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ORIGINAL PAPER

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Genesis and progress of virtual power principle

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Abstract The virtual power principle (VPP) of continuum mechanics states a celebrated variational equality between external and internal virtual powers for any virtual velocity field conforming with linear kinematic constraints. The topic is here addressed to investigate how the original ideas born in the early XIX century are modelled by modern formulations based on Functional Analysis and Differential Geometry. These notions are able to provide an effective mathematical context for proving existence of Lagrange multipliers associated with the constraint of rigidity on velocity fields. The VPP stands as privileged tool for giving to stress fields a consistent definition based on duality with conforming virtual stretching fields. By complementarity, the VPP generates a variational condition for integrability of stretching fields, with self-equilibrated stresses as test fields. Progress is got by the formulation of the rate virtual power principle (RVPP) by time derivation of the VPP along the motion, with internal virtual power integrated per unit mass. The basic distinction between spatial and material fields according to the geometric paradigm is prompted to replace the one previously adopted in the literature. The need for a non-redundant implicit formulation of the rigidity constraint is emphasised to contrast degeneracy. This logical demand avoids proliferation of multipliers, in the spirit of Ockham's Razor, a celebrated philosophical motto with multiform applications. The shining mathematical theory set out by Leonhard Euler, Jean-Baptiste Le Rond d'Alembert, Joseph Louis Lagrange, and Augustin Cauchy is in this respect a point of optimality. A geometric rate theory of elasticity meets the call for no-dissipation in push-closed elastic cycles, with non need of any finite strain elastic energy functional, thus leading to a proper statement of rate equilibrium problems, basilar for computational formulations and for investigations about instability phenomena and post-critical behaviours.

1 Introduction

Awareness of the basic role of the rigidity constraint in the mathematical definition of the notion of equilibrium of a continuum body immersed in Euclid space-time, dates back at least to the end of the XVI century.

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At that time the Flemish scientist Simon Stevin (1548–1620), in investigating about the statics of fluids,[1] enunciated the *principle of solidification*:

"*The state of equilibrium of a deformable body is not altered if any part of it is replaced with a rigid body of the same geometry.*"

In XVIII century, scientists like Johann Bernoulli, Alexis Clairaut, Leonhard Euler, Jean-Baptiste le Rond d'Alembert, and later Louis Poinsot, made this statement more precise by means of effective mathematical treatments. These investigations were confined to connected 3D bodies, for which rigid motions can be given a fairly simple characterisation, expressed by the statement:[2]

"*Every rigid-body motion can be represented as a combination of a translation along a line and a rotation about that line*."

More pertinent to our theme is the qualification of the instantaneous rigid velocity field in a motion. The relevant mathematical treatment requires the definition of a twice covariant metric tensor field $\mathbf{g}_{\text{SPA}}$ in the ambient space and of the metric tensor field $\mathbf{g}_{\text{MAT}}$ induced in the body manifold.

Existence of a metric tensor field in Euclid space can be proven on the basis of validity of the parallelogram rule for norms of tangent vectors.[3]

The instantaneous rigidity is expressed by requiring that the convective (Lie) derivative of the material metric tensor field along the motion vanishes in the body configuration $\boldsymbol{\Omega}$.

A fairly general case, able to discuss most engineering applications, is modelled by assuming the body subject to time-dependent kinematic constraints such that at each configuration $\boldsymbol{\Omega}$ the spatial velocity field belongs to a prescribed set of admissible spatial vector fields.[4]

In the majority of preliminary engineering investigations bilateral frictionless constraints are considered. In this case the admissible set is an affine variety composed of the sum of a singleton describing an imposed velocity field and of a linear subspace of conforming velocities.

When the conforming set of velocities is a Hilbert space which includes as a dense subset the space of smooth velocity fields vanishing in a boundary layer, the kinematic data are said to give rise to a *boundary value problem*. Indeed in practice this condition is fulfilled by imposing velocity fields taking given values on the boundary manifold.

The test fields in the VPP are virtual velocities which are assumed to belong to a linear conformity space. Therefore, they are effective only to within a proportionality factor and cannot be endowed with a definite physical dimension. Some scholars prefer to assume dimension of a length and refer to the test fields as virtual displacements and to the virtual power as virtual work. This choice is however arbitrary and possibly misleading even in the Euclid context since the true displacement fields are not tangent to the space slice containing the current configuration.

2 Continuum kinematics

The simplest 3D splitting of rigid virtual velocities into translational and rotational components requires a proper mathematical treatment [8].

In the special context of 3D body manifolds resident in the 3D ambient Euclid space, the rate of distortion is effectively measured by the *natural stretching* [9], see Eq. (8) below.

To put this notion in mathematical terms, we consider in the Euclid (3+1)D spacetime $\mathcal{E}$ the motion $\boldsymbol{\varphi}_\alpha : \mathcal{T}_\mathcal{E} \mapsto \mathcal{T}_\mathcal{E}$ with $\alpha \in \mathcal{Z}$ a time lapse along a dynamical trajectory $\mathcal{T}_\mathcal{E}$ submanifold of $\mathcal{E}$.

Each observer is characterized by a vector field of future-pointing *spacetime arrows* $\mathbf{Z} : \mathcal{E} \mapsto T\mathcal{E}$ and by the differential $dt_\mathcal{E} : T\mathcal{E} \mapsto T\mathcal{Z}$ of a time-bundle projection $t_\mathcal{E} : \mathcal{E} \mapsto \mathcal{Z}$[5] normalised by setting:

$$\langle dt_\mathcal{E}, \mathbf{Z} \rangle = 1 \circ t_\mathcal{E} \, . \tag{1}$$

[1] "De Beginselen des Waterwichts" *the principles of water balance* [1].

[2] First proved in [2] by Giulio Giuseppe Mozzi (1730–1813) and later [3] by Michel Floréal Chasles (1793–1880).

[3] In normed spaces this non-trivial but simple result is due to Maurice Fréchet, János von Neumann, and Pascual Jordan [4].

[4] This mathematical definition is suitable to discuss most engineering models adopted in structural applications. On the other side, the usual definition of boundary constraints, based on a partition of the bounding surface $\partial\boldsymbol{\Omega}$ into complementary parts on which tensions and velocities are, respectively, prescribed, as proposed in [5] and acritically adopted in the literature (see e.g. [6, Eq. (5.55)]), is needlessly restrictive and also source of delicate issues at lines of separation [7].

[5] A projection is a morphism which is surjective with its tangent map. We denote by T the tangent functor, by $\mathcal{S}$ a space slice, and by $\mathcal{Z}$ the time-axis (In German Zeit=time).

This position defines a linear projector:

$$\mathbf{R} := dt_{\mathcal{E}} \otimes \mathbf{Z} : T\mathcal{E} \mapsto T\mathcal{E}\,, \tag{2}$$

given by:

$$(dt_{\mathcal{E}} \otimes \mathbf{Z}) \cdot \mathbf{X} = \langle dt_{\mathcal{E}}, \mathbf{X} \rangle\, \mathbf{Z}\,, \quad \forall\, \mathbf{X} \in T\mathcal{E}\,, \tag{3}$$

so that by Eq. (1) $\mathbf{R} \cdot \mathbf{R} = \mathbf{R}$. The complementary projector on space slices $\mathcal{S}$, which are integral manifolds of the constraint $dt_{\mathcal{E}} = \mathbf{0}$, is $\mathbf{P} = \mathbf{I} - \mathbf{R}$.

The motion preserves simultaneity of events in the trajectory $\mathcal{T}_{\mathcal{E}}$ and fulfils the group composition property:

$$\boldsymbol{\varphi}_\alpha \circ \boldsymbol{\varphi}_\beta = \boldsymbol{\varphi}_\beta \circ \boldsymbol{\varphi}_\alpha = \boldsymbol{\varphi}_{(\alpha+\beta)}\,, \quad \forall\, \alpha, \beta \in \mathcal{Z}\,. \tag{4}$$

The velocity may be split into space and time components by setting $\mathbf{v}_{\boldsymbol{\varphi}}^{\mathrm{SPA}} := \mathbf{P}\mathbf{v}_{\boldsymbol{\varphi}}$, so that:

$$\mathbf{v}_{\boldsymbol{\varphi}} := \partial_{\alpha=0}\, \boldsymbol{\varphi}_\alpha = \mathbf{v}_{\boldsymbol{\varphi}}^{\mathrm{SPA}} + \mathbf{Z}\,, \quad \text{with } \langle dt_{\mathcal{E}}, \mathbf{v}_{\boldsymbol{\varphi}}^{\mathrm{SPA}} \rangle = 0\,. \tag{5}$$

The spatial configuration $\boldsymbol{\Omega}_{\mathcal{E}} \subset \mathcal{T}_{\mathcal{E}}$ at time $t \in \mathcal{Z}$ is the corresponding fiber of the trajectory time-bundle.[6]

In Euclid spacetime the spatial metric $\mathbf{g} = \mathbf{g}_{\mathrm{SPA}} : T_{\boldsymbol{\Omega}_{\mathcal{E}}}\mathcal{S} \mapsto T^*_{\boldsymbol{\Omega}_{\mathcal{E}}}\mathcal{S}$ is a symmetric positive definite covariant spatial tensor acting in the spatial slices.[7] Time-independence of the spatial metric is expressed by the condition $\mathcal{L}_{\mathbf{Z}}(\mathbf{g}_{\mathrm{SPA}}) = \mathbf{0}$.

