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Enhanced strength of cyclically preloaded arrays of pillars

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Abstract Under compression, a cyclically precompressed nanopillar supports greater load than its as-fabricated counterpart. Such an improvement on mechanical properties takes place only when the preloading process is tuned carefully with regard to a particular pillar being tested. This experimental evidence raises a question: does a cyclic preloading applied simultaneously to an ensemble of nanopillars enhance the overall strength of the system? To answer this question, we simulate numerically cyclic loadings of pillars assembled into an array. Assuming that the pillars are characterized by random strength-thresholds $\{\sigma_{th}\}$, we demonstrate that quasi-statically compressed arrays with initial cycling process support higher load than the corresponding ones with no precompression. By applying the fibre bundle model, we evolve $\{\sigma_{th}\}$ and estimate that the mean strengthening may attain 7–9% for an optimally tailored cycling.

1 Introduction

The accessibility of reliable parts of multicomponent systems is a common concern of manufacturers and customers. From the manufacturer's point of view, a substantial portion of fabrication errors originates from material defects and thus any procedure which reduces imperfections is of primary interest for engineering purposes. This is also the case with nanopillars assembled into different sub-micron structures because of a wide range of their applicability in nanotechnology. Among the accessible procedures is mechanical annealing, known as one of the most effective measures for reducing the dislocation density, which has a serious drawback resulting in a considerable shape change of the pillar subjected to high monotonic load [1]. Contrary to the load applied monotonically, the low-amplitude cycling loading removes dislocations effectively with a little change of the pillar's shape [2].

When a single sub-micrometer pillar is subjected to a carefully tuned low-amplitude cyclic preloading, it acquires a significantly higher strength [3–5]. Is such tuning feasible with respect to a set of pillars and does this enhance the overall strength of the system? To answer these questions, the present paper focuses on computer modelling and simulation of strengthening of cyclically loaded sets of pillars. By mimicking experiments conducted on a single pillar, we have designed a numerical experiment that involves an array of pillars. Analogously to the single-pillar experiments, our simulation goes through two steps: (a) cyclic preloading that should strengthen the array followed by (b) quasi-static loading to detect the maximal load supported by the system. If such a load is higher than the one corresponding to the same array exempted from cycling preloading, then the step (a) induces a desired strengthening.

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We limit ourselves to a statistical characterization of the system. This means that the simulation is neither nanoscaled molecular dynamics [6] nor sub-micron-scaled continuum mechanics. It handles, however, distributions of appropriate microscopic quantities, such as pillar-strength-thresholds, in order to keep a link between the load applied on a system with resulting stresses felt individually by the pillars. Our minimalist model includes two key ingredients of the cyclic stage of simulation: (i) the load transfer rule and (ii) the functional link between the local on-pillar load and corresponding change of its strength-threshold. Specifically, to go along with results of experiments reported in [4] closely, we apply the following rule with regard to (ii): if at a given cycle τ the on-pillar load f is smaller than 80% of its actual-load threshold σ_{th} , then the threshold increases by an amount $\delta\sigma$ that mainly depends on σ_{th} , f and τ , otherwise it decreases to $\sigma_{th} - \delta\sigma$. Additionally, the increment $\delta\sigma$ smoothly attenuates to become negligible around $\tau \sim 200$.

Based on conducted simulations, we estimate that a cyclically precompressed array of pillars attains an overall mean strengthening of the order of 10% with respect to its as-fabricated counterpart.

2 Simulating cyclic preloading and the quasi-static compression test

The idea of our model stems from experiments on a cyclically loaded sub-micron Ni pillar [4]. These experiments clearly show that a carefully tuned cyclic preloading induces a significant strengthening of the pillar. Contrary to a single-pillar system, our model handles an ensemble of pillars.

2.1 Preliminary

We assume that the pillars are either intact or crushed irreversibly, where the term ‘intact’ means ‘fully functional’. A pillar that becomes overloaded starts to collapse, and its load has to be taken over by the intact pillars. This means that each pillar-crushing event triggers the resulting-load transfer. An important feature of such a process is that the transferred and externally applied loads may together activate subsequent crushes in a form of so-called bursts of failures. The bursts may become self-sustained, and they either destroy the entire system or stop by freezing the system in a stable configuration. Obviously, the number of intact pillars decreases when the system passes between consecutive stable configurations.

