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Explicit form of yield conditions dual to a class of dissipation potentials dependent on three invariants

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Abstract In the paper, a pragmatic approach to finding the dual formulation for isotropic perfectly plastic materials given a dissipation potential dependent on three cylindrical invariants and involving the Ottosen shape function is proposed and illustrated by examples. The main goal is to provide instructions on how to perform the Legendre transformation used when passing from a dissipation potential to its conjugate yield condition and offer some suggestions regarding calibration for particular potentials dependent on the trace of the strain rate tensor and the product of the norm of its deviator and the Ottosen shape function, which covers a wide class of engineering materials. The classic framework for constitutive modelling of thermodynamically consistent materials within the small deformation theory is used. First, general formulae connecting a dissipation potential dependent on three invariants of the strain rate tensor to the coupled yield condition are derived. Then, they are narrowed down for the aforementioned case of dissipation functions dependent on the Lode angle in a way proposed by Ottosen. Finally, three examples are given involving classical potentials: Beltrami's, Drucker–Prager's and Mises–Schleicher's generalised potential using the shape function. Detailed calculations exposing the introduced technique are performed. Also, a method of the calibration of such potentials leading to explicit mathematical formulae is demonstrated, based on the typical tests located on the tension and compression meridians.

1 Introduction

The approach to the constitutive modelling of dissipative materials is twofold. A surface describing the initiation of a phenomenon (plasticity, damage, etc.) and the evolution of the respective internal variables can be captured in functions referring exactly to the curves obtained in tests. Such a method is very effective when trying to recognise a problem. However, experimental data for some materials indicate the necessity of using non-associative evolution laws, which can cause the violation of laws of thermodynamics for particular boundary-valued problems. The issue is properly addressed by the alternative approach—the framework for constitutive modelling of thermodynamically consistent materials, where the fulfilment of the laws is ensured [1–4]. For non-associative plasticity, the first approach requires knowing a yield condition and a flow rule, which are autonomous, while in the second one, only a dissipation potential is needed, and this function solely describes the material's behaviour.

The framework for constitutive modelling of thermodynamically consistent materials is a very powerful tool. It is very convenient to establish just one potential governing plastic behaviour (and, possibly, behaviour due to other irreversible phenomena) [5]. However, a question arises how to choose such a function. Contrary to

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a yield condition, the features of a dissipation (strain rate) potential cannot be directly captured in experiments. The need to obtain the yield condition conjugated to the assumed dissipation function appears. The Legendre transformation has to be performed in order to find the dual formulation and estimate material parameters. The goal of this paper is to shed light on the technique of such calculations for a particular class of functions dependent on three independent invariants of a plastic strain rate tensor.

A rigorous exhaustive description of the used framework was given by Houlsby and Puzrin [2]. Herein, we are interested in a part of the procedure, that is in finding a dual yield condition given a dissipation potential. This seems the most troublesome issue of the framework, which in general encloses also elasticity and damage. An outline of the framework is given in [Appendix A](#), with a suitable reduction regarding the considered case. Performing the herein shown Legendre transformation is a non-trivial task. Examples of deriving dissipation potentials coupled with simple yield conditions of two main types are present in the relevant literature [2–8]. The first type consists of yield conditions represented in the Haigh–Westergaard space by plasticity surfaces of revolution (Huber–Mises, Beltrami, Drucker–Prager, Mises–Schleicher, Burzyński, Gurson etc.), while the second group concerns plasticity surfaces composed of intersecting planes with singularities at joint edges (Rankine, Tresca, Coulomb–Mohr, etc.). When the transformed function becomes dependent on the Lode angle, analytical results are rather scarce and incomplete [7, 9]. In the cited papers, dissipation potentials are sought for given yield conditions. A successful attempt to describe an algorithm of finding a convex conjugate of a function that is dependent only on the deviatoric part of a strain rate (or stress) tensor is shown in [9], but it leaves complex scalar equations to be inverted. When all three independent invariants are involved, the process becomes quite complicated. Unfortunately, this is the case for a vast majority of engineering materials. In this work, we propose a pragmatic method of finding the dual formulation when a dissipation potential of a particular form dependent on three cylindrical invariants of the strain rate tensor function is given. Special attention is paid to the functions involving the Ottosen-type shape function [10], as they are extremely useful for describing a wide variety of materials. For example, such an approach was applied to modelling plasticity in concrete and metals in [11, 12]. Emphasis is placed on the technique of calculations rather than on theoretical considerations. To narrow down the scope of our research, we consider only associative plasticity. The approach can be successfully adopted to fit a non-associative behaviour also for similar phenomena such as damage [5].

Although our attention is laid on finding a yield condition conjugated to a given dissipation potential, the inverse path (going from a yield condition to a dissipation function) can also be explored. Having a yield condition, its associated flow rule can be applied and then the strain rate tensor invariants can be expressed via the stress tensor invariants. The respective formulae are derived and presented in the paper completing the dual model formulation. Depending on the needs in modelling one or both dual potentials resulting in constitutive relationships for a rigid-plastic material can be used and possibly extended to elastoplasticity including damage as well [1–6].

The primary aim of the paper is to reveal the method of conducting calculations, which is commonly disregarded in publications on the subject. Trying to use mathematical tools, as simple as possible and to keep transparent notations, we obtain some general formulae helpful when seeking the dual formulation and then specify them for three dissipation potentials being generalised functions of Beltrami, Drucker–Prager and Mises–Schleicher. As a result, the obtained dual potentials and constitutive relationships open the door to the development of new models or extension of available models in the literature [2–8] to be dependent on all three invariants.

2 Dissipation potential dependent on three cylindrical invariants

Due to the representation of yield conditions for isotropic materials in the Haigh–Westergaard coordinate system, both dissipation potentials and yield functions have to be periodic repeating over intervals of $2\pi/3$ radians. There are many types of functions that satisfy this requirement. However, an obvious choice are trigonometric functions, cyclic by their very nature. For this reason, we restrict our considerations to dissipation potentials dependent on the cosine of the tripled Lode angle ϕ .

Let us assume an isotropic dissipation potential of the following form:

$$D(\dot{\mathbf{e}}_P) = D(p, q, \cos 3\phi), \quad (1)$$

where p , q and ϕ are cylindrical invariants of the plastic strain rate tensor $\dot{\mathbf{e}}_P$:

$$p = \frac{1}{\sqrt{3}} \text{tr} \dot{\mathbf{e}}_P, \quad q = \sqrt{\dot{\mathbf{e}}_P \cdot \dot{\mathbf{e}}_P} \quad \text{and} \quad \cos 3\phi = \frac{\sqrt{6} \text{tr} \dot{\mathbf{e}}_P^3}{\sqrt{\text{tr}^3 \dot{\mathbf{e}}_P^2}}, \quad (2)$$

$\dot{\mathbf{e}}_P = \dot{\mathbf{e}}_P - (p/\sqrt{3})\mathbf{I}$ being its deviator. $\mathbf{I}$ is a second-order unit tensor, tr denotes the operation of trace, $\cdot$ is a scalar product of two second-order tensors. We assume D is positive for all $\dot{\mathbf{e}}_P \neq \mathbf{0}$ with $D(\mathbf{0}) = 0$ and convex. The only argument of the function is the plastic strain rate tensor (or its three independent invariants for an isotropic scalar function). With no loss of generality, we omit hardening and other phenomena such as damage, see [Appendix A](#) for a brief introduction to these issues. Including the respective variables into the dissipation potential yields additional equations, but it does not change the structure of the problem solved [1–4], and thus, we leave it out for the sake of clarity and focus on the simplest case, yet complex enough.

The following Legendre transformation binds the yield condition $F(\sigma) = 0$ and the dissipation potential [2]:

$$D(\dot{\mathbf{e}}_P) + \lambda F(\sigma) = \sigma \cdot \dot{\mathbf{e}}_P, \quad (3)$$

where $\lambda \geq 0$ is a plastic multiplier. Since D is convex and non-negative, then the plastic potential is also convex and the following constitutive relations are in force:

$$\sigma = \frac{\partial D(\dot{\mathbf{e}}_P)}{\partial \dot{\mathbf{e}}_P} \quad \text{for } \dot{\mathbf{e}}_P \neq \mathbf{0} \quad (4)$$

and the dual relationship:

$$\dot{\mathbf{e}}_P = \lambda \frac{\partial F(\sigma)}{\partial \sigma}. \quad (5)$$

To reach the goal of this paper in a simple way, we limit ourselves to the case of a plastic flow $\dot{\mathbf{e}}_P \neq \mathbf{0}$ rather than to a rigid response $\dot{\mathbf{e}}_P = \mathbf{0}$. If needed, the reader can easily adjust our results to a full constitutive model or adapt them to the convex analysis formalism introducing the gauge function, the indicator function and subderivatives, etc.

The transformation (3) is singular, which entails the ambiguity of the plastic potential F . Regardless of an expression obtained for F , the resultant yield surface is always the same. This information can be useful when choosing a form of the function to perform a calibration.

As we have changed the argument of the isotropic dissipation function from the plastic strain rate tensor to its invariants with the help of representation theorems [1], (4) can be expressed using the chain rule:

$$\sigma = \frac{\partial D(\dot{\mathbf{e}}_P)}{\partial \dot{\mathbf{e}}_P} = \alpha_1 \mathbf{g}_1 + \alpha_2 \mathbf{g}_2 + \alpha_3 \mathbf{g}_3, \quad (6)$$

where

$$\alpha_1 = \frac{\partial D}{\partial p}, \quad \alpha_2 = \frac{\partial D}{\partial q}, \quad \alpha_3 = \frac{\partial D}{\partial \cos 3\phi}, \quad (7)$$

and

$$\mathbf{g}_1 = \frac{\partial p}{\partial \dot{\mathbf{e}}_P} = \frac{1}{\sqrt{3}}\mathbf{I}, \quad \mathbf{g}_2 = \frac{\partial q}{\partial \dot{\mathbf{e}}_P} = \frac{1}{q}\dot{\mathbf{e}}_P, \quad \mathbf{g}_3 = \frac{\partial \cos 3\phi}{\partial \dot{\mathbf{e}}_P} = \frac{3\sqrt{6}}{q^3} \left(\dot{\mathbf{e}}_P^2 - \frac{q \cos 3\phi}{\sqrt{6}} \dot{\mathbf{e}}_P - \frac{q^2}{3} \mathbf{I} \right). \quad (8)$$

$\mathbf{g}_i$ are orthogonal generators of the stress tensor σ , i.e. $\mathbf{g}_i \cdot \mathbf{g}_j = 0$ for $i \neq j$ where $i, j = 1, 2, 3$. Moreover, two first tensor generators are unit tensors $\|\mathbf{g}_1\| = 1$ and $\|\mathbf{g}_2\| = 1$.