The Lie derivative of the spatial metric along the motion is given by:

$$\mathcal{L}_{\mathbf{v}_{\boldsymbol{\varphi}}}(\mathbf{g}_{\mathrm{SPA}}) := \partial_{\alpha=0}\, (\boldsymbol{\varphi}_\alpha \!\downarrow\! \mathbf{g}_{\mathrm{SPA}}) = \mathcal{L}_{\mathbf{v}_{\boldsymbol{\varphi}}^{\mathrm{SPA}}}(\mathbf{g}_{\mathrm{SPA}}) + \cancel{\mathcal{L}_{\mathbf{Z}}(\mathbf{g}_{\mathrm{SPA}})}\,. \tag{6}$$

The invariance gap $\boldsymbol{\varphi}_\alpha \!\downarrow\! \mathbf{g}_{\mathrm{SPA}} - \mathbf{g}_{\mathrm{SPA}}$ of the spatial metric tensor $\mathbf{g}_{\mathrm{SPA}}$ is the motion-induced measure in $\boldsymbol{\Omega}_{\mathcal{E}}$ of the *distortion* performed during the time lapse $\alpha \in \mathcal{Z}$, with the pull-back to $\boldsymbol{\Omega}_{\mathcal{E}}$ defined by:[8]

$$(\boldsymbol{\varphi}_\alpha \!\downarrow\! \mathbf{g}_{\mathrm{SPA}})(\mathbf{u}\,, \mathbf{v}) = \mathbf{g}_{\mathrm{SPA}}(T\boldsymbol{\varphi}_\alpha \cdot \mathbf{u}\,, T\boldsymbol{\varphi}_\alpha \cdot \mathbf{v})\,, \quad \forall\, \mathbf{u}, \mathbf{v} \in T_{\boldsymbol{\Omega}_{\mathcal{E}}}\mathcal{S}\,. \tag{7}$$

The *natural stretching* is introduced as half the Lie derivative in Eq. (6):

$$\boldsymbol{\epsilon}(\mathbf{v}_{\boldsymbol{\varphi}}) := \tfrac{1}{2}\mathcal{L}_{\mathbf{v}_{\boldsymbol{\varphi}}}(\mathbf{g}_{\mathrm{SPA}})\,. \tag{8}$$

The Lie derivative of the spatial metric field can be expressed in terms of the derivative ∇ of the velocity field, according to the associated spatial Levi-Civita connection, which is metric preserving and symmetric (torsion free):

$$\begin{cases} \nabla(\mathbf{g}_{\mathrm{SPA}}) = \mathbf{0}\,, \\ \nabla_{\mathbf{u}}(\mathbf{v}) - \nabla_{\mathbf{v}}(\mathbf{u}) - [\mathbf{u}\,, \mathbf{v}] = \mathbf{0}\,. \end{cases} \tag{9}$$

The factor $\frac{1}{2}$ in Eq. (8) is motivated by the formula for the Euler *distortion rate*, mixed alteration of the *natural stretching* [9]:

$$\tfrac{1}{2}\mathcal{L}_{\mathbf{v}_{\boldsymbol{\varphi}}}(\mathbf{g}_{\mathrm{SPA}}) = \mathbf{g}_{\mathrm{SPA}} \cdot \mathrm{sym}\nabla(\mathbf{v}_{\boldsymbol{\varphi}})\,. \tag{10}$$

The covariant spatial metric tensor $\mathbf{g}_{\mathrm{SPA}}$ being positive definite, we have the equivalence:

$$\boldsymbol{\epsilon}(\mathbf{v}_{\boldsymbol{\varphi}}) = \mathbf{0} \iff \mathrm{sym}\nabla(\mathbf{v}_{\boldsymbol{\varphi}}) = \mathbf{0}\,. \tag{11}$$

The next step consists of proving that:[9]

Lemma 1 *(Euler) In a connected 3D body spatial uniformity of Euler* distortion rate $\nabla sym\nabla(\mathbf{v}_{\boldsymbol{\varphi}}) = \mathbf{0}$ *entails vanishing of the second derivative* $\nabla^2(\mathbf{v}_{\boldsymbol{\varphi}}) = \mathbf{0}$ *so that the velocity field is expressed by an affine law.*

6 Spacetime treatments are often motivated by a comparison between classical and relativistic descriptions [10]. We remark the very notion of trajectory requires a spacetime approach, and splitting into time and spatial velocities can be forbidden by the exit out of trajectory.

7 Covariant tensors are bilinear forms on pairs of tangent vectors $\mathbf{v} \in T\boldsymbol{\Omega}$. Contravariant tensors are bilinear form on pairs of cotangent vectors (covectors) $\mathbf{v}^* \in T^*\boldsymbol{\Omega}$. Mixed tensors are bilinear on vector-covector pairs.

8 The Green-St.Venant finite strain tensor $\frac{1}{2}(\boldsymbol{\varphi}_\alpha \!\downarrow\! \mathbf{g}_{\mathrm{SPA}} - \mathbf{g}_{\mathrm{SPA}})$ is got from Eq. (8) by time integration in $\boldsymbol{\Omega}_{\mathcal{E}}$.

9 The proof, exposed by Fichera in [11, p.384, fn.21] and sketched in [12], makes appeal to the regularity properties of the solution of the distributional Laplace equation.

2.1 Lower dimensional continua

For lower dimensional continua, such as membranes and wires, the special formula Eq. (10) doesn't apply and must be replaced with a more general formula [9].

Let $\mathcal{T}$ be the material trajectory with immersion map $\mathbf{i} : \mathcal{T} \mapsto \mathcal{E}$ in the Euclid spacetime $\mathcal{E}$, so that the immersed trajectory is the image $\mathcal{T}_{\mathcal{E}} := \mathbf{i}(\mathcal{T})$.[10]

For 3D bodies the injective immersion $\mathbf{i} : \mathcal{T} \mapsto \mathcal{E}$ and its injective tangent map $T\mathbf{i} : T\mathcal{T} \mapsto T\mathcal{E}$ are usually (but abusively) assimilated to identities.

The motion $\boldsymbol{\varphi}_\alpha^{\mathcal{T}} : \mathcal{T} \mapsto \mathcal{T}$ in the trajectory is related to the spacetime motion $\boldsymbol{\varphi}_\alpha : \mathcal{T}_{\mathcal{E}} \mapsto \mathcal{T}_{\mathcal{E}}$ in the immersed trajectory by:

$$\boldsymbol{\varphi}_\alpha^{\mathcal{T}} = \mathbf{i}\!\downarrow\!\boldsymbol{\varphi}_\alpha \iff \mathbf{i} \circ \boldsymbol{\varphi}_\alpha^{\mathcal{T}} = \boldsymbol{\varphi}_\alpha \circ \mathbf{i} \,. \tag{12}$$

To simplify, we will abusively adopt a single notation by putting $\boldsymbol{\varphi}_\alpha^{\mathcal{T}} = \boldsymbol{\varphi}_\alpha$ unless confusion may occur. The material metric field $\mathbf{g}_{\text{MAT}}$ is related to the spatial metric $\mathbf{g}_{\text{SPA}}$ by the pull back:

$$\mathbf{g}_{\text{MAT}} := \mathbf{i}\!\downarrow\!\mathbf{g}_{\text{SPA}} \,, \tag{13}$$

or explicitly, for all material vectors $\mathbf{u}, \mathbf{v} \in T\boldsymbol{\Omega}$:

$$\mathbf{g}_{\text{MAT}}(\mathbf{u}\,,\mathbf{v}) = \mathbf{g}_{\text{SPA}}(T\mathbf{i}\cdot\mathbf{u}\,, T\mathbf{i}\cdot\mathbf{v}) \circ \mathbf{i} \,. \tag{14}$$

The configuration $\boldsymbol{\Omega} \subset \mathcal{T}$ is a fiber of the time bundle $t_{\mathcal{T}} = t_{\mathcal{E}} \circ \mathbf{i} : \mathcal{T} \mapsto \mathcal{Z}$ induced on the trajectory $\mathcal{T}$, and $\boldsymbol{\Omega}_{\mathcal{E}} := \mathbf{i}(\boldsymbol{\Omega})$ is the corresponding fiber of the time bundle $t_{\mathcal{E}} : \mathcal{T}_{\mathcal{E}} \mapsto \mathcal{Z}$ on the spacetime immersed trajectory $\mathcal{T}_{\mathcal{E}}$.