The need of load redistribution among intact pillars raises numerous questions about how the load transfer is arranged [7–9]. Specifically, such questions are important for load-sharing parallel systems, as our arrays of pillars is. In this context, among the different approaches to load transfer, the fibre bundle model (FBM) framework is one of the most frequently employed [10–15]. The FBM includes a variety of load transfer rules that differ mainly in how they relate the distance between crushed and intact pillars with an amount of load addressed to the latter ones. In other words, a given rule reflects an assumed range of load transfer. From this point of view there are two rules, commonly named (i) global load sharing (GLS) and (ii) local load sharing (LLS), that correspond to two extremal ranges. In the GLS rule, a load originating from a crushed pillar is transferred equally to all the intact pillars, and thus, the range of transfer is maximal. The LLS rule, in turn, engages only the nearest intact neighbours of a crushed pillar, so the range of load transfer is minimal. As a consequence, the load distributed according to the GLS rule is the least harmful for the intact pillars, whereas the LLS represents the most damaging method of the load assignment.

In simulations, we apply the GLS and the LLS rules in order to estimate the upper and the lower bounds for possible strengthening of arrays of pillars, respectively. We also assume that the load transfer is an almost instantaneous process that happens simultaneously.

In the following, we consider an ensemble of vertical pillars located in nodes of a square grid Σ_0 and characterized by strength-thresholds $\{\sigma_{th}\}_{\Sigma_0}$. We denote by Σ_μ the subset of nodes occupied by intact pillars belonging to a given stable configuration μ . Since a growing on-pillar-load successively eliminates intact pillars, then $\Sigma_0 \supset \dots \Sigma_\mu \supset \Sigma_{\tilde{\mu}}$, where the configuration μ is followed by $\tilde{\mu}$. In our model, the strength-threshold is the only quantity that relates the state of a pillar to the load applied axially on the pillar. Because of various imperfections, the σ_{th} are random quantities. We assume that their values are drawn from the Weibull distribution [16, 17]. The cumulative function $P_{\rho,\lambda}(\sigma_{th}) = 1 - \exp[-(\sigma_{th}/\lambda)^\rho]$ of this distribution depends on two parameters: ρ and λ , known as the shape and scale parameters, respectively. In particular, ρ points to an amount of disorder. In all simulations, we assume that $\lambda = 1$, and thus, σ_{th} ’s distribution reads:

$$P_\rho(\sigma_{th}) = 1 - \exp[-(\sigma_{th})^\rho]. \quad (1)$$

Values of ρ span from 2 to 10 to keep track of a system's strong to a system's weak disorder, respectively. The term 'system disorder' concerns variations of pillar-strength-thresholds within a given array.

As we have already mentioned, our numerical experiment is composed of two steps: cyclic preloading and quasi-static loading. Below, we specify these steps in detail.

2.2 Cyclic preloading

Take a square array of pillars and let $\{\sigma_{th}\}_{\Sigma_0} = \{\sigma_{th}^i(0)\}_{i=1,\dots,N}$, be the set of initially assigned pillar-strength-thresholds. After choosing a desirable load transfer rule, the array is subjected to n cycles of a sudden load $F_{cyc} = N \cdot f_{cyc}$, which is kept constant during each cycle. Within a given cycle τ ($\tau = 1, \dots, n$), a particular pillar i whose actual $\sigma_{th}^i(\tau)$ is smaller than its local load f_i becomes crushed inducing an appropriate load redistribution among intact pillars and, possibly, further crushing. All these pillars whose $\sigma_{th} \geq f$ survive and their strength-thresholds change to new values. Once the system attains an accessible stable configuration, F_{cyc} is released and the cycle ends.

