Florinda Capone · Roberta De Luca · Giuliana Massa

Effect of anisotropy on the onset of convection in rotating bi-disperse Brinkman porous media

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Abstract Thermal convection in a horizontally isotropic bi-disperse porous medium (BDPM) uniformly heated from below is analysed. The combined effects of uniform vertical rotation and Brinkman law on the stability of the steady state of the momentum equations in a BDPM are investigated. Linear and nonlinear stability analysis of the conduction solution is performed, and the coincidence between linear instability and nonlinear stability thresholds in the L^2 -norm is obtained.

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1 Introduction

Thermal convection in single porous layers has attracted—in the past as nowadays—the interest of many authors due to its numerous applications in industrial processes, in geophysics and in astronomy, for instance. Starting from a detailed theoretical analysis [20, 26, 29–31], several mathematical models have been introduced in order to better describe different aspects of the phenomenon (see [1–4, 8–11, 14, 17, 18, 23–25] and the references therein). In particular, in [2] mixed convection flow has been numerically studied; in [4, 11], the effect of an external magnetic field induced in an electrically conducting fluid has been investigated; in [17] double-diffusive convection is analysed, while [24] deals with double-diffusive convection according to the Brinkman model.

Recently, the attention of researchers has turned to the onset of thermal convection in bi-disperse porous media (BDPM). A BDPM, as defined in [13], is composed of *clusters* of large particles that are agglomerations of small particles, there are macropores between the clusters and micropores within them, and in particular, the macropores are referred to as f-phase, while the remainder of the structure is referred to as p-phase. The reason of this nomenclature is that one can think of the f-phase as being a fracture phase, the p-phase as a porous phase. The fundamental theory for thermal convection in BDPM can be found in [22, 26]. In particular, as regards the porosity, a bi-disperse porous medium is a double porosity material. Precisely, denoting by φ the porosity of the macropores and by ϵ the porosity of the micropores, $(1 - \varphi)\epsilon$ represents the fraction of volume occupied by the micropores, $\varphi + (1 - \varphi)\epsilon$ is the fraction of volume occupied by the fluid, and $(1 - \epsilon)(1 - \varphi)$ is

Florinda Capone (✉) · Roberta De Luca · Giuliana Massa
Dipartimento di Matematica e Applicazioni 'R.Caccioppoli', Università degli Studi di Napoli Federico II, Via Cintia, Monte S. Angelo, 80126 Naples, Italy
E-mail: f.capone@unina.it

Roberta De Luca
E-mail: roberta.deluca@unina.it

Giuliana Massa
E-mail: giuliana.massa@unina.it

the fraction of volume occupied by the solid skeleton. Double porosity materials are employed in engineering, medicine, chemistry, since brain, bones or heat pipes can be modelled as bi-disperse porous media (see [15,26] and references therein). In particular, the use of anisotropic bi-disperse porous materials may have much more potential, compared to the anisotropic single porous media, due to many possibilities to design man-made materials for heat transfer or insulation problems (see [7,8,10,26–28] and references therein).

The study of fluid flow in rotating porous media is motivated by its theoretical significance and practical applications (see [3,5–8,14,29] and the references therein) like, for example, in geophysics and in engineering (food process industry, chemical process industry, centrifugal filtration processes, rotating machinery). For instance, in [6], Capone et al. analyse the influence of vertical rotation on the onset of convection in a single temperature BDPM, according to Darcy's law, while in [8], the authors investigate the onset of convection in a horizontal, vertical rotating, anisotropic porous layer assuming a local thermal non-equilibrium.

The goal of the present paper is to study the onset of thermal convection in an anisotropic bi-disperse porous medium uniformly rotating about a vertical axis, assuming the validity of the Brinkman law both for the micropores and the macropores. In particular, the use of Brinkman law in both micropores and macropores for a bi-disperse porous medium has been introduced by Nield and Kuznetsov [21] accounting for the discussion on the dispersion in a BDPM made by Moutsopolous and Koch in [19]. In fact, in [19] the authors prove a good agreement between theoretical predictions and experimental measurements when one considers a dilute array of large spheres in a Brinkman medium and the flow around the large spheres is modelled using Brinkman's equation.

The plan of the paper is as follows. In Sect. 2, we introduce the mathematical model and, in order to study the stability of the conduction solution, we introduce the dimensionless equations for the evolution (in time) of the perturbation fields. In Sect. 3, we perform the instability analysis of the conduction solution and we prove the validity of the strong form of the principle of exchange of stabilities, which means that if the convection sets in, it arises via a stationary state. Section 4 deals with the nonlinear stability analysis of the conduction solution, and the coincidence between instability and (global) nonlinear stability thresholds in the L^2 -norm is proved. Numerical simulations concerning the asymptotic behaviour of the instability threshold with respect to the meaningful parameters of the model are performed in Sect. 5. The paper ends with a concluding section (Sect. 6), in which all the obtained results are collected.

2 Mathematical model

Let $Oxyz$ be a reference frame with fundamental unit vectors $\mathbf{i}, \mathbf{j}, \mathbf{k}$ ($\mathbf{k}$ pointing vertically upward). Let L be a bi-disperse porous layer of thickness d uniformly heated from below and rotating about the vertical axis z , with constant angular velocity $\Omega = \Omega \mathbf{k}$. Let L be saturated by a homogeneous incompressible fluid at rest state, and let us assume the validity of the local thermal equilibrium between the f-phase and the p-phase, i.e. $T^f = T^p = T$. The saturated bi-disperse porous medium is also supposed to be *horizontally isotropic*. Let the axes (x, y, z) be the *principal axes* of the permeability, so the macropermeability tensor and the micropermeability tensor may be written as:

$$\begin{aligned}\mathbf{K}^f &= \text{diag}(K_x^f, K_y^f, K_z^f) = K_z^f \mathbf{K}^{f*}, \\ \mathbf{K}^p &= \text{diag}(K_x^p, K_y^p, K_z^p) = K_z^p \mathbf{K}^{p*}, \\ \mathbf{K}^{f*} &= \text{diag}(k, k, 1), \quad \mathbf{K}^{p*} = \text{diag}(h, h, 1),\end{aligned}$$

where

$$k = \frac{K_x^f}{K_z^f} = \frac{K_y^f}{K_z^f}, \quad h = \frac{K_x^p}{K_z^p} = \frac{K_y^p}{K_z^p}.$$