The projection $\boldsymbol{\Pi} : T_{\boldsymbol{\Omega}_{\mathcal{E}}}\mathcal{S} \mapsto T\boldsymbol{\Omega}$ is defined by assuming the composition with the tangent map $T\mathbf{i} : T\boldsymbol{\Omega} \mapsto T\boldsymbol{\Omega}_{\mathcal{E}}$, given by:

$$T\mathbf{i} \circ \boldsymbol{\Pi} : T_{\boldsymbol{\Omega}_{\mathcal{E}}}\mathcal{S} \mapsto T\boldsymbol{\Omega}_{\mathcal{E}} \,, \tag{15}$$

to be the $\mathbf{g}_{\text{SPA}}$-orthogonal projector over the identity $\mathbf{Id}_{\boldsymbol{\Omega}_{\mathcal{E}}} : \boldsymbol{\Omega}_{\mathcal{E}} \mapsto \boldsymbol{\Omega}_{\mathcal{E}}$.

This projection property yields, for any pair of *spatial field* $\mathbf{s} : \boldsymbol{\Omega} \mapsto T_{\boldsymbol{\Omega}_{\mathcal{E}}}\mathcal{S}$ over the immersion $\mathbf{i} : \boldsymbol{\Omega} \mapsto \boldsymbol{\Omega}_{\mathcal{E}}$, and *material field* $\mathbf{v} : \boldsymbol{\Omega} \mapsto T\boldsymbol{\Omega}$, having a common base point, the equality:

$$\mathbf{g}_{\text{MAT}}(\boldsymbol{\Pi}\cdot\mathbf{s}\,,\mathbf{v}) = \mathbf{g}_{\text{SPA}}(T\mathbf{i}\cdot\boldsymbol{\Pi}\cdot\mathbf{s}\,, T\mathbf{i}\cdot\mathbf{v}) \circ \mathbf{i} = \mathbf{g}_{\text{SPA}}(\mathbf{s}\,, T\mathbf{i}\cdot\mathbf{v}) \circ \mathbf{i} \,. \tag{16}$$

On the other hand, the $(\mathbf{g}_{\text{MAT}}, \mathbf{g}_{\text{SPA}})$-adjoint $\boldsymbol{\Pi}^A : T\boldsymbol{\Omega} \mapsto T_{\boldsymbol{\Omega}}\mathcal{S}$ is defined by the identity:

$$\mathbf{g}_{\text{MAT}}(\boldsymbol{\Pi}\cdot\mathbf{s}\,,\mathbf{v}) = \mathbf{g}_{\text{SPA}}(\mathbf{s}\,,\boldsymbol{\Pi}^A\cdot\mathbf{v}) \circ \mathbf{i} \,. \tag{17}$$

Comparing Eq. (16) with Eq. (17) we get the relations:

$$\begin{cases} \boldsymbol{\Pi}^A = T\mathbf{i} : T\boldsymbol{\Omega} \mapsto T_{\boldsymbol{\Omega}_{\mathcal{E}}}\mathcal{S} \,, \\ \boldsymbol{\Pi} \circ T\mathbf{i} : T\boldsymbol{\Omega} \mapsto T\boldsymbol{\Omega} = \mathbf{Id}_{T\boldsymbol{\Omega}} \,, \\ T\mathbf{i} \circ \boldsymbol{\Pi} : T\boldsymbol{\Omega}_{\mathcal{E}} \mapsto T\boldsymbol{\Omega}_{\mathcal{E}} = \mathbf{Id}_{T\boldsymbol{\Omega}_{\mathcal{E}}} \,. \end{cases} \tag{18}$$

The material stretching may then be expressed by means of Eq. (10) to get:

$$\begin{aligned} \boldsymbol{\epsilon}(\mathbf{v}_{\boldsymbol{\varphi}}) &= \tfrac{1}{2}\,\mathcal{L}_{\mathbf{i}\downarrow\mathbf{v}_{\boldsymbol{\varphi}}}(\mathbf{g}_{\text{MAT}}) = \tfrac{1}{2}\,\mathbf{i}\!\downarrow \mathcal{L}_{\mathbf{v}_{\boldsymbol{\varphi}}}(\mathbf{g}_{\text{SPA}}) = \mathbf{i}\!\downarrow\!\Big(\mathbf{g}_{\text{SPA}}\cdot \text{sym}\nabla(\mathbf{v}_{\boldsymbol{\varphi}})\Big) \\ &= \mathbf{g}_{\text{MAT}}\Big(\boldsymbol{\Pi}\cdot\text{sym}\nabla(\mathbf{v}_{\boldsymbol{\varphi}})\cdot\boldsymbol{\Pi}^A\Big) = \mathbf{g}_{\text{MAT}}\cdot\mathbf{D}(\mathbf{v}_{\boldsymbol{\varphi}}) \,, \end{aligned} \tag{19}$$

so that $\mathbf{D}(\mathbf{v}_{\boldsymbol{\varphi}}) := \boldsymbol{\Pi}\cdot\text{sym}\nabla(\mathbf{v}_{\boldsymbol{\varphi}})\cdot\boldsymbol{\Pi}^A$ is the mixed alteration of the material geometric stretching $\boldsymbol{\epsilon}(\mathbf{v}_{\boldsymbol{\varphi}})$.

The features discussed above explain why treatments proposing extensions of the VPP deal exclusively with 3D bodies, see François and Eugène Cosserat [13] and Paul Germain [14–16].

[10] The co-restricted map $\mathbf{i} : \mathcal{T} \mapsto \mathcal{T}_{\mathcal{E}}$ is a diffeomorphism.

3 Genesis

The brilliant idea of applying Lagrange method of multipliers to impose the rigidity constraint on virtual velocity fields opened the way for the introduction of the basilar notion of duality in continuum mechanics.[11] Stress fields so become entities subordinate to the kinematic definition of a continuous model.

At difference with the classical local approach based on Cauchy tetrahedron, the duality method introduces the stress as a field in the body configuration $\boldsymbol{\Omega}$.

The characterisation of virtual velocity fields and the implicit representation of the rigidity constraint are the key tools for building the whole edifice of continuum mechanics.

The design is carried out by exploiting the mathematical theory of duality between linear functional spaces and of linear operators between dual spaces.

The role of duality is certainly not confined to continuum mechanics but rather pervades the modelling of the whole context of Mathematical Physics.

The kinematics of a structural model is defined by a linear kinematical space $\mathcal{V}$ of all possible velocity fields and by the linear subspace $\mathcal{L} \subset \mathcal{V}$ of conforming spatial velocity fields which fulfil linear constraint conditions.

The kinematical space $\mathcal{V}$ is conveniently endowed with the topology of a pre-Hilbert space whose norm is given by the mean square value of the vector fields and of their derivatives on the elements of a regularity patchwork, with finite cardinality, drawn in the configuration $\boldsymbol{\Omega}$.

Force systems $\mathbf{f} \in \mathcal{F}$, acting on the structural model are continuous linear functionals on $\mathcal{V}$, i.e. elements of the dual Hilbert space $\mathcal{F} = \mathcal{V}^*$.

The closed conformity subspace $\mathcal{L} \subset \mathcal{V}$ is a Hilbert space of vector fields on $\boldsymbol{\Omega}$ sharing a common regularity patchwork $\mathcal{P}(\boldsymbol{\Omega})$ with imposed interface conditions between elements.

Constraint reactions $\mathbf{r} \in \mathcal{L}^\circ \subset \mathcal{V}^*$ are force systems characterised by the property of being *friction-free*:

$$\langle \mathbf{r}, \delta\mathbf{v} \rangle_{\boldsymbol{\Omega}_{\mathcal{E}}} = 0 \,, \quad \forall\, \delta\mathbf{v} \in \mathcal{L} \,. \tag{20}$$

In other words, reactive systems are characterised by the property of performing a null virtual power for any conforming spatial velocity field.