Analogous to (2), the following set of cylindrical invariants of the stress tensor is considered:

$$\xi = \frac{1}{\sqrt{3}} \text{tr} \sigma, \quad r = \sqrt{\mathbf{s} \cdot \mathbf{s}} \quad \text{and} \quad \cos 3\theta = \frac{\sqrt{6} \text{tr} \mathbf{s}^3}{\sqrt{\text{tr}^3 \mathbf{s}^2}}, \quad (9)$$

$\mathbf{s} = \sigma - (\xi/\sqrt{3})\mathbf{I}$ and θ is the Lode angle for the stress tensor.

Making use of the isotropic decomposition of both stress and strain rate tensors, the constitutive relationship (6) yields:

$$\xi \mathbf{g}_1 = \alpha_1 \mathbf{g}_1 \quad \text{and} \quad \mathbf{s} = \alpha_2 \mathbf{g}_2 + \alpha_3 \mathbf{g}_3. \quad (10)$$

Now, Eqs. (8) and (9) can be taken into account. Finally, we obtain three scalar relationships:

$$\begin{aligned}\xi &= \alpha_1, \quad r^2 = \alpha_2^2 + \frac{9 \sin^2 3\phi}{q^2} \alpha_3^2, \\ r^3 \cos 3\theta &= \alpha_2^3 \cos 3\phi + \frac{9 \sin^2 3\phi}{q} \alpha_2^2 \alpha_3 - \frac{27 \cos 3\phi \sin^2 3\phi}{q^2} \alpha_2 \alpha_3^2 - \frac{27 \sin^4 3\phi}{q^3} \alpha_3^3.\end{aligned}\quad (11)$$

The above equations define a parametric form of the sought yield condition $F(\sigma) = 0$. Eliminating kinematic invariants (p , q and ϕ) allows to obtain a direct form of the condition. Note that functions α_i defined by (7) are generally functions of invariants (2). As the operation involves inverting some parts of the scalar Eqs. (11), it is not always possible to find an explicit form of $F(\xi, r, \theta)$, which in general seems to be the main inconvenience of using the Legendre transformation and dissipation function (strain rate potentials) for the development of more advanced constitutive models. Examples of existing constitutive models proposed in [3, 5, 6] are limited to potentials independent of the Lode angles.

In the dual formulation, the flow rule (5) can be written as:

$$\dot{\epsilon}_P = \lambda \frac{\partial F}{\partial \sigma} = \lambda (\tilde{\alpha}_1 \tilde{\mathbf{g}}_1 + \tilde{\alpha}_2 \tilde{\mathbf{g}}_2 + \tilde{\alpha}_3 \tilde{\mathbf{g}}_3), \quad (12)$$

where similarly to (7) and (8):

$$\tilde{\alpha}_1 = \frac{\partial F}{\partial \xi}, \quad \tilde{\alpha}_2 = \frac{\partial F}{\partial r}, \quad \tilde{\alpha}_3 = \frac{\partial F}{\partial \cos 3\theta}, \quad (13)$$

$$\tilde{\mathbf{g}}_1 = \frac{\partial \xi}{\partial \sigma} = \frac{1}{\sqrt{3}} \mathbf{I}, \quad \tilde{\mathbf{g}}_2 = \frac{\partial r}{\partial \sigma} = \frac{1}{r} \mathbf{s}, \quad \tilde{\mathbf{g}}_3 = \frac{\partial \cos 3\theta}{\partial \sigma} = \frac{3\sqrt{6}}{r^3} \left(\mathbf{s}^2 - \frac{r \cos 3\theta}{\sqrt{6}} \mathbf{s} - \frac{r^2}{3} \mathbf{I} \right). \quad (14)$$

Analogously to (11), the following relationships between the invariants of $\dot{\epsilon}_P$ and σ remain in force:

$$\begin{aligned}p &= \lambda \tilde{\alpha}_1, \quad q^2 = \lambda^2 \left(\tilde{\alpha}_2^2 + \frac{9 \sin^2 3\theta}{r^2} \tilde{\alpha}_3^2 \right), \\ q^3 \cos 3\phi &= \lambda^3 \left(\tilde{\alpha}_2^3 \cos 3\theta + \frac{9 \sin^2 3\theta}{r} \tilde{\alpha}_2^2 \tilde{\alpha}_3 - \frac{27 \cos 3\theta \sin^2 3\theta}{r^2} \tilde{\alpha}_2 \tilde{\alpha}_3^2 - \frac{27 \sin^4 3\theta}{r^3} \tilde{\alpha}_3^3 \right).\end{aligned}\quad (15)$$

3 Dissipation potential involving Ottosen deviatoric shape function

In this section, some general formulae useful when entangling Eq. (11) are derived. The following dissipation potentials are of our interest:

$$D(\dot{\epsilon}_P) = D(p, q, h(\cos 3\phi)), \quad (16)$$

where the deviatoric shape function h is the above-mentioned Ottosen-type function [10] of the form:

$$h(\cos 3\phi) = \cos \psi = \cos \left[\frac{1}{3} \arccos(\gamma \cos 3\phi) \right], \quad |\gamma| < 1. \quad (17)$$

The function was chosen because of its flexibility and great capacity to describe many classes of engineering materials. Parameter γ allows to capture a broad spectrum of possible deviatoric cross-section shapes: from circular ($\gamma = 0$) to triangular ($\gamma = \pm 1$). In formula (17), the range of ψ is $0 \leq \psi \leq \pi/3$ and the range of the values of h function is $1/2 \leq h \leq 1$ for $|\gamma| \leq 1$. Generally, D has to be convex, positive for $\dot{\epsilon}_P \neq \mathbf{0}$ and homogenous of degree 1 with respect to p and q . The invariants ϕ or $\cos 3\phi$ are homogenous of degree 0 with respect to the plastic strain rate tensor.

Denoted by α , the following derivative of (16) with respect to $q \cos \psi$ is related to the shape of a meridian:

$$\alpha = \frac{\partial D}{\partial (q \cos \psi)}. \quad (18)$$

Then according to (7) and (18), we obtain the partial derivatives of D :

$$\alpha_2 = \frac{\partial D}{\partial q} = \frac{\partial D}{\partial (q \cos \psi)} \cos \psi = \alpha \cos \psi, \quad \alpha_3 = \frac{\partial D}{\partial \cos 3\phi} = \frac{\gamma \alpha q}{3(1 + 2 \cos 2\psi)}. \quad (19)$$

As a consequence, (6) reduces to the following relationship:

$$\boldsymbol{\sigma} = \frac{\partial D(\dot{\boldsymbol{\varepsilon}}_P)}{\partial \dot{\boldsymbol{\varepsilon}}_P} = \alpha_1 \mathbf{g}_1 + \alpha \left[\cos \psi \mathbf{g}_2 + \frac{\gamma q}{3(1 + 2 \cos 2\psi)} \mathbf{g}_3 \right], \quad (20)$$

where α_1 and the generators are described by (7)₁ and (8), respectively.

Now, (11)₂ is brought to the form

$$r^2 = \alpha^2 \left(1 - \frac{1 - \gamma^2}{(1 + 2 \cos 2\psi)^2} \right), \quad (21)$$

while successively (11)₃ is transformed to

$$2\gamma \left[1 - \frac{1 - \gamma^2}{(1 + 2 \cos 2\psi)^2} \right]^{\frac{3}{2}} \cos 3\theta = 1 + \gamma^2 - \frac{3(1 - \gamma^2)}{(1 + 2 \cos 2\psi)^2} - \frac{2(1 - \gamma^2)^2}{(1 + 2 \cos 2\psi)^3}. \quad (22)$$

The above relation between the Lode angles ϕ and θ is shown in Fig. 1, compare with [9]. Independently of γ , the compression meridian ($\phi = 0$) and the tension meridian ($\phi = \pi/3$) of a constant dissipation surface transform into the respective meridians of the conjugated yield surface. All the shown curves have common points for $\phi = 0 = \theta$ and $\phi = \pi/3 = \theta$. This property is effectively used in the calibration of material parameters.

If α and α_1 are simple enough, we can combine (11)₁ and (21) and then make use of the fact that the expression for $1 + 2 \cos 2\psi$ appears in both Eqs. (21) and (22), which is illustrated by examples given in the next section. Let us assume that as a result of merging (11)₁ and (21), the following convenient invariant connected to the Ottosen shape function:

$$\delta = \sqrt{1 - \frac{1 - \gamma^2}{(1 + 2 \cos 2\psi)^2}}, \quad \frac{1}{2} \leq \delta \leq 1, \quad (23)$$

compare Fig. 1, can be expressed via the stress tensor invariants (9). Then Eq. (22) becomes:

$$3\delta^2 - 2\sqrt{1 - \gamma^2}(\sqrt{1 - \delta^2})^3 - 2\gamma\delta^3 \cos 3\theta - 2 + \gamma^2 = 0, \quad (24)$$

which is one of many possible forms of the yield condition conjugated to the dissipation potential (16).

The flow rule (12), definitions (13), (14) and relations (15) remain unchanged.

Now, some remarks are made, which are useful when systematising the derivations for the subsequent examples.