To quantify the variation of σ_{th} , we employ the following strategy. Under a given cyclic load F_{cyc} , after each cycle, we compare local loads f_i with actual strength-thresholds σ_{th}^i for all working pillars. It is important to notice that f_i comprises f_{cyc} as well as loads that are accumulated on the i th pillar due to transfers from previously crushed pillars. If $f_i < \eta \cdot \sigma_{th}^i$ then $\sigma_{th}^i \rightarrow \sigma_{th}^i + \delta\sigma(f_i, \sigma_{th}^i, \tau)$, otherwise σ_{th}^i decreases. This can be expressed by the following iterative equation:

$$\sigma_{th}^i(\tau + 1) = \sigma_{th}^i(\tau) + 2\frac{\epsilon}{\eta} \cdot \gamma(\tau) \begin{cases} \frac{1}{2}\eta \cdot \sigma_{th}^i - \left| f_i - \frac{1}{2}\eta \cdot \sigma_{th}^i \right|, & f_i \leq \eta \cdot \sigma_{th}^i, \\ \eta \cdot \sigma_{th}^i - f_i, & f_i > \eta \cdot \sigma_{th}^i, \end{cases} \quad (2)$$

where $\tau \in \{1, \dots, n-1\}$ and $\gamma(\tau)$ serves as the cycle-attenuation factor. Its presence in (2) is due to an experimentally observed vanishing variation of the pillar strength with an increasing number of cycling. Numerous mathematical functions may reflect such a behaviour. For our purpose, we choose the following form:

$$\gamma(\tau) = \left[1 - \frac{2}{\pi} \arctan(\tau - 1) \right] \sqrt[3]{\tau}. \quad (3)$$

Rule (2), with $\epsilon \sim 0.02$ and $\eta \sim 0.8$, recovers the strengthening effect observed in single-pillar experiments, as, for example, the effect displayed in Fig. 3(b) of [4].

2.3 Quasi-static compression test

The aim of this test is to estimate the highest load safely supported by a given array of pillars represented by $\{\sigma_{th}\}_\Sigma = \{\sigma_{th}^i\}_{i \in \Sigma}$. To achieve this goal, we apply to the pillars a growing load F that increases stepwise. Subsequent load increments δF are tuned according to the following way: if the array is in a stable configuration under the load F , then the load $F + \delta F$, with the increment $\delta F = \text{Min}(f_i - \sigma_{th}^i)$, either drives the system to another stable configuration or triggers its ultimate destruction. In the latter case, F is the maximal load F_c supported by the array.

It is worth mentioning that at each passage between subsequent stable configurations, at least one pillar is eliminated from the system, and thus, the quasi-static compression test enables us to determine the greatest load F_c supported by the smallest number of pillars. To be precise, during this test the system is driven to a marginally stable configuration Σ_c , i.e. any removal of pillars from Σ_c will initiate destruction of the entire system.

2.4 Measure of strengthening effect

The most important question we address in this work concerns the impact of cyclic preloading on the system strength. To answer this question, we need a suitable measure to quantify the strength of an array of pillars. We can formulate such a measure by combining the highest load $F_c^\Sigma = F_c(\{\sigma_{th}(n, f_{cyc})\}_{\Sigma \subset \Sigma_0})$, supported

by the array that was previously cycled, with $F_c^0 = F_c(\{\sigma_{th}\}_{\Sigma_0})$ that refers to the same array prior to cycling. Specifically, we detect the strengthening when $F_c^\Sigma / F_c^0 > 1$.

3 Results and discussion

We have developed a computer code in Wolfram Mathematica to simulate the cyclic loading and compression test described in the previous section. Using this code, we have collected data that characterize distributions of different quantities related to arrays of pillars after the cyclic preloading. By comparing them with analogous data related to as-fabricated arrays of pillars, we can detect and quantify, e.g., the system's strengthening. Specifically, we analyse how the array size N and distribution (1) of pillar-strength-thresholds $\{\sigma_{th}\}_{\Sigma_0}$ together with the number of cycles n and the range of load transfer change the overall strength of the array of pillars.

Our numerical experiment is built around the following scheme. For a given number of pillars (N) and a chosen amount of disorder (ρ), we generate and store at least $M = 10^4$ sets $\{\sigma_{th}\}_{\Sigma_0(N,\rho)}^{(j)}$, $j \in \{1, \dots, M\}$, each with N pillar-strength-thresholds drawn from (1). Then, picking $\{\sigma_{th}\}_{\Sigma_0(N,\rho)}^{(j)}$ up one by one, we engage the quasi-static compression test to compute as-fabricated $F_c^0(N, \rho)^{(j)}$, in correspondence with the GLS and the LLS rules. After that, for all $\{\sigma_{th}\}_{\Sigma_0(N,\rho)}^{(j)}$ taken by the same order, we simulate the cyclic preloading, store the resulting $\{\tilde{\sigma}_{th}\}_{\Sigma \subset \Sigma_0(N,\rho)}^{(j)}$ and finally execute the quasi-static compression test in order to get $F_c(\{\tilde{\sigma}_{th}\}_{\Sigma \subset \Sigma_0(N,\rho)}^{(j)})$. The presence of the corresponding load-transfer rules is implied.

