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Doppler effect described by the solutions of the Cattaneo telegraph equation

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Abstract The Cattaneo telegraph equation for temperature with moving time-harmonic source is studied on the line and the half-line domain. The Laplace and Fourier transforms are used. Expressions which show the wave fronts and elucidate the Doppler effect are obtained. Several particular cases of the considered problem including the heat conduction equation and the wave equation are investigated. The quasi-steady-state solutions are also examined for the case of non-moving time-harmonic source and time-harmonic boundary condition for temperature.

1 Introduction

The classical parabolic diffusion equation and hyperbolic wave equation have their own characteristic features. For example, dissipation and infinite velocity of the disturbance propagation are inherent to the diffusion equation, while wave fronts, finite speeds of propagation and the Doppler effect [1–3] are peculiar to the solutions of the wave equation.

The time-fractional diffusion-wave equation (see [4–6] and the references therein)

$$\frac{\partial^\alpha T}{\partial t^\alpha} = a \Delta T, \quad 0 < \alpha \leq 2, \quad (1)$$

with the Caputo time-fractional derivative describes different important physical phenomena in bodies with complex internal structure, and interpolates between the diffusion equation when $\alpha = 1$ and the wave equation when $\alpha = 2$. Overall, Eq. (1) exhibits inherent features of both types of equations [6–8].

The term “diffusion-wave” is also used in another context. Ångström [9] was the first to investigate the standard parabolic diffusion equation (heat conduction equation) under time-harmonic (wave) impact. This pioneering study had aroused considerable interest of researchers and was even translated into English [10]. To describe this type of physical phenomenon the term “oscillatory diffusion” is used, in parallel with the term “diffusion-wave”. An extensive review of literature on this subject can be found in [11–13].

There are two possibilities to introduce oscillations into the parabolic diffusion equation. The first one consists in adding the harmonic source term [14, 15]; the second one involves time-harmonic boundary conditions [16, 17]. Often, in previous studies, the quasi-steady-state oscillations were investigated when the solution was

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represented as a product of a function of the spatial coordinates and time-harmonic term $e^{i\omega t}$ with the angular frequency ω .

In 1948 Cattaneo [18] proposed the evolution equation for the heat flux

$$\mathbf{q} + \tau_0 \frac{\partial \mathbf{q}}{\partial t} = -k \text{grad } T, \quad (2)$$

where τ_0 is a constant relaxation time (see also [19–24] and references therein). In combination with the law of conservation of energy, the constitutive equation (2) leads to the hyperbolic Cattaneo telegraph equation for temperature

$$\frac{\partial T}{\partial t} + \tau_0 \frac{\partial^2 T}{\partial t^2} = a \Delta T. \quad (3)$$

The heat waves as solutions of the telegraph equation are also called the “second sound” [20, 25, 26].

We would like to mention very interesting unified approach based on random walks proposed by Takayasu [27]. For the inertial random walk in which the mean-free-path is finite, the probability $\rho(x, t)$ that the particle exists at a point (x, t) obeys the telegraph equation. When the mean-free-path becomes infinite, the probability $\rho(x, t)$ satisfies the wave equation; when the mean-free-path equals zero, the standard diffusion equation is obtained.

In this paper, we study the Cattaneo telegraph equation for temperature with moving time-harmonic source on a line and on a half-line domain. Expressions which show the wave fronts and elucidate the Doppler effect are obtained. Several particular cases of the considered problem including the heat conduction equation and the wave equation are investigated. The quasi-steady-state solutions are also considered for the case of non-moving time-harmonic source and time-harmonic boundary condition for temperature. The paper develops the results of previous authors’ investigations [22, 28–31].

2 Telegraph equation on a real line with moving time-harmonic source

Consider the telegraph equation

$$\frac{\partial T}{\partial t} + \tau_0 \frac{\partial^2 T}{\partial t^2} = a \frac{\partial^2 T}{\partial x^2} + Q_0 \delta(x - vt) e^{i\omega t}, \quad -\infty < x < \infty, \quad (4)$$

under zero initial conditions

$$t = 0 : \quad T(x, t) = 0, \quad (5)$$

$$t = 0 : \quad \frac{\partial T(x, t)}{\partial t} = 0, \quad (6)$$

where $\delta(x)$ is the Dirac delta function, v denotes the velocity of the moving source, ω is the angular frequency.

The Laplace transform with respect to time t and the exponential Fourier transform with respect to the spatial coordinate x result in

$$\tilde{T}^*(\xi, s) = \frac{Q_0}{\sqrt{2\pi}} \frac{1}{s - i(\omega + v\xi)} \frac{1}{a\xi^2 + s + \tau_0 s^2}. \quad (7)$$

Here, the asterisk denotes the Laplace transform; s is the transform variable; the tilde marks the Fourier transform; ξ is the transform variable.