Remark 1 The tension meridian $\delta_t(\xi, r)$ and the compression meridian $\delta_c(\xi, r)$ can be described by the following equations:

$$\delta_t(\xi, r) = \cos\left(\frac{1}{3} \arccos \gamma\right) \equiv \Delta_t \quad \text{and} \quad \delta_c(\xi, r) = \cos\left(\frac{1}{3} \arccos(-\gamma)\right) \equiv \Delta_c. \quad (25)$$

Fig. 1 Relations between the Lode angle for the stress tensor θ and the Lode angle for the strain rate tensor ϕ (left) and between the invariants δ and ϕ (right)

Equations (25) are the solutions to (24) with respect to δ in (23) when $\theta = 0$ (the tension meridian) and $\theta = \pi/3$ (the compression meridian), respectively. These representations of the curves are extremely helpful when calibrating the considered yield conditions. Details are given in [Appendix C](#).

Remark 2 Let δ be a function of the form

$$\delta(\xi, r) = \frac{r}{C(\xi)}, \quad (26)$$

where $C(\xi) > 0$ for almost all ξ (the exceptions can be, for example, vertices). Then, the yield condition (24) can be converted to

$$F(\xi, r, \theta) = 3C(\xi)r^2 - 2\sqrt{1-\gamma^2}\left(\sqrt{C(\xi)^2-r^2}\right)^3 - (2-\gamma^2)C(\xi)^3 - 2\gamma r^3 \cos 3\theta = 0. \quad (27)$$

Equation (27) establishes a very clear formula for a yield locus when (26) based on (11)₁ and (21) is true.

$|\gamma| = 1$ are excluded from our deliberations in order to reduce distractions caused by investigation of special (singular) cases and concentrate on the intermediate values of γ , although the respective yield conditions can be obtained calculating proper limits ($\gamma \rightarrow \pm 1$) of the given results. Also, for $\gamma = 0$ the considered potential does not depend on the Lode angle, thus some of the final formulae for the classic functions shown below can be found in the relevant literature. Cases analogous to $\gamma = \pm 1$ for the Coulomb–Mohr yield function were regarded in [4, 7], among others.

Implementing the formulae presented in this section, based on a dissipation potential one can obtain its dual yield condition and the associated flow rule. The given algorithm is useful for some forms of dissipation potential only and it is rather a method of conduct than a strict procedure.

4 Examples of dual potentials and relationships

In this section, three illustrative examples of finding a dual representation of a dissipation potential of the form (16) are given. We analyse dissipation functions conjugated to the classic yield conditions of Beltrami, Drucker–Prager and Mises–Schleicher, modified here by the Ottosen shape function. Calibrations of the considered potentials are performed using tests located on characteristic meridians ($\theta = 0$ and $\theta = \pi/3$), which facilitates their further application. Based on experimental data for metallic foams and concrete, we show illustrations of both the dissipation potential contours and the connected yield surfaces to give the reader some intuition about the workings of the singular Legendre transformation.

4.1 Generalised Beltrami dissipation potential

As the first example, the following dissipation potential is considered:

$$D(p, q, \phi) = \sqrt{A^2 p^2 + B^2 q^2 \cos^2 \psi}, \quad (28)$$

where $\cos \psi$ is defined by (17). Equation (28) is an extension of the potential conjugated to the Beltrami yield condition. The original function can be recovered by substituting $\gamma = 0$, whose contour defines an ellipsoidal surface of revolution in the Haigh–Westergaard space. $A > 0$ and $B > 0$ are material parameters. Clearly, the potential is non-negative, convex and homogeneous of degree 1 with respect to p and q . A special case of dissipation function (28) for $A = 0$ (a generalised Huber–Mises dissipation potential) applicable to incompressible metals with strength differential effect was investigated in [12].

Formulae (7)₁ and (18) yield:

$$\alpha_1 = \frac{\partial D}{\partial p} = \frac{A^2 p}{\sqrt{A^2 p^2 + B^2 q^2 \cos^2 \psi}}, \quad \alpha = \frac{\partial D}{\partial (q \cos \psi)} = \frac{B^2 q \cos \psi}{\sqrt{A^2 p^2 + B^2 q^2 \cos^2 \psi}}. \quad (29)$$

At this point, we can obtain the constitutive relationship (20):

$$\sigma = \frac{(\sqrt{6} A^2 p \sin 3\psi - \gamma B^2 q \sin 2\psi) \mathbf{I} + 2\sqrt{6} B^2 \sin \psi \cos^2 \psi \dot{\mathbf{e}}_p + \frac{3\gamma B^2 \sin 2\psi}{q} \dot{\mathbf{e}}_p^2}{\sqrt{6} \sin 3\psi \sqrt{A^2 p^2 + B^2 q^2 \cos^2 \psi}}. \quad (30)$$

Equations (11)₁ and (21) lead to the expressions

$$\xi^2 = \frac{A^4 p^2}{A^2 p^2 + B^2 q^2 \cos^2 \psi} \quad \text{and} \quad r^2 = \frac{B^4 q^2 \cos^2 \psi}{A^2 p^2 + B^2 q^2 \cos^2 \psi} \left(1 - \frac{1 - \gamma^2}{(1 + 2 \cos 2\psi)^2} \right). \quad (31)$$

Let us notice that based on (31), the following is true:

$$\frac{\xi^2}{A^2} + \frac{r^2}{B^2 \delta^2} = 1, \quad \text{where} \quad \delta^2 = 1 - \frac{1 - \gamma^2}{(1 + 2 \cos 2\psi)^2}. \quad (32)$$

This observation allows to make ψ dependent on ξ and r only:

$$\frac{1}{1 + 2 \cos 2\psi} = \sqrt{\frac{B^2(A^2 - \xi^2) - A^2 r^2}{(1 - \gamma^2)B^2(A^2 - \xi^2)}}. \quad (33)$$

Now, according to (23) or (32), δ is derived in the sequence of forms

$$\delta(\xi, r) = \sqrt{1 - \frac{1 - \gamma^2}{(1 + 2 \cos 2\psi)^2}} = \sqrt{\frac{A^2 r^2}{B^2(A^2 - \xi^2)}} = \frac{Ar}{B\sqrt{A^2 - \xi^2}} \equiv \frac{r}{C(\xi)}, \quad (34)$$

As δ satisfies requirement (26), the sought yield condition in the compact form (27) leads to:

$$\begin{aligned} F(\xi, r, \theta) = & \sqrt{B^2(A^2 - \xi^2)} [3A^2 r^2 - (2 - \gamma^2)B^2(A^2 - \xi^2)] \\ & - 2\sqrt{1 - \gamma^2} \left(\sqrt{B^2(A^2 - \xi^2) - A^2 r^2} \right)^3 - 2\gamma A^3 r^3 \cos 3\theta = 0. \end{aligned} \quad (35)$$

The functions (13) appearing in flow rule (12) and relationships (15) are then:

$$\begin{aligned} \tilde{\alpha}_1 &= \frac{\partial F}{\partial \xi} = 6B^2 \xi \left(\frac{(2 - \gamma^2)B^2(A^2 - \xi^2) - A^2 r^2}{2\sqrt{B^2(A^2 - \xi^2)}} + \sqrt{1 - \gamma^2} \sqrt{B^2(A^2 - \xi^2) - A^2 r^2} \right), \\ \tilde{\alpha}_2 &= \frac{\partial F}{\partial r} = 6A^2 r \left(\sqrt{B^2(A^2 - \xi^2)} + \sqrt{1 - \gamma^2} \sqrt{B^2(A^2 - \xi^2) - A^2 r^2} - \gamma A r \cos 3\theta \right), \\ \tilde{\alpha}_3 &= \frac{\partial F}{\partial \cos 3\theta} = -2\gamma A^3 r^3 \quad \text{for } r \neq 0 \text{ and } |\xi| \neq A. \end{aligned} \quad (36)$$

Let us emphasise that the dual yield condition (35) is used to formulate the constitutive equations of plastic flow without searching a direct inverse of (30), which generates complicated relations between stress and strain rate invariants.

The parameters A , B and γ are yet unknown. γ determines the shape of the deviatoric cross section of the yield surface, while A and B define its overall dimensions. From the equation of the ellipse (32), we can deduce that A defines a semi-axis in the ξ direction and $B\delta$ defines a semi-axis in the r direction. The semi-axis $B\delta$ through δ depends on the angle ϕ and the parameter γ as shown in Fig. 1. Consequently, the semi-axis $B\delta$ can be expressed by the angle θ and the parameter γ using the implicit relation (22). A is the location of the vertices, i.e. $\xi_V = \pm A$ for $r = 0$. For the purpose of this calibration, γ is chosen arbitrarily. Then, the remaining two unknowns can be calculated based on the yield limits for the uniaxial tension and the uniaxial compression tests, σ_T and σ_C , respectively. For the uniaxial tests, the cylindrical invariants are as follows:

$$\xi_T = \frac{\sigma_T}{\sqrt{3}}, \quad r_T = \sqrt{\frac{2}{3}} \sigma_T, \quad \theta_T = 0 \quad \text{and} \quad \xi_C = -\frac{\sigma_C}{\sqrt{3}}, \quad r_C = \sqrt{\frac{2}{3}} \sigma_C, \quad \theta_C = \frac{\pi}{3}. \quad (37)$$

The tension meridian and the compression meridian are described by the very simple formulae given in (25). Taking into account (37), the following set of equations determines the constants A and B , compare Appendix C:

$$\frac{Ar_T}{B\sqrt{A^2 - \xi_T^2}} = \Delta_t \quad \text{and} \quad \frac{Ar_C}{B\sqrt{A^2 - \xi_C^2}} \equiv \Delta_c. \quad (38)$$

The solution to the system is:

$$A^2 = \frac{(\Delta_t^2 - \Delta_c^2)\sigma_T^2\sigma_C^2}{3(\Delta_t^2\sigma_C^2 - \Delta_c^2\sigma_T^2)} \quad \text{and} \quad B^2 = \frac{2(\Delta_c^2 - \Delta_t^2)\sigma_T^2\sigma_C^2}{3\Delta_t^2\Delta_c^2(\sigma_C^2 - \sigma_T^2)}. \quad (39)$$

The above right-hand-side expressions for the parameters have to result in positive numbers. If not, there is no real solution for the parameters A and B related to the semi-axes of the ellipse (32). This requirement is met only for some sets of the input data. In Fig. 2, the restrictions of the proposed calibration are illustrated. As the plot indicates, the considered yield surface can accommodate limited differences between the uniaxial tension and compression yield limits, as is observed in some materials, for example metallic foams [13].