In simulations, we have varied four quantities. Two of them, $\rho = 2, 3, \dots, 10$ and $N = L \times L$ with $L \in \{40, 60, 80, 100\}$, are related to the arrays themselves, whereas $n \in \{10, 20, 50, 100\}$ and $f_{cyc} < f_{cyc}^*$ represent the cyclic preloading. The upper bound f_{cyc}^* is discussed in Sect. 3.1. For each combination of these four parameters, we have employed the GLS and the LLS rules for transferring loads from crushed pillars to the intact ones. The results are presented and discussed in the following aspects: (i) strengthening, (ii) survival analysis, (iii) fraction of broken pillars and (iv) post-cyclic pillar-strength-threshold distribution.

3.1 Strengthening F_c^Σ / F_c^0 and optimal cyclic load $\hat{f}_{cyc}$

In Sect. 2.4, the highest load $F_c^\Sigma = F_c(\{\sigma_{th}(n, f_{cyc})\}_{\Sigma \subset \Sigma_0})$ supported by a previously cycled array was combined with $F_c^0 = F_c(\{\sigma_{th}\}_{\Sigma_0})$ that refers to the same array with no cycling to formulate a measure of the strengthening. Specifically, as a measure of the strengthening, we employ the ratio

$$\xi(n, f_{cyc}) = F_c^\Sigma / F_c^0. \quad (4)$$

At this point, we introduce a scaling $F \rightarrow f = F/N$ which enables us to compare transparently and through data gathered from arrays with a different number of pillars. Therefore, the strengthening takes place when

$$\xi(n, f_{cyc}) = f_c^\Sigma / f_c^0 > 1. \quad (5)$$

We know from experiments realized with a single pillar that when the cycling amplitude is bounded from above by about $0.8\sigma_{th}$, then the pillar gets stronger. Obviously, a similar upper bound (f_{cyc}^*) should exist when an array of pillars undergoes a cyclic loading. In this case, two requirements, related to f_{cyc} , have to be simultaneously taken into account: (i) a desired level of strengthening should be achieved; (ii) it is essential to preserve the integrity of the array because the cycling eliminates a majority of pillars whose $\sigma_{th} \leq f_{cyc}$. To satisfy the requirement (i), we use condition (5), whose lower bound is equal to 1 and thus $f_c^0 < f_c(f_{cyc}^{*(i)})$, i.e. we allow f_{cyc} to grow until the detected strengthening disappears. With regard to (ii), we tune $f_{cyc}^{*(ii)}$ in such a way that the probability of destruction of the entire array is smaller than a given number p , which is set to 0.01 in our simulations. Since both requirements hold together then $f_{cyc}^* = \text{Min}(f_{cyc}^{*(i)}, f_{cyc}^{*(ii)})$. Based on the results presented in Fig. 1, it turns out that $f_{cyc}^* = f_{cyc}^{*(ii)}$ when the LLS rule is applied, contrary to $f_{cyc}^* = f_{cyc}^{*(i)}$ induced by the GLS rule.

Figure 1 illustrates how the mean strengthening varies with f_{cyc} for a different ρ when the LLS (left panel) or the GLS (right panel) rule are applied. For the GLS rule, $\langle \xi(n, f_{cyc}) \rangle$ increases with a growing f_{cyc} until

Fig. 1 The mean strengthening for chosen values of Weibull shape parameter: $\rho = 2$ (circles), $\rho = 3$ (squares) and $\rho = 5$ (diamonds). The number of cycling $n = 50$. Left panel—the LLS rule, right panel—the GLS rule. Results were obtained from 10^4 samples for each data point. Continuous lines represent strengthening when $0 < p \leq 0.01$, whereas data points reflect cycling when all arrays survive ($p = 0$)