Equation (7) can be rewritten as

$$\tilde{T}^*(\xi, s) = \frac{Q_0}{\sqrt{2\pi}(-iva)} \frac{\xi - i\left(\frac{s - i\omega}{v}\right)}{\xi^2 + \left(\frac{s - i\omega}{v}\right)^2} \frac{1}{\xi^2 + \frac{1}{c^2}\left(s^2 + \frac{s}{\tau_0}\right)}, \quad (8)$$

where for the sake of convenience and in relation to the wave equation,

$$c = \sqrt{\frac{a}{\tau_0}} \quad (9)$$

denotes the velocity of the heat wave.

The expression (8) for $\tilde{T}^*(\xi, s)$ can be decomposed into partial fractions, and the inverse Fourier transform gives

$$T^*(x, s) = \frac{Q_0}{2a(1-\gamma^2)} B(s) \left\{ \left[\frac{c(s-i\omega)}{\sqrt{s^2 + s/\tau_0}} + v \operatorname{sign} x \right] \exp\left(-\frac{|x|}{c} \sqrt{s^2 + \frac{s}{\tau_0}}\right) - v(1 + \operatorname{sign} x) \exp\left[-\frac{|x|}{v}(s-i\omega)\right] \right\}, \quad (10)$$

where the coefficient $B(s)$ is written as

$$B(s) = \frac{1}{s^2 - \frac{1}{1-\gamma^2} \left(\frac{\gamma^2}{\tau_0} + 2i\omega \right) s - \frac{\omega^2}{1-\gamma^2}}, \quad (11)$$

the parameter γ denotes the ratio of the source velocity v and the wave velocity c :

$$\gamma = \frac{v}{c}, \quad (12)$$

and the integrals (69) and (70) from Appendix A have been used.

Taking into account the convolution theorem, the attenuation theorem [32] and the equations for the inverse Laplace transform (77)–(81) from Appendix B, we get the solution

$$\begin{aligned} T(x, t) = & \frac{Q_0 c}{2a(1-\gamma^2)\beta} \exp\left(\frac{i\omega t}{1-v^2/c^2}\right) \left\{ (1 + \gamma \operatorname{sign} x) \sinh\left[\beta\left(t - \frac{|x|}{c}\right)\right] \right. \\ & \times \exp\left[\frac{\gamma^2 t - (1 + 2i\omega\tau_0)|x|/c}{2\tau_0(1-\gamma^2)}\right] H\left(t - \frac{|x|}{c}\right) \\ & - \gamma(1 + \operatorname{sign} x) \sinh\left[\beta\left(t - \frac{|x|}{v}\right)\right] \\ & \times \exp\left[\gamma^2 \frac{t - (1 + 2i\omega\tau_0)|x|/v}{2\tau_0(1-\gamma^2)}\right] H\left(t - \frac{|x|}{v}\right) \\ & + \int_{\frac{|x|}{c}}^t \exp\left[\frac{\gamma^2 t - (1 + 2i\omega\tau_0)y}{2\tau_0(1-\gamma^2)}\right] \sinh[\beta(t-y)] \\ & \times \left[\frac{1}{2\tau_0} \left(y + \gamma \operatorname{sign} x \frac{|x|}{c} \right) \frac{1}{\sqrt{y^2 - |x|^2/c^2}} I_1\left(\frac{1}{2\tau_0} \sqrt{y^2 - \frac{|x|^2}{c^2}}\right) \right. \\ & \left. \left. - \left(\frac{1}{2\tau_0} + i\omega \right) I_0\left(\frac{1}{2\tau_0} \sqrt{y^2 - \frac{|x|^2}{c^2}}\right) \right] dy H\left(t - \frac{|x|}{c}\right) \right\}. \quad (13) \end{aligned}$$

Here

$$\beta = \sqrt{\frac{1}{4(1-\gamma^2)^2} \left(\frac{\gamma^2}{\tau_0} + 2i\omega \right)^2 + \frac{\omega^2}{1-\gamma^2}}, \quad (14)$$

and $H(x)$ is the Heaviside step function.

In the solution (13), the coefficient

$$\exp\left(\frac{i\omega t}{1-v^2/c^2}\right) \quad (15)$$

demonstrates the Doppler effect, and the Heaviside step function $H\left(t - \frac{|x|}{c}\right)$ describes two wave fronts at the points $x = \pm ct$.

Now, we consider two particular cases of the Cattaneo telegraph equation (4). For the standard heat conduction equation with the moving time-harmonic source term

$$\frac{\partial T}{\partial t} = a \frac{\partial^2 T}{\partial x^2} + Q_0 \delta(x - vt) e^{i\omega t}, \quad -\infty < x < \infty, \quad (16)$$

under zero initial condition (5), the integral transform technique allows us to obtain the corresponding expression for the sought-for temperature in the transform domain

$$\tilde{T}^*(\xi, s) = \frac{Q_0}{\sqrt{2\pi}} \frac{1}{s - i(\omega + v\xi)} \frac{1}{s + a\xi^2} \quad (17)$$

and the solution

$$T(x, t) = \frac{Q_0}{2\sqrt{\pi a}} \int_0^t e^{i\omega(t-u)} \frac{1}{\sqrt{u}} \exp\left\{-\frac{[x - v(t-u)]^2}{4au}\right\} du. \quad (18)$$

Here the integral (72) from Appendix A has been taken into account.