If one would like to make use of the pure shear yield limit σ_S located on shear meridian $\theta = \pi/6$, the relation between parameters B , γ and the respective yield limit was:

$$B^3\sqrt{1-\gamma^2} = 2\sigma_S^2\sqrt{3B^2 - 2\sigma_S^2} - \left(\sqrt{B^2 - 2\sigma_S^2}\right)^3. \quad (40)$$

Equation (40) is inconvenient for calibration when explicit mathematical formulae for A , B and γ are sought.

In Figs. 3, 4, 5 and 6, the overall shape and cross sections of a surface of the constant dissipation potential and of the conjugated yield condition are given. It is assumed that $\gamma = -0.50$, $\sigma_T = 0.440$ MPa and $\sigma_C = 0.534$ MPa, resulting in $A = 1.17$ MPa and $B = 0.481$ MPa. The constant dissipation surface is shown for $D = D_0 = 1.17$ MW/m³.

Fig. 2 Regions of positive expressions for the right-hand side of Eq. (39)

Fig. 3 Constant dissipation surface (left) and conjugated yield surface (right) in Haigh–Westergaard strain rate/stress space

Fig. 4 Meridional cross sections of constant dissipation surface (left) and conjugated yield surface (right)

Fig. 5 Deviatoric cross sections of constant dissipation surface (left) and conjugated yield surface (right)

Fig. 6 Plane stress cross section of meridional cross sections of considered yield surface versus experimental data for aluminium replicated foams [13]

Three-dimensional plots of the surfaces (28) and (35) are presented in Fig. 3. In the considered case, the Legendre transformation changes the orientation of the deviatoric cross sections but not the character of the curves. For both the yield surface and the constant dissipation surface, ellipses appear in the meridional cross sections, although they “swap” the semi-minor and the semi-major axes, see Fig. 4. If the semi-major axis of the dissipation meridian goes along the q direction, then its counterpart for the yield surface is located on the ξ axis, the case of $B\delta > A$. Similarly, for the deviatoric cross sections, the triangular curves are reflected by the distortion (shear) axes, compare Fig. 5. To sum up the features of the modified Beltrami potentials, if the constant dissipation surface is an oblate ellipsoid of a triangular deviatoric cross section, then the conjugate yield locus is a prolate ellipsoid with its deviatoric cross section of inverted triangle, and vice versa, see Appendix B for more graphical interpretations.

In Fig. 6, the cross sections of the Beltrami condition are compared to the experimental data for aluminium replicated foams [13]. The plane stress cross section bears a resemblance to the experimental curves of the material, but the compatibility of the meridional cross sections is poor at best. Still, the modification improves the yield condition, which in [13] initially was represented by a spheroid. It can be seen as a rough approximation only for this material.

4.2 Generalised Drucker–Prager dissipation potential

In this subsection, the conjugate yield condition and the respective flow rule are sought for the following dissipation function:

$$D(p, q, \phi) = \beta p - \sqrt{A^2 p^2 - B^2 q^2 \cos^2 \psi} \quad \text{for } A p > B q \cos \psi, \quad (41)$$

which is a generalised Drucker–Prager potential including four parameters: $\beta > A > 0$, $B > 0$ and $|\gamma| < 1$ for the Ottosen shape function. The potential is non-negative, convex and homogeneous of degree 1 with respect to p and q . A prototype dissipation function independent of ϕ ($\gamma = 0$) used for generalisation (41) is briefly described in [Appendix B](#), based on [8].

The required partial derivatives of function (41) according to (7)₁ and (18) are:

$$\alpha_1 = \frac{\partial D}{\partial p} = \beta - \frac{A^2 p}{\sqrt{A^2 p^2 - B^2 q^2 \cos^2 \psi}}, \quad \alpha = \frac{\partial D}{\partial (q \cos \psi)} = \frac{B^2 q \cos \psi}{\sqrt{A^2 p^2 - B^2 q^2 \cos^2 \psi}}. \quad (42)$$

As a result, we obtain the constitutive relation (20) in the form

$$\begin{aligned} \sigma = \frac{\partial D(\dot{\epsilon}_P)}{\partial \dot{\epsilon}_P} &= \frac{1}{\sqrt{3}} \left(\beta - \frac{A^2 p}{\sqrt{A^2 p^2 - B^2 q^2 \cos^2 \psi}} \right) \mathbf{I} + \frac{B^2 \cos^2 \psi}{\sqrt{A^2 p^2 - B^2 q^2 \cos^2 \psi}} \dot{\epsilon}_P \\ &+ \frac{\sqrt{6} \gamma B^2 \sin 2\psi}{2 q \sin 3\psi \sqrt{A^2 p^2 - B^2 q^2 \cos^2 \psi}} \left(\dot{\epsilon}_P^2 - \frac{q \cos 3\psi}{\sqrt{6}} \dot{\epsilon}_P - \frac{q^2}{3} \mathbf{I} \right). \end{aligned} \quad (43)$$

Now, (11)₁ and (21) yield the following:

$$\xi = \beta - \frac{A^2 p}{\sqrt{A^2 p^2 - B^2 q^2 \cos^2 \psi}}, \quad r^2 = \frac{B^4 q^2 \cos^2 \psi}{A^2 p^2 - B^2 q^2 \cos^2 \psi} \left(1 - \frac{1 - \gamma^2}{(1 + 2 \cos 2\psi)^2} \right), \quad (44)$$

which allows to draw the following conclusion:

$$\frac{(\beta - \xi)^2}{A^2} - \frac{r^2}{B^2 \delta^2} = 1. \quad (45)$$

As in the previous example, (45) is solved for the expression dependent on ψ :

$$\frac{1}{1 + 2 \cos 2\psi} = \sqrt{\frac{B^2 (\beta - \xi)^2 - A^2 (r^2 + B^2)}{(1 - \gamma^2) B^2 [(\beta - \xi)^2 - A^2]}} \quad (46)$$

or directly for $\delta(\xi, r)$:

$$\delta(\xi, r) = \sqrt{1 - \frac{1 - \gamma^2}{(1 + 2 \cos 2\psi)^2}} = \sqrt{\frac{A^2 r^2}{B^2 [(\beta - \xi)^2 - A^2]}} = \frac{A r}{B \sqrt{(\beta - \xi)^2 - A^2}} \equiv \frac{r}{C(\xi)}. \quad (47)$$

The primary form of the sought yield condition is described by (27), which in this case reduces to

$$\begin{aligned} F(\xi, r, \theta) &= \sqrt{B^2 [(\beta - \xi)^2 - A^2]} [3 A^2 r^2 - (2 - \gamma^2) B^2 [(\beta - \xi)^2 - A^2]] \\ &- 2 \gamma A^3 r^3 \cos 3\theta - 2 \sqrt{1 - \gamma^2} \left(\sqrt{B^2 (\beta^2 - \xi^2) - A^2 (r^2 + B^2)} \right)^3 = 0 \end{aligned} \quad (48)$$

for $\xi \leq \beta - A = \xi_V$.

Based on (48) $\tilde{\alpha}_i$ in flow rule (12) and the relations between the invariants (15) are defined by the following functions:

$$\tilde{\alpha}_1 = \frac{\partial F}{\partial \xi} = 3 B^2 (\beta - \xi) \left(\frac{W(\xi, r)^2}{B \sqrt{(\beta - \xi)^2 - A^2}} + (1 - \gamma^2) B \sqrt{(\beta - \xi)^2 - A^2} + 2 \sqrt{1 - \gamma^2} W(\xi, r) \right),$$

$$\begin{aligned}\tilde{\alpha}_2 &= \frac{\partial F}{\partial r} = 6A^2r \left(\sqrt{1 - \gamma^2} W(\xi, r) + \sqrt{B^2(A^2 - \xi^2) - A^2r^2} - \gamma A r \cos 3\theta \right), \\ \tilde{\alpha}_3 &= \frac{\partial F}{\partial \cos 3\theta} = -2\gamma A^3r^3 \text{ for } r \neq 0 \text{ and } |\xi| \neq A,\end{aligned}\quad (49)$$

where $W(\xi, r) \equiv \sqrt{B^2(\beta - \xi)^2 - A^2(B^2 + r^2)}$.

As the dual formulae have been derived, the calibration comes into focus. Four parameters are sought, namely A , B , β and γ . The obtained yield condition is suitable for quasibrittle materials, so we make use of experiments commonly performed for those type of materials: the uniaxial compression test (C), the uniaxial tension test (T), the equibiaxial compression test (BC) and the triaxial compression test (TC). For the two first tests, the cylindrical invariants are given by (37), while for the remaining two the corresponding formulae are:

$$\begin{aligned}\xi_{BC} &= -\frac{2\sigma_{BC}}{\sqrt{3}}, \quad r_{BC} = \sqrt{\frac{2}{3}}\sigma_{BC}, \quad \theta_{BC} = 0, \\ \xi_{TC} &= -\frac{(\eta + 2)\sigma_{TC}}{\sqrt{3}}, \quad r_{TC} = \sqrt{\frac{2}{3}}(\eta - 1)\sigma_{TC}, \quad \theta_{TC} = \frac{\pi}{3},\end{aligned}\quad (50)$$

where σ_{BC} and σ_{TC} with $\eta > 1$ are the yield limits for the equibiaxial and the triaxial compression tests, respectively.