A loading system $\ell \in \Lambda$ is a continuous functional on the conformity subspace $\mathcal{L}$, i.e. an element of the dual Hilbert space $\Lambda = \mathcal{L}^*$. By the identification:

$$\mathcal{L}^* = \mathcal{V}^* / \mathcal{L}^\circ \,, \tag{21}$$

a loading system may be interpreted as an equivalence class of force systems defined to within the addition of any reaction system.

The foundation of the equilibrium edifice is the variational condition stated by Johann Bernoulli in his famous letter to Pierre Varignon in 1717, which in modern terms can be written as:

$$\langle \ell, \delta\mathbf{v} \rangle_{\boldsymbol{\Omega}} = \mathbf{0} \,, \tag{22}$$

for all conforming rigid virtual velocities $\delta\mathbf{v} \in \mathcal{L} \cap \mathcal{V}_o$.

The rigidity constraint $\delta\mathbf{v} \in \mathcal{V}_o$ is implicitly expressed by vanishing of the *virtual natural stretching* in each element of the regularity patchwork $\mathcal{P}(\boldsymbol{\Omega})$:

$$\delta\mathbf{v} \in \mathcal{V}_o \iff \boldsymbol{\epsilon}(\delta\mathbf{v}) = \mathbf{i}{\downarrow}\Big(\tfrac{1}{2} \mathcal{L}_{\delta\mathbf{v}}(\mathbf{g}_{\mathrm{SPA}}) \Big) = \mathbf{0} \,. \tag{23}$$

A steep step is necessary in passing from the equilibrium condition expressed by Eqs. (22–23) to the variational condition expressed by the VPP in the form:

$$\langle \ell, \delta\mathbf{v} \rangle_{\boldsymbol{\Omega}_{\mathcal{E}}} = \int_{\boldsymbol{\Omega}} \langle \boldsymbol{\sigma}, \boldsymbol{\epsilon}(\delta\mathbf{v}) \rangle \, \mathbf{m} \,, \quad \forall\, \delta\mathbf{v} \in \mathcal{L} \,. \tag{24}$$

The contravariant field of *natural stress* $\boldsymbol{\sigma}$ in Eq. (24), is in duality with the symmetric covariant tensor field $\boldsymbol{\epsilon}(\delta\mathbf{v})$ of *natural stretching*.

A key point is that a covariant field can be seen as a linear map $\boldsymbol{\epsilon} : T\boldsymbol{\Omega} \mapsto T^*\boldsymbol{\Omega}$ while a contravariant one is a linear map $\boldsymbol{\sigma} : T^*\boldsymbol{\Omega} \mapsto T\boldsymbol{\Omega}$, so that their composition $\boldsymbol{\sigma} \cdot \boldsymbol{\epsilon} : T\boldsymbol{\Omega} \mapsto T\boldsymbol{\Omega}$ is a mixed tensor.

[11] An idea attributed to Gabrio Piola [17–20] by Truesdell and Toupin [21, Ch.V, p.594]. See also [22].

The duality pairing: $\langle \boldsymbol{\sigma}, \boldsymbol{\epsilon} \rangle = J_1(\boldsymbol{\sigma} \cdot \boldsymbol{\epsilon})$ is the linear invariant J_1 of the composition. Consequently, the pairing vanishes when one tensor is symmetric while the other one is skew-symmetric.

Contravariant stress tensor fields $\boldsymbol{\sigma}$ are then defined on the configuration $\boldsymbol{\Omega}$ only to within addition of an arbitrary skew-symmetric tensor field.

A convenient choice is to take them symmetric. Alleged proofs of symmetry of stress tensors can be proven to be tautological [23].

The skew-symmetric material maximal form:

$$\mathbf{m} = \rho\, \boldsymbol{\mu} \tag{25}$$

is the product of the scalar mass density ρ times the material pull-back:

$$\boldsymbol{\mu} := \mathbf{i}{\downarrow}\boldsymbol{\mu}_{\mathbf{g}}\,, \tag{26}$$

of the spatial volume form $\boldsymbol{\mu}_{\mathbf{g}}$ compatible with the metric $\mathbf{g}$,[12] so that duality in Eq. (24) is expressed per unit mass.

The importance of considering the duality per unit mass instead of per unit volume stands in the Principle of Conservation of Mass along the body motion:

$$\mathbf{m} = \boldsymbol{\varphi}_{\alpha}{\downarrow}\mathbf{m} \iff \mathcal{L}_{\mathbf{v}_{\varphi}}(\mathbf{m}) = \mathbf{0}\,. \tag{27}$$

Basic consequences of Conservation of Mass are:

- Equivalence between Euler law of Dynamics in terms of momentum rate and d'Alembert formulation in terms of acceleration field.
- Conservation of elastic energy in any cyclic process of elastic states.

This last is a characteristic property of elasticity [24].

The mixed alteration

$$\mathbf{S} = \boldsymbol{\sigma} \cdot \mathbf{g}_{\mathrm{MAT}}\,, \tag{28}$$

of the contravariant *natural stress* is related to the Cauchy *true stress* tensor $\mathbf{T}$ by proportionality according to the scalar mass density:

$$\mathbf{T} = \rho\, \mathbf{S}\,. \tag{29}$$

Mathematical and engineering presentations of 3D continuum mechanics are used to adopt a formulation in terms of Cauchy stress $\mathbf{T}$ and Euler stretching:

$$\mathbf{D}(\delta\mathbf{v}) = \mathbf{g}_{\mathrm{MAT}}^{-1} \cdot \boldsymbol{\epsilon}(\delta\mathbf{v})\,, \tag{30}$$

which perform internal virtual power per unit volume, for all conforming virtual velocities $\delta\mathbf{v} \in \mathcal{L}$:

$$\langle \ell, \delta\mathbf{v} \rangle_{\boldsymbol{\Omega}_{\mathcal{E}}} = \int_{\boldsymbol{\Omega}} \langle \mathbf{T}, \mathbf{D}(\delta\mathbf{v}) \rangle\, \boldsymbol{\mu}\,. \tag{31}$$

The comparison with Eq. (24) is explicated by the relations:

$$\begin{aligned} \langle \boldsymbol{\sigma}, \boldsymbol{\epsilon}(\delta\mathbf{v}) \rangle &= \langle \boldsymbol{\sigma} \cdot \mathbf{g}_{\mathrm{MAT}}, \mathbf{g}_{\mathrm{MAT}}^{-1} \cdot \boldsymbol{\epsilon}(\delta\mathbf{v}) \rangle \\ &= \langle \mathbf{S}, \mathbf{D}(\delta\mathbf{v}) \rangle \\ &= \rho^{-1} \langle \mathbf{T}, \mathbf{D}(\delta\mathbf{v}) \rangle\,. \end{aligned} \tag{32}$$

Motivation for this preference is to be found in the effective representation of external forces in equilibrium as bulk $\mathbf{b} = \operatorname{div}(\mathbf{T})$ and contact $\mathbf{t} = \mathbf{Tn}$ contributions, as provided by George Green's formula:

$$\begin{aligned} \langle \ell, \delta\mathbf{v} \rangle_{\boldsymbol{\Omega}_{\mathcal{E}}} &= \int_{\boldsymbol{\Omega}} \langle \mathbf{T}, \mathbf{D}(\delta\mathbf{v}) \rangle\, \boldsymbol{\mu} \\ &= \int_{\boldsymbol{\Omega}} \langle -\operatorname{div}(\mathbf{T}), \delta\mathbf{v} \rangle\, \boldsymbol{\mu} + \int_{\partial\boldsymbol{\Omega}} \langle \mathbf{Tn}, \delta\mathbf{v} \rangle\, \partial\boldsymbol{\mu}\,, \quad \forall\, \delta\mathbf{v} \in \mathcal{L}\,. \end{aligned} \tag{33}$$

[12] This means that a space cube with edges of unitary length has unitary volume. Since maximal forms are proportional one another, this condition yields a well-definite volume form.

However, for ensuring absence of dissipation in cyclic elastic processes [24], rate elastic constitutive relations have to be expressed in terms of natural stress $\boldsymbol{\sigma}$ and stretching $\boldsymbol{\epsilon}(\delta\mathbf{v})$, interacting per unit mass, according to Eq. (24).

In fact, in the finite deformation range, for 3D rate elasticity models such as hypo-elasticity [25], formulated in terms of co-rotational rate of Cauchy *true stress*, dissipation in cyclic elastic processes cannot be ruled out [26].

Coming back to the equilibrium conditions, the fact that VPP Eq. (24) implies equilibrium in the sense of Eqs. (22–23), is clear.

The converse implication requires a proof of existence for symmetric stress fields $\boldsymbol{\sigma}$ playing the role of Lagrange multipliers for the rigidity constraint Eq. (23), and this is a deep mathematical result.