Table 1 Estimated $\hat{f}_{\text{cyc}}$ and corresponding mean strengthening $\langle \xi(n, \hat{f}_{\text{cyc}}) \rangle$ of arrays of 80×80 pillars with $\{\sigma_{\text{th}}\}_{\Sigma_0}$ distributed according to (1) with different values of ρ

GLS	$\rho = 2$	$\rho = 3$	$\rho = 5$	$\rho = 8$	$\rho = 10$
$\hat{f}_{\text{cyc}}$	0.26	0.28	0.30	0.32	0.34
$\langle \xi(50, \hat{f}_{\text{cyc}}) \rangle$	1.0892	1.0902	1.0904	1.0904	1.0898
LLS	$\rho = 2$	$\rho = 3$	$\rho = 5$	$\rho = 8$	$\rho = 10$
$\hat{f}_{\text{cyc}}$	0.23	0.26	0.26	0.28	0.30
$\langle \xi(50, \hat{f}_{\text{cyc}}) \rangle$	1.0760	1.0754	1.0747	1.0777	1.0797

Results computed from 10^4 arrays, each cycled 50 times and then compressed quasi-statically, in the presence of either the GLS or the LLS rule of load transfer

$f_{\text{cyc}}^* \sim 0.3$ and then decreases. When $\rho \geq 5$, a relatively large $f_{\text{cyc}} > 0.55$ weakens the system while still conserving its integrity. When the LLS rule is used, the arrays are strengthened for all $f_{\text{cyc}} \leq f_{\text{cyc}}^*$.

Based on the above observations, it turns out that for a given array of pillars there exists a particular $\tilde{f}_{\text{cyc}} \leq f_{\text{cyc}}^*$ that maximizes the strengthening. In the following, we call such a load the optimal cyclic load ($\hat{f}_{\text{cyc}}$). Exemplary information about $\hat{f}_{\text{cyc}}$ is provided in Table 1. This table shows that for all examined values of ρ , the estimated mean strengthening attains $\sim 9\%$ when the GLS rule operates. When dealing with the LLS rule, the effect of cyclic preloading is less pronounced, namely the highest mean strengthening is about 7.5–8% depending on ρ . This is mainly because the GLS rule engages all pillars in sharing the load uniformly, whereas the LLS rule, by accumulating loads in neighbourhoods of crushed pillars, generates inhomogeneities in the stress distribution. In consequence, the LLS rule facilitates an overloading of intact pillars.

According to experiments, a few early stages of cycling dominate the strengthening process [4]. Our model perfectly recovers this effect due to the cycle-attenuation factor (3) included in (2). Quantitative details are presented in Figs. 2 and 3. Specifically, Fig. 2 shows that the impact of growing n on ξ is most pronounced when $f_{\text{cyc}} \sim \hat{f}_{\text{cyc}}$.

The above-presented results refer to the mean strengthening of pillars that are cyclically preloaded with a given n and f_{cyc} . Precisely speaking, these $\langle \xi(n, f_{\text{cyc}}) \rangle$ are based on empirical probability distributions of $f_c^{\Sigma}(n, f_{\text{cyc}})$ computed from a preloaded population of 10^4 arrays with pillars whose $\{\sigma_{\text{th}}\}_{\Sigma_0}$ are distributed according to (1). In Fig. 4, we present results related to the LLS rule, $\rho = 8$ and $n = 50$. Additionally, the distribution of $f_c^{\Sigma_0}$ is shown in order to have a clear view on how the cycling shifts this as-fabricated distribution.

Fig. 2 The mean strengthening for increasing number of cycles: $n = 10$ (circles), $n = 20$ (squares), $n = 50$ (diamonds) and $n = 100$ (triangles) computed in the presence of the LLS (left panel) and the GLS (right panel) rules of load transfer. Each data point corresponds to averaging over 10^4 arrays of 80×80 pillars with $\{\sigma_{th}\}_{\Sigma_0}$ distributed according to (1) with $\rho = 8$. Continuous lines represent strengthening when $0 < p \leq 0.01$, whereas data points reflect cycling when all arrays survive ($p = 0$)

