CORRECTION

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Correction to: Eshelby force and power for uniform bodies

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In the original paper, we erroneously used the *total* potential energy (Helmholtz free energy of the material *plus* potential energy of the external forces) in the entropy inequality. This is because we incautiously merged two sets of calculations: in the first set, W denoted the *total* potential energy and, in the second, W denoted the Helmholtz free energy. This confusion caused the appearance of incorrect terms, featuring the kinetic energy density K and its time derivative $\dot{K}$. Thus, we must correct the text between Eqs. (43) and (45) in Sect. 2.4 on the Lagrangian for a thermoelastic body, Remark 3 and the entire Sect. 4.6 on the entropy inequality [note that, in the corrected version of Sect. 4.6, there are no Eqs. (147)–(153)]. In order to reduce the number of corrections to a minimum, we renamed the total energy and the Helmholtz free energy, but not the potential energy of the external forces, as per the scheme below:

	Original paper	This erratum
Total potential energy	W	W_{tot}
Helmholtz free energy	W_{el}	W
Potential energy of the external forces	W_{ext}	W_{ext}

Moreover, in Sect. 2.3 on the material balance equations, when writing the most general form of the balance of energy and of the entropy inequality, we erroneously omitted the non-compliant surface contribution to the internal energy. This had no consequences in the later calculations because we neglected all non-compliant terms. However, for the sake of completeness and consistency, we felt that it was necessary to amend these equations.

Finally, we take this chance to provide some other minor corrections. Note that, although the most serious corrections are to Sects. 2.4 and 4.6, we report the corrections to Sect. 2.3 first, in order to maintain the order in the original paper.

The original article can be found online at <https://doi.org/10.1007/s00707-018-2353-6>.

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Corrections to Section 2.3

In Sect. 2.3, Eqs. (25)–(30), as well as Eq. (33), should have featured the non-compliant surface contribution to the internal energy, which we denote $\mathbf{U}$, and should have read:

$$\begin{aligned} \partial_t \int_{\mathcal{R}} \left(\frac{1}{2} \rho_R \dot{\phi} \cdot \dot{\phi} + \rho_R \mathcal{E} \right) = & \int_{\mathcal{R}} \left(\frac{1}{2} \Pi \dot{\phi} \cdot \dot{\phi} + \Pi \mathcal{E} + \langle \mathbf{f} | \dot{\phi} \rangle + \mathcal{H} + \mathcal{U} + \langle \mathbf{z} | \dot{\phi} \rangle \right) \\ & + \int_{\partial \mathcal{R}} \langle [\dot{\phi} (\mathbf{T} + \mathbf{A}) + (\mathcal{E} + \frac{1}{2} \dot{\phi} \cdot \dot{\phi}) \mathbf{M} + \mathbf{U} + \mathbf{Q}] | \mathbf{N} \rangle, \end{aligned} \quad (25 \text{ corr.})$$

$$\begin{aligned} \partial_t \left(\frac{1}{2} \rho_R \dot{\phi} \cdot \dot{\phi} + \rho_R \mathcal{E} \right) = & \frac{1}{2} \Pi \dot{\phi} \cdot \dot{\phi} + \Pi \mathcal{E} + \langle \mathbf{f} | \dot{\phi} \rangle + \mathcal{H} + \mathcal{U} + \langle \mathbf{z} | \dot{\phi} \rangle + \\ & + \text{Div} [\dot{\phi} (\mathbf{T} + \mathbf{A}) + (\mathcal{E} + \frac{1}{2} \dot{\phi} \cdot \dot{\phi}) \mathbf{M} + \mathbf{U} + \mathbf{Q}], \end{aligned} \quad (26 \text{ corr.})$$

$$\rho_R \dot{\mathcal{E}} = \mathcal{H} + \mathcal{U} + (\mathbf{T} + \mathbf{A}) : \dot{\mathbf{F}} + \langle \text{Grad } \mathcal{E} | \mathbf{M} \rangle + \text{Div} (\mathbf{Q} + \mathbf{U}). \quad (27 \text{ corr.})$$

$$\partial_t \int_{\mathcal{R}} \rho_R \mathcal{S} \geq \int_{\mathcal{R}} \left[\Pi \mathcal{S} + \frac{\mathcal{H}}{\Theta} + \frac{\mathcal{U}}{\Theta} + \mathcal{Z} \right] + \int_{\partial \mathcal{R}} \left\langle \left[\frac{\mathbf{Q} + \mathbf{U}}{\Theta} + \mathcal{S} \mathbf{M} \right] | \mathbf{N} \right\rangle, \quad (28 \text{ corr.})$$