For 3D continua the proof can be attained on the basis of Korn's second inequality and of Banach's closed range theorem [8].

For lower dimensional continuous models, validity of Korn's second inequality is ruled out due to the infinite dimensionality of the space of rigid velocity fields. In fact it can be proven that Korn's inequality implies finite dimensionality of the kernel of the stretching operator [27–29].

To the authors' knowledge, the existence proof of stress fields in lower dimensional continua is still a main open mathematical problem.

A naïve argument can be resorted to to overcome this difficulty from an engineering point of view by observing the equilibrium condition Eq. (22) means that the load virtual power may be alternatively expressed not in terms of conforming kinematisms but in terms of the corresponding stretching fields.

Constraint reactions may be explicitly included in the VPP by enlarging the space of test fields from the subspace $\mathcal{L}$ of conforming virtual velocities to a larger subspace $\bar{\mathcal{L}} \supset \mathcal{L}$ of the kinematical space $\mathcal{V}$, so that:

$$\langle \ell, \delta\mathbf{v} \rangle_{\boldsymbol{\Omega}_{\mathcal{E}}} + \langle \mathbf{r}, \delta\mathbf{v} \rangle_{\boldsymbol{\Omega}_{\mathcal{E}}} = \int_{\boldsymbol{\Omega}} \langle \mathbf{T}, \mathbf{D}(\delta\mathbf{v}) \rangle \, \boldsymbol{\mu} \,, \tag{34}$$

for all velocity fields $\delta\mathbf{v} \in \bar{\mathcal{L}} \subset \mathcal{V}$.

Let us observe that Eq. (34) is in fact the defining variational expression of constraint reactions. Indeed the friction-free requirement Eq. (20) is fulfilled if the VPP equilibrium condition Eq. (31) is satisfied.

3.1 Degeneracy

As reported in Sect.V (*Variational principles*) of [21, Ch.D], the expression of the VPP was generalised to include higher order derivatives of the velocity.

The treatment there reproduced, intended to reach a higher level of generality, is however mechanically not motivated and confusing.

In preserving the parental relation between the proper equilibrium law Eqs. (22–23) and the variational condition formulating the VPP Eq. (31), a careful attention must be paid to the requirement of *non-redundancy*.

To put in evidence this basic point we observe that, in accord to Stevin's *Principle of Solidification*, the equilibrium notion is strictly connected to the concept of *rigid act of motion*.

Rigid virtual velocities are in fact the proper test fields for equilibrium.

Rigidity entails that equilibrium is independent of any modelling of the involved material body by means of constitutive relations, and this is a fundamental feature of the equilibrium notion in continuum mechanics.

The decisive step forward for introducing the notion of stress fields in a body in equilibrium consists in providing a *non-redundant* implicit formulation of the rigidity requirement on the acts of virtual motion to be adopted as test fields.

In this regard, the material metric tensor field $\mathbf{g}_{\mathrm{MAT}}$ Eq. (13), plays the role of a principal actor which permits to express the rigidity property by means of the vanishing of a field of distortion rates, as evidenced in Eq. (19).

The qualification of *non-redundancy* is in the spirit of the *Lex Parsimoniae* or *Novacula Occami* [30,31].[13] The very same concept is resorted to in Optimisation Theory by the requirement of *constraint qualification* [32].

[13] The "Ockham's Razor" was so named by Scottish metaphysician Sir William Hamilton (9th-baronet) (1788–1856), not to be confused with the Irish physicist, astronomer and mathematician Sir William Rowan Hamilton (1805–1865).

Let us here provide an elementary description of the phenomenon of *redundancy* by considering the case in which a set of admissible tensor fields on a manifold $\boldsymbol{\Omega}$ is qualified by the fulfilment of a list composed of a finite number of differential conditions.

If the same set of tensor fields can be qualified by the fulfilment of a sublist of differential conditions, strictly included in the original list, we may say that the original list was *redundant*.

The redundancy issue can be detected in polar and micromorphic continuum theories [31]. The concern is the implicit representation of the subspace of rigid acts of motion of these structured continua for which Euler formula Eq. (23) doesn't suffice.

Violations of the logical and philosophical dictation of *non-redundancy* set out by sapient Franciscan friars of XIII-XIV century, led to generate a multitude of stress-like parameters of questionable physical meaning and consequently to develop elasticity theories involving a large number of constitutive moduli (hundreds or even more) whose experimental evaluation looms like a prohibitive task.

The Ockham razor applies to early proposals made by Toupin [33–35], Mindlin [36–39], and Green et al. [40–43].

Indeed, in a classical *non-structured* continuum, stretching gradients of any order are manifestly redundant.

On the other side, polar continuum theories, initiated by the brothers Cosserat [13] and brought back to light by Truesdell and Toupin in [21], were diffusely investigated by Eringen and coauthors [44–47], and couple stress models were discussed by Capriz [48] and by Del Piero [49,50].

These formulations of *structured* 3D continua were carried out in unaware but manifest violation of the *Lex Parsimoniae*, as discussed in [30,31]. The proof of redundancy makes recourse to Lemma 1, a basic mathematical result in Continuum Kinematics which appears to be neither exposed nor quoted as it deserves to be.

4 Progress

The space $\mathcal{H}_{\mathcal{D}}$ of stretching tensor fields is Hilbert when endowed with the norm topology of square integrable tensor fields according to the volume form $\boldsymbol{\mu}_{\mathbf{g}}$ induced by the material metric $\mathbf{g}_{\text{MAT}}$, or according to the associated mass-form:

$$\mathbf{m} = \rho\, \boldsymbol{\mu}_{\mathbf{g}} \,. \tag{35}$$

The topological dual $\mathcal{H}_{\mathcal{S}} = \mathcal{H}_{\mathcal{D}}^*$ is the Hilbert space of stress fields. Although these dual topological spaces are usually identified in mathematical treatments, to a unique pivot space, we will keep for them distinct notations for reasons of different physical dimension.

The bounded linear operators between Hilbert spaces:

$$\begin{cases} \mathbf{D}_{\mathcal{L}} \; : \mathcal{L} \mapsto \mathcal{H}_{\mathcal{D}} \,, & \text{distortion rate} \\ \mathbf{D}'_{\mathcal{L}} \; : \mathcal{H}_{\mathcal{S}} \mapsto \mathcal{L}^* \,, & \text{equilibrium} \end{cases} \tag{36}$$

where the subscript $\mathcal{L}$ emphasises restriction to conforming kinematics, are in duality, which means fulfilment for all $\mathbf{v} \in \mathcal{L}$ and $\mathbf{T} \in \mathcal{H}_{\mathcal{S}}$ of the equality:

$$\langle \mathbf{T}, \mathbf{D}_{\mathcal{L}}(\mathbf{v}) \rangle_{\boldsymbol{\Omega}} = \langle \mathbf{D}'_{\mathcal{L}}(\mathbf{T}), \mathbf{v} \rangle_{\boldsymbol{\Omega}} \,. \tag{37}$$

The distortion rate operator is $\mathbf{D} : \mathcal{V} \mapsto \mathcal{H}_{\mathcal{D}}$ so that rigid kinematic fields belong to $\mathbf{Ker}\mathbf{D}$. Conforming and rigid kinematic fields belong to $\mathbf{Ker}\mathbf{D}_{\mathcal{L}}$ given by:

$$\mathbf{Ker}\mathbf{D}_{\mathcal{L}} = \mathcal{L} \cap \mathbf{Ker}\mathbf{D} \,. \tag{38}$$

Duality Eq. (37) clearly implies the conditions:

$$\begin{cases} \mathbf{Ker}\mathbf{D}_{\mathcal{L}} & = {}^{\circ}(\mathbf{Im}\mathbf{D}'_{\mathcal{L}}) \subset \mathcal{L} \,, \\ \mathbf{Ker}\mathbf{D}'_{\mathcal{L}} & = (\mathbf{Im}\mathbf{D}_{\mathcal{L}})^{\circ} \subset \mathcal{H}_{\mathcal{S}} \,, \end{cases} \tag{39}$$

with the notations:

- The symbol $()^{\circ}$ denotes the annihilator in $\mathcal{L}^*$ according to the duality between $\mathcal{L}$ and $\mathcal{L}^*$.
- The symbol ${}^{\circ}()$ denotes the annihilator in $\mathcal{L}$ according to the duality between $\mathcal{L}$ and $\mathcal{L}^*$.

