation of the parameter. These results concern the structural stability of the problem. We also compare the solutions of the MGT equation with the Maxwell-Cattaneo heat conduction equation and the type III heat equation (limit cases for the first two previous parameters) and we show how the difference between the solutions can be controlled by a term depending on the square of the limit parameter. This result gives a measure of the convergence between the solutions for the different theories.

Mathematics subject classification 35B30 · 35L30 · 35Q79 · 80A99

1 Introduction

In the recent paper [39] the author proposed the so-called Moore–Gibson–Thompson heat equation. We recall that the MGT heat equation has the form

$$c(\tau\dddot{\alpha}+\ddot{\alpha})=k^{*}\Delta\alpha+k\Delta\dot{\alpha} \tag{1.1}$$

where α is the thermal diplacement, and c is the thermal capacity, τ is the relaxation parameter, k^* is the conductivity rate and k is the thermal conductivity. The motivation for considering (1.1) as a heat equation came from the fact that the heat conduction of type III, that was proposed by Green and Nagdhi [19–21], implies the unbounded rate of propagation of the thermals waves [18,40]. Since it violates the causality principle, this is not well accepted from the physical point of view. Hence, it is natural to introduce a relaxation parameter (similar to the suggestion of Cattaneo and Maxwell) to obtain a heat equation with a finite rate of propagation for the thermal waves (see [39] for more details).

The Moore–Gibson–Thompson heat equation has deserved much interest in the last years, both when it is only considered by itself [12–14,23,27,28,30,32,35,36] either in the context of acoustics, mechanics, or heat transfer or also when it is assumed as the heat conduction equation for a thermoelastic theory [5,6,11,22,26,34]. On the other hand, studies concerning structural stability of different problems have deserved a big interest

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in recent years. That is, to study the continuous dependence of the solutions on the constitutive parameters defining the problem or equation.

In this paper we are concerned with the structural stability of the Moore–Gibson–Thompson equation for small changes in the relaxation parameter, the conductivity rate parameter, and the thermal conductivity parameter. What we expect is that the difference in the solutions can be controlled by a term determined by the variation of the parameter. We believe that it is important to give an approximation to clarify its relevance. This is our motivation because we believe that the mathematical analysis of this theory helps to reveal its applicability. Furthermore, continuous dependence results are important because of the inevitable flaws arising in the numerical computation as well as the physical measurement of data. In this sense, we believe that it is important to know the magnitude of the effect of such errors on the solutions of the problems.

A second aspect that we are interested in this paper is the difference that we can find in the case that we were working with different thermal conduction theories and how the corresponding solutions are related. It will be relevant to determine a measure of the error we find depending upon the particular heat conduction theory we consider. We are interested in this comparison because we note that in order to define the MGT-equation (1.1) we need four parameters. But, dropping some of them, we obtain equations of alternative heat conduction theories. For instance, if we drop the relaxation parameter (that is, we consider $\tau = 0$) our equation becomes

$$c\ddot{\alpha} = k^* \Delta\alpha + k\Delta\dot{\alpha} \tag{1.2}$$

which corresponds to the type III heat conduction (also known as strongly damped wave equation) (see [19–21]). On the other side if we consider the case where the conductivity rate parameter vanishes (that is, $k^* = 0$) we then obtain

$$c(\tau\ddot{\theta} + \dot{\theta}) = k\Delta\theta, \quad \dot{\alpha} = \theta, \tag{1.3}$$

which is known as Maxwell-Cattaneo heat equation (also known as weakly damped wave equation) (see [9]).

As we obtain a different theory by dropping a parameter it will be relevant to obtain a measure for the difference of the solutions of the MGT-equation and its corresponding limit problem which depends upon the dropped parameter. This will be the second aim of this paper.

It is worth recalling that both aspects commented previously have deserved much attention in the literature concerning heat conduction, thermoelasticity and porous materials. In this sense we can recall the contributions [2–4,10,16,17,29,31,33,37,38,41], among others. Also the limit for vanishing relaxation time τ has been considered for the nonlinear Jordan-Moore–Gibson–Thompson equation in [24,25].

It is worth noting that a study concerning the convergence when the relaxation parameter is vanishing was developed in the references [7,8] or also in the even more recent work [1]. In fact, the study was presented in [7,8] in a more general context than in our case (abstract point of view, stronger topologies and nonlinear problem). However, we here provide some alternative results giving a complementary result. In this sense, the estimates we present are faster, as we will see in the corresponding sections.

On a more technical side, let us summarize some facts that will be considered in the whole paper. First, we will denote by B a three dimensional bounded domain with the boundary smooth enough to apply the divergence theorem.

Second, to obtain our estimates it will be suitable to recall the following results, which say that there exists a constant $C > 0$ such that the following inequalities

$$\int_B |\nabla f|^2 dx \geq C \int_B |f|^2 dx$$

and

$$\int_B |\Delta f|^2 dx \geq C \int_B |\nabla f|^2 dx$$

hold for every smooth function $f(x)$ vanishing at the boundary of B.

And, finally, we are going to assume that the different equations considered along the paper also hold for the initial time $t = 0$. Moreover we also assume that the time derivatives also satisfy this requirement.

The structure of the paper is the following. In Sect. 2 we obtain a structural stability result with respect to the relaxation parameter τ, and a convergence result in the case that this relaxation parameter vanishes. This convergence result allows us to compare the solutions of the Moore–Gibson–Thompson equation with the ones of the type III heat conduction theory, which turns outs to be limit equation in this case. In Sect. 3, we do a similar study with respect to the conductivity rate parameter k^*, obtaining also a structural stability result and a convergence result when this parameter vanishes. Therefore, we will be able to compare the Moore–Gibson–Thompson heat transfer theory with the Maxwell-Cattaneo one (limit equation in this case). Finally, in Sect. 4, we prove a structural stability result with respect to the thermal conductivity parameter k. The limit case $k \to 0$ leads to an ill-posed problem (in the Hadamard sense). Hence, it does not make sense to consider this limit for the Moore–Gibson–Thompson equation.

In the next sections, we will use the energy arguments to prove the above-mentioned results. However, it is worth noting that we will need the combination of different functionals (defined on the solutions) in order to overcome the difficulties that arised from the usual arguments concerning this kind of approach.

2 Variation of the relaxation parameter

In this section, we obtain estimates for the difference of the solutions of the Moore–Gibson–Thompson equation (1.1) when the relaxation parameter τ is modified. First, we study this difference in the case of a small variation of this parameter. Second, we compare the solution of the MGT heat equation when τ is small with the solution of the type III heat conduction (1.2) (limit equation when $\tau \to 0$).

2.1 Small variation of the relaxation parameter τ

In this section we will compare the solution of the problem determined by the equation

$$c(\tau_1 \dddot{\alpha}_1 + \ddot{\alpha}_1) = k^* \Delta \alpha_1 + k \Delta \dot{\alpha}_1 \tag{2.1}$$

with the boundary condition

$$\alpha_1(x,t) = 0, \quad x \in \partial B, \quad t > 0, \tag{2.2}$$

and the initial conditions

$$\alpha_1(x,0) = \alpha^0(x), \quad \dot{\alpha}_1(x,0) = \theta^0(x), \quad \ddot{\alpha}_1(x,0) = \eta^0(x), \quad x \in B, \tag{2.3}$$

with the solution of the problem

$$c(\tau_2 \dddot{\alpha}_2 + \ddot{\alpha}_2) = k^* \Delta \alpha_2 + k \Delta \dot{\alpha}_2 \tag{2.4}$$

also with the boundary condition

$$\alpha_2(x,t) = 0, \quad x \in \partial B, \quad t > 0, \tag{2.5}$$

and the initial conditions

$$\alpha_2(x,0) = \alpha^0(x), \quad \dot{\alpha}_2(x,0) = \theta^0(x), \quad \ddot{\alpha}_2(x,0) = \eta^0(x), \quad x \in B, \tag{2.6}$$