In the case of the wave equation with the moving time-harmonic source term

$$\frac{\partial^2 T}{\partial t^2} = c^2 \frac{\partial^2 T}{\partial x^2} + Q_0 \delta(x - vt) e^{i\omega t}, \quad -\infty < x < \infty, \quad (19)$$

under zero initial conditions (5) and (6), the solution reads

$$\begin{aligned} T(x, t) = & \frac{Q_0}{2v\omega} \left\{ (1 + \gamma \operatorname{sign} x) \exp\left[\frac{i\omega}{1-\gamma^2} \left(t - \frac{|x|}{c}\right)\right] \right. \\ & \times \sin\left[\frac{\gamma\omega}{1-\gamma^2} \left(t - \frac{|x|}{c}\right)\right] H\left(t - \frac{|x|}{c}\right) \\ & - \gamma (1 + \operatorname{sign} x) \exp\left[\frac{i\omega}{1-\gamma^2} \left(t - \gamma^2 \frac{|x|}{v}\right)\right] \\ & \times \sin\left[\frac{\gamma\omega}{1-\gamma^2} \left(t - \frac{|x|}{v}\right)\right] H\left(t - \frac{|x|}{v}\right) \\ & \left. - i\omega \int_{\frac{|x|}{c}}^t \exp\left[\frac{i\omega(t-u)}{1-\gamma^2}\right] \sin\left[\frac{\gamma\omega(t-u)}{1-\gamma^2}\right] du H\left(t - \frac{|x|}{c}\right) \right\}. \quad (20) \end{aligned}$$

The real part of the solution (20) takes the following form:

$$\Re T(x, t) = \begin{cases} 0, & -\infty < x < -ct, \\ \frac{Q_0}{2c\omega} \sin\left(\frac{c}{c+v}\omega t + \frac{\omega x}{c+v}\right), & -ct < x < vt, \\ \frac{Q_0}{2c\omega} \sin\left(\frac{c}{c-v}\omega t - \frac{\omega x}{c-v}\right), & vt < x < ct, \\ 0, & ct < x < \infty. \end{cases} \quad (21)$$

From the expression (21), it is evident that there are two wave fronts at $x = -ct$ and at $x = ct$, and the obtained solution describes the Doppler effect: at one side

$$\omega_1 = \frac{c}{c+v} \omega, \quad (22)$$

at the other side

$$\omega_2 = \frac{c}{c-v} \omega. \quad (23)$$

In what follows, we will also use an equivalent form of the solution to the problem (4)–(6) which is obtained by changing the order of inverse integral transforms applied to Eq. (7): first we invert the Laplace transform with respect to time and then the Fourier transform.

Fig. 1 Solution to the telegraph Eq. (4) with moving time-harmonic source ($\bar{t} = 1$, $\bar{\tau}_0 = 0.1$, $\bar{\omega} = \pi/4$)

Then we get

$$T(x, t) = \frac{Q_0 t}{\pi \tau_0 c} \int_0^1 e^{i\omega t(1-u)} \exp\left(-\frac{ut}{2\tau_0}\right) du \\ \times \left\langle \int_0^1 \frac{1}{\sqrt{1-\xi^2}} \sinh\left(\frac{ut}{2\tau_0} \sqrt{1-\xi^2}\right) \cos\left\{[x - vt(1-u)] \frac{\xi}{2\tau_0 c}\right\} d\xi \right. \\ \left. + \int_1^\infty \frac{1}{\sqrt{\xi^2-1}} \sin\left(\frac{ut}{2\tau_0} \sqrt{\xi^2-1}\right) \cos\left\{[x - vt(1-u)] \frac{\xi}{2\tau_0 c}\right\} d\xi \right\rangle. \quad (24)$$

Equation (24) is simpler than Eq. (13) and more amenable to numerical computations, but does not point clearly to the distinguishing features of the solution—the wave fronts at the points $x = -ct$ and at $x = ct$ and the Doppler effect.

The solution to the problem (4)–(6) for the Cattaneo telegraph equation with a non-moving time-harmonic source

$$\frac{\partial T}{\partial t} + \tau_0 \frac{\partial^2 T}{\partial t^2} = a \frac{\partial^2 T}{\partial x^2} + Q_0 \delta(x) e^{i\omega t}, \quad -\infty < x < \infty, \quad (25)$$

$$t = 0 : T(x, t) = 0, \quad (26)$$

$$t = 0 : \frac{\partial T(x, t)}{\partial t} = 0, \quad (27)$$

can be obtained as a particular case of the general solution (13) under the condition $v \rightarrow 0$ and has the following form:

$$T(x, t) = \begin{cases} \frac{Q_0}{2\sqrt{a\tau_0}} \int_{\frac{|x|}{c}}^t e^{i\omega(t-y)} e^{-y/(2\tau_0)} I_0\left(\frac{1}{2\tau_0} \sqrt{y^2 - \frac{x^2}{c^2}}\right) dy, & |x| < ct, \\ 0, & |x| > ct. \end{cases} \quad (28)$$

The real part of the solution (24) is presented in Fig. 1 for various values of the velocity v . In numerical computations, we have used the non-dimensional quantities marked by a bar.