A Boussinesq approximation is used, whereby the density is constant except in the buoyancy forces, which are linear in temperature. Taking into account the Coriolis terms due to the uniform rotation of the layer about the vertical axis z for the micropores and the macropores [6] and extending the Brinkman model for a simple porous medium to BDPM [21], the relevant equations are:

$$\begin{cases} \mathbf{v}^f = \mu^{-1} \mathbf{K}^f \cdot \left[-\zeta(\mathbf{v}^f - \mathbf{v}^p) - \nabla p^f + \varrho_F \alpha g T \mathbf{k} - \frac{2\varrho_F \Omega}{\varphi} \mathbf{k} \times \mathbf{v}^f + \tilde{\mu}_f \Delta \mathbf{v}^f \right], \\ \mathbf{v}^p = \mu^{-1} \mathbf{K}^p \cdot \left[-\zeta(\mathbf{v}^p - \mathbf{v}^f) - \nabla p^p + \varrho_F \alpha g T \mathbf{k} - \frac{2\varrho_F \Omega}{\epsilon} \mathbf{k} \times \mathbf{v}^p + \tilde{\mu}_p \Delta \mathbf{v}^p \right], \\ \nabla \cdot \mathbf{v}^f = 0, \\ \nabla \cdot \mathbf{v}^p = 0, \\ (\varrho c)_m T_{,t} + (\varrho c)_f (\mathbf{v}^f + \mathbf{v}^p) \cdot \nabla T = k_m \Delta T, \end{cases} \quad (1)$$

where

$$p^s = P^s - \frac{\varrho_F}{2} |\Omega \times \mathbf{x}|^2, \quad s = \{f, p\}$$

is the reduced pressure, with $\mathbf{x} = (x, y, z)$, $\mathbf{v}^s$ = seepage velocity, P^s = pressure, ϱ = density, ζ = interaction coefficient between the f-phase and the p-phase, $\mathbf{g} = -g\mathbf{k}$ = gravity, μ = fluid viscosity, $\tilde{\mu}_s$ = effective viscosity, ϱ_F = reference constant density, α = thermal expansion coefficient, c = specific heat, c_p = specific heat at a constant pressure, $(\varrho c)_m = (1 - \varphi)(1 - \epsilon)(\varrho c)_{sol} + \varphi(\varrho c)_f + \epsilon(1 - \varphi)(\varrho c)_p$, $k_m = (1 - \varphi)(1 - \epsilon)k_{sol} + \varphi k_f + \epsilon(1 - \varphi)k_p$ = thermal conductivity (The subscript *sol* refers to the solid skeleton).

To (1), the following boundary conditions are appended:

$$\begin{aligned} \mathbf{v}^s \cdot \mathbf{n} &= 0, \quad s = \{f, p\} \quad \text{on } z = 0, d, \\ T &= T_L, \quad \text{on } z = 0, \quad T = T_U, \quad \text{on } z = d \end{aligned} \quad (2)$$

where $\mathbf{n}$ is the unit outward normal to the impermeable horizontal planes delimiting the layer and $T_L > T_U$. The problem (1)–(2) admits the stationary conduction solution:

$$\bar{\mathbf{v}}^f = \mathbf{0}, \quad \bar{\mathbf{v}}^p = \mathbf{0}, \quad \bar{T} = -\beta z + T_L,$$

where $\beta = \frac{T_L - T_U}{d}$ is the temperature gradient. Denoting by $\{\mathbf{u}^f, \mathbf{u}^p, \theta, \pi^f, \pi^p\}$ a perturbation to the steady solution, one recovers that the evolutionary system governing the perturbation fields is given by

$$\begin{cases} \mathbf{u}^f = \mu^{-1} \mathbf{K}^f \cdot \left[-\zeta(\mathbf{u}^f - \mathbf{u}^p) - \nabla \pi^f + \varrho_F \alpha g \theta \mathbf{k} - \frac{2\varrho_F \Omega}{\varphi} \mathbf{k} \times \mathbf{u}^f + \tilde{\mu}_f \Delta \mathbf{u}^f \right], \\ \mathbf{u}^p = \mu^{-1} \mathbf{K}^p \cdot \left[-\zeta(\mathbf{u}^p - \mathbf{u}^f) - \nabla \pi^p + \varrho_F \alpha g \theta \mathbf{k} - \frac{2\varrho_F \Omega}{\epsilon} \mathbf{k} \times \mathbf{u}^p + \tilde{\mu}_p \Delta \mathbf{u}^p \right], \\ \nabla \cdot \mathbf{u}^f = 0, \\ \nabla \cdot \mathbf{u}^p = 0, \\ (\varrho c)_m \theta_{,t} + (\varrho c)_f (\mathbf{u}^f + \mathbf{u}^p) \cdot \nabla \theta = (\varrho c)_f \beta (w^f + w^p) + k_m \Delta \theta, \end{cases} \quad (3)$$

where $\mathbf{u}^f = (u^f, v^f, w^f)$, $\mathbf{u}^p = (u^p, v^p, w^p)$. Introducing the non-dimensional parameters

$$\begin{aligned} \mathbf{x}^* &= \frac{\mathbf{x}}{d}, \quad t^* = \frac{t}{\tilde{t}}, \quad \theta^* = \frac{\theta}{\tilde{T}}, \quad \mathbf{u}^{s*} = \frac{\mathbf{u}^s}{\tilde{u}}, \quad \pi^{s*} = \frac{\pi^s}{\tilde{P}}, \quad \text{for } s = \{f, p\}, \\ \eta &= \frac{\varphi}{\epsilon}, \quad \sigma = \frac{\tilde{\mu}_p}{\tilde{\mu}_f}, \quad \gamma_1 = \frac{\mu}{K_z^f \zeta}, \quad \gamma_2 = \frac{\mu}{K_z^p \zeta}, \end{aligned}$$

where the scales are given by

$$\tilde{u} = \frac{k_m}{(\varrho c)_f d}, \quad \tilde{t} = \frac{d^2 (\varrho c)_m}{k_m}, \quad \tilde{P} = \frac{\zeta k_m}{(\varrho c)_f}, \quad \tilde{T} = \sqrt{\frac{\beta k_m \zeta}{(\varrho c)_f \varrho_F \alpha g}},$$

and setting

$$\mathcal{T} = \frac{2\varrho_F \Omega K_z^f}{\varphi \mu}, \quad Da_f = \frac{\tilde{\mu}_f K_z^f}{d^2 \mu}, \quad R = \sqrt{\frac{\beta d^2 (\varrho c)_f \varrho_F \alpha g}{k_m \zeta}},$$

which are the Taylor number $\mathcal{T}$, the Darcy number Da_f , and the thermal Rayleigh number R , respectively, and the resulting non-dimensional perturbation equations, dropping all the asterisks, are

$$\begin{cases} \gamma_1(\mathbf{K}^f)^{-1}\mathbf{u}^f + (\mathbf{u}^f - \mathbf{u}^p) = -\nabla\pi^f + R\theta\mathbf{k} - \gamma_1\mathcal{T}\mathbf{k} \times \mathbf{u}^f + Da_f\gamma_1\Delta\mathbf{u}^f, \\ \gamma_2(\mathbf{K}^p)^{-1}\mathbf{u}^p - (\mathbf{u}^f - \mathbf{u}^p) = -\nabla\pi^p + R\theta\mathbf{k} - \eta\gamma_1\mathcal{T}\mathbf{k} \times \mathbf{u}^p + Da_f\gamma_1\sigma\Delta\mathbf{u}^p, \\ \nabla \cdot \mathbf{u}^f = 0, \\ \nabla \cdot \mathbf{u}^p = 0, \\ \theta_{,t} + (\mathbf{u}^f + \mathbf{u}^p) \cdot \nabla\theta = R(w^f + w^p) + \Delta\theta, \end{cases} \quad (4)$$

under the initial conditions

$$\mathbf{u}^s(\mathbf{x}, 0) = \mathbf{u}_0^s(\mathbf{x}), \quad \varphi^s(\mathbf{x}, 0) = \varphi_0(\mathbf{x}), \quad \theta(\mathbf{x}, 0) = \theta_0(\mathbf{x})$$

with $\nabla \cdot \mathbf{u}_0^s = 0$, $s = \{f, p\}$, and the stress-free boundary conditions [12]

$$u_{,z}^f = v_{,z}^f = u_{,z}^p = v_{,z}^p = w^f = w^p = \theta = 0 \quad \text{on } z = 0, 1. \quad (5)$$

Moreover, according to experimental results, let us assume that the perturbation fields are periodic functions in the x, y directions and denote by

$$V = \left[0, \frac{2\pi}{l}\right] \times \left[0, \frac{2\pi}{m}\right] \times [0, 1]$$

the periodicity cell.