If we consider the function $w = \alpha_1 - \alpha_2$, it satisfies the problem

$$c(\tau_1 \dddot{w} + \ddot{w}) = k^* \Delta w + k \Delta \dot{w} + c(\tau_2 - \tau_1) \dddot{\alpha}_2 \tag{2.7}$$

with homogeneous Dirichlet boundary conditions and homogeneous initial conditions, that is

$$w(x,t) = 0, \quad x \in \partial B, \quad t > 0, \tag{2.8}$$

and

$$w(x,0) = 0, \quad \dot{w}(x,0) = 0 \quad \ddot{w}(x,0) = 0. \quad x \in B. \tag{2.9}$$

The problem (2.7)–(2.9) can be written as the following first order evolution equation

$$\frac{dW}{dt} = \mathcal{A}_\tau W + \mathcal{F}$$

and the corresponding initial conditions, where $W = (w, u, v)^T$ and the linear operator $\mathcal{A}_\tau : D(\mathcal{A}_\tau) \subset \mathcal{H} \longrightarrow \mathcal{H}$ and $\mathcal{F} : \mathcal{H} \longrightarrow \mathcal{H}$ are given by

$$\mathcal{A}_\tau \begin{pmatrix} w \\ u \\ v \end{pmatrix} = \begin{pmatrix} u \\ v \\ \dfrac{1}{c\tau_1}\left(-cv + \Delta(k^* w + ku)\right) \end{pmatrix}, \quad \mathcal{F} = \begin{pmatrix} 0 \\ 0 \\ \dfrac{\tau_2 - \tau_1}{\tau_1} \dddot{\alpha}_2 \end{pmatrix}$$

with α_2 being the solution of problem (2.4)–(2.6). The space and domain where this operator is defined are the following

$$\mathcal{H} = \left\{(w, u, v) \in H_0^1(B) \times H_0^1(B) \times L^2(B)\right\}$$

with the usual norm:

$$\|(w, u, v)\|_{\mathcal{H}} = \int_B |\nabla w|^2 \, dv + \int_B |\nabla u|^2 \, dv + \int_B |\, v|^2 \, dv \tag{2.10}$$

and

$$D(\mathcal{A}_\tau) = \left\{(w, u, v) \in \mathcal{H},\, v \in H_0^1(B),\ k^* w + ku \in H^2(B)\right\}.$$

This problem is well-posed if the initial conditions for α_2 given in (2.6) are smooth enough. Since it is not the purpose of this work, we omit the details (here and in the rest of the paper).

Remark 2.1 The operators, spaces, and domains in the subsequent sections can be defined in an analogous way to the ones above. For the sake of readability, we will omit this part in the rest of the paper.

Let us consider the following functions

$$E_1(t) = \frac{1}{2} \int_B \left(c(\dot{w} + \tau_1 \ddot{w})^2 + k^* |\nabla(w + \tau_1 \dot{w})|^2 + \tau_1 K_1 |\nabla \dot{w}|^2\right) dv, \tag{2.11}$$

(where $K_i = k - \tau_i k^* > 0,\ i = 1, 2$)

$$E_2(t) = \frac{1}{2} \int_B \left(c\tau_1 (\ddot{w})^2 + k|\nabla \dot{w}|^2\right) dv + \int_B k^* \nabla w \nabla \dot{w} dv \tag{2.12}$$

and

$$E_3(t) = \frac{1}{2} \int_B k|\nabla w|^2 \, dv + \int_B c\,(\tau_1 \ddot{w} + \dot{w})\, w \, dv. \tag{2.13}$$

We now define

$$E(t) = E_1(t) + m_2 E_2(t) + m_3 E_3(t) \tag{2.14}$$

for some constants $m_2, m_3 > 0$ to be chosen below.

Lemma 2.2 *The function $E_1(t)$ is an energy equivalent to the usual norm in $\mathcal{H}$ given in (2.10). Hence, if $m_2, m_3 > 0$ are small enough, $E(t)$ is equivalent to the usual norm in $\mathcal{H}$ too.*

Proof The first part of the lemma is straightforward. The second part follows from the fact that if we take m_2 small enough and, later, take $m_3 > 0$ much smaller (for instance, of $O(m_2^2)$), $E(t)$ is equivalent to $E_1(t)$. □

Theorem 2.3 *Let w be a solution of the problem (2.7)–(2.9) and $\varepsilon_\tau = \tau_2 - \tau_1$. Then, there exist computable constants $M > 0$, $\lambda - \lambda^*$ (all of them of order 1 if ε_τ is small enough) such that the energy $E(t)$ defined in (2.14) satisfies that*

$$E(t) \le \frac{M}{\lambda - \lambda^*} J(0)\, \varepsilon_\tau^2 \left(e^{-\lambda^* t} - e^{-\lambda t}\right)$$

with

$$J(0) = \frac{1}{2}\int_B \left(c(\eta^0 + \tau_2\xi^0)^2 + k^*|\nabla(\theta^0 + \tau_2\eta^0)|^2 + \tau_2 K_2|\nabla\eta^0|^2\right) dv$$
$$+\frac{1}{2}n_2\int_B (c\tau_2|\xi^0|^2 + k|\nabla\eta^0|^2)dv + n_2\int_B k^*\nabla\theta^0\nabla\eta^0 dv$$
$$+\frac{1}{2}n_3\int_B k|\nabla\theta^0|^2 dv + n_3\int_B c(\tau_2 + \xi^0 + \eta^0)\theta^0 dv$$

where $n_2, n_3 > 0$ *are small enough constants,* $\alpha^0, \theta^0, \eta^0$ *are the initial conditions given in* (2.6)*, and*

$$\xi^0 = \frac{1}{c\tau_2}\left(-\eta^0 + k^*\Delta\alpha^0 + k\Delta\theta^0\right).$$

Remark 2.4 With the result given in this theorem, we have proved the structural stability of the problem (2.1)–(2.3) with respect to the parameter τ. That is we have proved that the error we have in the measure of the norm of the solution when we modify the relaxation parameter can be controlled by an expression of the order of $O(\varepsilon_\tau^2)$, where ε_τ is the variation of the parameter.

Also, we observe that the energy $E(t)$ is analogous to $J(t)$, which is actually defined in a similar way for $\dot{\alpha_2}$ (see (2.19)–(2.22) in the proof of the theorem below).

Remark 2.5 Notice that the amplitude term (that is, everything that is multiplying to ε_τ^2) vanishes at $t = 0$ and tends to 0 when t tends to infinity even when $\varepsilon_\tau \neq 0$. In particular, this gives the convergence in t when $t \to \infty$. The same will happen in all the analogous results of the rest of the paper, so we will omit this comment for the sake of this work.

Remark 2.6 Finally, in the proof of the theorem we will see that in order to compute (and optimize) the value of the constants M, λ, λ^* we should solve some nonlinear equations, which turn out to be a bit complicated. Hence, and as it is not the purpose of this work, we are not going to compute them (neither here nor in the rest of the sections) and we will simply recall their existence.

Proof Let us fix $\tau_1 > 0$ and consider $\tau_2 > 0$ varying (but being of the same order of τ_1). We consider the functions $E_1(t), E_2(t), E_3(t)$ given in (2.11), (2.12), and (2.13), respectively. After derivation and use of equations (2.7)–(2.9), an easy calculation leads to

$$\frac{dE_1}{dt} = -\int_B K_1|\nabla\dot{w}|^2 dv + c\,\varepsilon_\tau\int_B \dddot{\alpha}_2(\tau_1\ddot{w} + \dot{w})dv \tag{2.15}$$
$$\frac{dE_2}{dt} = -\int_B c|\ddot{w}|^2 dv + k^*\int_B |\nabla\dot{w}|^2 + c\,\varepsilon_\tau\int_B \dddot{\alpha}_2\ddot{w}dv \tag{2.16}$$
$$\frac{dE_3}{dt} = -k^*\int_B |\nabla w|^2\,dv + \int_B c\,(\tau_1\ddot{w} + \dot{w})\,\dot{w}\,dv + c\,\varepsilon_\tau\int_B \dddot{\alpha}_2 w\,dv \tag{2.17}$$

where α_2 is the solution of the problem (2.4)–(2.6). Now, we consider the function $E(t)$ given in (2.14). By (2.15)–(2.17), we have

$$\frac{dE}{dt} = -\int_B \left(K_1|\nabla\dot{w}|^2 + c\,m_2|\ddot{w}|^2 + k^* m_3|\nabla w|^2\right)\,dv +$$
$$+\int_B \left(m_2\,k^*|\nabla\dot{w}|^2 + m_3\,c\,(\tau_1\ddot{w} + \dot{w})\,\dot{w}\right)\,dv +$$
$$+c\,\varepsilon_\tau\int_B \dddot{\alpha}_2\,(\tau_1\ddot{w} + \dot{w} + m_2\ddot{w} + m_3 w)\,dv.$$