Let L be some characteristic length. In this case, we have

$$\bar{T} = \frac{a}{Q_0 L} T, \quad \bar{x} = \frac{x}{L}, \quad \bar{t} = \frac{a}{L^2} t, \quad \bar{\tau}_0 = \frac{a}{L^2} \tau_0, \quad \bar{\omega} = \frac{L^2}{a} \omega, \quad \bar{v} = \frac{v}{c}. \quad (29)$$

The bullet points in Fig. 1 indicate the wave fronts at $x = -ct$ and $x = ct$.

Equation (25) allows considering the quasi-steady-state solution corresponding to the assumption that the temperature is represented as a product of a function of the spatial coordinate $U(x)$ and time-harmonic term:

$$T(x, t) = U(x) e^{i\omega t}. \quad (30)$$

For the function $U(x)$ we have the ordinary differential equation

$$(i\omega - \tau_0\omega^2) U(x) = a \frac{d^2 U}{dx^2} + Q_0 \delta(x), \quad (31)$$

and the exponential Fourier transform gives in the transform domain

$$\tilde{U}(\xi) = \frac{Q_0}{\sqrt{2\pi}} \frac{1}{a\xi^2 + (i\omega - \tau_0\omega^2)}. \quad (32)$$

Taking into account Eq. (69) from Appendix A, after inverting the transform we obtain the quasi-steady-state solution

$$T(x, t) = \frac{Q_0}{2\sqrt{a}\sqrt{i\omega - \tau_0\omega^2}} \exp\left(i\omega t - \frac{|x|}{\sqrt{a}} \sqrt{i\omega - \tau_0\omega^2}\right). \quad (33)$$

For $\tau_0 = 0$, the solution (33) coincides with that obtained by Nowacki [14, 15].

The real part of the solution (33) can be written as follows:

$$\begin{aligned} \Re T(x, t) = & \frac{Q_0}{2\sqrt{a\omega}\sqrt{1 + \tau_0^2\omega^2}} \exp\left\{-\frac{|x|\sqrt{\omega}}{\sqrt{a}} \sqrt[4]{1 + \tau_0^2\omega^2}\right. \\ & \times \cos\left[\frac{\pi}{4} + \frac{1}{2} \arctan(\tau_0\omega)\right]\bigg\} \cos\left\{\omega t - \frac{\pi}{4} - \frac{1}{2} \arctan(\tau_0\omega)\right. \\ & \left. - \frac{|x|\sqrt{\omega}}{\sqrt{a}} \sqrt[4]{1 + \tau_0^2\omega^2} \sin\left[\frac{\pi}{4} + \frac{1}{2} \arctan(\tau_0\omega)\right]\right\}. \end{aligned} \quad (34)$$

The results of numerical calculations according to the quasi-steady-state solution (34) for the non-moving time-harmonic source are presented in Fig. 2, whereas Fig. 3 shows the dependence of temperature on time for different types of solution. The bullet point in Fig. 3 indicates time when the wave front arrives at the point $\bar{x} = 4$.

3 Telegraph equation on a half-line

3.1 Time-harmonic source

First, we consider the Cattaneo telegraph equation with the moving time-harmonic source in the half-infinite domain:

$$\frac{\partial T}{\partial t} + \tau_0 \frac{\partial^2 T}{\partial t^2} = a \frac{\partial^2 T}{\partial x^2} + Q_0 \delta(x - vt) e^{i\omega t}, \quad 0 < x < \infty, \quad (35)$$

under zero initial conditions

$$t = 0 : \quad T(x, t) = 0, \quad (36)$$

$$t = 0 : \quad \frac{\partial T(x, t)}{\partial t} = 0, \quad (37)$$

and zero boundary condition for temperature

$$x = 0 : \quad T(x, t) = 0. \quad (38)$$

Fig. 2 The quasi-steady-state solution for the telegraph equation on a real line with the time-harmonic source. Dependence of temperature on distance: **a**— $\bar{i} = 0.5$, $\bar{\tau}_0 = 0.1$; **b**— $\bar{i} = 1$, $\bar{\tau}_0 = 0.1$; **c**— $\bar{i} = 0.5$, $\bar{\tau}_0 = 1$; **d**— $\bar{i} = 1$, $\bar{\tau}_0 = 1$

The Laplace transform with respect to time t and the Fourier sine transform with respect to the spatial coordinate x give the expression for temperature in the transform domain:

$$\tilde{T}^*(\xi, s) = \frac{Q_0 v \xi}{(s - i\omega)^2 + v^2 \xi^2} \frac{1}{a \xi^2 + s + \tau_0 s^2}, \quad (39)$$

where Eq. (75) from Appendix A has been used.