Fig. 3 The mean strengthening as a function of number of cycles: $f_{cyc} = 0.08$ (squares), $f_{cyc} = 0.18$ (triangles) and $f_{cyc} = 0.28$ (diamonds) computed in the presence of the LLS rule of load transfer. Each data point corresponds to an averaging over 10^4 arrays of 80×80 pillars with $\{\sigma_{th}\}_{\Sigma_0}$ distributed according to (1) with $\rho = 8$

Employing the Cramer-von Mises and Anderson-Darling goodness-of-fit tests, we have found that empirical distributions of $f_c^\Sigma(n, f_{cyc})$, whose representative examples are displayed in Fig. 4, can be fitted by a three-parameter skew-normal distribution (SND):

$$\text{SND}(x) = \frac{\text{erfc}\left(-\theta \frac{x-\zeta}{\sqrt{2}\omega}\right)}{\sqrt{2\pi}\omega} \exp\left[-\left(\frac{x-\zeta}{\sqrt{2}\omega}\right)^2\right], \quad (6)$$

where ζ , ω and θ are the location, scale and shape parameters, respectively. Collected data enable us to relate ζ , ω and θ with f_{cyc} , n and ρ .

Fig. 4 Empirical distributions of f_c for different values of f_{cyc} and $\rho = 8$: $f_{cyc} = 0$ (circles), $f_{cyc} = 0.08$ (squares), $f_{cyc} = 0.18$ (triangles), $f_{cyc} = 0.28$ (diamonds). The LLS rule is employed for load transfers. For each f_{cyc} , the distribution is obtained from 10^4 arrays of $N = 80 \times 80$ pillars. Solid lines represent skew normal distributions (6) with parameters estimated from the simulations

3.2 Survival analysis

Although cycling should strengthen systems, it happens, however, that highly disordered arrays with the LLS rule become critically damaged when the value of f_{cyc} is slightly bigger than the corresponding near-optimal one. Therefore, it is worth including in our study an estimate of system's survival under a given f_{cyc} . To achieve this goal, we compare data sets related to cycling in the presence of the LLS and the GLS rules when f_{cyc} , n , ρ and N vary. Based on these data sets, we draw resulting empirical survival functions S_{LLS} and S_{GLS} . Then, through an appropriate nonlinear fitting we retrieve adequate analytical forms of these functions. Figure 5 shows how S_{LLS} (left panel) and S_{GLS} (right panel) vary under growing number of pillars when keeping ρ constant. A size effect is seen from the left panel in Fig. 5, i.e. the greater N , the smaller probability that the array survives cycling with a given value of f_{cyc} .

In the LLS rule case, the complementary of the cumulative distribution function (CDF) of a skew-normal distribution is the best fit for the survival function:

$$S_{LLS}(f_{cyc}) = 1 - \frac{1}{2} \operatorname{erfc} \left(-\frac{f_{cyc} - \zeta}{\sqrt{2}\omega} \right) + 2T \left(\frac{f_{cyc} - \zeta}{\omega}, \theta \right), \quad (7)$$

where ζ , ω , θ are the same parameters as in (6). All of them depend on ρ and N . Symbols: $\operatorname{erfc}(\cdot)$ and $T(\cdot)$ stand for the complementary error and the Owen's function, respectively.

A different fit corresponds to the GLS scenario with the survival functions being nicely approximated by the complementary of the CDF of a normal distribution:

$$S_{GLS}(f_{cyc}) = \frac{1}{2} \left[1 + \operatorname{erf} \left(\frac{\mu - f_{cyc}}{\sqrt{2}\delta} \right) \right], \quad (8)$$

which is parameterized by μ and δ , the mean and the standard deviation, respectively. Both are functions of n , ρ and N . Obviously, δ decreases with growing N , and thus, S_{GLS} approaches the Heaviside step function for a large number of pillars. In such a limit, a cut-off cycling amplitude $f_{cyc}^*(\rho)$ separates cycling that preserves the system's integrity from a cycling that crushes all the pillars.

When comparing cycling that involves the GLS rule to one involving the LLS rule, we notice two facts: (i) under the former cycling, an array may support significantly higher values of f_{cyc} ; (ii) when the LLS rule is engaged, the transition from being almost surely survived to almost surely destroyed takes place within a range of f_{cyc} that is wider than that corresponding to the GLS rule. We have also analysed how the number of cycles n changes the survival function S . As is seen in Fig. 6, S decreases with growing values of n when the GLS rule is applied. Contrary to that, an impact of n on S is negligibly small when the LLS rule is engaged.

