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On basic equations and kinematic-wave theory of separation processes in suspensions with gravity, centrifugal and Coriolis forces

This paper is dedicated to the memory of Franz Ziegler

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Abstract Neglecting inertial and viscosity effects in the bulk flow is a common assumption in the analysis of separation processes in suspensions under the action of gravity or centrifugal and Coriolis forces. While there is a number of examples of particular solutions, the general form of the basic equations for three space dimensions, together with the appropriate boundary and initial conditions, is still uncertain and, with regard to certain aspects, even controversial. An essential point is a proper choice of the variables. Here it is proposed to introduce the mass density of the mixture, the mean mass velocity of the mixture and the total volume flux as a set of dependent variables. After some manipulations, a complete set of basic equations is obtained. It consists of two continuity equations, a generalized drift-flux relation, and two linearly independent components of a vector equation describing the total body force as irrotational. Then, by eliminating the mean mass velocity of the mixture from the set of unknowns, a generalized kinematic-wave equation is derived. It describes kinematic waves that are embedded in a bulk flow that may be one-, two- or three-dimensional. Concerning boundary conditions at solid walls, one has to ascertain whether the total body force at the wall points into the suspension or out of it. In the former case, a thin boundary layer of clear liquid is formed at the wall, whereas in the latter case a thin sediment layer may either stick at the wall or slide along it. Each of those three possibilities leads to a particular boundary condition for the bulk flow in terms of the dependent variables. In addition, initial conditions and kinematic shock relations are briefly discussed. Finally, the application of the kinematic-wave theory to the settling process in rotating tubes is outlined.

1 Introduction

The process of solid particles settling slowly due to gravity in a liquid-filled vessel with vertical walls is not as simple as it might appear at first glance. Even if the particles are spherical and uniform in both size and material, there is a variety of physical phenomena that affect the process. To be mentioned, above all, are the mutual hindering of the suspended particles and the counter flow of the liquid in case of an impermeable bottom of the vessel. In a truly pioneering paper, Kynch [1] took those effects into account and developed a theory that is now known as kinematic-wave theory [2–5]. In its original version, the kinematic-wave theory was based on one space coordinate and time as independent variables. Strictly speaking, that implies vessels of constant cross section, but gravity settling in vessels with varying cross sections has been analyzed in a one-dimensional approximation [6]. Retaining one spatial coordinate, it is also possible to describe particle separation due to centrifugal forces in cylindrical centrifuges [7–10]. Remarkably, the radial dependence of

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both the centrifugal force and the cross section can lead to kinematic shock waves with unusual properties [6–8]; cf. also [11].

As early as 1920, it was already observed by the medical doctor Boycott [12] that gravity settling of particles in a tube of constant cross section is enhanced if the axis of the tube is inclined to the vertical. This effect, which is now named after Boycott, is due to the buoyancy-driven flow of clear liquid in a thin layer attached to the downward facing wall of a tube or vessel. It took, however, more than 60 years until a kinematic-wave theory became available that is able to cope with the Boycott effect [13]. It required, of course, a generalization to more than one spatial coordinate. The Boycott effect occurs also in centrifuges with compartments [14, 15] and in rotating buckets [16]; cf. also the survey articles [17, 18] and the monograph [19]. Generalizations to more than one spatial coordinate are also required to account for viscosity effects in the bulk flow [20], or to describe enhanced settling in centrifuges [21].

In course of further generalizations of the kinematic-wave theory of suspensions, a variety of problems of practical interest has been considered. To be mentioned are, among others, sedimentation of particles with two or more different sizes [22–29] and the associated experiments [24, 30, 31]; sedimentation and centrifugation of flocculated suspensions [32–35]; accounting for consolidation processes [36, 37] with associated experiments [38, 39]; and accounting for the flow of suspension into the vessel and out from it (e.g., thickeners under start-up conditions) [40]. With regard to the aim of the present analysis, as described below, it may be of interest to note that wall effects, like the Boycott effect as well as deposition of particles at the sidewalls or gliding of particles along the sidewalls, are *not* taken into account in the one-dimensional, unified analysis of batch and continuous thickening in vessels with varying cross section given in [34]. For applying the kinematic-wave theory to applications, there is the problem of determining the parameters for given materials; Refs. [41–43] provide answers. For reviews of the development of that particular branch of fluid mechanics, the reader is referred to [44, 45].