The corresponding counterparts of the solutions (13) and (24) for the telegraph equation on the real line with the moving source term read

$$\begin{aligned} T(x, t) = & \frac{Q_0 v}{a(1 - \gamma^2) \beta} \exp\left(\frac{i\omega t}{1 - v^2/c^2}\right) \\ & \times \left\{ \exp\left[\frac{\gamma^2 t - (1 + 2i\omega\tau_0)x/c}{2\tau_0(1 - \gamma^2)}\right] \sinh\left[\beta\left(t - \frac{x}{c}\right)\right] H\left(t - \frac{x}{c}\right) \right. \\ & \left. - \exp\left[\gamma^2 \frac{t - (1 + 2i\omega\tau_0)x/v}{2\tau_0(1 - \gamma^2)}\right] \sinh\left[\beta\left(t - \frac{x}{v}\right)\right] H\left(t - \frac{x}{v}\right) \right\} \end{aligned}$$

Fig. 3 Dependence of temperature on time ($\bar{\tau}_0 = 0.1$, $\bar{x} = 4$, $\bar{\omega} = \pi/4$). The black line represents the real part of the solution (24); the blue line corresponds to the quasi-steady-state solution (34); the red line shows the real part of the solution (28)

$$\begin{aligned}
 & + \frac{x}{2\tau_0 c} \int_{\frac{x}{c}}^t \exp \left[\frac{\gamma^2 t - (1 + 2i\omega\tau_0)y}{2\tau_0(1 - \gamma^2)} \right] \\
 & \times \frac{1}{\sqrt{y^2 - x^2/c^2}} I_1 \left(\frac{1}{2\tau_0} \sqrt{y^2 - \frac{x^2}{c^2}} \right) \sinh [\beta(t - y)] dy H \left(t - \frac{x}{c} \right) \Big\} \quad (40)
 \end{aligned}$$

and

$$\begin{aligned}
 T(x, t) = & \frac{2Q_0 t}{\pi \tau_0 c} \int_0^1 e^{i\omega t(1-u)} \exp \left(-\frac{ut}{2\tau_0} \right) du \\
 & \times \left\{ \int_0^1 \frac{1}{\sqrt{1 - \xi^2}} \sinh \left(\frac{ut}{2\tau_0} \sqrt{1 - \xi^2} \right) \sin \left[\frac{\gamma t(1-u)\xi}{2\tau_0} \right] \sin \left(\frac{x\xi}{2\tau_0 c} \right) d\xi \right. \\
 & \left. + \int_1^\infty \frac{1}{\sqrt{\xi^2 - 1}} \sin \left(\frac{ut}{2\tau_0} \sqrt{\xi^2 - 1} \right) \sin \left[\frac{\gamma t(1-u)\xi}{2\tau_0} \right] \sin \left(\frac{x\xi}{2\tau_0 c} \right) d\xi \right\}. \quad (41)
 \end{aligned}$$

Here we have used the necessary integrals from Appendix A and formulae for the inverse Laplace transform given in Appendix B.

For completeness, we also present two particular cases of the considered problem. In the case of the classical heat conduction equation, the expression for temperature reads

$$\begin{aligned}
 T(x, t) = & \frac{Q_0}{2\sqrt{\pi a}} \int_0^t e^{i\omega(t-u)} \frac{1}{\sqrt{u}} \\
 & \times \left\langle \exp \left\{ -\frac{[x - v(t-u)]^2}{4au} \right\} - \exp \left\{ -\frac{[x + v(t-u)]^2}{4au} \right\} \right\rangle du, \quad (42)
 \end{aligned}$$

whereas for the wave equation we have

$$T(x, t) = \frac{Q_0}{c\omega} \left\{ \exp \left[\frac{i\omega}{1 - \gamma^2} \left(t - \frac{x}{c} \right) \right] \sin \left[\frac{\gamma\omega}{1 - \gamma^2} \left(t - \frac{x}{c} \right) \right] H \left(t - \frac{x}{c} \right) \right.$$

$$- \exp \left[\frac{i\omega}{1-\gamma^2} \left(t - \gamma^2 \frac{x}{v} \right) \right] \sin \left[\frac{\gamma\omega}{1-\gamma^2} \left(t - \frac{x}{v} \right) \right] H \left(t - \frac{x}{v} \right) \}. \quad (43)$$

The real part of the solution (43) takes the form

$$\Re T(x, t) = \begin{cases} \frac{Q_0}{c\omega} \cos \left(\frac{c}{c+v} \omega t \right) \sin \left(\frac{\omega x}{c+v} \right), & 0 \leq x < vt, \\ \frac{Q_0}{2c\omega} \left[\sin \left(\frac{c}{c-v} \omega t - \frac{\omega x}{c-v} \right) - \sin \left(\frac{c}{c+v} \omega t - \frac{\omega x}{c+v} \right) \right], & vt < x < ct, \\ 0, & ct < x < \infty. \end{cases} \quad (44)$$