The points representing all the mentioned experiments lay on either the tension meridian $r_t(\xi) = C\delta_t$ or the compression meridian $r_c(\xi) = C\delta_c$, according to (26). Taking into account (25) with specification (47), the meridians are described by the following curves:

$$r_t(\xi) = \frac{B\Delta_t}{A}\sqrt{(\beta - \xi)^2 - A^2} \quad \text{and} \quad r_c(\xi) = \frac{B\Delta_c}{A}\sqrt{(\beta - \xi)^2 - A^2}. \quad (51)$$

The system of equations defining the free parameters becomes:

$$\begin{aligned}r_T &= \frac{B\Delta_t}{A}\sqrt{(\beta - \xi_T)^2 - A^2}, & r_C &= \frac{B\Delta_c}{A}\sqrt{(\beta - \xi_C)^2 - A^2}, \\ &\text{and} & & \\ r_{BC} &= \frac{B\Delta_t}{A}\sqrt{(\beta - \xi_{BC})^2 - A^2}, & r_{TC} &= \frac{B\Delta_c}{A}\sqrt{(\beta - \xi_{TC})^2 - A^2}.\end{aligned}\quad (52)$$

The equations are nonlinear, but an analytical solution can be obtained. Denoting:

$$t = \frac{\Delta_c}{\Delta_t}, \quad a = \frac{\xi_{BC}r_T^2 - \xi_T r_{BC}^2}{r_{BC}^2 - r_T^2}, \quad b = \frac{\xi_{TC}r_C^2 - \xi_C r_{TC}^2}{r_{TC}^2 - r_C^2}, \quad c = \frac{\xi_{BC}^2 r_T^2 - \xi_T^2 r_{BC}^2}{r_{BC}^2 - r_T^2}, \quad d = \frac{\xi_{TC}^2 r_C^2 - \xi_C^2 r_{TC}^2}{r_{TC}^2 - r_C^2}, \quad (53)$$

the formulae describing the material constants are:

$$\begin{aligned}\beta &= \frac{c - d}{2(a - b)}, \quad A = \sqrt{\left[\frac{c - d}{2(a - b)} \right]^2 + \frac{a d - b c}{b - a}}, \quad t = \frac{r_C}{r_T} \sqrt{\frac{(b - a)\xi_T^2 + (c - d)\xi_T - a d + b c}{(b - a)\xi_C^2 + (c - d)\xi_C - a d + b c}}, \\ \gamma &= \frac{3\sqrt{3}t(1 - t)}{2(\sqrt{t^2 - t + 1})^3}, \quad B = \frac{r_T}{2\Delta_t} \sqrt{\frac{(c - d)^2 + 4(a d - b c)(b - a)}{(b - a)[(b - a)\xi_T^2 + (c - d)\xi_T - a d + b c]}}.\end{aligned}\quad (54)$$

Of course the presented calibration is limited to the input data that generates positive results for A , B , β and $|\gamma| < 1$, fulfilling also $\beta > A$. Note that according to (37) and (50) hardening can be conveniently included in the definitions of σ_T , σ_C , σ_{BC} and σ_{TC} since the calibration formulae are explicit. In the simplest case, the yield limit $\sigma_C(\epsilon_P)$ is selected for description of hardening, while remaining limits are scaled by this limit. For isotropic hardening usually an equivalent plastic strain defined as invariant $\epsilon_P = \sqrt{\epsilon_P \cdot \epsilon_P}$ is chosen as an internal variable.

Denoting ratios $\alpha_T = \sigma_T/\sigma_C$, $\alpha_{BC} = \sigma_{BC}/\sigma_C$ and $\alpha_{TC} = \sigma_{TC}/\sigma_C$, we can illustrate the allowed proportions of parameters using a contour plot, see Fig. 7. Based on experimental data [10, 14, 15] for triaxial

compression tests, assuming function $\alpha_{TC}(\eta)$ suitable for concrete as depicted on the plot, we can indicate the regions where values of α_T , α_{BC} and η result in $\beta > A$, $B > 0$ and $|\gamma| < 1$. It seems that for concrete the key parameter in calibration of the regarded yield condition (48) is the yield limit for the uniaxial tension test σ_T (or more precisely ratio α_T).

Assuming $\sigma_C = 20$ MPa, $\sigma_T = 2$ MPa, $\sigma_{BC} = 23.2$ MPa, $\sigma_{TC} = 25.1$ MPa for $\eta = 4.91$ (data typical for concrete [10, 14]), (53) and (54) yield $\beta = 16.1$ MPa, $A = 14.5$ MPa, $B = 10.3$ MPa and $\gamma = -0.824$. Plots of the constant dissipation surface, i.e. $D = D_0 = 14$ MW/m³, the yield condition and their cross sections are presented in Figs. 8 and 9.

The three-dimensional views of the surfaces given in Fig. 8 reveal that the constant dissipation surface has an ellipsoidal shape. However, the conjugate yield surface is hyperboloidal, which is caused by the shift of the centre of the ellipsoid from the origin of the coordinate system clearly visible for the meridional cross sections in Fig. 9. The vertex of the yield surface is at $\xi_V = 1.57$ MPa. The meridians of the constant dissipation surface are sections of ellipses cut on one side (the requirement of convexity of (41), see Appendix B), while the meridians of the yield condition are hyperboles. As in the previous case, the triangular deviatoric cross sections have inverted positions characteristic for the Ottosen shape function, compare Appendix B for more details.

The comparison of the yield condition with experimental data for concrete [14, 15] is given in Fig. 10. The compatibility is satisfactory, better for the meridional cross sections, worse but acceptable for the plane stress cross section.

Fig. 7 Limits on material parameters for concrete: assumed function $\alpha_{TC}(\eta)$ (left) and admissible regions (grey) of α_T and η for chosen values of α_{BC} (right)

Fig. 8 Constant dissipation surface (left) and conjugated yield surface (right) in Haigh–Westergaard strain rate/stress space

Fig. 9 Meridional cross sections of constant dissipation surface (left) and conjugated yield surface (right), tension and compression meridians

Fig. 10 Meridional and plane stress cross sections of generalised Drucker–Prager yield condition versus experimental data [14, 15]

4.3 Generalised Mises–Schleicher dissipation potential

Consider the following function being a generalisation of the Mises–Schleicher dissipation potential:

$$D(p, q, \phi) = p \left[A + B \left(\frac{q \cos \psi}{p} \right)^K \right]. \quad (55)$$

D is non-negative and convex for $A > 0$, $B > 0$, $K > 1$ and $|\gamma| < 1$. Also, it is homogeneous of degree 1 with respect to p and q . A special case of dissipation (55) with $\gamma = 0$ and $K = 2$ was investigated in [4] leading to the simplest case of the yield condition, whose graph is a paraboloid of revolution. The Mises–Schleicher-based dissipation with another deviatoric shape function was investigated in [11].

To formulate the constitutive law and the conjugate yield condition, the appropriate derivatives given by (7)₁ and (18) are calculated:

$$\alpha_1 = \frac{\partial D}{\partial p} = A - B(K-1) \left(\frac{q \cos \psi}{p} \right)^K, \quad \alpha = \frac{\partial D}{\partial (q \cos \psi)} = B K \left(\frac{q \cos \psi}{p} \right)^{K-1}. \quad (56)$$

Now, the constitutive relation (20) is expressed as follows:

$$\begin{aligned} \boldsymbol{\sigma} = \frac{\partial D(\dot{\boldsymbol{\epsilon}}_P)}{\partial \dot{\boldsymbol{\epsilon}}_P} = \frac{1}{\sqrt{3}} & \left\{ A - B \left(\frac{q \cos \psi}{p} \right)^K \left[K - 1 + \frac{3\sqrt{2}Kp}{q^2} \left(1 - \frac{2}{1 + 2 \cos 2\psi} \right) \right] \right\} \mathbf{I} \\ & + \frac{BK}{q(1 + 2 \cos 2\psi)} \left(\frac{q \cos \psi}{p} \right)^{K-1} \left(\frac{\sqrt{6}\gamma}{q} \dot{\boldsymbol{\epsilon}}_P^2 + 2 \cos \psi \dot{\boldsymbol{\epsilon}}_P \right). \end{aligned} \quad (57)$$

The relationships between the invariants (11)₁ and (21) reduce to:

$$\xi = A - B(K-1) \left(\frac{q \cos \psi}{p} \right)^K, \quad r^2 = B^2 K^2 \left(\frac{q \cos \psi}{p} \right)^{2(K-1)} \left(1 - \frac{1 - \gamma^2}{(1 + 2 \cos 2\psi)^2} \right), \quad (58)$$

which lead to the identity

$$\left[\frac{A - \xi}{B(K-1)} \right]^{\frac{1}{K}} = \left(\frac{r}{BK\delta} \right)^{\frac{1}{K-1}}. \quad (59)$$

The next step is to solve (59) for $\delta(\xi, r)$ according to its definition (23):

$$\delta(\xi, r) = \frac{r}{BK} \left[\frac{A - \xi}{B(K-1)} \right]^{-\frac{K-1}{K}} \equiv \frac{r}{C(\xi)}. \quad (60)$$

Using (60) in (27) results in the following yield condition:

$$\begin{aligned} F(\xi, r, \theta) = 3BK & \left[\frac{A - \xi}{B(K-1)} \right]^{\frac{K-1}{K}} r^2 - 2\gamma r^3 \cos 3\theta \\ & - (2 - \gamma^2) B^3 K^3 \left[\frac{A - \xi}{B(K-1)} \right]^{\frac{3(K-1)}{K}} \\ & - 2 \sqrt{(1 - \gamma^2) \left\{ B^2 K^2 \left[\frac{A - \xi}{B(K-1)} \right]^{\frac{2(K-1)}{K}} - r^2 \right\}^3} = 0 \end{aligned} \quad (61)$$

for $\xi \leq A = \xi_V$, where ξ_V is a vertex of the yield loci.