3 Onset of convection

To analyse the onset of convection, i.e., to find the linear instability threshold of the conduction solution, we consider the linear version of (4) and seek for solutions in which $\mathbf{u}^f, \mathbf{u}^p, \theta, \pi^f, \pi^p$ have time dependence like $e^{\bar{\sigma}t}$, i.e.

$$\begin{cases} \gamma_1(\mathbf{K}^f)^{-1}\mathbf{u}^f + (\mathbf{u}^f - \mathbf{u}^p) = -\nabla\pi^f + R\theta\mathbf{k} - \gamma_1\mathcal{T}\mathbf{k} \times \mathbf{u}^f + Da_f\gamma_1\Delta\mathbf{u}^f, \\ \gamma_2(\mathbf{K}^p)^{-1}\mathbf{u}^p - (\mathbf{u}^f - \mathbf{u}^p) = -\nabla\pi^p + R\theta\mathbf{k} - \eta\gamma_1\mathcal{T}\mathbf{k} \times \mathbf{u}^p + Da_f\gamma_1\sigma\Delta\mathbf{u}^p, \\ \bar{\sigma}\theta = R(w^f + w^p) + \Delta\theta. \end{cases} \quad (6)$$

Let $(\cdot, \cdot)$ and $\|\cdot\|$ denote inner product and norm on the complex Hilbert space $L^2(V)$, respectively, and let $*$ denote the complex conjugate of a field. Multiplying (6)₁ by $\mathbf{u}^{f*}$, (6)₂ by $\mathbf{u}^{p*}$ and (6)₃ by θ^* , the integration over V leads to:

$$\begin{aligned} \bar{\sigma}\|\theta\|^2 &= -\gamma_1(\mathbf{K}^f)^{-1}\|\mathbf{u}^f\|^2 - \gamma_2(\mathbf{K}^p)^{-1}\|\mathbf{u}^p\|^2 - \|\mathbf{u}^f - \mathbf{u}^p\|^2 - \|\nabla\theta\|^2 \\ &\quad + R[(\theta, w^{f*} + w^{p*}) + (w^f + w^p, \theta^*)] \\ &\quad - \gamma_1\mathcal{T}(\mathbf{k} \times \mathbf{u}^f, \mathbf{u}^{f*}) - \eta\gamma_1\mathcal{T}(\mathbf{k} \times \mathbf{u}^p, \mathbf{u}^{p*}) \\ &\quad - Da_f\gamma_1\|\nabla\mathbf{u}^f\|^2 - Da_f\sigma\gamma_1\|\nabla\mathbf{u}^p\|^2. \end{aligned} \quad (7)$$

If $\bar{\sigma} = \bar{\sigma}_r + i\bar{\sigma}_i$, the imaginary part of (7) is

$$\bar{\sigma}_i\|\theta\|^2 = -\gamma_1\mathcal{T}(\mathbf{k} \times \mathbf{u}^f, \mathbf{u}^{f*}) - \eta\gamma_1\mathcal{T}(\mathbf{k} \times \mathbf{u}^p, \mathbf{u}^{p*}). \quad (8)$$

On the other hand, applying the same procedure to the complex conjugate of (6) and multiplying by $\mathbf{u}^f, \mathbf{u}^p, \theta$, one gets:

$$\bar{\sigma}_i\|\theta\|^2 = -\gamma_1\mathcal{T}(\mathbf{k} \times \mathbf{u}^{f*}, \mathbf{u}^f) - \eta\gamma_1\mathcal{T}(\mathbf{k} \times \mathbf{u}^{p*}, \mathbf{u}^p). \quad (9)$$

Adding (8) and (9), one obtains:

$$2\bar{\sigma}_i\|\theta\|^2 = 0 \quad (10)$$

and hence necessarily $\bar{\sigma}_i = 0$, i.e., $\bar{\sigma} \in \mathbb{R}$ and the strong form of the principle of exchange of stability holds, i.e., the oscillatory convection cannot arise.

Denoting by

$$\Delta_1 f = f_{,xx} + f_{,yy}, \quad \Delta^m \equiv \underbrace{\Delta \Delta \cdots \Delta}_m, \quad \omega_3^s = (\nabla \times \mathbf{u}^s) \cdot \mathbf{k}, \quad s = \{f, p\},$$

$$\bar{a} = \frac{\gamma_1}{k} + 1, \quad \bar{b} = \frac{\gamma_2}{h} + 1$$

and defining the operators

$$A \equiv \bar{a} - Da_f \gamma_1 \Delta, \quad B \equiv \bar{b} - Da_f \sigma \gamma_1 \Delta, \quad \Psi \equiv (AB - 1), \quad (11)$$

the third components of curl and of double curl of (4)_{1,2} are, respectively, given by

$$\begin{cases} A\omega_3^f - \omega_3^p = \gamma_1 \mathcal{T} w_{,z}^f, \\ -\omega_3^f + B\omega_3^p = \eta \gamma_1 \mathcal{T} w_{,z}^p \end{cases} \quad (12)$$

and

$$\begin{cases} -\frac{\gamma_1}{k} w_{,zz}^f - \gamma_1 \Delta_1 w^f - \Delta w^f + \Delta w^p = -R \Delta_1 \theta + \gamma_1 \mathcal{T} \omega_{3,z}^f - Da_f \gamma_1 \Delta^2 w^f, \\ -\frac{\gamma_2}{h} w_{,zz}^p - \gamma_2 \Delta_1 w^p + \Delta w^f - \Delta w^p = -R \Delta_1 \theta + \eta \gamma_1 \mathcal{T} \omega_{3,z}^p - Da_f \gamma_1 \sigma \Delta^2 w^p. \end{cases} \quad (13)$$

Applying the operator B to (12)₁, by virtue of (12)₂, one has

$$\Psi \omega_3^f = \gamma_1 \mathcal{T} B w_{,z}^f + \eta \gamma_1 \mathcal{T} w_{,z}^p.$$

This equation, together with that one obtained by applying the operator Ψ to (12)₂, leads to

$$\begin{cases} \Psi \omega_3^f = \gamma_1 \mathcal{T} B w_{,z}^f + \eta \gamma_1 \mathcal{T} w_{,z}^p, \\ \Psi B \omega_3^p = \gamma_1 \mathcal{T} B w_{,z}^f + \eta \gamma_1 \mathcal{T} A B w_{,z}^p. \end{cases} \quad (14)$$

Applying the operator Ψ to (13)₁ and ΨB to (13)₂, one gets

$$\begin{cases} -\bar{a} \Psi w_{,zz}^f - \hat{\gamma}_1 \Psi \Delta_1 w^f + \Psi \Delta_1 w^p + \Psi w_{,zz}^p \\ \quad = -R \Psi \Delta_1 \theta + \gamma_1 \mathcal{T} \Psi \omega_{3,z}^f - Da_f \gamma_1 \Psi \Delta^2 w^f, \\ -\bar{b} \Psi B w_{,zz}^p - \hat{\gamma}_2 \Psi B \Delta_1 w^p + \Psi B \Delta_1 w^f + \Psi B w_{,zz}^f \\ \quad = -R \Psi B \Delta_1 \theta + \eta \gamma_1 \mathcal{T} \Psi B \omega_{3,z}^p - Da_f \sigma \gamma_1 \Psi B \Delta^2 w^p, \end{cases} \quad (15)$$

with $\hat{\gamma}_r = \gamma_r + 1$, for $r = 1, 2$.