It is interesting to compare the solutions (20) and (43) (the solutions (21) and (44), respectively) with the solution of the similar problem for the wave equation on a half-line domain under zero Neumann boundary condition when instead of the Fourier exponential transform and sine transform, the Fourier cosine transform is used:

$$\begin{aligned} T(x, t) = \frac{Q_0}{\omega v} & \left\{ \exp \left[\frac{i\omega}{1-\gamma^2} \left(t - \frac{x}{c} \right) \right] \sin \left[\frac{\gamma\omega}{1-\gamma^2} \left(t - \frac{x}{c} \right) \right] H \left(t - \frac{x}{c} \right) \right. \\ & - \gamma \exp \left[\frac{i\omega}{1-\gamma^2} \left(t - \gamma^2 \frac{x}{v} \right) \right] \sin \left[\frac{\gamma\omega}{1-\gamma^2} \left(t - \frac{x}{v} \right) \right] H \left(t - \frac{x}{v} \right) \\ & \left. - i\omega \int_{\frac{x}{c}}^t \exp \left[\frac{i\omega(t-u)}{1-\gamma^2} \right] \sin \left[\frac{\gamma\omega(t-u)}{1-\gamma^2} \right] du H \left(t - \frac{x}{c} \right) \right\}. \end{aligned} \quad (45)$$

The real part of the solution (45) takes the form

$$\Re T(x, t) = \begin{cases} \frac{Q_0}{c\omega} \sin \left(\frac{c}{c+v} \omega t \right) \cos \left(\frac{\omega x}{c+v} \right), & 0 \leq x < vt, \\ \frac{Q_0}{2c\omega} \left[\sin \left(\frac{c}{c-v} \omega t - \frac{\omega x}{c-v} \right) + \sin \left(\frac{c}{c+v} \omega t - \frac{\omega x}{c+v} \right) \right], & vt < x < ct, \\ 0, & ct < x < \infty. \end{cases} \quad (46)$$

Figure 4 shows the dependence of the real part of the solution (41) on distance for different values of the velocity v , whereas the dependence of temperature on time is presented in Fig. 5. The bullet point indicates the wave front at $x = ct$.

3.2 The time-harmonic Dirichlet boundary condition

Here we consider the following initial-boundary-value problem:

$$\frac{\partial T}{\partial t} + \tau_0 \frac{\partial^2 T}{\partial t^2} = a \frac{\partial^2 T}{\partial x^2}, \quad 0 < x < \infty, \quad (47)$$

$$x = 0 : T(x, t) = T_0 e^{i\omega t}, \quad (48)$$

$$t = 0 : T(x, t) = 0, \quad (49)$$

$$t = 0 : \frac{\partial T(x, t)}{\partial t} = 0. \quad (50)$$

The Laplace transform with respect to time and the Fourier sine transform with respect to the spatial coordinate result in the solution in the transform domain

$$\tilde{T}^*(\xi, s) = aT_0 \xi \frac{1}{a\xi^2 + s + \tau_0 s^2} \frac{1}{s - i\omega}. \quad (51)$$

Fig. 4 Solution to the telegraph equation with moving time-harmonic source on the half-line domain under zero temperature boundary condition ($\bar{t} = 1$, $\bar{\tau}_0 = 0.1$, $\bar{\omega} = \pi/4$)

Fig. 5 The telegraph equation on a half-line with moving source term. Dependence of temperature on time ($\bar{\tau}_0 = 0.1$, $\bar{x} = 4$, $\bar{\omega} = \pi/4$)

The inverse Fourier sine transform gives

$$T^*(x, s) = T_0 \frac{1}{s - i\omega} \exp\left(-\frac{x}{c} \sqrt{s^2 + \frac{s}{\tau_0}}\right). \quad (52)$$

Taking into account the convolution theorem, the attenuation theorem [32] and the formulae for the inverse Laplace transform (see Appendix B), we obtain the solution

$$T(x, t) = \begin{cases} T_0 \exp\left[i\omega\left(t - \frac{x}{c}\right)\right] \exp\left(-\frac{x}{2c\tau_0}\right) + T_0 \frac{x}{2c\tau_0} \int_{\frac{x}{c}}^t e^{i\omega(t-u)} e^{-u/(2\tau_0)} \\ \times \frac{1}{\sqrt{u^2 - \frac{x^2}{c^2}}} I_1\left(\frac{1}{2\tau_0} \sqrt{u^2 - \frac{x^2}{c^2}}\right) du, & x < ct, \\ 0, & x > ct. \end{cases} \quad (53)$$

Fig. 6 The telegraph equation on a half-line with the time-harmonic Dirichlet boundary condition. Dependence of temperature on distance ($\bar{\tau}_0 = 0.1$, $\bar{t} = 1$)

For the classical heat conduction equation, we have

$$T(x, t) = \frac{T_0 x}{2\sqrt{\pi a}} \int_0^t e^{i\omega(t-u)} \frac{1}{u^{3/2}} \exp\left(-\frac{x^2}{4au}\right) du. \quad (54)$$

The real part of the integral representation (54) coincides with that given by Carslaw and Jaeger [16] and Morse and Feshbach [17].