To specify the associated flow rule (12), $\tilde{\alpha}_i$ have to be determined:

$$\begin{aligned} \tilde{\alpha}_1 = \frac{\partial F}{\partial \xi} &= 3 \left[\frac{A - \xi}{B(K-1)} \right]^{\frac{1}{K}} \left[(2 - \gamma^2) C(\xi)^2 + 2C(\xi) \sqrt{(1 - \gamma^2)(C(\xi)^2 - r^2)} - r^2 \right], \\ \tilde{\alpha}_2 = \frac{\partial F}{\partial r} &= 6r \left(C(\xi) + \sqrt{(1 - \gamma^2)(C(\xi)^2 - r^2)} - \gamma r \cos 3\theta \right), \\ \tilde{\alpha}_3 = \frac{\partial F}{\partial \cos 3\theta} &= -2\gamma r^3 \text{ for } |\xi| \neq A. \end{aligned} \quad (62)$$

Again, the calibration of the free parameters is based on the results of three representative experiments: the uniaxial compression test (*C*), the uniaxial tension test (*T*) and the equibiaxial compression test (*BC*). Additionally, the location of the surface's vertex ($r = 0$) is set as ξ_V . Following the previously used procedure

involving formulae (25), (37) and (50)₁ to describe the tension and compression meridians, the system of equations for the sought constants is:

$$\begin{aligned} r_T &= \Delta_t B K \left[\frac{A - \xi_T}{B(K-1)} \right]^{\frac{K-1}{K}}, & r_C &= \Delta_c B K \left[\frac{A - \xi_C}{B(K-1)} \right]^{\frac{K-1}{K}}, \\ &\text{and} & & \\ r_{BC} &= \Delta_t B K \left[\frac{A - \xi_{BC}}{B(K-1)} \right]^{\frac{K-1}{K}}, & A &= \xi_V. \end{aligned} \quad (63)$$

Denoting

$$t = \frac{\Delta_c}{\Delta_t}, \quad x_0 = \frac{\sqrt{3}\xi_V - \sigma_T}{\sqrt{3}\xi_V + 2\sigma_{BC}}, \quad y_0 = \log_{x_0} \frac{\sigma_T}{\sigma_{BC}}, \quad (64)$$

Eq. (63) yields the following solution:

$$\begin{aligned} A &= \xi_V, \quad K = \frac{1}{1 - y_0}, \quad t = \frac{\sigma_C}{\sigma_T} \left(\frac{\sqrt{3}\xi_V - \sigma_T}{\sqrt{3}\xi_V + \sigma_C} \right)^{y_0}, \\ \gamma &= \frac{3\sqrt{3}t(1-t)}{2(\sqrt{t^2 - t + 1})^3}, \quad B = \frac{\sqrt{2}(1-y_0)\sigma_C}{\sqrt{3}\Delta_c} \left[\frac{\sqrt{2}y_0\sigma_C}{\Delta_c(\sqrt{3}\xi_V + \sigma_C)} \right]^{\frac{y_0}{1-y_0}}. \end{aligned} \quad (65)$$

The input data are restricted to the test results returning $A > 0$, $B > 0$, $K > 1$ and $|\gamma| < 1$. Denoting $\alpha_T = \sigma_T/\sigma_C$, $\alpha_{BC} = \sigma_{BC}/\sigma_C$ and $\alpha_V = \xi_V/\sigma_T$, we can illustrate the range of admissible values of the material parameters, which is shown in Fig. 11. The considered yield condition allows to model a material in a wider range than it was for the Drucker–Prager case. This extended flexibility is due to the class of the dissipation function used (K th degree).

In order to show some characteristic features of both the dissipation potential and the conjugate yield condition, the data for the typical concrete are used [14, 15]: $\sigma_C = 20$ MPa, $\sigma_T = 2$ MPa, $\sigma_{BC} = 23.2$ MPa and $\xi_V = \sqrt{3}\sigma_T = 3.46$ MPa, resulting in the following values of the constants: $A = 3.46$ MPa, $x_0 = 0.0763$, $K = 21.2$, $t = 1.68$, $\gamma = -0.947$ and $B = 1.80$ MPa. The constant dissipation surface is presented for $D = D_0 = 30$ MW/m³.

The modified Mises–Schleicher yield condition constitutes an open surface of a triangular deviatoric cross section in the Haigh–Westergaard stress space, while the constant dissipation surface is closed also with a triangular deviatoric cross section, compare Figs. 12, 13 and 14. Translating the vertex of the yield condition in the direction of the origin of the coordinate system (lowering ξ_V) causes the constant dissipation surface to

Fig. 11 Limits on material parameters α_T and α_V for concrete: the admissible regions (grey) expand from the black marked curves to line $\alpha_T = 0.5$

Fig. 12 Constant dissipation surface and conjugated yield surface in Haigh–Westergaard strain rate/stress space

Fig. 13 Meridional cross sections of constant dissipation surface and conjugated yield surface, tension and compression meridians

Fig. 14 Deviatoric cross sections of constant dissipation surface and conjugated yield surface. Left: I: $p = 0.5p_0$, II: $p = 0.75p_0$, III: $p = p_0$, $p_0 = 8.25 s^{-1}$. Right: 1: $\xi = 0.5\xi_V$, 2: $\xi = -0.5\xi_V$, 3: $\xi = -3\xi_V$

Fig. 15 Comparison of the obtained yield condition with experimental data [14, 15]

extend along the p axis (its meridians extend from $p = 0$ to $p = D_0/A = D_0/\xi_V$, see Fig. 13). The “inversion” of the deviatoric cross section is present as it was for the previous examples, see Fig. 14 and Appendix B.

Checking the compatibility of the yield condition and experimental data for concrete of various uniaxial compressive strengths [14, 15] is possible when the unit yield limit $\sigma_C = 1$ MPa is set. Then $\sigma_T = 0.1$ MPa, $\sigma_{BC} = 1.16$ MPa and $\xi_V = 0.12$ MPa. The calibration results in the following values of the parameters: $A = 0.12$ MPa, $x_0 = 0.0427$, $K = 4.84$, $t = 1.53$, $\gamma = -0.864$ and $B = 0.143$ MPa. As formulae (64) and (65) are highly nonlinear, the change in K and γ is considerable compared to the previous data set.

The obtained yield condition fits the experimental data for concrete quite nicely. Various meridians shown in Fig. 15 prove a good compatibility with the results of the tests [14, 15]. When it comes to the plane stress cross section, the curve matches the experiments in the zones of tension–tension or tension–compression, although when both principal stresses are compressive the compatibility is acceptable for rough calculations only.

5 Conclusions

To propose and calibrate a dissipation potential, one has to obtain the conjugate yield condition via Legendre transformation, which, as shown above, is a cumbersome task. The paper deals with the case of an isotropic material, which subjected to plastic deformations shows the dependency on all three independent invariants of the strain rate tensor and involves the Ottosen-type shape function, describing many commonly used engineering materials, i.e. quasibrittle materials, concrete included, but also metals and foams. A method of conduct in such a case is formulated in the text, including remarks on performing an effective calibration of the function’s parameters.

The purpose of giving the details of calculations is to offer some suggestions as to the technique of obtaining the dual formulation. An indirect form of the sought yield condition (11) is easily derived. However, entangling it poses a challenge. The strategy to do it is based on the observation of the appearance of the same term, namely $1 + 2 \cos 2\psi$, in the constitutive relationships, which enables their partial inversion and expressing the yield condition via the stress tensor invariants (24). As the reader noticed, we give some indications of how to proceed, but demonstrating a precise algorithm is not possible, as an inversion of a function is not an automatic process. The generalisations of the three classical functions of Beltrami, Drucker–Prager and Mises–Schleicher chosen to illustrate the problem are quite representative examples, which allows to anticipate that the procedure can be successfully applied to similar dissipation potentials, i.e. to relatively simple functions composed of polynomial expressions. As a result, three new generalisations of the classical yield conditions were obtained.

The presented examples of dissipation functions dependent on all three independent invariants of the strain rate tensor open the possible paths to the development of explicit forms of dual potentials and relationships of perfectly plastic materials. The described procedure allows to take into account the dependence on the Lode angle in both new and modified constitutive models. Such models can be then included in thermodynamically consistent modelling of a coupled elastic, plastic and damage response of engineering materials, compare Appendix A.

It seems that analogous general formulae can be obtained if the Podgórski's shape function [7] is used instead of Ottosen's, which is a useful generalisation of (17). However, a rough inspection of the problem reveals that the output formulae could be more complex. To maintain the simplicity of calculations, we omit this extension, although it is an issue worth raising in future work.

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Appendix A: Constitutive modelling of elastoplastic damaged material using Helmholtz energy and dissipation potential

In the paper, we present a method of finding a yield condition when a dissipation function of a specific kind (16) is given. This is one of the most problematic parts of a wider issue, which is thermodynamically consistent constitutive modelling of elastoplastic damaged materials. We employ the framework developed by Collins and Houlsby [2] and Houlsby and Puzrin [3]. Its extension for damage is due to Einav et al. [5]. The goal of this Appendix is to briefly present the consecutive steps of the method of obtaining a full set of equations defining the behaviour of an elastoplastic damaged material in the range of small displacement theory.

First, we need to choose internal variables describing the analysed irreversible phenomena, i.e. plasticity and damage. Let $\boldsymbol{\varepsilon}_P$ and $\mathbf{M}$ be two second-order tensors, a plastic strain tensor and a damage tensor, respectively. The concept of building of the constitutive relationships is based on two scalar-valued functions. Let us assume that we have the following two potentials.

1. Helmholtz energy H is strictly convex with respect to $\boldsymbol{\varepsilon}$ and non-negative. Its arguments are as follows:

$$H = H(\boldsymbol{\varepsilon}, \boldsymbol{\varepsilon}_P, \mathbf{M}) = \tilde{H}(\boldsymbol{\varepsilon} - \boldsymbol{\varepsilon}_P, \mathbf{M}), \quad (66)$$

which means that the strain tensors act through their difference (or: the argument is an elastic strain tensor $\boldsymbol{\varepsilon}_E = \boldsymbol{\varepsilon} - \boldsymbol{\varepsilon}_P$). This applies to materials with no kinematic hardening. For the particulars of that case, we refer the reader to the cited papers.