$$\begin{aligned} \Theta \partial_t (\rho_R \mathcal{S}) \geq & \Pi \Theta \mathcal{S} + \mathcal{H} + \mathcal{U} + \Theta \mathcal{Z} \\ & - \frac{1}{\Theta} \langle \text{Grad } \Theta | \mathbf{Q} + \mathbf{U} \rangle + \text{Div} (\mathbf{Q} + \mathbf{U}) + \Theta \langle \text{Grad } \mathcal{S} | \mathbf{M} \rangle + \Theta \mathcal{S} \text{Div } \mathbf{M}. \end{aligned} \quad (29 \text{ corr.})$$

$$\rho_R \Theta \dot{\mathcal{S}} \geq \rho_R \dot{\mathcal{E}} - (\mathbf{T} + \mathbf{A}) : \dot{\mathbf{F}} + \Theta \mathcal{Z} - \frac{1}{\Theta} \langle \text{Grad } \Theta | \mathbf{Q} + \mathbf{U} \rangle - \langle \text{Grad } \mathcal{E} - \Theta \text{Grad } \mathcal{S} | \mathbf{M} \rangle. \quad (30 \text{ corr.})$$

$$-\rho_R \dot{\Psi} - \rho_R \mathcal{S} \dot{\Theta} - \Theta \mathcal{Z} + (\mathbf{T} + \mathbf{A}) : \dot{\mathbf{F}} + \frac{1}{\Theta} \langle \mathbf{Q} + \mathbf{U} | \text{Grad } \Theta \rangle + \langle \text{Grad } \mathcal{E} - \Theta \text{Grad } \mathcal{S} | \mathbf{M} \rangle \geq 0. \quad (33 \text{ corr.})$$

Moreover, in the text following Eq. (25), the rate of non-mechanical energy supply per unit volume, $\mathcal{H}$, is actually per unit *volume* and not per unit mass, as erroneously indicated.

Correction to Section 2.4. Lagrangian density of a thermoelastic body

The Lagrangian density (per unit reference volume) $\mathcal{L}$ is defined as

$$\mathcal{L} = K - W_{\text{tot}}, \quad \Rightarrow \quad \hat{\mathcal{L}} \circ (\phi, \dot{\phi}, \Theta, \mathbf{F}, \mathcal{X}, \tau) = \hat{K} \circ (\dot{\phi}, \mathcal{X}, \tau) - \hat{W}_{\text{tot}} \circ (\phi, \Theta, \mathbf{F}, \mathcal{X}, \tau), \quad (43)$$

In Eq. (43), K is the kinetic energy density (per unit reference volume), i.e.,

$$K = \frac{1}{2} \rho_R \dot{\phi} \cdot \dot{\phi}; \quad \rho_R = \hat{\rho}_R \circ (\mathcal{X}, \tau) \quad \Rightarrow \quad \hat{K} \circ (\dot{\phi}, \mathcal{X}, \tau) = \frac{1}{2} \hat{\rho}_R \circ (\mathcal{X}, \tau) \dot{\phi} \cdot \dot{\phi}, \quad (44)$$

where $\rho_R = \hat{\rho}_R \circ (\mathcal{X}, \tau)$ denotes the referential mass density (which could depend explicitly not only on the point X , but also on time t , in the case of volumetric growth), and $\hat{W}_{\text{tot}}$ is the total potential energy, given by the sum of the Helmholtz free energy per unit reference volume W and the potential energy of the external forces W_{ext} , i.e.

$$W_{\text{tot}} = \hat{W}_{\text{tot}} \circ (\phi, \mathbf{F}, \Theta, \mathcal{X}, \tau) = \hat{W} \circ (\mathbf{F}, \Theta, \mathcal{X}, \tau) + \hat{W}_{\text{ext}} \circ (\phi, \tau). \quad (45)$$

Note that the potential energy $\hat{W}_{\text{ext}}$ of the external forces (per unit reference volume) generally depends explicitly on time via the referential mass density ρ_R , much like the kinetic energy $\hat{K}$, as seen in Eq. (44).