In the case of the wave equation, the solution is

$$T(x, t) = T_0 \exp\left[i\omega\left(t - \frac{x}{c}\right)\right] H\left(t - \frac{x}{c}\right). \quad (55)$$

Dependence of the real part of the solution (53) on distance is shown in Fig. 6. The non-dimensional temperature is introduced as

$$\bar{T} = \frac{T}{T_0}, \quad (56)$$

other non-dimensional quantities are the same as in Eq. (29).

3.3 The quasi-steady-state solution

In this case the telegraph equation is investigated without imposing initial conditions, but under the assumption that temperature is represented as the product of a function in spatial variable and time-harmonic term:

$$\frac{\partial T}{\partial t} + \tau_0 \frac{\partial^2 T}{\partial t^2} = a \frac{\partial^2 T}{\partial x^2}, \quad 0 < x < \infty, \quad (57)$$

$$x = 0: \quad T(x, t) = T_0 e^{i\omega t}. \quad (58)$$

Let

$$T(x, t) = U(x) e^{i\omega t}. \quad (59)$$

For the function $U(x)$ we have the equation

$$(i\omega - \tau_0 \omega^2) U(x) = a \frac{d^2 U}{dx^2} \quad (60)$$

under the boundary condition

$$x = 0 : \quad U(x) = T_0. \quad (61)$$

The Fourier sine transform with respect to the spatial coordinate x gives

$$\tilde{U}(\xi) = aT_0 \frac{\xi}{a\xi^2 + (i\omega - \tau_0\omega^2)}, \quad (62)$$

and after inverting the transform we obtain the quasi-steady-state solution

$$T(x, t) = T_0 \exp\left(i\omega t - \frac{x}{\sqrt{a}} \sqrt{i\omega - \tau_0\omega^2}\right) \quad (63)$$

and

$$\begin{aligned} \Re e \, T(x, t) = T_0 \exp\left\{-\frac{x\sqrt{\omega}}{\sqrt{a}} \sqrt{1 + \tau_0^2 \omega^2} \cos\left[\frac{\pi}{4} + \frac{1}{2} \arctan(\tau_0 \omega)\right]\right\} \\ \times \cos\left\{\omega t - \frac{x\sqrt{\omega}}{\sqrt{a}} \sqrt{1 + \tau_0^2 \omega^2} \sin\left[\frac{\pi}{4} + \frac{1}{2} \arctan(\tau_0 \omega)\right]\right\}. \end{aligned} \quad (64)$$

The Fourier sine transform with respect to the spatial coordinate x gives

$$\tilde{U}(\xi) = aT_0 \frac{\xi}{a\xi^2 + (i\omega - \tau_0\omega^2)}. \quad (65)$$

In particular, the quasi-steady-state solution of the heat conduction equation reads

$$T(x, t) = T_0 \exp\left(i\omega t - x\sqrt{\frac{i\omega}{a}}\right), \quad (66)$$

whereas the corresponding result for the wave equation is (see Eq. (71) from Appendix A)

$$T(x, t) = T_0 e^{i\omega t} \cos\left(\frac{\omega}{c} x\right). \quad (67)$$

The results of numerical calculations based on Eq. (64) are presented in Fig. 7. Comparison of the solution with zero initial conditions and the quasi-steady-state solution is shown in Fig. 8.

4 Concluding remarks

It was shown in [30] that the telegraph equation has two harmonic wave solutions: temporally attenuated and spatially periodic (TASP) and spatially attenuated and temporally periodic (SATP). In the present paper, we have investigated the solutions to the telegraph equation with the time-harmonic source term and with the time-harmonic Dirichlet boundary condition. For moving time-harmonic source, the obtained solutions contain the wave fronts and describe the Doppler effect. Wave solutions of the telegraph equation are dissipative in contrast to solutions of the wave equation. In a graphical representation of the numerical results, we have used different scales due to large differences in wave amplitudes for various values of the non-dimensional parameters.

The obtained solutions can be successfully used when the source term or boundary value function $f(t)$ can be expanded into a Fourier series

$$f(t) = \sum_{n=-\infty}^{\infty} A_n e^{in\omega t}. \quad (68)$$

Fig. 7 The quasi-steady-state solution for the telegraph equation on a half-line with the time-harmonic Dirichlet boundary condition. Dependence of temperature on distance: **a**— $\bar{\tau} = 0.5$, $\bar{\tau}_0 = 0.1$; **b**— $\bar{\tau} = 1$, $\bar{\tau}_0 = 0.1$

Fig. 8 The telegraph equation on a half-line with the time-harmonic Dirichlet boundary condition. Dependence of temperature on time ($\bar{\omega} = \pi/4$; $\bar{x} = 4$; $\bar{\tau}_0 = 0.1$). The blue line corresponds to the quasi-steady-state solution (64); the red line shows the real part of the solution (53)

In this case the temperature can be represented as a superposition of harmonic terms. Similarly, the Fourier superposition can be carried out by integrating the derived solution with respect to the frequency ω (for the heat conduction equation such a possibility was discussed by Nowacki [14, 15] and Morse and Feshbach [17]).