2. Dissipation potential D is non-negative and convex with respect to $\dot{\boldsymbol{\varepsilon}}_P$ and $\dot{\mathbf{M}}$. For rate independent cases, it is a homogenous function of degree 1 of the rates of the internal variables:

$$D = D(\dot{\boldsymbol{\varepsilon}}_P, \dot{\mathbf{M}}). \quad (67)$$

The second step towards achieving the desired constitutive equations is to define the following stress tensors:

1. the tensors related to the Helmholtz energy:

- the Cauchy stress tensor:

$$\boldsymbol{\sigma} = \frac{\partial H}{\partial \boldsymbol{\varepsilon}}, \quad (68)$$

- the generalised stress tensors connected to plasticity ($\boldsymbol{\sigma}_P$) and damage ($\boldsymbol{\sigma}_M$):

$$\boldsymbol{\sigma}_P = -\frac{\partial H}{\partial \boldsymbol{\varepsilon}_P} \quad \text{and} \quad \boldsymbol{\sigma}_M = -\frac{\partial H}{\partial \dot{\mathbf{M}}}. \quad (69)$$

2. the dissipative stress tensors related to the dissipation potential:

$$\bar{\boldsymbol{\sigma}}_P = \frac{\partial D}{\partial \dot{\boldsymbol{\varepsilon}}_P} \quad \text{and} \quad \bar{\boldsymbol{\sigma}}_M = \frac{\partial D}{\partial \dot{\mathbf{M}}}. \quad (70)$$

According to the Euler's theorem for the regarded homogenous function, the dissipation potential can be written as:

$$D(\dot{\boldsymbol{\varepsilon}}_P, \dot{\mathbf{M}}) = \frac{\partial D}{\partial \dot{\boldsymbol{\varepsilon}}_P} \cdot \dot{\boldsymbol{\varepsilon}}_P + \frac{\partial D}{\partial \dot{\mathbf{M}}} \cdot \dot{\mathbf{M}} = \bar{\boldsymbol{\sigma}}_P \cdot \dot{\boldsymbol{\varepsilon}}_P + \bar{\boldsymbol{\sigma}}_M \cdot \dot{\mathbf{M}}. \quad (71)$$

The yield/damage condition and the dissipation function are bound by the following singular Legendre transformation:

$$D(\dot{\boldsymbol{\varepsilon}}_P, \dot{\mathbf{M}}) + \lambda \bar{F}(\bar{\boldsymbol{\sigma}}_P, \bar{\boldsymbol{\sigma}}_M) = \bar{\boldsymbol{\sigma}}_P \cdot \dot{\boldsymbol{\varepsilon}}_P + \bar{\boldsymbol{\sigma}}_M \cdot \dot{\mathbf{M}}, \quad (72)$$

which allows us to find $\bar{F}$ dual to D . $\lambda \geq 0$ is a plastic/damage multiplier. (72) enables the derivation of the yield/damage condition:

$$\bar{F}(\bar{\boldsymbol{\sigma}}_P, \bar{\boldsymbol{\sigma}}_M) = 0, \quad (73)$$

which is in detail shown in the paper. Then, the multiplier can be obtained using the consistency condition:

$$\dot{\bar{F}}(\bar{\boldsymbol{\sigma}}_P, \bar{\boldsymbol{\sigma}}_M) = 0, \quad (74)$$

and the evolution laws of the internal variables are:

$$\dot{\boldsymbol{\varepsilon}}_P = \lambda \frac{\partial \bar{F}}{\partial \bar{\boldsymbol{\sigma}}_P} \quad \text{and} \quad \dot{\mathbf{M}} = \lambda \frac{\partial \bar{F}}{\partial \bar{\boldsymbol{\sigma}}_M}. \quad (75)$$

In order to find material parameters included in dissipation, condition (73) has to be expressed via the Cauchy stress tensor. This transition is possible due to Ziegler's orthogonality hypothesis [2]. The hypothesis plays a classifying role: it is suitable for a wide range of engineering materials, for particulars see [3]. The hypothesis is based on equalisation of mechanical dissipation D_{mech} due to the rate of the Helmholtz energy and the proposed dissipation potential D :

$$D_{mech} = \boldsymbol{\sigma} \cdot \dot{\boldsymbol{\varepsilon}} - \dot{H} = D. \quad (76)$$

Calculation of time derivative of (66), then application of (68) and (69) and finally comparison with (71) lead to the orthogonality principle stating that:

$$\bar{\boldsymbol{\sigma}}_P = \boldsymbol{\sigma}_P \quad \text{and} \quad \bar{\boldsymbol{\sigma}}_M = \boldsymbol{\sigma}_M. \quad (77)$$

The direct consequence of H being expressed as a function of the difference of total and plastic strain tensors (66) is that:

$$\boldsymbol{\sigma}_P = \boldsymbol{\sigma}. \quad (78)$$

Also, for classic forms of Helmholtz energy it is possible to make $\bar{\boldsymbol{\sigma}}_M$ dependent on $\boldsymbol{\sigma}$. This is not an automatic process; therefore, every form of a chosen Helmholtz function has to be considered separately.

As a result of the above considerations, we obtain the yield/damage condition expressed via Cauchy stress tensor and the respective consistency condition:

$$F(\boldsymbol{\sigma}) = 0, \quad \dot{F}(\boldsymbol{\sigma}) = 0. \quad (79)$$

Now, the evolution laws can be transformed to depend on σ instead of $\bar{\sigma}_P$ and $\bar{\sigma}_M$, and consequently the constitutive fourth-order stiffness tensor can be derived to include plasticity and damage. Examples of constitutive models developed according to the described approach are presented in [2, 3, 5], among others.

Let us emphasise that changing the arguments in the evolution laws (75) in general can imply:

$$\dot{\epsilon}_P = \lambda \frac{\partial \bar{F}}{\partial \bar{\sigma}_P} \neq \lambda \frac{\partial F}{\partial \sigma_P} \quad \text{and} \quad \dot{\mathbf{M}} = \lambda \frac{\partial \bar{F}}{\partial \bar{\sigma}_M} \neq \lambda \frac{\partial F}{\partial \sigma_M}, \quad (80)$$

which means that the evolution laws are not necessarily associated with the yield/damage condition when expressed in the generalised stresses spaces, although they are associated in the dissipative stresses spaces. This occurs when we take into account additional stress-type arguments in dissipation function, i.e., $D = D(\dot{\epsilon}_P, \dot{\mathbf{M}}; \sigma_P)$. The feature is very useful when modelling materials such as concrete and rocks or soils.

Hardening can be easily incorporated in the described framework. It suffices to state that material parameters are functions of, i.e., ϵ_P , so in turn $D = D(\dot{\epsilon}_P, \dot{\mathbf{M}}; \epsilon_P)$. This does not violate the equations presented above. The change influences strongly consistency condition (79) and it affects directly the calculations of multiplier λ , because we have to differentiate the plastic strain tensor with respect to time, so an additional term including $\dot{\epsilon}_P$ appears. In the remaining relations, the parameters dependent on ϵ_P need to be substituted only.

In the paper, we focus on finding a dual potential when $D = D(\dot{\epsilon}_P)$, so there is no damage involved, and there is no supplementary arguments and the Helmholtz potential is of form (66), which results in

$$\bar{\sigma}_P = \sigma_P = \sigma, \quad (81)$$

hence the three stress tensors are fully equivalent, also during differentiations. The Legendre transformation (72) then reduces to the form

$$D(\dot{\epsilon}_P) + \lambda F(\sigma) = \sigma \cdot \dot{\epsilon}_P, \quad \text{where} \quad \sigma = \frac{\partial D}{\partial \dot{\epsilon}_P} \quad (82)$$

and the respective yield condition, consistency condition and flow rule are:

$$F(\sigma) = 0, \quad \dot{F}(\sigma) = 0, \quad \dot{\epsilon}_P = \lambda \frac{\partial F}{\partial \sigma}. \quad (83)$$

In the general case, all the results obtained in the paper are valid in the dissipative stresses spaces.

Appendix B: Graphical interpretation of Legendre transformation

Let us show a graphical representation of the Legendre transformation to clarify the shape of the yield surface conjugated to a given dissipation potential, and vice versa. In the case of a dissipation function dependent on all three cylindrical invariants, due to isotropic decomposition of a tensor, it suffices to split the representing surface into meridional and deviatoric cross sections. The respective components of every conjugate quantities can be independently interpreted and then put together to form sought counterparts. We will consider a constant dissipation surface described by the following equation:

$$D(\dot{\epsilon}_P) = \sigma \cdot \dot{\epsilon}_P = D_0 = \text{const.} \quad (84)$$

The formulae determining the singular Legendre transformation read:

$$D(\dot{\epsilon}_P) + \lambda F(\sigma) = \sigma \cdot \dot{\epsilon}_P, \quad \sigma = \frac{\partial D}{\partial \dot{\epsilon}_P}, \quad \dot{\epsilon}_P = \lambda \frac{\partial F}{\partial \sigma}. \quad (85)$$

Now, we consider a constant dissipation surface that is only dependent on invariants p and q (independent of the Lode angle) and is an ellipse on the pq plane, as shown in Fig. 16. Our goal is to construct a meridian of its dual yield surface.

Let X and Y be the semi-minor and the semi-major axes of the original ellipse $D = D_0$ according to definition (28) with $\cos \psi = 1$. We know the direction of σ (here represented as vectors) in the Haigh–Westergaard space, it is normal to the constant dissipation surface according to (85)₂. For the vertex points 1, 2, 3 and 4 in Fig. 16, the following occurs:

$$\sigma_1 \cdot \dot{\epsilon}_{P1} = \sigma_2 \cdot \dot{\epsilon}_{P2} = \sigma_3 \cdot \dot{\epsilon}_{P3} = \sigma_4 \cdot \dot{\epsilon}_{P4} = D_0. \quad (86)$$

Fig. 17 Legendre transformation of a function whose representation in the Haigh–Westergaard space is a regular triangle in deviatoric cross section

described by $\dot{\epsilon}_{PA}$. As the dissipation stays constant, all the stress tensors (vectors) having the same projection on the $\dot{\epsilon}_{PA}$ direction represent valid duals, so

$$\sigma_{AB} \cdot \dot{\epsilon}_{PA} = \sigma_{AC} \cdot \dot{\epsilon}_{PA} = \dots = D_0. \quad (92)$$

This means that all the counterparts of A are located on line segment KM and the counterparts of B and C are located on line segments KN and MN .