Note that we can select $m_2 > 0$ small enough and, later, select $m_3 > 0$ much smaller (for instance, of $O(m_2^2)$) to guarantee the existence of a constant $C_1 > 0$ such that

$$-\int_B \left(K_1|\nabla\dot{w}|^2 + c\,m_2|\ddot{w}|^2 + k^* m_3|\nabla w|^2\right)\,dv$$
$$+\int_B \left(m_2\,k^*|\nabla\dot{w}|^2 + m_3\,c\,(\tau_1\ddot{w} + \dot{w})\,\dot{w}\right)\,dv$$
$$\leq -C_1\int_B \left(K_1|\nabla\dot{w}|^2 + c\,m_2|\ddot{w}|^2 + k^* m_3|\nabla w|^2\right)\,dv.$$

Therefore,

$$\frac{dE}{dt} \leq -C_1 \int_B \left(K_1 |\nabla \dot{w}|^2 + c\, m_2 |\ddot{w}|^2 + k^* m_3 |\nabla w|^2 \right) dv$$
$$+ \frac{\delta}{2} \int_B (\tau_1 \dddot{w} + \dot{w} + m_2 \ddot{w} + m_3 w)^2 \, dv + \frac{c^2\, \varepsilon_\tau^2}{2\delta} \int_B |\ddot{\alpha}_2|^2 \, dv$$

for an arbitrary $\delta > 0$. We can select this δ in such a way that

$$\delta \int_B (\tau_1 \dddot{w} + \dot{w} + m_2 \ddot{w} + m_3 w)^2 \, dv \leq C_1 \int_B \left(K_1 |\nabla \dot{w}|^2 + c\, m_2 |\ddot{w}|^2 + k^* m_3 |\nabla w|^2 \right) dv.$$

It then follows

$$\frac{dE}{dt} \leq -\frac{C_1}{2} \int_B \left(K_1 |\nabla \dot{w}|^2 + c\, m_2 |\ddot{w}|^2 + k^* m_3 |\nabla w|^2 \right) dv + \frac{c^2\, \varepsilon_\tau^2}{2\delta} \int_B |\ddot{\alpha}_2|^2 \, dv.$$

Note also that we can choose a constant $C_2 > 0$ such that

$$\int_B \left(K_1 |\nabla \dot{w}|^2 + c\, m_2 |\ddot{w}|^2 + k^* m_3 |\nabla w|^2 \right) dv \geq C_2 E(t)$$

Combining these previous two inequalities, we obtain

$$\frac{dE(t)}{dt} + \frac{C_1 C_2}{2} E(t) \leq \frac{c^2\, \varepsilon_\tau^2}{2\delta} \int_B |\ddot{\alpha}_2|^2 \, dv. \tag{2.18}$$

At this point, we need an estimate for the integral of the right hand side of the previous inequality. In the same spirit as (2.11)–(2.13), we consider

$$J_1(t) = \frac{1}{2} \int_B \left(c(\ddot{\alpha}_2 + \tau_2 \dddot{\alpha}_2)^2 + k^* |\nabla(\dot{\alpha}_2 + \tau_2 \ddot{\alpha}_2)|^2 + \tau_2 K_2 |\nabla \ddot{\alpha}_2|^2 \right) dv, \tag{2.19}$$

$$J_2(t) = \frac{1}{2} \int_B \left(c \tau_2 (\ddot{\alpha}_2)^2 + k |\nabla \dot{\alpha}_2|^2 \right) dv + \int_B k^* \nabla \dot{\alpha}_2 \nabla \ddot{\alpha}_2 dv, \tag{2.20}$$

$$J_3(t) = \frac{1}{2} \int_B k |\nabla \dot{\alpha}_2|^2 \, dv + \int_B c \left(\tau_2 \ddot{\alpha}_2 + \dot{\alpha}_2 \right) \dot{\alpha}_2 \, dv \tag{2.21}$$

and

$$J(t) = J_1(t) + n_2 J_2(t) + n_3 J_3(t) \tag{2.22}$$

for n_2, n_3 to be choosen below. We recall that α_2 is the solution of (2.4)–(2.6) (we will consider α_2 regular enough to compute the previous integrals).

As in the first part of this proof, after derivation and making use of (2.4)–(2.6) we have

$$\frac{dJ}{dt} = -\int_B \left(K_2 |\nabla \ddot{\alpha}_2|^2 + c\, n_2 |\dddot{\alpha}_2|^2 + k^* n_3 |\nabla \dot{\alpha}_2|^2 \right) dv +$$
$$+ \int_B \left(n_2\, k^* |\nabla \ddot{\alpha}_2|^2 + n_3\, c \left(\tau_2 \dddot{\alpha}_2 + \ddot{\alpha}_2 \right) \ddot{\alpha}_2 \right) dv.$$

Following the same steps to obtain (2.18), we can find $n_2 > 0$ small enough and $n_3 > 0$ much smaller such that there exist $B_1, B_2 > 0$ constant that

$$\frac{dJ(t)}{dt} + \frac{B_1 B_2}{2} J(t) \leq 0.$$

Therefore, by Gronwall's inequality, we have

$$J(t) \leq J(0) e^{-\lambda^* t} \tag{2.23}$$

where $\lambda^* = \frac{B_1 B_2}{2}$. We now observe that there exists a constant $C_3 > 0$ such that

$$\int_B |\ddot{\alpha}_2|^2 \, dv \leq C_3 J(t)$$

Hence, using (2.23) in (2.18) we have

$$\frac{dE}{dt} + \lambda E(t) \leq \frac{C_3}{2\delta} J(0)\, \varepsilon_\tau^2 \, e^{-\lambda^* t}.$$

where $\lambda = \frac{C_1 C_2}{2}$. After integration (and recalling that $E(0) = 0$), we obtain

$$E(t) \leq \frac{M}{\lambda - \lambda^*} J(0)\, \varepsilon_\tau^2 \left(e^{-\lambda^* t} - e^{-\lambda t} \right).$$

for a certain constant $M > 0$. Observe that λ, λ^* depend on the constants C_1, C_2, B_1, B_2, that have chosen conveniently large or small. That means we have a certain margin to choose them and, therefore, we can take them in a form that $\lambda \neq \lambda^*$. However, if $\lambda = \lambda^*$ similar estimates can be obtained.

Now, as τ_1 is fixed and τ_2 varies in the same order of τ_1, the dependence of these parameters on the above constants is not relevant. Hence, we obtain the result given in the statement of the theorem. □

2.2 Convergence as $\tau \to 0$

The aim of this section is to compare the solutions corresponding to the MGT-heat equation of (1.1) with solutions for the type III heat equation (1.2) (or strongly damped wave equation). This corresponds to see what happens in the case that the relaxation parameter τ tends to zero in (1.1) and compare the solution with the corresponding solutions.

More concretely, we are going to compare the solution α_1 of the problem (2.1)–(2.3) (with τ instead of τ_1) with the solution of the problem given by the equation

$$c\ddot{\alpha}_2 = k^* \Delta \alpha_2 + k \Delta \dot{\alpha}_2 \tag{2.24}$$

with the boundary condition (2.5) and the initial conditions

$$\alpha_2(x, 0) = \alpha^0(x), \quad \dot{\alpha}_2(x, 0) = \theta^0(x), \quad x \in B. \tag{2.25}$$

Observe that the problem above is the formal limit problem for (2.1)–(2.3) when $\tau \to 0$. Also observe that, deriving (2.24) we have

$$c\dddot{\alpha}_2 = k^* \Delta \dot{\alpha}_2 + k \Delta \ddot{\alpha}_2 \tag{2.26}$$

that we are going to use below.