Fig. 5 Empirical survival function of arrays with growing number of pillars $N = L \times L$: $L = 40$ (circles), $L = 60$ (squares), $L = 80$ (triangles) and $L = 100$ (diamonds) computed in the presence of the LLS (left panel) and the GLS (right panel) rule and $\rho = 2$. Lines are given by (7) and (8), respectively, for the left and right panel with corresponding parameters estimated from the data

Fig. 6 Empirical survival function S for arrays of pillars cycled with f_{cyc} for growing number of cycles: $n = 10$ (circles), $n = 20$ (squares), $n = 50$ (triangles) and $n = 100$ (diamonds) computed in the presence of the GLS rule. Prior to cycling, each array hosted 6400 pillars with strength-thresholds distributed according to (1) with $\rho = 2$ (strong disorder—left panel) or $\rho = 8$ (weak disorder—right panel). Lines are drawn according to (8)

3.3 Fraction of crushed pillars N_{cr}/N

Independently of which load transfer rule operates, fractures of pillars during cycling affect the weakest pillars mainly or pillars located in regions of high stress concentration that appear when the LLS rule is applied. We have traced numbers of pillars that were crushed during each cycling (N_{cr}) and then have analysed resulting statistics.

A useful characteristic is seen in Fig. 7 which displays an average fraction of crushed pillars N_{cr}/N in a linear relation to f_{cyc} for different values of ρ . These log-log plots indicate that fractions of crushed pillars are power-law functions of f_{cyc} and that N_{cr} are ranked in descending order of ρ . This means that $N_{cr}/N \sim (f_{cyc})^{\kappa(\rho)}$, where the exponent turns out to be almost a linear function of ρ , i.e. $\kappa(\rho) \sim 1.03\rho$. For example, under the LLS rule, if $\rho = 2$ and the cyclic load f_{cyc} is close to the corresponding optimal value $\hat{f}_{cyc} = 0.25$ then N_{cr}/N reaches 0.086. When a bigger value of ρ is taken into account, e.g. $\rho = 5$ along with $f_{cyc} \sim \hat{f}_{cyc} = 0.26$ then $N_{cr}/N \sim 0.001$. Slight deviations from linearities, seen in both panels of Fig. 7,

Fig. 7 Log-log plots of mean fraction of broken pillars N_{cr}/N for different values of ρ : $\rho = 2$ (circles), $\rho = 3$ (white squares), $\rho = 5$ (diamonds), $\rho = 8$ (triangles) and $\rho = 10$ (black squares) in correspondence with the LLS (left panel) and the GLS (right panel) rules. Each data point represents an average over 10^4 arrays with 6400 pillars cycled 50 times

appear for loads $f_{cyc} \sim f_{cyc}^*$ that are substantially higher than the optimal load $\hat{f}_{cyc}$, i.e. for cycling at the limit of applicability.

Qualitatively similar characteristics hold when the GLS rule governs the load transfer. The main quantitative difference between the LLS and GLS rules is that a given array resists a higher f_{cyc} under the GLS rule than that under the former one. It is worth mentioning that under both rules of load transfer, the resulting fractions of crushed pillars N_{cr}/N are weakly dependent on number of cycling n and almost insensitive to N . The lack of sensitivity to N does not refer to size effects observed in nanopillars [18, 19]. This is because in our simulations a given value of N points to the size of an array with this particular number of pillars and not to the pillars themselves.

3.4 Post-cyclic distribution of pillar-strength-thresholds

Prior to cycling, the strength-thresholds $\{\sigma_{th}\}_{\Sigma_0}$ are distributed according to (1). After n cycles of loading by f_{cyc} , the strength-thresholds of intact pillars become $\{\sigma_{th}\}_{\Sigma}$. In this section, we discuss how preloading affects the distribution of pillar strength-thresholds (1). In the following, we restrict our discussion to results gathered from simulations involving the LLS rule.