In view of (14), (15) can be written as

$$\begin{cases} [-\bar{a} \Psi - (\gamma_1 \mathcal{T})^2 B] w_{,zz}^f - \hat{\gamma}_1 \Psi \Delta_1 w^f + \Psi \Delta_1 w^p \\ \quad + [\Psi - \eta (\gamma_1 \mathcal{T})^2] w_{,zz}^p + Da_f \gamma_1 \Psi \Delta^2 w^f = -R \Psi \Delta_1 \theta, \\ [-\bar{b} \Psi B - (\eta \gamma_1 \mathcal{T})^2 A B] w_{,zz}^p - \hat{\gamma}_2 \Psi B \Delta_1 w^p + \Psi B \Delta_1 w^f \\ \quad + [\Psi B - \eta (\gamma_1 \mathcal{T})^2 B] w_{,zz}^f + Da_f \sigma \gamma_1 \Psi B \Delta^2 w^p = -R \Psi B \Delta_1 \theta. \end{cases} \quad (16)$$

Therefore, to find the linear instability threshold, we consider (6)₃, (16)₁ and (16)₂ with $\bar{\sigma} = 0$, i.e.

$$\begin{cases} [-\bar{a} \Psi - (\gamma_1 \mathcal{T})^2 B] w_{,zz}^f - \hat{\gamma}_1 \Psi \Delta_1 w^f + \Psi \Delta_1 w^p \\ \quad + [\Psi - \eta (\gamma_1 \mathcal{T})^2] w_{,zz}^p + Da_f \gamma_1 \Psi \Delta^2 w^f = -R \Psi \Delta_1 \theta, \\ [-\bar{b} \Psi B - (\eta \gamma_1 \mathcal{T})^2 A B] w_{,zz}^p - \hat{\gamma}_2 \Psi B \Delta_1 w^p + \Psi B \Delta_1 w^f \\ \quad + [\Psi B - \eta (\gamma_1 \mathcal{T})^2 B] w_{,zz}^f + Da_f \sigma \gamma_1 \Psi B \Delta^2 w^p = -R \Psi B \Delta_1 \theta, \\ R(w^f + w^p) + \Delta \theta = 0. \end{cases} \quad (17)$$

Employing normal modes in (17), i.e. assuming the following representation [12]

$$\begin{aligned} w^f &= W_0^f \sin(n\pi z) e^{i(lx+my)}, \\ w^p &= W_0^p \sin(n\pi z) e^{i(lx+my)}, \\ \theta &= \Theta_0 \sin(n\pi z) e^{i(lx+my)}, \end{aligned} \quad (18)$$

with W_0^f, W_0^p, Θ_0 real constants, setting $a^2 = l^2 + m^2$ and $\Lambda_n = a^2 + n^2\pi^2$, from (17), it turns out that

$$\left\{ \begin{aligned} & \left[\Lambda_n e(A_1 M + \sigma f n^2 \pi^2) + \Lambda_n^2 e(M e \sigma + B_1) + f \bar{b} n^2 \pi^2 + B_1 M \right. \\ & \quad \left. + e^2 \Lambda_n^3 A_1 + e^3 \sigma \Lambda_n^4 \right] W_0^f + \left[-B_1 \Lambda_n - e A_1 \Lambda_n^2 - e^2 \sigma \Lambda_n^3 + \eta f n^2 \pi^2 \right] W_0^p \\ & \quad - R a^2 \left[B_1 + e \Lambda_n A_1 + e^2 \sigma \Lambda_n^2 \right] \Theta_0 = 0, \\ & \left[\Lambda_n (e \sigma n^2 \pi^2 \eta f - \bar{b} B_1) + \eta f n^2 \pi^2 \bar{b} - \Lambda_n^2 e C \right. \\ & \quad \left. - \Lambda_n^3 e^2 \sigma (A_1 + \bar{b}) - \Lambda_n^4 \sigma^2 e^3 \right] W_0^f \\ & \quad + \left\{ \Lambda_n e(C N + \eta^2 f A_1 n^2 \pi^2) + \Lambda_n^2 e \sigma [e(A_1 + \bar{b}) N + \bar{b} B_1 + e \eta^2 f n^2 \pi^2] \right. \\ & \quad \left. + B_1 \bar{b} N + e^2 \Lambda_n^3 \sigma (C + e \sigma N) + \Lambda_n^4 e^3 \sigma^2 (A_1 + \bar{b}) + \Lambda_n^5 e^4 \sigma^3 \right. \\ & \quad \left. + \eta^2 f n^2 \pi^2 \bar{a} \bar{b} \right\} W_0^p - R a^2 \left[\bar{b} B_1 + e \Lambda_n C + \Lambda_n^2 e^2 \sigma (A_1 + \bar{b}) + e^3 \sigma^2 \Lambda_n^3 \right] \Theta_0 = 0, \\ & R W_0^f + R W_0^p - \Lambda_n \Theta_0 = 0, \end{aligned} \right. \quad (19)$$

where

$$\begin{aligned} A_1 &= \sigma \bar{a} + \bar{b}, \quad B_1 = \frac{\gamma_1}{k} \frac{\gamma_2}{h} + \frac{\gamma_1}{k} + \frac{\gamma_2}{h}, \quad C = \sigma(2B_1 + 1) + \bar{b}^2, \\ M &= \frac{\gamma_1}{k} n^2 \pi^2 + \gamma_1 a^2 + \Lambda_n, \quad N = \frac{\gamma_2}{h} n^2 \pi^2 + \gamma_2 a^2 + \Lambda_n, \\ e &= D a_f \gamma_1, \quad f = (\gamma_1 T)^2. \end{aligned} \quad (20)$$

Setting

$$\begin{aligned} h_{11} &= \Lambda_n e(A_1 M + \sigma f n^2 \pi^2) + \Lambda_n^2 e(M e \sigma + B_1) + f \bar{b} n^2 \pi^2 + B_1 M + e^2 \Lambda_n^3 A_1 + e^3 \sigma \Lambda_n^4, \\ h_{12} &= -B_1 \Lambda_n - e A_1 \Lambda_n^2 - e^2 \sigma \Lambda_n^3 + \eta f n^2 \pi^2, \\ h_{13} &= B_1 + e \Lambda_n A_1 + e^2 \sigma \Lambda_n^2, \\ h_{21} &= \Lambda_n (e \sigma n^2 \pi^2 \eta f - \bar{b} B_1) + \eta f n^2 \pi^2 \bar{b} - \Lambda_n^2 e C - \Lambda_n^3 e^2 \sigma (A_1 + \bar{b}) - \Lambda_n^4 \sigma^2 e^3, \\ h_{22} &= \Lambda_n e(C N + \eta^2 f A_1 n^2 \pi^2) + \Lambda_n^2 e \sigma [e(A_1 + \bar{b}) N + \bar{b} B_1 + e \eta^2 f n^2 \pi^2] \\ & \quad + B_1 \bar{b} N + e^2 \Lambda_n^3 \sigma (C + e \sigma N) + \Lambda_n^4 e^3 \sigma^2 (A_1 + \bar{b}) + \Lambda_n^5 e^4 \sigma^3 + \eta^2 f n^2 \pi^2 \bar{a} \bar{b}, \\ h_{23} &= \bar{b} B_1 + e \Lambda_n C + \Lambda_n^2 e^2 \sigma (A_1 + \bar{b}) + e^3 \sigma^2 \Lambda_n^3, \end{aligned}$$