We first note that the problem for α_1 involves three initial conditions, but the one for α_2 only involves two initial conditions. Therefore, it will be sufficient to restrict our attention to the particular case were we impose that

$$\eta^0(x) = c^{-1}(k^* \Delta \alpha^0 + k \Delta \theta^0). \tag{2.27}$$

Estimates for the other cases (different η^0) could be obtained after the addition of the continuous dependence with respect to initial conditions.

Remark 2.7 In Theorem 2.8 it seems that we are comparing α_1, which is the solution of the third order in time problem (2.1)–(2.3) with α_2, solution of the second order in time problem (2.24)–(2.5)–(2.25).

Actually, the complete proof will consist of the two following steps: first, compare α_1 solution of (2.1)–(2.3) with $\tilde{\alpha}_1$ solution of the same problem with η^0 satisfying (2.27) (hence, a problem with two initial conditions); and, second, compare $\tilde{\alpha}_1$ with α_2, solution of (2.24)–(2.5)–(2.25) (that is, a problem with two initial conditions since η^0 is satisfying (2.27)).

As the first step follows directly from the well-known exponential decay results of the corresponding problems, we have omitted it in the proof below, which only contains the details of the second step described above.

In order to compare both problems we denote

$$w(x,t)=\alpha_1-\alpha_2.$$

The function $w(x,t)$ satisfies the problem determined by the equation

$$c(\tau\dddot{w}+\ddot{w})=k^*\Delta w+k\Delta\dot{w}+c\tau\dddot{\alpha}_2 \tag{2.28}$$

with the homogeneous initial and boundary conditions given in (2.8) and (2.9).

Theorem 2.8 *Let w be a solution of the problem (2.28), with initial and boundary conditions given by (2.8) and (2.9), respectively. Then, there exist a constant $M>0$ independent of τ if τ is small enough such that $E_1(t)$ defined in (2.11) (with τ instead of τ_1) satisfies that*

$$E_1(t)\le M\,\tau^2\,t^2\,J^*(0)$$

with

$$J^*(0)=\frac{1}{2}\int_B\left(c|\xi^0|^2+k^*|\nabla\eta^0|^2\right)dv$$

for

$$\xi^0=\frac{1}{c}\left(k^*\Delta\theta^0+k\Delta\eta^0\right), \tag{2.29}$$

where θ^0, η^0 are the initial conditions given in (2.25) and (2.27), respectively.

Remark 2.9 Regarding the regularity required for the initial conditions, there are some differences between our hypotheses and those required in Theorem 2.4 of [7]. Essentially, we ask the initial conditions to be as regular as needed to fulfill the above results. To be more specific, from (2.25), (2.27), and (2.29) this happens if $k^*\alpha^0+k\theta^0\in H^3(B)$ and $k^*\theta^0+kc^{-1}(k^*\Delta\alpha^0+k\Delta\theta^0)\in H^2(B)$. However, in [7] the needed regularity is just $\alpha^0,\theta^0\in H^2(B)$ and $\eta^0\in H^1(B)$.

Proof We consider the function $E_1(t)$ given in (2.11) (with τ instead of τ_1). After derivation and use of equations (2.28) and the boundary condition (2.8), it is easy to see that

$$\frac{dE_1}{dt}=-\int_B K|\nabla\dot{w}|^2dv+c\,\tau\int_B\dddot{\alpha}_2(\tau\dddot{w}+\ddot{w})dv. \tag{2.30}$$

where $K=k-\tau k^*>0$ and α_2 is the solution of the problem (2.24)–(2.5)–(2.25).

As $K>0$ if τ is small enough, from the inequality above we have

$$\frac{dE_1(t)}{dt}\le c\tau\int_B\dddot{\alpha}_2(\tau\dddot{w}+\ddot{w})\,dv\le c^{1/2}\tau\left(\int_B|\dddot{\alpha}_2|^2\,dv\right)^{1/2}E_1^{1/2}(t).$$

Hence,

$$\frac{\dot{E}_1(t)}{E_1^{1/2}(t)}\le c^{1/2}\tau\left(\int_B|\dddot{\alpha}_2|^2\,dv\right)^{1/2}$$

which integrated leads to

$$2E_1(t)^{1/2}\le c^{1/2}\tau t^{1/2}\left(\int_0^t\int_B|\dddot{\alpha}_2|^2\,dv\,d\tau\right)^{1/2}. \tag{2.31}$$

To estimate the integral on the right hand side we now consider

$$J^*(t)=\frac{1}{2}\int_B\left(c|\dddot{\alpha}_2|^2+k^*|\nabla\ddot{\alpha}_2|^2\right)dv \tag{2.32}$$

After derivation, and making use of Eqs. (2.24), (2.5) and (2.25), we arrive at

$$\frac{dJ^*(t)}{dt} = -k\int_B |\nabla \ddot{\alpha}_2|^2\,dv \leq 0.$$

We now observe that there is a constant $C^* > 0$ such that

$$\int_B |\ddot{\alpha}_2|^2\,dv \leq C^* J^*(t) \leq C^* J^*(0)$$

as $J^*(t)$ is a decreasing function. Combining this with (2.31), we arrive at the result given in this theorem. □

The estimate obtained in Theorem 2.8 is valid whenever (2.27) holds. But following the same steps as in Theorem 2.8, we can obtain a similar estimate for the general case. Also, in order to clarify the convergence between the solutions of the MGT-heat equation(1.1) and the type III heat equation(1.2) (and also to compare these results with those obtained in Theorem 2.4 in [7]) it is convenient to give an estimate for a measure of the solution independent of τ. Results in these directions are given in Theorem 2.10 below.

Theorem 2.10 *Let w be a solution of the problem (2.28), with initial and boundary conditions given by (2.8) and (2.9), respectively. Then, there exists a constant $M > 0$ independent of τ if τ is small enough such that $E_1(t)$ defined in (2.11) (with τ instead of τ_1) satisfies that*

$$E_1(t) \leq 2\tau^2 H(0) + 2M\tau^2 t^2 J^{**}(0) \tag{2.33}$$

where

$$H(0) = \frac{1}{2}\int_B c(\eta^0 - \chi^0)^2\,dv$$

and

$$J^{**}(0) = \frac{1}{2}\int_B \left(c|\xi_*^0|^2 + k^*|\nabla\chi^0|^2\right)\,dv$$

where

$$\chi^0 = c^{-1}(k^*\Delta\alpha^0 + k\Delta\theta^0)$$

and

$$\xi_*^0 = c^{-1}(k^*\Delta\theta^0 + k\Delta\chi^0).$$

Also, we can obtain the following convergence of order τ^2 for the H^1-norm of w when $\tau \to 0$:

$$\int_B |\nabla w|^2\,dv \leq \frac{4}{k^*}\tau^2 H(0) + \frac{4M}{k^*}\tau^2 t^2 J^{**}(0). \tag{2.34}$$

Remark 2.11 For the estimate (2.33), we need $k^*\alpha^0 + k\theta^0 \in H^3(B)$, $k^*\theta^0 + kc^{-1}(k^*\Delta\alpha^0 + k\Delta\theta^0) \in H^2(B)$ and $c^{-1}(k^*\Delta\alpha^0 + k\Delta\theta^0) - \eta^0 \in L^2(B)$.

Also, observe that if we impose some more regularity on the initial conditions we could obtain estimates for the H^1-norm of the time derivatives of w similar to (2.34). This would allow us to obtain a measure of the solution independent of τ, as desired.

Proof As we said, following the same steps as in Theorem 2.8, we can obtain estimate (2.33).