We have shown that the dual to a dissipation function whose contour lines form a regular triangle is a yield condition that is a regular triangle with inverted orientation as its deviatoric cross section. We have chosen a simple shape, although singular, to depict a general feature of the Legendre transformation. This procedure can be repeated for the Ottosen shape function, which is of very similar smooth shape, but calculations are a bit more complicated, which unnecessarily clouds the issue.

The third example concerns derivation of the dissipation function (41) being independent of the third invariant θ . Let us start with the following yield condition for smoothed Drucker–Prager condition [8]:

$$F(\xi, r) = \frac{A}{B} \sqrt{B^2 + r^2} + \xi - \beta = 0 \quad (93)$$

with restrictions on the parameters $\beta > A > 0$, $B > 0$ and $F(0, 0) < 0$. Using the yield law (5), we obtain

$$\dot{\epsilon}_P = \lambda \frac{\partial F}{\partial \sigma} = \lambda \left(\frac{1}{\sqrt{3}} \mathbf{I} + \frac{A}{B \sqrt{B^2 + r^2}} \mathbf{s} \right), \quad (94)$$

which leads to the following relations between the invariants (2) and (9):

$$p = \lambda, \quad q = \lambda \frac{Ar}{B \sqrt{B^2 + r^2}}. \quad (95)$$

Eliminating λ between Eqs. (95) and using the yield condition (93), we can find the inverse relations:

$$\xi = \beta - \frac{A^2 p}{\sqrt{A^2 p^2 - B^2 q^2}}, \quad r = \frac{B^2 q}{\sqrt{A^2 p^2 - B^2 q^2}} \quad \text{for } A^2 p^2 \neq B^2 q^2. \quad (96)$$

Then, the dissipation function (84)₁ can be derived:

$$D(p, q) = \xi p + r q = \beta p - \sqrt{A^2 p^2 - B^2 q^2}, \quad (97)$$

Fig. 18 Dual potentials: yield condition and contour of dissipation function for smooth Drucker–Prager model

Fig. 19 Contours and axonometric view of non-negative, convex cone of dissipation function for smooth Drucker–Prager model

which is limited to $Ap > Bq$, since $p > 0$ for the regarded yield condition (93). The obtained function is non-negative, convex and homogeneous of degree one with respect to the strain rate tensor. Formula (97) is a prototype to its generalisation to the form (41).

Graphical representations of the yield condition (93) and the conjugated dissipation (97) are shown in Fig. 18. One branch of hyperbolic yield condition (93) separates the safe $F < 0$ from the failure $F > 0$ stress states in a unique way. The asymptote of the hyperbola branch, $r = B(\beta - \xi)/A$, represents the original Drucker–Prager yield condition. Parameters involved in definition of yield condition (93) are explained in Fig. 18. Note that for stress paths fulfilling condition $Ar + B\xi \leq 0$ there is no failure of material; therefore, to these stress states no dual dissipation is assigned. Contour $D = D_0$ of dissipation function (97) which represents a segment of an ellipse is shown in Fig. 18. The line tangent to the ellipse is given by equation $Ap = Bq$, the point of tangency represents stress states satisfying $Ar + B\xi = 0$. The semi-axes a_p and a_q of the ellipse with the other characteristic parameters p_C , p_A , q_A have the following values:

$$a_p = \frac{AD_0}{\beta^2 - A^2}, \quad a_q = \frac{AD_0}{B\sqrt{\beta^2 - A^2}}, \quad p_C = \frac{\beta D_0}{\beta^2 - A^2}, \quad p_A = \frac{D_0}{\beta}, \quad q_A = \frac{AD_0}{\beta B}. \quad (98)$$

The non-negative, convex and homogeneous degree one dissipation function (97) is shown in Fig. 19. The contours $D_1 < \dots < D_5$ form segments of ellipses belonging to a convex skew cone of the dissipation function. The remaining part of the cone (cut off by the plane $D = \beta p$ including tangent line $Ap = Bq$) forms a concave surface so it is excluded from the definition of dissipation function.

Appendix C: Calibration of proposed yield conditions

The calibration of the proposed yield conditions relies upon a relatively simple structure of equations for the compression ($\theta = \pi/3$, $\phi = \pi/3$) and tension ($\theta = 0$, $\phi = 0$) meridians. Let us consider the primary form of the considered yield conditions (24):

$$3\delta^2 - 2\sqrt{1-\gamma^2}\left(\sqrt{1-\delta^2}\right)^3 - 2\gamma\delta^3 \cos 3\theta - 2 + \gamma^2 = 0 \quad (99)$$

with the assumption that $\delta = \delta(\xi, r)$ defined by (23). Consequently, for the tension meridian and the compression meridian Eq. (99) reduces to the following:

$$\theta = 0 \Rightarrow 3\delta_t^2 - 2\gamma\delta_t^3 - 2 + \gamma^2 = 2\sqrt{1-\gamma^2}\left(\sqrt{1-\delta_t^2}\right)^3, \quad (100)$$

$$\theta = \frac{\pi}{3} \Rightarrow 3\delta_c^2 + 2\gamma\delta_c^3 - 2 + \gamma^2 = 2\sqrt{1-\gamma^2}\left(\sqrt{1-\delta_c^2}\right)^3. \quad (101)$$

Squaring the sides of both Eqs. (100) and (101), after factoring polynomials, we obtain:

$$(\delta_t - \gamma)^3(-4\delta_t^3 + 3\delta_t + \gamma) = 0 \Leftrightarrow \delta_t = \gamma \vee -4\delta_t^3 + 3\delta_t + \gamma = 0, \quad (102)$$

$$(\delta_c + \gamma)^3(4\delta_c^3 - 3\delta_c + \gamma) = 0 \Leftrightarrow \delta_c = -\gamma \vee 4\delta_c^3 - 3\delta_c + \gamma = 0. \quad (103)$$

$\delta_t = \gamma$ and $\delta_c = -\gamma$ are the solutions of (102) and (103) resulting from the squaring and the reader can check by a direct substitution that they do not satisfy initial equations, neither (100) nor (101). The remaining equations are cubic, so it is always possible to solve them using the trigonometric substitution method. The valid, real, positive solutions are:

$$\delta_t(\xi, r) = \cos\left(\frac{1}{3} \arccos \gamma\right) \equiv \Delta_t, \quad (104)$$

$$\delta_c(\xi, r) = \cos\left(\frac{1}{3} \arccos(-\gamma)\right) \equiv \Delta_c. \quad (105)$$

Equations (104) and (105) define the tension and compression meridians of the considered yield condition. Note that $\delta > 0$ due to definition (23). Moreover, relation (26) implies $\delta_t(\xi, r) = r_t C(\xi)$ and $\delta_c(\xi, r) = r_c C(\xi)$, where $r_t(\xi)$ and $r_c(\xi)$ determine curves of the tension and compression meridians, respectively.

Now, we will show in detail the determination of the parameters for the generalised Beltrami condition. δ , according to (34), is as follows:

$$\delta(\xi, r) = \frac{Ar}{B\sqrt{A^2 - \xi^2}}. \quad (106)$$

We regard an arbitrary value of γ , so constants Δ_t and Δ_c are known and we are looking for A and B , which requires two equations (two experimental tests). Let us choose the uniaxial tension tests and the uniaxial compression test, where the yield limits are σ_T and σ_C , respectively. The stress tensor cylindrical invariants calculated according to (9) are:

$$\xi_T = \frac{\sigma_T}{\sqrt{3}}, \quad r_T = \sqrt{\frac{2}{3}} \sigma_T, \quad \theta_T = 0 \quad \text{and} \quad \xi_C = -\frac{\sigma_C}{\sqrt{3}}, \quad r_C = \sqrt{\frac{2}{3}} \sigma_C, \quad \theta_C = \frac{\pi}{3}, \quad (107)$$

so the point representing the uniaxial tension test is located on the tension meridian, while the point representing the uniaxial compression test lays on the compression meridian in the Haigh–Westergaard space. Then in conformity with (104), (105) and (106), the respective values of δ are:

$$\delta(\xi_T, r_T) = \frac{Ar_T}{B\sqrt{A^2 - \xi_T^2}} = \frac{\sqrt{2} A \sigma_T}{B\sqrt{3A^2 - \sigma_T^2}} = \delta_t(\xi_T, r_T) = \Delta_t, \quad (108)$$

$$\delta(\xi_C, r_C) = \frac{Ar_C}{B\sqrt{A^2 - \xi_C^2}} = \frac{\sqrt{2} A \sigma_C}{B\sqrt{3A^2 - \sigma_C^2}} = \delta_c(\xi_C, r_C) = \Delta_c. \quad (109)$$

The system of equations (108) and (109) yields the following solutions:

$$A^2 = \frac{(\Delta_t^2 - \Delta_c^2)\sigma_T^2\sigma_C^2}{3(\Delta_t^2\sigma_C^2 - \Delta_c^2\sigma_T^2)} \quad \text{and} \quad B^2 = \frac{2(\Delta_c^2 - \Delta_t^2)\sigma_T^2\sigma_C^2}{3\Delta_t^2\Delta_c^2(\sigma_C^2 - \sigma_T^2)}. \quad (110)$$

The calibration is more complicated if we are also interested in calculating γ , because the system of equations leading to the determination of parameters becomes quite complex. However, in the cases of the Drucker–Prager and Mises–Schleicher potentials it is possible to find an analytical solution due to the simplicity of formulae (104) and (105). In case of the Ottosen shape function, only tests represented by points located on tension and compression meridians lead to a convenient calibration. The feature may be seen as the disadvantage. However, for the calibration of the regarded yield conditions with three or more parameters we achieved closed-form expressions, which is rarely met in the literature.

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