(19) can be written as

$$\left\{ \begin{aligned} & h_{11} W_0^f + h_{12} W_0^p - R a^2 h_{13} \Theta_0 = 0, \\ & h_{21} W_0^f + h_{22} W_0^p - R a^2 h_{23} \Theta_0 = 0, \\ & R W_0^f + R W_0^p - \Lambda_n \Theta_0 = 0. \end{aligned} \right. \quad (21)$$

The smallest value of R^2 which vanishes the determinant of (21) is the critical Rayleigh number for the onset of instability, i.e.

$$R_L^2 = \min_{(n, a^2) \in \mathbb{N} \times \mathbb{R}^+} f_L^2(n, a^2) \quad (22)$$

with

$$f_L^2(n, a^2) = \frac{\Lambda_n}{a^2} \frac{(h_{11} h_{22} - h_{12} h_{21})}{h_{13} h_{22} - h_{12} h_{23} + h_{11} h_{23} - h_{21} h_{13}}. \quad (23)$$

We have proved that: (i) both numerator and denominator of (23) are strictly positive; (ii) the minimum of f_L^2 with respect to n , by using numerical computations, is attained at $n = 1$. Hence,

$$R_L^2 = \min_{a^2 \in \mathbb{R}^+} f_L^2(1, a^2). \quad (24)$$

The minimum of $f_L^2(1, a^2)$ with respect to a^2 is analysed in Sect. 5.

Remark 1 Let us observe that:

- (i) if one assumes the validity of the Darcy's law ($Da_f = 0$), from (22) one gets:

$$R_L^2 = \min_{n, a^2} \frac{\Lambda_n}{a^2} \frac{B_1 M N - B_1 \Lambda_n^2 + \eta^2 f^2 n^4 \pi^4 + f n^2 \pi^2 (\bar{b} N + \eta^2 \bar{a} M + 2\eta \Lambda_n)}{B_1 (M + N + 2\Lambda_n) + f n^2 \pi^2 (\bar{a} \eta^2 - 2\eta + \bar{b})} \quad (25)$$

which coincides with the critical threshold found in [7];

- (ii) if $Da_f = 0$ and $h = k = 1$ (isotropic case), from (22) one obtains the critical Rayleigh number found in [6], i.e.

$$R_L^2 = \min_{n, a^2} \frac{\Lambda_n}{a^2} \frac{\Gamma^2 \Lambda_n^2 + \gamma_1^2 T^2 n^2 \pi^2 \Lambda_n [\eta^2 (\gamma_1 + 1)^2 + (\gamma_2 + 1)^2 + 2\eta] + \gamma_1^4 T^4 n^4 \pi^4 \eta^2}{\Gamma \Lambda_n (\gamma_1 + \gamma_2 + 4) + \gamma_1^2 T^2 n^2 \pi^2 [\eta^2 \gamma_1 + \gamma_2 + (\eta - 1)^2]}$$

where $\Gamma = \gamma_1 \gamma_2 + \gamma_1 + \gamma_2$;

- (iii) if $h = k = 1$, $\tilde{\mu}_p = 0$, in the limit as $T \rightarrow 0$, i.e. assuming the validity of the Brinkman's law only in the momentum equation for the macropores and in the absence of rotation, from (22) one obtains

$$R^2 = \min_{n, a^2} \frac{\Lambda_n^2}{a^2} \frac{\gamma_1 \gamma_2 + \gamma_1 + \gamma_2 + e(\gamma_2 + 1) \Lambda_n}{e \Lambda_n + \gamma_1 + \gamma_2 + 4} \quad (26)$$

and (26) coincides with the critical threshold found in [16].

4 Nonlinear stability

In order to study the nonlinear stability of the conduction solution, by virtue of (11), since

$$\Psi \equiv \bar{a}\bar{b} - eA_1\Delta + e^2\sigma\Delta^2 - 1, \quad \Psi B \equiv \bar{b}B_1 - eC\Delta + e^2\sigma(A_1 + \bar{b})\Delta^2 - e^3\sigma^2\Delta^3$$

(16) and (4)₅ can be written as:

$$\begin{cases} L_1 w^f + L_2 w^p + RL_3 \theta = 0, \\ M_1 w^f + M_2 w^p + RM_3 \theta = 0, \\ \theta_{,t} + (\mathbf{u}^f + \mathbf{u}^p) \cdot \nabla \theta = R(w^f + w^p) + \Delta \theta, \end{cases} \quad (27)$$

where the following differential operators have been defined:

$$\begin{aligned} L_1 &\equiv -\bar{a}B_1\partial_{zz} + e(\bar{a}A_1 + \sigma f)\partial_{zz}\Delta - f\bar{b}\partial_{zz} - \hat{\gamma}_1 B_1 \Delta_1 - \bar{a}\sigma e^2 \Delta^2 \partial_{zz} \\ &\quad + \hat{\gamma}_1 e A_1 \Delta \Delta_1 - \hat{\gamma}_1 e^2 \sigma \Delta^2 \Delta_1 + e B_1 \Delta^2 - e^2 A_1 \Delta^3 + e^3 \sigma \Delta^4, \\ L_2 &\equiv B_1 \Delta_1 + B_1 \partial_{zz} - e A_1 \Delta^2 + e^2 \sigma \Delta^3 - \eta f \partial_{zz}, \\ L_3 &\equiv B_1 \Delta_1 - e A_1 \Delta \Delta_1 + e^2 \sigma \Delta^2 \Delta_1 \\ M_1 &\equiv \bar{b} B_1 \Delta - e C \Delta^2 + e^2 \sigma (A_1 + \bar{b}) \Delta^3 - e^3 \sigma^2 \Delta^4 - \bar{b} \eta f \partial_{zz} + e \eta f \sigma \Delta \partial_{zz}, \\ M_2 &\equiv -\bar{b}(\bar{b} B_1 + \bar{a} \eta^2 f) \partial_{zz} + \sigma e \bar{b} B_1 \Delta^2 - \hat{\gamma}_2 \bar{b} B_1 \Delta_1 \\ &\quad + e(\bar{b} C + \eta^2 f A_1) \Delta \partial_{zz} - e^2 \sigma [\bar{b}(A_1 + \bar{b}) + \eta^2 f] \Delta^2 \partial_{zz} + e^3 \bar{b} \sigma \Delta^3 \partial_{zz} \\ &\quad + \hat{\gamma}_2 e C \Delta \Delta_1 - \hat{\gamma}_2 e^2 \sigma (A_1 + \bar{b}) \Delta^2 \Delta_1 \\ &\quad + \hat{\gamma}_2 e^3 \sigma^2 \Delta^3 \Delta_1 - \sigma e^2 C \Delta^3 + \sigma^2 e^3 (A_1 + \bar{b}) \Delta^4 - e^4 \sigma^3 \Delta^5, \\ M_3 &\equiv \bar{b} B_1 \Delta_1 - e C \Delta \Delta_1 + e^2 \sigma (A_1 + \bar{b}) \Delta^2 \Delta_1 - e^3 \sigma^2 \Delta^3 \Delta_1. \end{aligned}$$