To obtain estimate (2.34), we note that

$$E_1(t) \geq \int_B \left(k^*|\nabla(w + \tau\dot{w})|^2 + \tau K|\nabla\dot{w}|^2\right)\,dv.$$

But

$$\begin{aligned} k^*|\nabla(w+\tau\dot{w})|^2+\tau K|\nabla\dot{w}|^2 &= k^*\left(|\nabla w|^2+2\tau\nabla w\nabla\dot{w}+\tau^2|\nabla\dot{w}|^2\right)+\tau K|\nabla\dot{w}|^2 \\ &\geq k^*\left(|\nabla w|^2-\frac{1}{2}|\nabla w|^2-2\tau^2|\nabla\dot{w}|^2+\tau^2|\nabla\dot{w}|^2\right)+\tau K|\nabla\dot{w}|^2 \\ &\geq \frac{k^*}{2}|\nabla w|^2+(\tau K-\tau^2k^*)|\nabla\dot{w}|^2 \\ &\geq \frac{k^*}{2}|\nabla w|^2 \end{aligned} \tag{2.35}$$

whenever τ is small enough. Therefore we obtain

$$k^*\int_B|\nabla w|^2\,dv\leq 2E_1(t)$$

that, combined with (2.33) gives us (2.34).

□

Remark 2.12 Theorem 2.8 says that the difference between the solutions for the problem determined by the MGT heat conduction theory and the type III heat equation can be controlled by an expression of the order of $O(\tau^2)$, where τ is the relaxation parameter. It is worth recalling that the MGT heat equation can be seen after the introduction of the relaxation parameter in the type III heat equation (see [11,39]). Therefore, as far as the relaxation parameter is small, the difference of solutions between both theories is also small. The same idea applies to the result given in Theorem 2.10 for the general initial conditions case.

Finally, it is worth recalling the contributions [7,8], which give another result concerning the convergence we study here. In fact, they obtain their results in an abstract way, with stronger topologies and even in a nonlinear context. However, our Theorem 2.8 improves these results in the finite time case. More concretely, we obtain a rate of convergence that is quadratic in the relaxation parameter τ when t is bounded, while in [7,8] this rate of convergence is linear in τ. Notice that our results combined with the same arguments used in [7] allow us to obtain the same strong convergence result given in Theorem 2.4b of [7] (we do not include them here as it is neither the purpose nor the spirit of the present work). Maybe similar results with different regularity on the solutions could be derived from the use of $E(t)$, but this would be part of a future work.

Remark 2.13 In Theorems 2.8 and 2.10 we have used $E_1(t)$ instead of $E(t)$, as $E(t)$ is not an equivalent norm to the usual one in $\mathcal{H}$ when $\tau\to 0$ (see definitions (2.11)–(2.14)). However, we would like to say that one could also extend the arguments of Theorems 2.8 and 2.10 to consider stronger topologies. If in Theorem 2.8 we define

$$\hat{E}_1(t)=\frac{1}{2}\int_B\left(c|\nabla(\dot{w}+\tau\ddot{w})|^2+k^*|\Delta(w+\tau_1\dot{w})|^2+\tau K_1|\Delta\dot{w}|^2\right)dv, \tag{2.36}$$

we have that

$$\frac{d\hat{E}_1}{dt}\leq c\,\tau\int_B\nabla\ddot{\alpha}_2\nabla(\tau\ddot{w}+\dot{w})dv\leq c^{1/2}\tau\left(\int_B|\nabla\ddot{\alpha}_2|^2dv\right)^{1/2}\hat{E}_1(t)^{1/2}.$$

Therefore we obtain that

$$\hat{E}_1(t)\leq M\tau^2t^2\hat{J}^*(0).$$

where $\hat{J}^*(0)$ can be obtained in terms of the initial conditions. That is the convergence that would be given in a stronger topology. Again, stronger regularity conditions on the initial data will be needed.

Similarly, an estimate analogous at (2.34) could be obtained applying the same kind of the above arguments to Theorem 2.10.

3 Variation of the conductivity rate parameter

In this section, we study continuous dependence and convergence of the solutions of (1.1) with respect to the conductivity rate parameter k^*. We recall that in case that $k^*=0$ the heat equation becomes the Maxwell-Cattaneo heat equation (1.3).

3.1 Small change of the parameter k^*

In this section, we compare the solutions of the problem determined by (2.1)–(2.3) (with τ and k_1^* instead of τ_1 and k^*) with the solutions of the problem defined by the equation

$$c(\tau\dddot{\alpha}_2 + \ddot{\alpha}_2) = k_2^*\Delta\alpha_2 + k\Delta\dot{\alpha}_2 \tag{3.1}$$

with the boundary and initial conditions (2.5) and (2.6), respectively.

Denote (again) by w the difference of the solutions, that is $w = \alpha_1 - \alpha_2$, which is easy to see that satisfies the equation

$$c(\tau\dddot{w} + \ddot{w}) = k_1^*\Delta w + k\Delta\dot{w} + (k_1^* - k_2^*)\Delta\alpha_2, \tag{3.2}$$

with homogeneous Dirichlet boundary conditions and homogeneous initial conditions ((2.8) and (2.9), respectively).

Theorem 3.1 *Let w be a solution of the problem (3.2)–(2.8)–(2.9) for $k_1^* > 0$ fixed and a varying $k_2^* > 0$ and $\varepsilon_{k^*} = k_1^* - k_2^*$. Then, there exist three computable constants $M > 0$, $\lambda - \lambda^*$ (all of them of order 1 if ε_{k^*} is small enough) such that the energy $E(t)$ defined in (2.14) (with τ and k_1^* instead of τ_1 and k^*) satisfies that*

$$E(t) \le \frac{M}{\lambda - \lambda^*} F(0)\, \varepsilon_{k^*}^2 \left(e^{-\lambda^* t} - e^{-\lambda t}\right)$$

with

$$F(0) = \frac{1}{2}\int_B \left(c|\nabla(\theta^0 + \tau\eta^0)|^2 + k_2^*|\Delta(\alpha^0 + \tau\theta^0)|^2 + \tau(k - \tau k_2^*)|\Delta\theta^0|^2\right) dv \tag{3.3}$$

and where $\alpha^0, \theta^0, \eta^0$ are the initial conditions given in (2.6).

Remark 3.2 We see that the difference between both solutions can be controlled by an expression of the order of $O(\varepsilon_{k^*}^2)$ where ε_{k^*} is the difference between the conductivity rate parameters. This proves the structural stability of the problem (2.1)–(2.3) with respect to the conductivity rate parameter k^*.

Also, observe that $F(t)$ (which will be defined in the proof of the theorem, see (3.5)) is analogous to the usual H^2 energy.

Proof The proof will follow the same steps as the proofs of Theorems 2.3 or 2.8 in the previous section.

We now fix $k_1^* > 0$ and let $k_2^* > 0$ vary. Again, we consider the functions $E_1(t), E_2(t), E_3(t), E(t)$ given in (2.11), (2.12), (2.13), and (2.14) (with τ and k_1^* instead of τ_1 and k^*, respectively). After derivation and use of Eqs. (3.2)–(2.8)–(2.9), an easy calculation leads to

$$\begin{aligned}\frac{dE}{dt} &= -\int_B \left(K_1^*|\nabla\dot{w}|^2 + c\,m_2|\ddot{w}|^2 + k_1^* m_3|\nabla w|^2\right) dv + \\ &+ \int_B \left(m_2\, k_1^*|\nabla\dot{w}|^2 + m_3\, c\,(\tau\ddot{w} + \dot{w})\,\dot{w}\right) dv + \\ &+ \varepsilon_{k^*}\int_B \Delta\alpha_2\,(\tau\ddot{w} + \dot{w} + m_2\ddot{w} + m_3 w)\; dv\end{aligned}$$

with now $K_i^* = k - \tau k_i^* > 0$, $i = 1, 2$, and where α_2 is the solution of problem (3.1)–(2.5)–(2.6).

Following the same steps as in the proof of Theorem 2.3 or Theorem 2.8, we can see that

$$\frac{dE(t)}{dt} + \frac{C_1 C_2}{2} E(t) \le \frac{\varepsilon_{k^*}^2}{2\delta}\int_B |\Delta\alpha_2|^2\, dv \tag{3.4}$$

for $m_2 > 0$ small enough, $m_3 > 0$ much smaller, and $C_1, C_2, \delta > 0$.