After cyclic preloading, the vast majority of pillars have strength-thresholds greater than the applied f_{cyc} . Even though all the pillars with initial $\sigma_{th} < f_{cyc}$ are crushed during the first cycle of preloading, there may exist a small fraction of pillars whose varying strength-thresholds will drop below f_{cyc} in the course of preloading. Having an array of pillars with known distribution of $\{\sigma_{th}\}_{\Sigma_0}$ after a cycling with given values of n and f_{cyc} , we get the resulting post-cyclic distribution $\{\sigma_{th}\}_{\Sigma}$. Interestingly, we were able to model the randomness of $\{\sigma_{th}\}_{\Sigma}$ by a family of left-truncated two-parameter Weibull distributions defined by

$$P(\sigma_{th}) = 1 - \exp \left[\left(\frac{\sigma_{min}}{\beta} \right)^\alpha - \left(\frac{\sigma_{th}}{\beta} \right)^\alpha \right], \quad (9)$$

where α and β denote the shape and scale parameters, respectively [20]. The truncation point $0 < \sigma_{min} = \text{Min}(\{\sigma_{th}\}_{\Sigma}) \sim f_{cyc}$ is a random variable. By applying a number of goodness-of-fit tests, we have rigorously compared data gathered from simulations with those provided by (9) and have retained (9) as the statistical model correctly characterizing distributions $\{\sigma_{th}\}_{\Sigma}$. It is worth noting that model (9) is not unexpected given that $\{\sigma_{th}\}_{\Sigma_0}$ are governed by the Weibull distribution (1).

We have also estimated values of the parameters α and β by employing the maximum likelihood procedure. Both these parameters are normally distributed. Their mean values are reported in Fig. 8. To facilitate comparison of arrays of pillars referred to a different ρ , the mean $\langle \alpha \rangle$ is scaled by a corresponding value of ρ , i.e. $\langle \alpha \rangle \rightarrow \langle \alpha \rangle / \rho$, and presented in the left panel of Fig. 8. For all employed values of ρ , both $\langle \alpha \rangle / \rho$ and $\langle \beta \rangle$

Fig. 8 Estimated mean values of α/ρ and β parameters from distribution (9) as functions of f_{cyc} for exemplary values of ρ : $\rho = 2$ (circles), $\rho = 3$ (squares), $\rho = 5$ (diamonds), $\rho = 8$ (white triangles) and $\rho = 10$ (black triangles). Each data point represents an average over 10^4 arrays with 6400 pillars cycled 50 times

increase as long as $f_{cyc} \leq \hat{f}_{cyc}$. Passing $\hat{f}_{cyc}$ the mean $\langle \beta \rangle$ still grows, whereas $\langle \alpha \rangle / \rho$ decreases with growing values of f_{cyc} when $\rho > 3$.

It is important to notice that the distribution (9) differs profoundly from the one that would be formally derived from (1) by a left truncation at σ_{min} . Even though both these distributions refer to a smaller population of pillars within a given array, the latter keeps all $\sigma_{th} > \sigma_{min}$ unchanged, whereas (9) involves thresholds $\{\sigma_{th}\}_{\Sigma \subset \Sigma_0}$ that evolved from $\{\sigma_{th}\}_{\Sigma_0}$ during n cycles of the preloading process.

4 Summary

We have investigated how cyclic preloading improves the strength of arrays of pillars that become compressed ulteriorly.

Assuming that pillars are represented by random strength-thresholds drawn from the Weibull distribution, we have performed simulations that mimic analogous experiments with a single pillar that reveal the strengthening due to the tuned cyclic preloading. Our simulations are solely governed by two factors: (a) the relation describing how the distribution of pillar-strength-thresholds evolves under different regimes of cycling and (b) the load transfer rule chosen to manage allocation of load from crushed to intact pillars. We obtained statistics of the following quantities characterizing arrays with a given number of pillars and known Weibull shape parameter: (i) strengthening as function of number of cycles and of preloading amplitude, including the estimated optimal values of this amplitude, (ii) survival function, (iii) fraction of crushed pillars and (iv) post-cycling distribution of pillar-strength-thresholds. The simulations show that the most significant strengthening appears during the first dozen cycles of preloading.

Our simulations would add new insight to similarities between mechanical properties of a single-component system and its multicomponent counterpart.

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