To determine the nonlinear stability threshold, in order to evaluate the rotation effect, it is not possible to proceed via a standard energy analysis, since the Coriolis terms in the momentum equations are antisymmetric. For this reason, we employ the differential constraint approach (see [8, 17, 25]) and set

$$\begin{aligned} E(t) &= \frac{1}{2} \|\theta\|^2, \\ I(t) &= (w^f + w^p, \theta), \quad D(t) = \|\nabla \theta\|^2. \end{aligned} \quad (28)$$

Retaining (27)_{1,2} as constraints, multiplying (27)₃ by θ and integrating over the periodicity cell, one gets

$$\frac{dE}{dt} = I - D \leq -D \left(1 - \frac{R}{R_E}\right), \quad (29)$$

where

$$\frac{1}{R_E} = \max_{\mathcal{H}^*} \frac{I}{D} \quad (30)$$

and

$$\begin{aligned} \mathcal{H}^* &= \{(w^f, w^p, \theta) \in (H^1)^3 \mid w^f = w^p = \theta = 0 \text{ on } z = 0, 1; \text{ periodic in } x, y \\ &\quad \text{with periods } 2\pi/l, 2\pi/m; D < \infty; \text{ verifying (27)}_{1,2}\} \end{aligned}$$

is the space of kinematically admissible solutions.

The variational problem associated with the maximum problem (30) is equivalent to

$$\frac{1}{R_E} = \max_{\mathcal{H}} \frac{I + \int_V \lambda' g_1 dV + \int_V \lambda'' g_2 dV}{D}, \quad (31)$$

where $\lambda'(\mathbf{x})$ and $\lambda''(\mathbf{x})$ are *Lagrange multipliers* and

$$\begin{aligned} g_1 &= L_1 w^f + L_2 w^p + R L_3 \theta, \quad g_2 = M_1 w^f + M_2 w^p + R M_3 \theta, \\ \mathcal{H} &= \{(w^f, w^p, \theta) \in (H^1)^3 \mid w^f = w^p = \theta = 0 \text{ on } z = 0, 1; \text{ periodic in } x, y \\ &\quad \text{with periods } 2\pi/l, 2\pi/m, \text{ respectively; } D < \infty\}. \end{aligned}$$

Applying the Poincaré inequality, one obtains that $D \geq \pi^2 \|\theta\|^2$, and hence, from (29) it follows that the condition $R < R_E$ implies $E(t) \rightarrow 0$ at least exponentially.

Remark 2 Multiplying (4)₁ by $\mathbf{u}^f$, (4)₂ by $\mathbf{u}^p$, integrating over the period cell V and adding the resulting equations, one finds

$$\begin{aligned} &\gamma_1 \int_V \left\{ k^{-1} [(u^f)^2 + (v^f)^2] + (w^f)^2 \right\} dV + \gamma_2 \int_V \left\{ h^{-1} [(u^p)^2 + (v^p)^2] + (w^p)^2 \right\} dV \\ &\quad + \|\mathbf{u}^f - \mathbf{u}^p\|^2 = R(\theta, w^f + w^p) - Da_f \gamma_1 \|\nabla \mathbf{u}^f\|^2 - \sigma Da_f \gamma_1 \|\nabla \mathbf{u}^p\|^2. \end{aligned} \quad (32)$$

Setting $\hat{k} = \max(k, 1)$ and $\hat{h} = \max(h, 1)$, by virtue of generalized Cauchy inequality and Poincaré-like inequality, from (32) one obtains

$$\left(\frac{\gamma_1}{\hat{k}} + 2Da_f \gamma_1 c_1 \right) \|\mathbf{u}^f\|^2 + \left(\frac{\gamma_2}{\hat{h}} + 2Da_f \gamma_1 \sigma c_2 \right) \|\mathbf{u}^p\|^2 \leq R^2 \left(\frac{\hat{k}}{\gamma_1} + \frac{\hat{h}}{\gamma_2} \right) \|\theta\|^2, \quad (33)$$

where c_1, c_2 are positive constants depending on the domain V . By virtue of (33), the condition $R < R_E$ also guarantees the exponential decay of $\mathbf{u}^f$ and $\mathbf{u}^p$.

In order to determine the critical Rayleigh number R_E^2 by solving the variational problem (31), we need to solve the associated Euler–Lagrange equations given by

$$\begin{cases} R_E(w^f + w^p) + R_E L_3 \lambda' + R_E M_3 \lambda'' + 2\Delta\theta = 0, \\ R_E \theta + L_1 \lambda' + M_1 \lambda'' = 0, \\ R_E \theta + L_2 \lambda' + M_2 \lambda'' = 0, \\ L_1 w^f + L_2 w^p + R L_3 \theta = 0, \\ M_1 w^f + M_2 w^p + R M_3 \theta = 0. \end{cases} \quad (34)$$

Eliminating the variable θ in (34), it turns out that

$$\begin{cases} -R_E^2 w^f - R_E^2 w^p + (2\Delta L_1 - R_E^2 L_3) \lambda' + (2\Delta M_1 - R_E^2 M_3) \lambda'' = 0, \\ -R_E^2 w^f - R_E^2 w^p + (2\Delta L_2 - R_E^2 L_3) \lambda' + (2\Delta M_2 - R_E^2 M_3) \lambda'' = 0, \\ (2\Delta L_1 - R_E^2 L_3) w^f + (2\Delta L_2 - R_E^2 L_3) w^p - R_E^2 L_3^2 \lambda' - R_E^2 L_3 M_3 \lambda'' = 0, \\ (2\Delta M_1 - R_E^2 M_3) w^f + (2\Delta M_2 - R_E^2 M_3) w^p - R_E^2 L_3 M_3 \lambda' - R_E^2 M_3^2 \lambda'' = 0. \end{cases} \quad (35)$$

By using (18)₁ and (18)₂ and choosing [17]

$$\begin{aligned} \lambda' &= \lambda'_0 \sin(n\pi z) e^{i(lx+my)}, \\ \lambda'' &= \lambda''_0 \sin(n\pi z) e^{i(lx+my)}, \end{aligned} \quad (36)$$

from (35), one obtains

$$\begin{cases} -R_E^2 W_0^f - R_E^2 W_0^p + (-2\Delta_n h_{11} + R_E^2 a^2 h_{13}) \lambda'_0 \\ \quad + (-2\Delta_n h_{21} + R_E^2 a^2 h_{23}) \lambda''_0 = 0, \\ -R_E^2 W_0^f - R_E^2 W_0^p + (-2\Delta_n h_{12} + R_E^2 a^2 h_{13}) \lambda'_0 \\ \quad + (-2\Delta_n h_{22} + R_E^2 a^2 h_{23}) \lambda''_0 = 0, \\ (-2\Delta_n h_{11} + R_E^2 a^2 h_{13}) W_0^f + (-2\Delta_n h_{12} + R_E^2 a^2 h_{13}) W_0^p \\ \quad - R_E^2 a^4 h_{13}^2 \lambda'_0 - R_E^2 a^4 h_{13} h_{23} \lambda''_0 = 0, \\ (-2\Delta_n h_{21} + R_E^2 a^2 h_{23}) W_0^f + (-2\Delta_n h_{22} + R_E^2 a^2 h_{23}) W_0^p \\ \quad - R_E^2 a^4 h_{13} h_{23} \lambda'_0 - R_E^2 a^4 h_{23}^2 \lambda''_0 = 0. \end{cases} \quad (37)$$

Requiring a zero determinant for (37), we get

$$R_E^2 = R_L^2;$$

hence, we have the coincidence between the instability threshold and the global nonlinear stability threshold with respect to the L^2 -norm (subcritical instabilities do not exist).