To estimate the right-hand side of the previous inequality we consider

$$F(t) = \frac{1}{2}\int_B \left(c|\nabla(\dot{\alpha}_2 + \tau\ddot{\alpha}_2)|^2 + k_2^*|\Delta(\alpha_2 + \tau\dot{\alpha}_2)|^2 + \tau K_2^*|\Delta\dot{\alpha}_2|^2\right) dv. \tag{3.5}$$

As before, we can see that there exists $C_3 > 0$ such that

$$\int_B |\Delta\alpha_2|^2\, dv \leq C_3 F(t). \tag{3.6}$$

As we will proof below, it can be seen that there exist a constant $\lambda^* > 0$ such that

$$\frac{dF(t)}{dt} \leq -\lambda^* F(t) \tag{3.7}$$

and, hence,

$$F(t) \leq F(0)e^{-\lambda^* t}$$

with $F(0)$ given in (3.3). Thus, following the same steps as in Theorems 2.3 or 2.8 we arrive at

$$E(t) \leq \frac{C_3\, C_4}{2\delta(\lambda - \lambda^*)} F(0)\, \varepsilon_{k^*}^2 \left(e^{-\lambda^* t} - e^{-\lambda t}\right).$$

Therefore, we obtain the result given in the statement of the theorem.

It only remains to prove inequality (3.7). To this end, we define

$$G_1(t) = \frac{1}{2}\int_B \left(c\tau|\nabla\ddot{\alpha}_2|^2 + k|\Delta\dot{\alpha}_2|^2 + 2k_2^*\Delta\alpha_2\Delta\dot{\alpha}_2\right)\, dv$$

$$G_2(t) = \frac{1}{2}\int_B \left(k|\Delta\alpha_2|^2 + 2c\nabla(\tau\ddot{\alpha}_2 + \dot{\alpha}_2)\nabla\alpha_2\right)\, dv.$$

Using (3.1), (2.5) and (2.6), we can see that

$$\frac{dG_1}{dt} = -\int_B c|\nabla\ddot{\alpha}_2|^2\, dv + k_2^*\int_B |\Delta\dot{\alpha}_2|^2\, dv$$

$$\frac{dG_2}{dt} = -\int_B k_2^*|\Delta\alpha_2|^2\, dv + \int_B c\nabla(\tau\ddot{\alpha}_2 + \dot{\alpha}_2)\nabla\dot{\alpha}_2\, dv$$

and also

$$\frac{dF}{dt} = -K_2^*\int_B |\Delta\dot{\alpha}_2|^2\, dv$$

Now, we can see that, for $n_1, n_2 > 0$ small enough, we have

$$F(t) + n_1 G_1(t) + n_2 G_2(t)$$

is equivalent to $F(t)$ and also that there exists a constant $C_5 > 0$ such that

$$\frac{d}{dt}\left(F(t) + n_1 G_1(t) + n_2 G_2(t)\right) \leq C_5 F(t)$$

This proves inequality (3.7), which completes the proof of the present theorem. □

3.2 Case $k^* \to 0$

Now, we compare the solutions of the problem determined by (2.1)-(2.3) (with τ instead of τ_1) with the solutions of the problem defined by the equation

$$c(\tau \dddot{\alpha}_2 + \ddot{\alpha}_2) = k\Delta\dot{\alpha}_2 \tag{3.8}$$

with the boundary and initial conditions (2.5) and (2.6), respectively.

But, as (3.8) is, in fact, a second order problem in $\dot{\alpha}_2$, only two initial conditions would be needed. Hence in this case we assume that

$$\alpha^0 = \frac{1}{k}\Delta^{-1}(c\tau\eta^0 + c\theta^0). \tag{3.9}$$

That is we can select α^0 in terms of θ^0 and η^0. Observe that the boundary conditions assumed in (2.5) guarantee that α^0 is well defined.

With these assumptions, we also observe that α_2 satisfies the equation

$$c(\tau\ddot{\alpha}_2 + \dot{\alpha}_2) = k\Delta\alpha_2 \tag{3.10}$$

(together with (2.5), (2.6), and (3.9)). This equation is obtained by integrating (3.8) and imposing the initial conditions discussed above.

The difference of the solutions $w = \alpha_1 - \alpha_2$ satisfies the equation

$$c(\tau\dddot{w} + \ddot{w}) = k^*\Delta w + k\Delta\dot{w} + k^*\Delta\alpha_2, \tag{3.11}$$

with homogenous Dirichlet boundary conditions and homogeneous initial conditions (that is, (2.8) and (2.9), respectively).

Remark 3.3 From definitions (2.11)–(2.14) we can see that $E(t)$ is not an equivalent norm to the usual one in $\mathcal{H}$ when $k^* \to 0$. But we can still work with $E_1(t)$ and obtain the previous results, which will allow us to obtain the convergence result when $k^* \to 0$ (see Remark 3.7).

A discussion on the regularity of the initial conditions can also be done, in the same spirit as Remark 2.7.

Theorem 3.4 *Let w be a solution of the problem* (3.11)*, with boundary and initial conditions given by* (2.8) *and* (2.9)*, respectively. Then, there exists a computable constant $M > 0$ independent of k^* if k^* is small enough such that $E_1(t)$ defined in* (2.11) *(with τ instead of τ_1) satisfies that*

$$E_1(t) \leq M\,(k^*)^2\,t^2 F^*(0)$$

with

$$F^*(0) = \frac{1}{2}\int_B \left(c\tau|\nabla\theta^0|^2 + k|\Delta\alpha^0|^2\right)\,dv$$

and where α^0, θ^0 are the initial conditions given in (2.6) *and* (3.9).

Remark 3.5 This estimate says that the difference of the solutions between the MGT heat equation and the Maxwell-Cattaneo heat equation is controlled by a term of the order of $O((k^*)^2)$ where k^* is the conductivity rate parameter. The solutions of these two problems would be close whenever this parameter is small. A strong convergence result in this case could be the purpose of a future work.

Also, observe that $F^*(t)$ (which will be defined in the proof of the theorem, see (3.13)) is analogous to the usual H^2 energy.

Remark 3.6 In Theorem 3.4 it seems that we are comparing α_1, which is the solution of the third order in time problem (2.1)–(2.3) with α_2, solution of the second order in time problem (3.8)–(3.9) and (2.5)–(2.6). But observe that a remark in the same spirit as Remark 2.7 can be done. We are comparing α_1 solution of (2.1)–(2.3) with $\tilde{\alpha}_1$ solution of the same problem with η^0 satisfying (3.9) (hence, a problem with two initial conditions); and, then, we compare $\tilde{\alpha}_1$ with α_2, solution of (3.8)–(3.9) and (2.5)–(2.6) (that is, a problem with two initial conditions since η^0 is satisfying (3.9)).

As before, we recall that the first part is a consequence of the well known exponential decay results of the corresponding problems. Hence, the proof below only focus on the details of the second step described above.

Proof The proof follows the same steps as the proof of Theorem 2.8. As in there, we consider the function $E_1(t)$ given in (2.11) (with τ instead of τ_1). After derivation and use of eqs. (3.11)–(2.8)–(2.9), an easy calculation leads to

$$\frac{dE_1}{dt} = -\int_B \left(K^*|\nabla \dot{w}|^2 + k^*\Delta\alpha_2(\tau\ddot{w} + \dot{w})\right) dv$$

with $K^* = k - \tau k^* > 0$, and α_2 being the solution of the problem (3.8)–(3.9) and (2.5)–(2.6). As $K^* > 0$ if k^* is small enough, we have

$$\frac{dE_1}{dt} \leq \int_B k^*\Delta\alpha_2(\tau\ddot{w} + \dot{w})\, dv \leq c^{-1/2}k^*\left(\int_B |\Delta\alpha_2|^2\, dv\right)^{1/2} E_1^{1/2}(t).$$

Following the same steps as in the proof of Theorem 2.8, we can se that

$$2E_1(t)^{1/2} \leq c^{1/2}k^*t^{1/2}\left(\int_0^t \int_B |\Delta\alpha_2|^2\, dv\, d\tau\right)^{1/2}. \tag{3.12}$$