The generalizations of the kinematic-wave theory of suspensions, as briefly described above, have received favorable attention in the scientific community; see, for instance, the monograph [19]. However, criticism can also be found in the literature. Two points appear to merit serious consideration. The first one concerns the general theory for more than one space coordinate. Irrespective of the good agreement of the theoretical predictions with measurements [24, 46, 47], it was argued from a mathematical point of view that the set of equations would be incomplete in the case of more than one space coordinate [37, 48–50]. A second criticism concerns the one-dimensional approximation that was applied in [8] to describe, among others, the settling process in tube centrifuges. Despite of the experimental support provided by [51–54] to the predictions based on the one-dimensional approximation, it is pointed out by M. Ungarish [55–57] that the particles, under the action of centrifugal forces, move strictly radially to hit the tube wall no matter how slender the tube may be, thereby violating the assumptions of one-dimensional flow. Ungarish's criticism seems to concern a rather special problem, but it raises the fundamental question of justification of one-dimensional flow approximations; cf. the monograph [2], which is devoted entirely to one-dimensional two-phase flow.

In view of the discussion and, in particular, the criticism concerning the kinematic-wave theory of separation processes in suspensions, it is thought that it may be of some interest to present, in what follows, a reformulation of the general theory given in [58] with the following aims:

- accounting for the criticism concerning the one-dimensional flow approximation following the ideas presented in [59, 60];
- justification of neglecting the Coriolis force in the momentum equation of the mixture, while retaining it in the drift-flux relation, as proposed in [14, 16];
- obtaining explicit equations (rather than the implicit relations given in [58]) for the unknown quantities in order to make the theory immediately applicable to particular problems. This may also foster various applications, e.g., pharmaceutical ones [61–63].

2 Equations of motion

We consider unsteady flow of an incompressible mixture of a fluid (subscript 1) and solid particles of uniform shape and size (subscript 2). The independent variables are time t and three spatial coordinates. For the dependent variables cf. below. The set of fundamental equations then consists of two continuity equations and two momentum equations.

2.1 Continuity equations

A common set of continuity equations consists of mass conservation equations of fluid and particle phases, respectively. For the present purpose it is more convenient, however, to use an equivalent set of continuity equations, consisting of the volume and mass balances, respectively, of the incompressible mixture with variable mass density, ρ_m . The volume balance of the mixture simply reads

$$\nabla \cdot \vec{j} = 0, \quad (2.1)$$

where \vec{j} is the total volume flux (i.e., the volumetric flow rate per unit area) of the mixture. In terms of the volume fluxes (fictitious velocities) \vec{j}_1 and \vec{j}_2 of the fluid and particle phases, respectively, the total volume flux becomes

$$\vec{j} = \vec{j}_1 + \vec{j}_2. \quad (2.2)$$

As the second continuity equation, the mass balance of the mixture is written as

$$\frac{\partial \rho_m}{\partial t} + \nabla \cdot (\rho_m \vec{v}_m) = 0, \quad (2.3)$$

where \vec{v}_m is the mean mass velocity of the mixture.

For later use, the following relationships between the variables can easily be derived:

$$\vec{j} = (1 - \alpha) \vec{v}_1 + \alpha \vec{v}_2; \quad (2.4)$$

$$\rho_m = (1 - \alpha) \rho_1 + \alpha \rho_2; \quad (2.5)$$

$$\rho_m \vec{v}_m = (1 - \alpha) \rho_1 \vec{v}_1 + \alpha \rho_2 \vec{v}_2, \quad (2.6)$$

where α is the volume fraction of the particles (often also addressed as “particle concentration”), ρ_1 and ρ_2 are the mass densities of the pure liquid and the particle material, respectively, while the velocities of the liquid and solid phases are denoted by \vec{v}_1 and \vec{v}_2 , respectively.