Banach's closed range theorem [4,51] states that closedness of the image of the operator $\mathbf{D}_{\mathcal{L}} : \mathcal{L} \mapsto \mathcal{H}_{\mathcal{D}}$ implies for the dual operator $\mathbf{D}'_{\mathcal{L}} : \mathcal{H}_{\mathcal{S}} \mapsto \mathcal{L}^*$ the same property and hence validity of the properties complementary to those in Eq. (39):

$$\begin{cases} \mathbf{ImD}'_{\mathcal{L}} = (\mathbf{KerD}_{\mathcal{L}})^{\circ} \subset \mathcal{L}^* \,, \\ \mathbf{ImD}_{\mathcal{L}} = {}^{\circ}(\mathbf{KerD}'_{\mathcal{L}}) \subset \mathcal{H}_{\mathcal{D}} \,. \end{cases} \tag{40}$$

These deep existence results are basic for proving the propositions exposed in the next Section.

4.1 Complementary principles

A loading system is in equilibrium under kinematic constraint defined by a linear subspace $\mathcal{L}$ of conforming kinematic fields if it fulfils the condition:

$$\ell \in (\mathbf{KerD}_{\mathcal{L}})^{\circ} \subset \Lambda := \mathcal{L}^* = \mathcal{V}^*/\mathcal{L}^{\circ} \,. \tag{41}$$

The last equality means that loading systems are force systems in $\mathcal{F} := \mathcal{V}^*$ defined to within addition of arbitrary reaction systems $\mathbf{r} \in \mathcal{L}^{\circ}$.

A stretching field $\boldsymbol{\Delta} \in \mathcal{H}_{\mathcal{D}}$ is *compatible* with the kinematic constraint if it fulfils the condition:

$$\boldsymbol{\Delta} \in \mathbf{ImD}_{\mathcal{L}} = \mathbf{D}\mathcal{L} \subset \mathcal{H}_{\mathcal{D}} \,. \tag{42}$$

The tensor fields $\mathbf{T} \in \mathbf{KerD}'_{\mathcal{L}}$ are *self-equilibrated* Cauchy stress fields, briefly *self-stresses*, in equilibrium with a null load functional.

The variational condition of kinematic compatibility is therefore the counterpart of the variational condition of equilibrium according to the following static-kinematic complementarity.

Rigid body acts of motion play the role of test fields for the variational equilibrium condition (a fact well-known in Statics to explain functioning of lever or seesaw and directly connected with the meaning of the term *equi-libria*).

Self-equilibrated stress fields play the role of test fields in the variational kinematic condition of compatibility (a definitely less evident statement, often neither proven nor quoted in text-books on Structural Mechanics).

A marked difference is mainly due to the fact that, in the standard theory of 3D continua, rigid body acts of motion form a finite dimensional linear space, so that the equilibrium condition requires only a finite number of verifications.

On the contrary, the space of self-equilibrated stress fields is not finite dimensional, in general.

A significant exception is provided by the theory of structural systems composed by beam models where self-stress fields are finitely generated and hence the variational kinematic compatibility condition provides a useful method of analysis, well-known to structural engineers as a necessary condition, widely adopted in applications.

On the basis of the deep existence results provided by Functional Analysis, as reported above in Eq. (40), two complementary principles, the Static Virtual Power Principle (SVPP), the one usually simply referred to as Virtual Power Principle (VPP), and the Kinematic Virtual Power Principle (KVPP), may be stated.

Theorem 1 (Static Virtual Power Principle) *Existence of at least a square integrable Cauchy stress tensor field* $\mathbf{T} \in \mathcal{S}$ *fulfilling the static equilibrium variational condition Eqs. (24–31):*

$$\langle \ell, \delta\mathbf{v} \rangle_{\boldsymbol{\Omega}_{\mathcal{E}}} = \int_{\boldsymbol{\Omega}} \langle \mathbf{T}, \mathbf{D}(\delta\mathbf{v}) \rangle \, \boldsymbol{\mu} \,, \quad \forall \; \delta\mathbf{v} \in \mathcal{L} \,, \tag{43}$$

is assured iff the prescribed load $\ell \in \Lambda = \mathcal{L}^*$ *fulfils the equilibrium condition in Eq. (41):*

$$\ell \in (\mathbf{KerD}_{\mathcal{L}})^{\circ} \,, \tag{44}$$

whose variational expression reads as a formulation of the Principle of Solidification:

$$\langle \ell, \delta\mathbf{v} \rangle_{\boldsymbol{\Omega}_{\mathcal{E}}} = 0 \,, \quad \forall \; \delta\mathbf{v} \in \mathbf{KerD}_{\mathcal{L}} \,. \tag{45}$$

Theorem 2 (Kinematic Virtual Power Principle) *Existence of at least a conforming kinematic vector field* $\mathbf{v} \in \mathcal{L}$ *fulfilling the compatibility variational equation:*

$$\langle \boldsymbol{\Delta}, \delta\mathbf{T} \rangle_{\boldsymbol{\Omega}} = \int_{\boldsymbol{\Omega}} \langle \mathbf{D}(\mathbf{v}), \delta\mathbf{T} \rangle \, \boldsymbol{\mu} \,, \quad \forall \, \delta\mathbf{T} \in \mathcal{H}_{\mathcal{S}} \,, \tag{46}$$

is assured iff the prescribed distortion rate $\boldsymbol{\Delta} \in \mathcal{H}_{\mathcal{D}}$ *fulfils the kinematic condition Eq. (42):*

$$\boldsymbol{\Delta} \in {}^{\circ}(\mathbf{KerD}'_{\mathcal{L}}) \subset \mathcal{H}_{\mathcal{D}} \,. \tag{47}$$

The equivalent variational formulation reads as a *Principle of Self-Equilibrium*:

"*The kinematic compatibility of a distortion rate is characterised by vanishing of the internal virtual power for any self-equilibrated stress field.*"

$$\langle \boldsymbol{\Delta}, \delta\mathbf{T} \rangle_{\boldsymbol{\Omega}} = 0 \,, \quad \forall \, \delta\mathbf{T} \in \mathbf{KerD}'_{\mathcal{L}} = (\mathbf{D}\mathcal{L})^{\circ} \,. \tag{48}$$

Thanks to Frigyes Riesz representation theorem in Hilbert spaces [4], the pairing at the l.h.s. of Eqs. (46) and (48) can be expressed as an integral of local inner products of square integrable tensor fields:

$$\langle \boldsymbol{\Delta}, \delta\mathbf{T} \rangle_{\boldsymbol{\Omega}} = \int_{\boldsymbol{\Omega}} \langle \boldsymbol{\Delta}, \delta\mathbf{T} \rangle \, \boldsymbol{\mu} \,. \tag{49}$$

5 Rate virtual power principle

For theoretical and applicative reasons, such as computation of incremental solutions of nonlinear structural problems and investigation on instability phenomena, it is essential to provide the expression of the time-derivative of the Virtual Power Principle Eq. (24) along the motion [52]:

$$\partial_{\alpha=0} \, \langle \ell, \delta\mathbf{v} \rangle_{\boldsymbol{\varphi}_{\alpha}(\boldsymbol{\Omega}_{\mathcal{E}})} = \partial_{\alpha=0} \int_{\boldsymbol{\varphi}_{\alpha}(\boldsymbol{\Omega})} \langle \boldsymbol{\sigma} \,, \boldsymbol{\epsilon}_{\delta\mathbf{v}} \rangle \, \mathbf{m} \,. \tag{50}$$

Here the virtual velocity field $\delta\mathbf{v} : \mathcal{T}_{\mathcal{E}} \mapsto V\mathcal{T}_{\mathcal{E}}$ is generated on the trajectory by extending a conforming spatial virtual velocity $\delta\mathbf{v} : \boldsymbol{\Omega}_{\mathcal{E}} \mapsto T_{\boldsymbol{\Omega}_{\mathcal{E}}}\mathcal{S} \in \mathcal{L}$ by means of parallel transport $\Uparrow$ along the motion according to the adopted connection ∇.

In fact it is essential that the *geometric paradigm* introduced in [23] be taken into account:[14]

- *Spatial fields*, have domain in the current spatial configuration and take values in tensor fibers of the ambient space. Therefore spatial fields can only be transformed by parallel transport along curves in the immersed trajectory.
- *Material fields*, have domain in the current configuration and take values in tensor fibers of the configuration manifold. Material fields can only be transformed along the trajectory by push-pull according to the motion.