To estimate the integral on the right hand side, we consider

$$F^*(t) = \frac{1}{2}\int_B \left(c\tau|\nabla\dot{\alpha}_2|^2 + k|\Delta\alpha_2|^2\right) dv, \tag{3.13}$$

We derive $F^*(t)$ and use that α_2 satisfies (3.10), together with (2.5)–(2.6), which leads to

$$\frac{dF^*(t)}{dt} = -c\int_B |\nabla\dot{\alpha}_2|^2\, dv.$$

Observe that we have $C^* > 0$ such that

$$\int_B |\Delta\alpha_2|^2\, dv \leq C^*F^*(t) \leq C^*F^*(0)$$

as $F^*(t)$ is a decreasing function. Combining this with (3.12) we arrive at the desired inequality. □

Remark 3.7 The inequality given in Theorem 3.4 allows us to prove the convergence to 0 of the H^1-norm of $\dot{w}$ when $k^* \to 0$. If we want to prove the convergence of the H^1-norm of w, we can observe that, if $\nabla w(0) = 0$, then

$$|\nabla w(t)|^2 = 2\int_0^t \nabla w\nabla\dot{w}\, ds \leq 2\left(\int_0^t |\nabla w|^2\, ds\right)^{1/2}\left(\int_0^t |\nabla\dot{w}|^2\, ds\right)^{1/2} \leq \frac{4t}{\pi}\left(\int_0^t |\nabla\dot{w}|^2\, ds\right)$$

(Poincaré's type inequalities). If we know that

$$\int_B |\nabla\dot{w}|^2\, ds \leq Mt^2(k^*)^2F^*(0)$$

then

$$\int_0^t \int_B |\nabla\dot{w}|^2\, dv\, ds \leq \frac{Mt^3}{3}(k^*)^2F^*(0).$$

Hence,

$$\int_B |\nabla w|^2\, dv \leq \frac{4Mt^4}{3\pi}(k^*)^2F^*(0).$$

4 Variation of the thermal conductivity parameter

In this section, we study continuous dependence of the solutions of (1.1) with respect the thermal conductivity parameter k.

We note that in the limit case $k = 0$ the problem (2.1)–(2.3) is ill-posed as it does not generate a strongly continuous semigroup in the usual space (see for example [15,23]). Hence, we will not study this convergence in this section.

4.1 Small change of the parameter k

In this section, we compare the solutions of the problem determined by (2.1)–(2.3) (now with $\tau_1 = \tau$ and $k = k_1$) with the solutions of the problem defined by the equation

$$c(\tau\dddot{\alpha}_2 + \ddot{\alpha}_2) = k^*\Delta\alpha_2 + k_2\Delta\dot{\alpha}_2 \tag{4.1}$$

with the boundary and initial conditions (2.5) and (2.6), respectively.

The difference $w = \alpha_1 - \alpha_2$ is a solution of the problem

$$c(\tau\dddot{w} + \ddot{w}) = k^*\Delta w + k_1\Delta\dot{w} + (k_1 - k_2)\Delta\dot{\alpha}_2 \tag{4.2}$$

with boundary and initial conditions (2.8)–(2.9), respectively.

Theorem 4.1 *Let w be a solution of the problem (4.2)–(2.8)–(2.9) for $k_1 > 0$ fixed and a varying $k_2 > 0$, and $\varepsilon_k = k_1 - k_2$. Then, there exist three computable constants $M > 0$, $\lambda - \lambda^*$ (all of them of order 1 if ε_k is small enough) such that the energy $E(t)$ defined in (2.14) (with τ and k_1 instead of τ_1 and k) satisfies that*

$$E(t) \leq \frac{M}{\lambda - \lambda^*} F(0)\, \varepsilon_k^2 \left(e^{-\lambda^* t} - e^{-\lambda t}\right)$$

with

$$F(0) = \frac{1}{2}\int_B \left(c|\nabla(\theta^0 + \tau\eta^0)|^2 + k^*|\Delta(\alpha^0 + \tau\theta^0)|^2 + \tau(k_2 - \tau k^*)|\Delta\theta^0|^2\right) dv$$

and where $\alpha^0, \theta^0, \eta^0$ are the initial conditions given in (2.6).

Remark 4.2 Again, as in the previous sections, the previous result proves the structural stability of the problem (2.1)–(2.3), now with respect to the thermal conductivity parameter k.

Also, $F(t)$ (which, as we will seen in the proof of the theorem, is the same one used in Theorem 3.1) is analogous to the usual H^2 energy.

Proof To prove this result, we can follow the same steps as in the previous sections.

In this case, we fix $k_1 > 0$ and let $k_2 > 0$ vary. Deriving the functions $E_1(t), E_2(t), E_3(t), E(t)$ given in (2.11)–(2.14) (with τ and k_1 instead of τ_1 and k), respectively), we obtain:

$$\begin{aligned}\frac{dE}{dt} &= -\int_B \left(K_1|\nabla\dot{w}|^2 + c\,m_2|\ddot{w}|^2 + k^* m_3|\nabla w|^2\right) dv + \\ &+ \int_B \left(m_2\, k^*|\nabla\dot{w}|^2 + m_3\, c\,(\tau\ddot{w} + \dot{w})\,\dot{w}\right) dv + \\ &+ \varepsilon_k \int_B \Delta\dot{\alpha}_2\,(\tau\ddot{w} + \dot{w} + m_2\ddot{w} + m_3 w)\, dv\end{aligned}$$

with now $K_i = k_i - \tau k^* > 0$, $i = 1, 2$, and where α_2 is the solution of problem (4.1)–(2.5)–(2.6).

As in the theorems of the previous sections, this means that there exist $C_1, C_2, \delta > 0$ such that Following the same steps as in the proof of Theorem 2.3 or Theorem 2.8, we can see that

$$\frac{dE(t)}{dt} + \frac{C_1 C_2}{2} E(t) \leq \frac{\varepsilon_k^2}{2\delta}\int_B |\Delta\dot{\alpha}_2|^2\, dv \tag{4.3}$$

for $m_2 > 0$ small enough, $m_3 > 0$ much smaller, and $C_1, C_2, \delta > 0$.

It only remains to estimate the right-hand side of the previous inequality. For that purpose, observe that we can use the same $F(t)$ defined in (3.5) (with, of course, k^* and k_2 instead of k_2^* and k). Indeed, we can see that there exists $C_3 > 0$ such that

$$\int_B |\Delta\dot{\alpha}_2|^2\, dv \leq C_3 F(t).$$

We recall that the exponential decay of $F(t)$ has been shown in Theorem 3.1. Hence, following the same steps as there, we can conclude the result given in Theorem 4.1 is true.

□

5 Conclusions

In this work we have been interested in the study of results on the continuous dependence and convergence with respect to differents parameters in the Moore–Gibson–Thompson heat equation.

More concretely, we have obtained the following results:

(1) continuous dependence on: relaxation parameter τ, conductivity rate parameter k^*, thermal conductivity parameter k;
(2) convergence on: relaxation parameter τ, conductivity rate parameter k^*.

Our estimators are of quadratic order in the continuity or convergence parameters, respectively.

We believe that our results can be extended to stronger topologies. And also that our point of view could be used in the abstract problem, in a similar way as in [7,8].

Acknowledgements The authors thank the anonymous referee for their useful remarks and comments.
M. Pellicer is part of the Catalan research group 2021 SGR 00087 funded by AGAUR (Generalitat de Catalunya) and is supported by the Spanish grants MTM2017-84214-C2-2-P and PID2021-123903NB-I00 funded by MCIN/AEI/10.13039/501100011033 and by ERDF "A way of making Europe", and RED2018-102650-T funded by MCIN/AEI/10.13039/501100011033 and by ERDF "A way of making Europe". The work of R. Quintanilla has been funded by the research project PID2019-105118GB-I00, funded by the Spanish Ministry of Science, Innovation and Universities and FEDER "A way to make Europe".

Funding Open Access funding provided thanks to the CRUE-CSIC agreement with Springer Nature.