Correction to Remark 3

Remark Some authors prefer to exclude the potential energy W_{ext} of the external forces from the Lagrangian. This results in the Lagrangian $K - W$, whose constitutive function $\hat{K} - \hat{W}$ does *not* depend on the configuration ϕ . Consequently, the negative of the pull-back of the body force features in the material balance. See, for instance, the works by Ericksen (see Eq. (4) in [8]) and Maugin (see Eq. (9) in [18]). While this choice is certainly legitimate, we prefer to follow Eshelby (see Eq. (3.2) in [11]) and Marsden and Hughes (see p. 279, Eq. (8) in [17]), who in turn follow the standard methods of Field Theory and, thus, we prefer to include *both* ϕ and $\mathbf{F} = T\phi$ among the arguments of our Lagrangian, so that the derivative of the Lagrangian with respect to ϕ is precisely the body force. In our approach, W_{tot} is the sum of the Helmholtz free energy W and of the potential energy W_{ext} of the external forces, as shown in Eq. (45).

Corrected Section 4.6. Entropy inequality

In this section, we handle the entropy inequality (Clausius–Duhem inequality) in Eq. (39b) in terms of the theory of uniformity. From the uniformity condition (95), we have $\hat{\mathcal{L}} \circ (\phi, \dot{\phi}, \mathbf{F}, \Theta, \mathcal{X}, \tau) = J_{\mathbf{P}}^{-1} \hat{\Lambda} \circ (\phi, \dot{\phi}, \mathbf{F}\mathbf{P}, \Theta)$, which we can also write for the Helmholtz free energy as

$$W = \hat{W} \circ (\mathbf{F}, \Theta, \mathcal{X}, \tau) = J_{\mathbf{P}}^{-1} \hat{\Omega} \circ (\mathbf{F}\mathbf{P}, \Theta), \quad (140)$$

where $\hat{\Omega}$ is the archetypal Helmholtz free energy. This implies that also the time derivatives are equal, i.e.

$$\dot{W} = \partial_t [\hat{W} \circ (\mathbf{F}, \Theta, \mathcal{X}, \tau)] = \partial_t [J_{\mathbf{P}}^{-1} \hat{\Omega} \circ (\mathbf{F}\mathbf{P}, \Theta)]. \quad (141)$$

From the definitions (43), (45), (46c), and (46d) of Lagrangian, total potential energy, entropy and first Piola–Kirchhoff stress, and the expression (86) of $\dot{J}_{\mathbf{P}}$, we have

$$\begin{aligned} \dot{W} = \mathbf{T} : \dot{\mathbf{F}} - S \dot{\Theta} + \frac{\partial \hat{W}}{\partial \tau} \circ (\mathbf{F}, \Theta, \mathcal{X}, \tau) &= -J_{\mathbf{P}}^{-1} [\hat{\Omega} \circ (\mathbf{F}\mathbf{P}, \Theta)] \text{tr}(\mathbf{L}_{\mathbf{P}}) + \\ &+ \left[J_{\mathbf{P}}^{-1} \frac{\partial \hat{\Omega}}{\partial \mathbf{F}\mathbf{P}} \circ (\mathbf{F}\mathbf{P}, \Theta) \right] : [\dot{\mathbf{F}}\mathbf{P} + \mathbf{F}\dot{\mathbf{P}}] + \left[J_{\mathbf{P}}^{-1} \frac{\partial \hat{\Omega}}{\partial \Theta} \circ (\mathbf{F}\mathbf{P}, \Theta) \right] \dot{\Theta}, \end{aligned} \quad (142)$$

which, with some manipulation of the double contractions, use of the definition (87) of $\mathbf{L}_{\mathbf{P}} = \dot{\mathbf{P}}\mathbf{P}^{-1}$ and some reordering of the terms in the right-hand side, becomes