Theorem 3 (Rate Virtual Power Principle) *Evaluating the time rates in Eq. (50) by Leibniz rule we get:*

$$\langle \dot{\ell}, \delta\mathbf{v} \rangle_{\boldsymbol{\Omega}_{\mathcal{E}}} = \int_{\boldsymbol{\Omega}} \langle \dot{\boldsymbol{\sigma}} \,, \boldsymbol{\epsilon}_{\delta\mathbf{v}} \rangle \, \mathbf{m} + \int_{\boldsymbol{\Omega}} \langle \boldsymbol{\sigma} \,, \dot{\boldsymbol{\epsilon}}(\mathbf{v}_{\boldsymbol{\varphi}} \,, \delta\mathbf{v}) \rangle \, \mathbf{m} \,, \tag{51}$$

for all $\delta\mathbf{v} \in \mathcal{L}$.

Proof Performing the rates of spatial and material fields according to the *geometric paradigm*, we get the (RVPP) Eq. (51), where, setting $\boldsymbol{\varphi}_{\alpha}\Downarrow := \boldsymbol{\varphi}_{-\alpha}\Uparrow$:

Loading rate (spatial):

$$\begin{aligned} \dot{\ell} := \nabla_{\mathbf{v}_{\boldsymbol{\varphi}}}(\ell) &= \partial_{\alpha=0} \, (\boldsymbol{\varphi}_{\alpha}\Downarrow\ell) \\ &= \partial_{\alpha=0} \, \boldsymbol{\varphi}_{\alpha}\Downarrow(\ell \circ \boldsymbol{\varphi}_{\alpha}) \,. \end{aligned} \tag{52}$$

[14] The definition of *spatial* and *material* tensor fields as specified in [21] is based on a diffeomorphic correspondence between the current configuration and a referential one, with referential fields named material. Our definition before Eq. (16) refers to field ranges, depending on whether their values are tensors on the space slice or on the configuration manifold. These fields must then be kept well distinct one another since there is no natural correspondence between them other than the one-way pull back of spatial tensors to material ones by immersion.

Virtual velocity rate (spatial) vanishing by *construction*:

$$\dot{\delta\mathbf{v}} := \nabla_{\mathbf{v}_{\boldsymbol{\varphi}}}(\delta\mathbf{v}) = \partial_{\alpha=0}\,(\boldsymbol{\varphi}_\alpha \Downarrow \delta\mathbf{v}) \\ = \partial_{\alpha=0}\,\boldsymbol{\varphi}_\alpha \Downarrow (\delta\mathbf{v} \circ \boldsymbol{\varphi}_\alpha)\,. \tag{53}$$

Mass rate (material) vanishing by *Mass Conservation*:

$$\dot{\mathbf{m}} := \mathcal{L}_{\mathbf{i}\downarrow\mathbf{v}_{\boldsymbol{\varphi}}}(\mathbf{m}) = \partial_{\alpha=0}\,(\boldsymbol{\varphi}_\alpha \downarrow \mathbf{m}) \\ = \partial_{\alpha=0}\,\boldsymbol{\varphi}_\alpha \downarrow (\mathbf{m} \circ \boldsymbol{\varphi}_\alpha) = \mathbf{0}\,. \tag{54}$$

Natural stress rate (material):

$$\dot{\boldsymbol{\sigma}} := \mathcal{L}_{\mathbf{i}\downarrow\mathbf{v}_{\boldsymbol{\varphi}}}(\boldsymbol{\sigma}) = \partial_{\alpha=0}\,(\boldsymbol{\varphi}_\alpha \downarrow \boldsymbol{\sigma}) \\ = \partial_{\alpha=0}\,\boldsymbol{\varphi}_\alpha \downarrow (\boldsymbol{\sigma} \circ \boldsymbol{\varphi}_\alpha)\,. \tag{55}$$

Virtual stretching rate (material):

$$\dot{\boldsymbol{\epsilon}}(\mathbf{v}_{\boldsymbol{\varphi}}, \delta\mathbf{v}) := \mathcal{L}_{\mathbf{i}\downarrow\mathbf{v}_{\boldsymbol{\varphi}}}(\boldsymbol{\epsilon}_{\delta\mathbf{v}}) = \mathcal{L}_{\mathbf{i}\downarrow\mathbf{v}_{\boldsymbol{\varphi}}}\Big(\mathbf{i}\downarrow\mathcal{L}_{\delta\mathbf{v}}(\mathbf{g}_{\mathrm{SPA}})\Big) \\ = \mathbf{i}\downarrow\Big(\mathcal{L}_{\mathbf{v}_{\boldsymbol{\varphi}}}\mathcal{L}_{\delta\mathbf{v}}(\mathbf{g}_{\mathrm{SPA}})\Big)\,. \tag{56}$$

□

The time rate of the rate of distortion in Eq. (56) can be evaluated in terms of derivatives according to a flat Levi-Civita connection to yield the formula [52]:

$$\dot{\boldsymbol{\epsilon}}(\mathbf{v}_{\boldsymbol{\varphi}}, \delta\mathbf{v}) = \mathbf{g}_{\mathrm{MAT}} \cdot \dot{\mathbf{D}}(\mathbf{v}_{\boldsymbol{\varphi}}, \delta\mathbf{v})\,, \tag{57}$$

with the mixed alteration of the material rate of stretching, given by:

$$\dot{\mathbf{D}}(\mathbf{v}_{\boldsymbol{\varphi}}, \delta\mathbf{v}) := \boldsymbol{\Pi} \cdot \mathrm{sym}_{\mathbf{g}}\Big((\nabla\mathbf{v}_{\boldsymbol{\varphi}})^A \cdot \nabla\delta\mathbf{v}\Big) \cdot \boldsymbol{\Pi}^A\,. \tag{58}$$

6 Rate elastic equilibrium

In continuum rate elastic problems, kinematic compatibility requires that the geometric natural stretching $\boldsymbol{\epsilon}(\mathbf{v}_{\boldsymbol{\varphi}})$ Eq. (10) is composed of elastic stretching $\dot{\mathbf{e}}$ and anelastic stretching $\boldsymbol{\theta}$:

$$\boldsymbol{\epsilon}(\mathbf{v}_{\boldsymbol{\varphi}}) = \dot{\mathbf{e}} + \boldsymbol{\theta}\,. \tag{59}$$

According to the imposed affine kinematic constraints, the motion velocity can be additively decomposed as sum of a prescribed constraint velocity $\mathbf{w} \in \mathcal{V}/\mathcal{L}$ and a conforming velocity $\mathbf{v} \in \mathcal{L}$:[15]

$$\mathbf{v}_{\boldsymbol{\varphi}} = \mathbf{w} + \mathbf{v}\,. \tag{60}$$

In the rate model the elastic stretching $\dot{\mathbf{e}}$, Lie derivative of the elastic state field $\mathbf{e}$ along the motion, is the constitutive response to the natural stress rate field $\dot{\boldsymbol{\sigma}}$ by the natural stress dependent tangent elastic compliance $\mathbf{C}(\boldsymbol{\sigma})$, as expressed by the rate elastic law [24]:

$$\dot{\mathbf{e}} = \mathbf{C}(\boldsymbol{\sigma}) \cdot \dot{\boldsymbol{\sigma}}\,, \tag{61}$$

with inverse:

$$\dot{\boldsymbol{\sigma}} = \mathbf{E}(\boldsymbol{\sigma}) \cdot \dot{\mathbf{e}}\,. \tag{62}$$

[15] We will not distinguish between the field $\mathbf{w} \in \mathcal{V}$ and the coset of the quotient space $\mathcal{V}/\mathcal{L}$ which it belongs to. In fact the solution $\mathbf{v}_{\boldsymbol{\varphi}} = \mathbf{v} + \mathbf{w}$ of the thermoelastic rate problem Eq. (64) is independent of the representative chosen therein.