5 Numerical results

In this section, we numerically analyse the asymptotic behaviour of R_L^2 with respect to the parameters h , k , T^2 , Da_f in order to study the influence of permeability, rotation and Brinkman law on the onset of convection. As pointed out in Sect. 3, in all the performed computations we set $n = 1$.

Tables 1 and 2 show that as h increases with k fixed, both Rayleigh and wave numbers decrease, so it is easier for convection to set in and the convection cells become wider; k increasing with h fixed leads to a similar trend, but in this case the Rayleigh number decreases *more slowly*, as we can also see from Fig. 1. These numerical simulations show that h and k have a destabilizing effect on the onset of convection (see also Fig. 4).

Table 1 Critical Rayleigh and wave numbers for $k = 1$, $\eta = 0.2$, $\sigma = 0.3$, $\gamma_1 = 0.9$, $\gamma_2 = 1.8$, $T^2 = 10$, $Da_f = 1$ at different h

R_L^2	a_c^2	h
326.5116	7.6377	0.1
211.6567	6.1968	0.5
190.6873	5.7021	1
171.6278	5.1756	5
169.0585	5.0986	10

Table 2 Critical Rayleigh and wave numbers for $h = 1$, $\eta = 0.2$, $\sigma = 0.3$, $\gamma_1 = 0.9$, $\gamma_2 = 1.8$, $T^2 = 10$, $Da_f = 1$ at different k

R_L^2	a_c^2	k
208.1330	6.2007	0.1
192.9895	5.7772	0.5
190.6873	5.7021	1
188.7616	5.6372	5
188.5154	5.6287	10

Fig. 1 **a** Asymptotic behaviour of R_L^2 with respect to h for $k = 1$, $\eta = 0.2$, $\sigma = 0.3$, $\gamma_1 = 0.9$, $\gamma_2 = 1.8$, $T^2 = 10$, $Da_f = 1$. **b** Asymptotic behaviour of R_L^2 with respect to k for $h = 1$, $\eta = 0.2$, $\sigma = 0.3$, $\gamma_1 = 0.9$, $\gamma_2 = 1.8$, $T^2 = 10$, $Da_f = 1$ **Table 3** Critical Rayleigh and wave numbers for $h = 10$, $k = 1$, $\eta = 0.2$, $\sigma = 0.3$, $\gamma_1 = 0.9$, $\gamma_2 = 1.8$, $T^2 = 10$ at different Darcy numbers

R_L^2	a_c^2	Da_f
56.3828	13.7917	0.001
100.9334	5.4036	0.5
169.0585	5.0986	1
716.2302	4.9566	5

A typical *small* Darcy number is $Da_f = 0.001$, while a typical Darcy number is $Da_f = 1$ [22]

Table 3 and Fig. 2 show an expected behaviour: the Brinkman term has a stabilizing effect on the onset of convection, i.e. as Da_f increases, the Rayleigh number increases and the system becomes more stable. Also, comparing the critical Rayleigh number R_L^2 in Table 4a for $Da_f = 0.001$ with the one in Table 4b for $Da_f = 1$, the stabilizing effect of Da_f becomes apparent.

Tables 4 and 5 and Fig. 3 display a similar trend, as the Taylor number T^2 increases, the critical Rayleigh number increases, so the heat transfer due to convection is inhibited and rotation has a stabilizing effect on the onset of convection, as we expected. As T^2 increases, the wavenumber a^2 also increases and this means that the convection cells become narrower.

Fig. 2 a Steady instability thresholds at $Da_f = 1, 5$ and $h = 0.1, k = 1, \mathcal{T}^2 = 10, \eta = 0.2, \sigma = 0.3, \gamma_1 = 0.9, \gamma_2 = 1.8$. **b** Steady instability thresholds at $h = 1, k = 0.1, \mathcal{T}^2 = 10, \eta = 0.2, \sigma = 0.3, \gamma_1 = 0.9, \gamma_2 = 1.8$

Table 4 Critical Rayleigh and wave numbers for increasing Taylor numbers

R_L^2	a_c^2	$\mathcal{T}^2$
(a) $Da_f = 0.001$		
52.2521	12.8790	0
52.7216	13.0444	0.1
54.5534	13.6882	0.5
56.7498	14.4563	1
71.7551	19.5421	5
86.7523	24.7523	10
211.5075	39.0022	100
(b) $Da_f = 1$		
323.9093	7.4974	0
323.9359	7.4988	0.1
324.0419	7.5046	0.5
324.1741	7.5118	1
325.2230	7.5685	5
326.5116	7.6377	10
346.4686	8.6407	100

(a) $h = 0.1, k = 1, \eta = 0.2, \sigma = 0.3, \gamma_1 = 0.9, \gamma_2 = 1.8, Da_f = 0.001$. (b) $h = 0.1, k = 1, \eta = 0.2, \sigma = 0.3, \gamma_1 = 0.9, \gamma_2 = 1.8, Da_f = 1$

Table 5 Critical Rayleigh and wave numbers for increasing Taylor numbers

R_L^2	a_c^2	$\mathcal{T}^2$
(a) $h = 1, k = 0.1$		
206.8022	6.1510	0
206.8155	6.1515	0.1
206.8690	6.1535	0.5
206.9358	6.1560	1
207.4692	6.1760	5
208.1330	6.2007	10
(b) $h = 0.1, k = 1$		
323.9093	7.4974	0
323.9359	7.4988	0.1
324.0419	7.5046	0.5
324.1741	7.5118	1
325.2230	7.5685	5
326.5116	7.6377	10

(a) $h = 1, k = 0.1, \eta = 0.2, \sigma = 0.3, \gamma_1 = 0.9, \gamma_2 = 1.8, Da_f = 1$. (b) $h = 0.1, k = 1, \eta = 0.2, \sigma = 0.3, \gamma_1 = 0.9, \gamma_2 = 1.8, Da_f = 1$

Fig. 3 **a** Steady instability thresholds at $T^2 = 0, 10, 100$ and for $h = 0.1, k = 10, Da_f = 1, \eta = 0.2, \sigma = 0.3, \gamma_1 = 0.9, \gamma_2 = 1.8$. **b** Asymptotic behaviour of R_L^2 with respect to T^2 for $h = 0.1, k = 1, \eta = 0.2, \sigma = 0.3, \gamma_1 = 0.9, \gamma_2 = 1.8, Da_f = 0.001$