$$\begin{aligned} \mathbf{T} : \dot{\mathbf{F}} - S \dot{\Theta} + \frac{\partial \hat{W}}{\partial \tau} \circ (\mathbf{F}, \Theta, \mathcal{X}, \tau) &= \left[J_{\mathbf{P}}^{-1} \frac{\partial \hat{\Omega}}{\partial \mathbf{F}\mathbf{P}} \circ (\mathbf{F}\mathbf{P}, \Theta) \mathbf{P}^T \right] : \dot{\mathbf{F}} + \left[J_{\mathbf{P}}^{-1} \frac{\partial \hat{\Omega}}{\partial \Theta} \circ (\mathbf{F}\mathbf{P}, \Theta) \right] \dot{\Theta} + \\ &- J_{\mathbf{P}}^{-1} [\hat{\Omega} \circ (\mathbf{F}\mathbf{P}, \Theta)] \text{tr}(\mathbf{L}_{\mathbf{P}}) + \mathbf{F}^T \left[J_{\mathbf{P}}^{-1} \frac{\partial \hat{\Omega}}{\partial \mathbf{F}\mathbf{P}} \circ (\mathbf{F}\mathbf{P}, \Theta) \mathbf{P}^T \right] : \mathbf{L}_{\mathbf{P}}. \end{aligned} \quad (143)$$

In the right-hand side of Eq. (143), we recognise the first Piola–Kirchhoff stress $\mathbf{T}$ (first term), the negative of the entropy S (second term), the negative of the Helmholtz free energy W (third term) and the Mandel stress $\mathfrak{M}$ of Eq. (73) (fourth term). Therefore, comparing left- and right-hand sides, we obtain the explicit time derivative as

$$\frac{\partial \hat{W}}{\partial \tau} \circ (\mathbf{F}, \Theta, \mathcal{X}, \tau) = -W \text{tr}(\mathbf{L}_{\mathbf{P}}) + \mathfrak{M} : \mathbf{L}_{\mathbf{P}}, \quad (144)$$

which, substituted in (142), gives

$$\dot{W} = \mathbf{T} : \dot{\mathbf{F}} - S \dot{\Theta} - W \text{tr}(\mathbf{L}_{\mathbf{P}}) + \mathfrak{M} : \mathbf{L}_{\mathbf{P}}. \quad (145)$$

Finally, substituting Eq. (145) and the expression $\Pi = -\rho_R \operatorname{tr}(\mathbf{L}\mathbf{P})$ of the volumetric growth rate [Eq. (110)] into Eq. (39b), we obtain the expression of the entropy inequality (Clausius–Duhem inequality) in the general framework of the theory of uniformity, i.e.

$$-\mathfrak{M} : \mathbf{L}\mathbf{P} + \frac{1}{\Theta} \langle \operatorname{Grad} \Theta | \mathbf{Q} \rangle \geq 0, \quad (146)$$

which shows that the Mandel stress is the *driving force* of the growth-remodelling process, whose kinematics is described by $\mathbf{L}\mathbf{P}$.

Minor corrections

Because of a glitch in our L^AT_EX code, some equation references were incorrect. In fact, there is no Eq. (2.2) and the correct references are as indicated below:

- In the text preceding Eq. (6), the reference is to Eq. (3);
- In the text preceding Eq. (13), the reference is to Eqs. (8) and (11);
- In the text following Eq. (78), the reference is to Eq. (13);
- In the text preceding Eq. (103), the reference is to Eq. (13).

Also, in the paragraph preceding Eq. (74), “body $\mathbf{B}$ ” should read “body $\mathcal{B}$ ”. Finally, because of an incautious copy-paste error on our part, the arguments of the referential Lagrangian $\hat{\mathcal{L}}$ were incorrect in Eqs. (96) and (97), which should read

$$\frac{\partial \hat{\mathcal{L}}}{\partial \mathbf{F}} \circ (\phi, \dot{\phi}, \mathbf{F}, \Theta, \mathcal{X}, \tau) = J_{\mathbf{P}}^{-1} \frac{\partial \hat{\mathcal{L}}}{\partial (\mathbf{F}\mathbf{P})} \circ (\phi, \dot{\phi}, \mathbf{F}\mathbf{P}, \Theta) \mathbf{P}^T, \quad (96 \text{ corr.})$$

and

$$\frac{\partial \hat{\mathcal{L}}}{\partial (\mathbf{F}\mathbf{P})} \circ (\phi, \dot{\phi}, \mathbf{F}\mathbf{P}, \Theta) = J_{\mathbf{P}} \frac{\partial \hat{\mathcal{L}}}{\partial \mathbf{F}} \circ (\phi, \dot{\phi}, \mathbf{F}, \Theta, \mathcal{X}, \tau) \mathbf{P}^{-T} = -J_{\mathbf{P}} \mathbf{T} \mathbf{P}^{-T}. \quad (97 \text{ corr.})$$

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