2.2 Momentum equation of the mixture

2.2.1 Pressure gradient in the mixture

If inertial and viscosity terms are negligibly small, the momentum equation of the mixture reduces to

$$\nabla p = \rho_m \vec{b}_m, \quad (2.7)$$

where p denotes the pressure and \vec{b}_m the body force per unit mass of the mixture. Neglecting inertial and viscosity terms in the momentum equation of the mixture is common practice in the theory of gravity settling and centrifugation. In case of gravity settling, power products of Reynolds number and Grashof number have to satisfy certain conditions [13], while in case of centrifugal settling the Ekman number and the Rossby number play analogous roles [15]. It should be mentioned, however, that inertial effects have been considered by Schafflinger [18,46,47], while the effects of viscosity on the bulk flow have been taken into account in the framework of kinematic-wave analysis by Smek [20]. Ekman layers have also been discussed [18]. With regard to viscosity effects in a more general context cf. [37].

In a frame of reference rotating with constant angular velocity $\vec{\omega}$, the gravity, centrifugal and Coriolis forces, respectively, contribute the following terms to \vec{b}_m :

$$\vec{b}_m = \vec{g} + \omega^2 \vec{r} - 2\vec{\omega} \times \vec{v}_m, \quad (2.8)$$

where \vec{g} is the acceleration of gravity, and \vec{r} is the radial distance from the axis of rotation (*not* the position vector).

2.2.2 The role of the Coriolis force in the mixture momentum equation

As mentioned in the Introduction, it is of interest under what conditions the Coriolis force should be neglected in the momentum equation of the mixture.

Eliminating the pressure gradient by applying the curl operator to the momentum Eq. (2.7) gives

$$\nabla \times (\rho_m \vec{b}_m) = 0, \quad (2.9)$$

i.e., the particles are distributed in the suspension during the settling process such that the volumetric body force is irrotational [58]. With \vec{b}_m according to (2.8), (2.9) gives

$$(\vec{g} + \omega^2 \vec{r}) \times (\nabla \rho_m) - \vec{C} = 0, \quad (2.10)$$

where \vec{C} is the curl of the Coriolis force per unit volume of the mixture, i.e.,

$$\vec{C} = -2\nabla \times [\vec{\omega} \times (\rho_m \vec{v}_m)]. \quad (2.11)$$

Using well-known identities for the curl of a vector product [64], (2.10) with (2.11) may be rewritten as

$$(\vec{g} + \omega^2 \vec{r}) \times (\nabla \rho_m) + 2[\vec{\omega} \nabla \cdot (\rho_m \vec{v}_m) - (\vec{\omega} \cdot \nabla) (\rho_m \vec{v}_m)] = 0, \quad (2.12)$$

where the term in brackets is due to the Coriolis force. Provided the Coriolis force is *not* neglected, the term $\nabla \cdot (\rho_m \vec{v}_m)$ in (2.12) can be replaced by $-\partial \rho_m / \partial t$ by means of the continuity Eq. (2.3) to obtain

$$2\vec{\omega} \frac{\partial \rho_m}{\partial t} = (\vec{g} + \omega^2 \vec{r}) \times (\nabla \rho_m) - 2(\vec{\omega} \cdot \nabla) (\rho_m \vec{v}_m). \quad (2.13)$$

(2.13) indicates that the time scale is $1/\omega$, with $\omega = |\vec{\omega}|$, provided the Coriolis force is not neglected. This, however, would imply that $\partial \vec{v}_m / \partial t$ and $\vec{\omega} \times \vec{v}_m$ are of the same order of magnitude. Thus, neglecting inertial terms while retaining the Coriolis force term in the momentum equation of the mixture is inconsistent.