Fig. 4 **a** Steady instability thresholds at $h = 0.1, 1, 10$ and $k = 1, T^2 = 10, \eta = 0.2, \sigma = 0.3, \gamma_1 = 0.9, \gamma_2 = 1.8, Da_f = 1$. **b** Steady instability thresholds at $k = 0.1, 1, 10$ and $h = 1, T^2 = 10, \eta = 0.2, \sigma = 0.3, \gamma_1 = 0.9, \gamma_2 = 1.8, Da_f = 1$

6 Conclusions

The onset of thermal convection in an anisotropic BDPM, uniformly rotating about a vertical axis and uniformly heated from below, has been analysed according to the Brinkman law in both micropores and macropores. In particular, it has been proved that:

- the strong form of the principle of exchange of stabilities holds and hence, when convection occurs, it sets in through a stationary motion;
- the linear instability threshold and the global nonlinear stability threshold in the L^2 -norm coincide: this is an optimal result since the stability threshold furnishes a necessary and sufficient condition to guarantee the global (i.e., for all initial data) nonlinear stability;
- the critical Rayleigh number for the onset of convection increases with the Taylor number, i.e., rotation has a stabilizing effect on the onset of convection;
- the critical Rayleigh number for the onset of convection increases with the Darcy number, i.e., the Brinkman law has a stabilizing effect.

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Declarations

Conflict of interest The authors declare that they have no conflict of interest.

Availability of data and material This article does not contain any additional data.

Code availability Not Applicable.

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References

1. Barletta, A., Magyari, E., Pop, I., et al.: Mixed convection with viscous dissipation in a vertical channel filled with a porous medium. *Acta Mech.* **194**, 123–140 (2007). <https://doi.org/10.1007/s00707-007-0459-3>
2. Barletta, A., Rees, D.A.S.: Unstable mixed convection flow in a horizontal porous channel with uniform wall heat flux. *Transp. Porous Media* **129**, 385–402 (2019). <https://doi.org/10.1007/s11242-019-01294-y>
3. Capone, F., De Luca, R.: Ultimately boundedness and stability of triply diffusive mixtures in rotating porous layers under the action of Brinkman law. *Int. J. Non-Linear Mech.* **47**(7), 799–805 (2012)
4. Capone, F., De Luca, R.: Porous MHD convection: effect of Vadasz inertia term. *Transp. Porous Media* **118**(3), 519–536 (2017)
5. Capone, F., De Luca, R.: The effect of the Vadasz number on the onset of thermal convection in rotating bidisperse porous media. *Fluids* **5**(4), 173 (2020). <https://doi.org/10.3390/fluids5040173>
6. Capone, F., De Luca, R., Gentile, M.: Coriolis effect on thermal convection in a rotating bidisperse porous layer. *Proc. R. Soc. A* (2020). <https://doi.org/10.1098/rspa.2019.0875>
7. Capone, F., De Luca, R., Gentile, M.: Thermal convection in rotating anisotropic bidisperse porous layers. *Mech. Res. Commun.* (2020). <https://doi.org/10.1016/j.mechrescom.2020.103601>
8. Capone, F., Gentile, M.: Sharp stability results in LTNE rotating anisotropic porous layer. *Int. J. Therm. Sci.* **134**, 661–664 (2018)
9. Capone, F., Gentile, M., Hill, A.A.: Anisotropy and symmetry in porous media convection. *Acta Mech.* **208**, 205 (2009). <https://doi.org/10.1007/s00707-008-0135-2>
10. Capone, F., Gentile, M., Hill, A.A.: Double-diffusive penetrative convection simulated via internal heating in an anisotropic porous layer with throughflow. *Int. J. Heat Mass Transf.* **54**(7–8), 1622–1626 (2011)
11. Capone, F., Rionero, S.: Porous MHD convection: stabilizing effect of magnetic field and bifurcation analysis. *Ricerche mat.* **65**(1), 163–186 (2016)
12. Chandrasekhar, S.: *Hydrodynamic and Hydromagnetic Stability*. Dover Publications, New York (1981)
13. Chen, Z.Q., Cheng, P., Hsu, C.T.: A theoretical and experimental study on stagnant thermal conductivity of bidispersed porous media. *Int. Commun. Heat Mass Transf.* **27**, 601–610 (2000)
14. De Luca, R., Rionero, S.: Steady and oscillatory convection in rotating fluid layers heated and salted from below. *Int. J. Non-Linear Mech.* **78**(1), 121–130 (2016)
15. Gentile, M., Straughan, B.: Bidisperse thermal convection. *Int. J. Heat Mass Transf.* **114**, 837–840 (2017)
16. Gentile, M., Straughan, B.: Bidisperse thermal convection with relatively large macropores. *J. Fluid Mech.* **898**, A14 (2020). <https://doi.org/10.1017/jfm.2020.411>
17. Hill, A.A.: A differential constraint approach to obtain global stability for radiation-induced double-diffusive convection in a porous medium. *Math. Methods Appl. Sci.* **32**(8), 914–921 (2009)
18. Kvernold, O., Tyvand, P.A.: Nonlinear thermal convection in anisotropic porous media. *Mech. Appl. Math.* **90**(4), 609–624 (1977)
19. Moutsopoulos, K.N., Koch, D.L.: Hydrodynamic and boundary-layer dispersion in bidisperse porous media. *J. Fluid Mech.* **385**, 359–379 (1999)
20. Nield, D.A., Bejan, A.: *Convection in Porous Media*, 5th edn. Springer, New York (2017)
21. Nield, D.A., Kuznetsov, A.V.: Heat transfer in bidisperse porous media. *Transp. Phenom. Porous Med. III* (2005). <https://doi.org/10.1016/B978-008044490-1/50006-5>
22. Nield, D.A., Kuznetsov, A.V.: The onset of convection in a bidisperse porous medium. *Int. J. Heat Mass Transf.* **49**(17–18), 3068–3074 (2006)
23. Nield, D.A., Kuznetsov, A.V.: The effects of combined horizontal and vertical heterogeneity on the onset of convection in a porous medium with vertical throughflow. *Transp. Porous Media* **90**, 465 (2011)

24. Rionero, S., Vergori, L.: Long-time behaviour of fluid motions in porous media according to the Brinkman model. *Acta Mech.* **210**, 221–240 (2010). <https://doi.org/10.1007/s00707-009-0205-0>
25. Straughan, B.: Global nonlinear stability in porous convection with a thermal non-equilibrium model. *Proc. R. Soc. A.* **462**, 409–418 (2006)
26. Straughan, B.: *Convection with Local Thermal Non-equilibrium and Microfluidic Effects*. Adv Mechanics and Matematics, vol. 32. Springer, Cham (2015)
27. Straughan, B.: Anisotropic bidispersive convection. *Proc. R. Soc. A.* **475**, 20190206 (2019)
28. Straughan, B.: Horizontally isotropic double porosity convection. *Proc. R. Soc. A* **475**, 20180672 (2019)
29. Vadasz, P.: Flow and thermal convection in rotating porous media. *Handbook of porous media*, 395–440 (2000)
30. Vadasz, P.: *Fluid Flow and Heat Transfer in Rotating Porous Media*, Springer Brief Thermal Engineering and Applied Sciences (eBook). Springer, Berlin (2016)
31. Vadasz, P.: Instability and convection in rotating porous media: a review. *Fluids* **4**, 1–31 (2019)

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