In [24] it is proven that the internal mechanical work vanishes in closed cycles of natural stress fields, with closedness evaluated by pull-back along the motion. Difficulties outlined in [26] and taken for granted in [53] were thus overcome. From Eqs. (59), (60), and (62) we get:

$$\dot{\boldsymbol{\sigma}} = \mathbf{E}(\boldsymbol{\sigma}) \cdot \Big(\boldsymbol{\epsilon}(\mathbf{v}) + \boldsymbol{\epsilon}(\mathbf{w}) - \boldsymbol{\theta} \Big) . \tag{63}$$

Insertion of Eq. (63) in the RVPP Eq. (51) yields the variational principle governing the Rate Elasticity Problem (REP), for all $\delta\mathbf{v} \in \mathcal{L}$:

$$\langle \dot{\ell} + \dot{\ell}_{\mathrm{EQ}}(\mathbf{w}\,,\boldsymbol{\theta}), \delta\mathbf{v} \rangle_{\boldsymbol{\Omega}_{\mathcal{E}}} = \int_{\boldsymbol{\Omega}} \langle \mathbf{E}(\boldsymbol{\sigma}) \cdot \boldsymbol{\epsilon}(\mathbf{v}), \boldsymbol{\epsilon}(\delta\mathbf{v}) \rangle \, \mathbf{m} + \int_{\boldsymbol{\Omega}} \langle \boldsymbol{\sigma}, \dot{\boldsymbol{\epsilon}}(\mathbf{v}\,, \delta\mathbf{v}) \rangle \, \mathbf{m} \,, \tag{64}$$

with the equivalent loading rate $\dot{\ell}_{\mathrm{EQ}}(\mathbf{w}\,,\boldsymbol{\theta})$ defined by:

$$\langle \dot{\ell}_{\mathrm{EQ}}(\mathbf{w}\,,\boldsymbol{\theta}), \delta\mathbf{v} \rangle_{\boldsymbol{\Omega}_{\mathcal{E}}} := \int_{\boldsymbol{\Omega}} \langle \mathbf{E}(\boldsymbol{\sigma}) \cdot (\boldsymbol{\theta} - \boldsymbol{\epsilon}(\mathbf{w})), \boldsymbol{\epsilon}(\delta\mathbf{v}) \rangle \, \mathbf{m} - \int_{\boldsymbol{\Omega}} \langle \boldsymbol{\sigma}, \dot{\boldsymbol{\epsilon}}(\mathbf{w}\,, \delta\mathbf{v}) \rangle \, \mathbf{m} \,. \tag{65}$$

This variational expression of the REP is the one suited for numerical computations since trial fields $\mathbf{v} \in \mathcal{L}$ and test fields $\delta\mathbf{v} \in \mathcal{L}$ belong to the same linear conformity subspace.

Inclusion of rate constitutive description of non-elastic behaviour is directly implementable and will be illustrated in detail in a forthcoming contribution.

In this context, of major interest is also a planned revisitation of classical treatments [54,55], respectively, concerning conservation laws in finite elasticity and invariance in solid mechanics.

7 Application to instability of elastic structures

The variational REP exposed in Eq. (64) finds a natural application in investigating phenomena of elastic instability.

In terms of a linear symmetric tangent stiffness:

$$\mathcal{K}(\boldsymbol{\sigma}) : \mathcal{L} \mapsto \mathcal{L}^* = \mathcal{V}^*/\mathcal{L}^\circ \,, \tag{66}$$

defined by:

$$\langle \mathcal{K}(\boldsymbol{\sigma}) \cdot \mathbf{v}, \delta\mathbf{v} \rangle_{\boldsymbol{\Omega}_{\mathcal{E}}} := \int_{\boldsymbol{\Omega}} \langle \mathbf{E}(\boldsymbol{\sigma}) \cdot \boldsymbol{\epsilon}(\mathbf{v}), \boldsymbol{\epsilon}(\delta\mathbf{v}) \rangle \, \mathbf{m} + \int_{\boldsymbol{\Omega}} \langle \boldsymbol{\sigma}, \dot{\boldsymbol{\epsilon}}(\mathbf{v}\,, \delta\mathbf{v}) \rangle \, \mathbf{m} \,, \tag{67}$$

the REP variational principle Eq. (64) may be rewritten in abridged form as:

$$\mathcal{K}(\boldsymbol{\sigma}) \cdot \mathbf{v} = \dot{\ell} + \dot{\ell}_{\mathrm{EQ}}(\mathbf{w}\,,\boldsymbol{\theta}) = \dot{\ell}_{\mathrm{EFF}} \,. \tag{68}$$

Singularity of the tangent stiffness $\mathcal{K}(\boldsymbol{\sigma})$ triggers instability phenomena. The kernel of the linear tangent operator $\mathcal{K}(\boldsymbol{\sigma})$ provides the instability modes. At a critical configuration the lowest principal value vanishes. In simple cases when:

$$\dim(\mathbf{Ker}(\mathcal{K}(\boldsymbol{\sigma}))) = 1 \,, \tag{69}$$

the equilibrium path either attains ca limit point or bifurcates, depending on whether the instability modes are annihilated by the loading rate $\dot{\ell} \in \mathcal{L}^*$ or do not.

A bifurcation occurs when:

$$\dot{\ell}_{\mathrm{EFF}} \in \Big(\mathbf{Ker}(\mathcal{K}(\boldsymbol{\sigma}))\Big)^\circ = \mathbf{Im}(\mathcal{K}(\boldsymbol{\sigma})) \,, \tag{70}$$

because the rate equilibrium problem Eq. (68) admits multiple solutions.

If the annihilation condition in Eq. (70) is not met, the linear problem Eq. (68) does not admit solution at all, and a limit point is arrived at.

A displacement driven perturbative analysis may be resorted to investigate on the stability of the subsequent post-critical static equilibrium path.

The seminal doctoral thesis of Warner Tjardus Koiter [56] written in Dutch, under the supervision of Cornelis Benjamin Biezeno, during the Nazi occupation of The Netherlands in World War II, was translated into English some 25 years later by Edward Riks [57], most-known for his arc length method [58–60].

The post-critical scenario depends on the subsequent behaviour of the lowest principal value along the continuation of the previous equilibrium path and along the new one, as first discussed, by means of a perturbative analysis of the eigenvalue problem in [61–64].

As briefly illustrated above, the description of elastic instability set out in terms of rate equilibrium does not involve the troublesome notion of elastic strain energy functional.

As a matter of fact, the very formulation of an elastic strain energy in terms of finite strains clashes against a conceptual difficulty because, at difference from the stress field, the elastic finite strain field is not a state variable because it relies upon a comparison between the actual and another reference state [24].

When chain decompositions of the deformation gradient are attempted with the aim of modelling elasto-thermo-plastic behaviours and similar ones [65,66], further issues are met. In fact a multiplicity of intermediate configurations have to be conceived and ordered by non-commutativity, without any physical basis.

The innovative treatment here outlined differs for basic features from standard ones, as exposed e.g. in [67, Ch.20], [68, Sec.12.3.1], [69, Ch.5]. It makes resort to a stress dependent rate elastic constitutive relation between natural stressing and elastic stretching Eqs. (61–62) which is mathematically simple, physically sound, and experimentally measurable.

The analysis is therefore applicable to continuous structural models of any geometric dimensionality and both to classical and structured continua with non-redundant formulations.

The rate theory offers moreover the possibility of including non-elastic rate constitutive relations by a well-defined combination of pertinent stretching tensors, by commutative addition, to yield the overall geometric stretching tensor.

8 Closing remarks

Notable are the Latin mottos of medieval Franciscan and Dominican friars:

- Roger BACON (Doctor Mirabilis) (1219–1292)
  "Prudens quaestio dimidium scientiae"
  (Opus Maius, 1267),
- Thomas Aquinas (Doctor Angelicus) (1225–1274)
  "Hominem unius libri timeo",
- John Duns Scotus (Doctor Subtilis) (1265–1308)
  "Pluralitas non est ponenda sine necessitate"
  (Opus Oxoniense, 1304),
- William of Ockham (Doctor Invincibilis) (1285–1347)
  "Frustra fit per plura quod potest fieri per pauciora"
  (Summa logicae, 1323).

The content of the latter is referred to as *lex parsimoniae* or *Novacula Occami* (Ockham's razor):

- *"It is vain to make with many what can be achieved with less"*.

This logical requirement is effective and widely adopted in many fields including literal expressions and technical design criteria in engineering.

In our opinion, Ockham's razor must be applied as a guiding principle in the construction of theoretical schemes for the mechanical modelling of real behaviours emerging from observation and experimental testing.

This warning is not trivial nor needless because proposed extensions of classical continuum mechanics, intended for applications to structured continua, have failed to discover and notice violations of this *golden rule*, thus bearing conceptual and applicative difficulties to the involved structural models, as discussed in Sect. 3.1.

Attention to this aspect in the formulation of the VPP is a distinctive feature of the present treatment in comparison with previous ones.

In continuum mechanics, a variational theory founded on the Rate Virtual Power Principle and on the Rate Theory of Elasticity, has natural extensions to formulations involving non-elastic constitutive schemes. Indeed, commutativity of the addition between rate tensor fields, with common domain a configuration manifold, makes this task performable in a physically meaningful way and with a mathematically consistent treatment.

The issue of impossibility of mechanical internal energy conservation in stress cycles of elastic materials formulated according to a rate theory, which was considered as obstruction in [25,26] and in subsequent literature, can be resolved by the above described geometric approach which is based on the adoption of natural-stress and stretching measures, and of their convective time derivatives along the motion, in the constitutive elastic relation in rate form, as discussed in Sect. 6.

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