In conclusion, the Coriolis force term in the momentum equation of the *mixture* is neglected in what follows, i.e., (2.8) is reduced to $\vec{b}_m = \vec{b}$ with

$$\vec{b} = \vec{g} + \omega^2 \vec{r}, \quad (2.14)$$

where the subscript m has been dropped in order to avoid confusion with the full body force according to (2.8).

2.2.3 Mixture density and concentration gradients

Replacing \vec{b}_m by \vec{b} in (2.7) and applying the curl operator gives

$$\nabla \times (\rho_m \vec{b}) = 0. \quad (2.15)$$

Note that in (2.15), as in any irrotational vector field, only two components are independent of each other, i.e., one component is superfluous. It follows from (2.14) that $\nabla \times \vec{b} = 0$. Thus, (2.15) can be written as

$$\vec{b} \times \nabla \rho_m = 0, \quad (2.16)$$

or

$$\vec{b} \times \nabla \alpha = 0, \quad (2.17)$$

i.e., the gradient of the particle concentration must either vanish or be parallel to the mass body force. Introducing unit vectors \vec{e}_b and \vec{e}_n in the direction of the body force \vec{b} and normal to it, respectively, one obtains the following relations for later use.

$$\nabla \alpha = \frac{\partial \alpha}{\partial s} \vec{e}_b; \quad (2.18)$$

$$\vec{e}_n \cdot \nabla \alpha = \frac{\partial \alpha}{\partial n} = 0, \quad (2.19)$$

where $\partial \alpha / \partial s$ and $\partial \alpha / \partial n$ are the derivatives of α in the direction of the body force \vec{b} and normal to it, respectively.

2.3 Momentum equation of the relative motion

2.3.1 Force balance

In general, the kinematic-wave theory of sedimentation is based on neglecting inertial effects on the particle motion, i.e., assume quasi-steady creeping flow [2]. The conditions for justifying that assumption are briefly discussed below, based on (2.21). Thus, the momentum equation of the relative motion reduces to the force balance for a particle in the mixture, which has the mass density ρ_m . The following forces contribute to the force balance.

- The *body force per volume of particle*, which is $\rho_2 \vec{b}_p$, with

$$\vec{b}_p = \vec{g} + \omega^2 \vec{r} - 2\vec{\omega} \times \vec{v}_2. \quad (2.20)$$

On the right-hand side of (2.20) there are the contributions of gravity, centrifugal force and Coriolis force, in this order.

- *The buoyancy force per volume of particle*. To determine the buoyancy force, the pressure gradient in the mixture, ∇p , is taken from (2.7) with (2.8). This implies that the contribution of the Coriolis force to the pressure gradient in the mixture is taken into account in the buoyancy, the reason being that the Coriolis force is not negligible in the drift-flux relation under certain conditions, cf. below.
- *The drag force*. For creeping motion, it is proportional to the drift velocity (relative velocity), $\vec{v}_{21} = \vec{v}_2 - \vec{v}_1$.

Added mass and Basset force (associated with time-dependent motion) are neglected; cf. [19], p. 28. The effect of Taylor columns is also discarded; cf. [65].

The force balance may then be written as the following generalized drift-flux relation:

$$\vec{v}_{21}/\tau = \varepsilon \vec{b} - 2\vec{\omega} \times (\vec{v}_{21} + \varepsilon \vec{v}_2), \quad (2.21)$$