References

1. Álvarez, E., Lizama, C.: Singular perturbation and initial layer for the abstract Moore-Gibson-Thompson equation. J. Mathemat. Analys. Appl. **516**(1), 12507 (2022)
2. Ames, K.A., Payne, L.E.: Continuous dependence for the initial time geometry for the thermoelastic system with sign-indefinite elasticity. J. Mathemat. Analys. Appl. **189**, 693–717 (1995)
3. Ames, K.A., Straughan, B.: Continuous dependence results for initially prestressed thermoelastic bodies. Int. J. Eng. Sci. **30**, 7–13 (1992)
4. Ames, K.A., Straughan, B.: Non-Standard and Improperly Posed Problems. Academic Press, USA (1997)
5. Bazarra, N., Fernández, J.R., Quintanilla, R.: Analysis of a Moore-Gibson-Thompson thermoelastic problem. J. Comput. Appl. Mathemat. **382**, 113058 (2021)
6. Bazarra, N.., Fernández, J..R.., Magaña, A., Quintanilla, R..: A poro-thermoelastic problem with dissipative heat conduction. J. Thermal Stresses **43**, 1415–1436 (2020)
7. Bongartii, M., Charoenphon, S., Lasiecka, I.: Singular thermal relaxation limit for the MGT equation arising in propagation of acoustic waves, Springer Proceedings in Mathematics and Statistics, vol. 325, Semigroups of Operators Theory and Applications. p. 147–182, 2020
8. Bongartii, M., Charoenphon, S., Lasiecka, I.: Vanishing relaxation time dynamics of the JMGT equation arising in nonlinear acoustics. J. Evol. Equ. **21**(3), 3553–3584 (2021)

9. Cattaneo, C.: On a form of heat equation which eliminates the paradox of instantaneous propagation. C. R. Acad. Sci. Paris **247**, 431–433 (1958)
10. Ciarletta, M., Straughan, B.: V Tibullo Structural stability for a rigid body with thermal microstructure. Int. J. Eng. Sci. **48**, 592–598 (2010)
11. Conti, M., Pata, V. Quintanilla., R.: Thermoelasticity of Moore-Gibson-Thompson type with history dependence in the temperature. Asymptotic Anal 120(1–2), 1–21 (2020)
12. Dell'Oro, F., Lasiecka, I., Pata, V.: The Moore-Gibson-Thompson equation with memory in the critical case. J. Diff. Equ. **261**, 4188–4222 (2016)
13. Dell'Oro, F., Pata, V.: On the Moore-Gibson-Thompson equation and its relation to linear viscoelasticity. Appl. Math. Optim. **76**, 641–655 (2017)
14. Dell'Oro, F. Pata, V. On a fourth-order equation of Moore-Gibson-Thompson type. Milan J. Math. 2017 ; 85, (215–234)
15. Dreher, M., Quintanilla, R., Racke, R.: Ill-posed problems in thermomechanics. Appl. Math. Lett. **22**(9), 1374–1379 (2009)
16. Franchi, F., Nibbi, R. Straughan, B. Continuous dependence on modelling for temperature-dependent bidispersive flow. Proc. Royal Soc. A: Mathemat. Phys. Eng. Sci. 2017 473, 20170485
17. Franchi, F., Straughan, B.: Continuous dependence and decay for the Forchheimer equations. Proc. Royal Soc. Mathemat. Phys. Eng. Sci. **459**, 3195–3202 (2003)
18. Giorgi, C., Grandi, D., Pata, V.: On the Green-Naghdi type III heat conduction model. Discrete Contin. Dyn. Syst. Ser. B **19**, 2133–2143 (2014)
19. Green, A.E., Naghdi, P.M.: On undamped heat waves in an elastic solid. J. Therm. Stresses **15**, 253–264 (1992)
20. Green, A.E., Naghdi, P.M.: Thermoelasticity without energy dissipation. J. Elast. **31**, 189–208 (1993)
21. Green, A. E. Naghdi, P. M.: A verified procedure for construction of theories of deformable media. I. Classical continuum physics, II. Generalized continua, III. Mixtures of interacting continua. Proc. Roy. Soc. London A **448**, 335–356, 357–377, 378–388 (1995)
22. Jangid, K., Mukhopadhyay, S.: A domain of influence theorem under MGT thermoelasticity theory. Math. Mech. Solids **26**(2), 285–295 (2021). https://doi.org/10.1177/1081286520946820
23. Kaltenbacher, B., Lasiecka, I., Marchand, R.: Wellposedness and exponential decay rates for the Moore-Gibson-Thompson equation arising in high intensity ultrasound. Control Cybernet. **40**, 971–988 (2011)
24. Kaltenbacher, B., Nikolić, V.: The Jordan-Moore-Gibson-Thompson equation: well-posedness with quadratic gradient nonlinearity and singular limit for vanishing relaxation time. Math. Models Methods Appl. Sci. **29**(13), 2523–2556 (2019)
25. Kaltenbacher, B., Nikolić, V.: Vanishing relaxation time limit of the Jordan-Moore-Gibson-Thompson wave equation with Neumann and absorbing boundary conditions, Pure Appl. Funct. Anal. **1**, 1–26 (2020)
26. Kumar, H., Mukhopadhyay, S.: Thermoelastic damping analysis in micro crobeam resonators based on Moore-Gibson-Thompson generalized thermoelasticity theory. Acta Mech. **231**, 3003–3015 (2020)
27. Lasiecka, I., Wang, X.: Moore-Gibson-Thompson equation with memory, part II: General decay of energy. J. Diff. Equ. **259**, 7610–7635 (2015)
28. Lasiecka, I., Wang, X.: Moore-Gibson-Thompson equation with memory, part I: exponential decay of energy. Z. Angew. Math. Phys. **67**, 17–67 (2016)
29. Liu, Y.: Convergence and continuous dependence for the Brinkman-Forchheimer equations. Mathemat. Comput. Model. **49**, 1401–1415 (2009)
30. Marchand, R., McDevitt, T., Triggiani, R.: An abstract semigroup approach to the third order Moore-Gibson-Thompson partial differential equation arising in high-intensity ultrasound: structural decomposition, spectral analysis, exponential stability. Math. Methods Appl. Sci. **35**, 1896–1929 (2012)
31. Morro, A., Payne, L.E., Straughan, B.: Decay, growth, continuous dependence and uniqueness results in generalized heat conduction theories. Appl. Anal. **31**, 231–243 (1990)
32. Ostoja-Starzewski, M., Quintanilla, R.: Spatial behavior of solutions of the Moore-Gibson-Thompson equation. J. Math. Fluid Mech. **23**, 105 (2021). https://doi.org/10.1007/s00021-021-00629-4
33. Payne, L.E., Payne, B.: Straughan Convergence and continuous dependence for the Brinkman-Forchheimer equations. Stud. Appl. Mathemat. **102**, 419–439 (1999)
34. Pellicer, M., Quintanilla, R.: On uniqueness and instability for some thermoemechanical problems involving the Moore-Gibson-Thompson equation. Z. Angew. Math. Phys. **71**, 84 (2020)
35. Pellicer, M., Said-Houari, B.: Wellposedness and decay rates for the Cauchy problem of the Moore-Gibson-Thompson equation arising in high intensity ultrasound. Appl. Math. Optim. **35**, 1–32 (2017)
36. Pellicer, M., Sola-Morales, J.: Optimal scalar products in the Moore-Gibson-Thompson equation. Evol. Equ. Control Theory **8**, 203–220 (2019)
37. Quintanilla, R.: Structural stability and continuous dependence of solutions of thermoelasticity of type III. Dyn. Continuous Discrete Syst. B **1**, 463–470 (2001)
38. Quintanilla, R.: Convergence and structural stability in thermoelasticity. Appl. Mathemat. Comput. **135**, 287–300 (2001)
39. Quintanilla, R.: Moore-Gibson-Thompson thermoelasticity. Math. Mech. Solids **24**, 4020–4031 (2019)
40. Renardy, M., Hrusa, W.J., Nohel, J.A.: Mathematical Problems in Viscoelasticity. John Wiley & Sons Inc, New York (1987)
41. Straughan, B.: Continuous dependence and convergence for a Kelvin-Voigt fluid of order one. Annali Univ. Ferrara **68**, 49–61 (2022)