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Compliance with ethical standards

Conflict of interest The authors declare that they have no conflict of interest.

Appendix A

We present integrals [33,34] used in the paper:

$$\int_0^\infty \frac{1}{x^2 + c^2} \cos(bx) dx = \frac{\pi}{2c} e^{-|b|c}, \quad \Re c > 0. \quad (69)$$

$$\int_0^\infty \frac{x}{x^2 + c^2} \sin(bx) dx = \frac{\pi}{2} e^{-|b|c} \operatorname{sign} b, \quad \Re c \geq 0. \quad (70)$$

$$\int_0^\infty \frac{x}{x^2 - c^2} \sin(bx) dx = \frac{\pi}{2} \cos(bc) \operatorname{sign} b, \quad \Re c \geq 0. \quad (71)$$

The integral (71) is understood as the Cauchy principal value.

$$\int_0^\infty e^{-cx^2} \cos(bx) dx = \frac{\sqrt{\pi}}{2\sqrt{c}} \exp\left(-\frac{b^2}{4c}\right), \quad \Re c > 0. \quad (72)$$

$$\int_0^\infty x e^{-cx^2} \sin(bx) dx = \frac{\sqrt{\pi} b}{4c^{3/2}} \exp\left(-\frac{b^2}{4c}\right), \quad \Re c > 0. \quad (73)$$

$$\int_0^\infty \frac{\sin(cx) \cos(bx)}{x} dx = \begin{cases} \frac{\pi}{2}, & c > b \geq 0, \\ 0, & b > c \geq 0. \end{cases} \quad (74)$$

$$\int_0^\infty e^{-px} \sin(bx) dx = \frac{b}{b^2 + p^2}, \quad \Re p > |\Im b|. \quad (75)$$

$$\int_0^\infty e^{-px} \cos(bx) dx = \frac{p}{b^2 + p^2}, \quad \Re p > |\Im b|. \quad (76)$$

Appendix B

Equations for the inverse Laplace transforms are taken from [32,35].

$$\mathcal{L}^{-1}\left\{\frac{1}{\sqrt{s^2 - c^2}} e^{-b\sqrt{s^2 - c^2}}\right\} = \begin{cases} I_0\left(c\sqrt{t^2 - b^2}\right), & t > b > 0, \\ 0, & 0 < t < b. \end{cases} \quad (77)$$

$$\mathcal{L}^{-1}\left\{e^{-bs}\right\} = \delta(t - b), \quad b > 0. \quad (78)$$

$$\mathcal{L}^{-1}\left\{\frac{1}{s} e^{-bs}\right\} = H(t - b), \quad b > 0. \quad (79)$$

$$\mathcal{L}^{-1}\left\{e^{-b\sqrt{s^2 - c^2}}\right\} = \delta(t - b) + \begin{cases} \frac{bc}{\sqrt{t^2 - b^2}} I_1\left(c\sqrt{t^2 - b^2}\right), & t > b > 0, \\ 0, & 0 < t < b. \end{cases} \quad (80)$$

$$\mathcal{L}^{-1}\left\{\frac{s}{\sqrt{s^2 - c^2}} e^{-b\sqrt{s^2 - c^2}}\right\} = \delta(t - b) + \begin{cases} \frac{ct}{\sqrt{t^2 - b^2}} I_1\left(c\sqrt{t^2 - b^2}\right), & t > b > 0, \\ 0, & 0 < t < b, \end{cases} \quad (81)$$

where $I_n(z)$ is the modified Bessel function, $\delta(x)$ is the Dirac delta function, and $H(x)$ is the Heaviside step function.

$$\mathcal{L}^{-1} \left\{ \frac{1}{s^2 - c^2} \right\} = \frac{1}{c} \sinh(ct), \quad \Re c > 0. \quad (82)$$

$$\mathcal{L}^{-1} \left\{ \frac{1}{s^2 + c^2} \right\} = \frac{1}{c} \sin(ct), \quad \Re c > 0. \quad (83)$$

$$\mathcal{L}^{-1} \left\{ \frac{s}{s^2 + c^2} \right\} = \cos(ct), \quad \Re c > 0. \quad (84)$$

$$\mathcal{L}^{-1} \left\{ e^{-b\sqrt{s}} \right\} = \frac{b}{2\sqrt{\pi}t^{3/2}} \exp\left(-\frac{b^2}{4t}\right), \quad b > 0. \quad (85)$$

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