with \vec{b} according to (2.14) and the density ratio ε defined as

$$\varepsilon = (\rho_2 - \rho_1)/\rho_1. \quad (2.22)$$

In accordance with common sedimentation or centrifugation processes, it will be assumed in what follows that the particles are specifically heavier than the liquid, i.e., $\varepsilon > 0$. Most relationships of the present analysis can easily be adjusted for the opposite case. Following [58], the time τ has been introduced in (2.21) such that $-\vec{v}_{21}/\tau$ represents the drag force of the particle per mass of displaced fluid, i.e., $-\vec{v}_{21}/\tau = \vec{F}_D/V_p \rho_1 (1 - \alpha)$, with drag force \vec{F}_D and particle volume V_p . The time τ characterizes velocity relaxation. In particular, it can easily be shown that the start-up time of a particle, i.e., the time it takes for a particle to reach its terminal velocity from rest, is of the order of τ . Thus, the quasi-steady flow approximation is justified if $\tau \ll L/|\vec{v}_{21}|$, or, in view of (2.21) with (2.14), $\tau^2 \ll L/\varepsilon(g + \omega^2 L)$, where L is a characteristic length of the (rotating) vessel.

2.3.2 Drift-flux relation

Since the drag force of a particle in the particle swarm depends on the concentration α (“hindered settling”), one may write

$$\tau = \tau_0 f_\tau(\alpha), \quad (2.23)$$

where τ_0 is the (translational) relaxation time of a single particle, e.g., $\tau_0 = d_p^2 Q_1/18\mu_1$ according to Stoke’s drag law for a spherical particle of diameter d_p in a fluid with viscosity μ_1 . For non-spherical particles, the translational relaxation time depends on direction, and additional rotational relaxation times may be important, cf. the survey article [66].

The function $f_\tau(\alpha)$ is to be determined from experiments (e.g., [2,46,47]) or suspension theory (e.g., [67]). The popular Richardson and Zaki correlation [68] gives

$$f_\tau(\alpha) = (1 - \alpha)^{n-1} \quad (n = \text{const.} > 1). \quad (2.24)$$

Good agreement with experimental data has been obtained with $n = 4.7$ [46,47].

It follows from (2.21) that the contribution of the Coriolis force to the drift velocity is of the order of $\tau\omega$. This leads to the following distinction:

- If $\tau\omega \ll 1$, the Coriolis force term can be neglected not only in the momentum equation of the mixture, but also in the drift-flux relation. Using $\tau \sim \tau_0 = d_p^2 \rho_1 / 18\mu_1$ according to Stoke's law and introducing the Taylor number $Ta = \omega d_p^2 \rho_1 / 18\mu_1$, one obtains the condition $Ta \ll 1$, which was already discussed in [9]. Note, however, that $\tau\omega \ll 1$ is not a sufficient condition for neglecting the *inertial* terms, as ε can be very large for heavy particles.
- If $\tau\omega = O(1)$, the Coriolis force has to be taken into account in the drift-flux relation.

Note also that accounting for the Coriolis force in the drift-flux relation is essential for describing the Boycott effect in cylindrical centrifuges with radial compartments [14].

Furthermore, it follows from (2.21) with (2.14) that the ratio of the inertial terms and the body force term in the momentum equation of the mixture is of the order of

$$v_{21}^2 / L |\vec{b}_m| = O(\varepsilon^2 \tau^2 \omega^2). \quad (2.25)$$

On the other hand, the ratio of the Coriolis force term and the centrifugal force term in the momentum equation of the mixture, (2.7) with (2.8), is of the order of

$$|\vec{v}_m| / \omega L = O(|\vec{v}_{21}| / \omega L) = O(\varepsilon \tau \omega). \quad (2.26)$$

Thus, neglecting the inertial terms and, consequently, the Coriolis term in the momentum equation of the mixture, while retaining the Coriolis term in the drift-flux relation, is justified only if the density difference between particles and fluid is sufficiently small to satisfy the relation

$$\varepsilon \ll 1. \quad (2.27)$$

This condition has been obtained previously for particular cases; cf. the survey [18].

In separation processes, the drift velocity, \vec{v}_{21} , is of the same order of magnitude as the velocity of the particle phase, \vec{v}_2 . This is true for one-dimensional as well as for two- or three-dimensional flows, see [2, 6, 8] and [13–15], respectively. Thus, the term $\varepsilon \vec{v}_2$ can be dropped in (2.21) under the condition (2.27), to obtain the following vector equation for \vec{v}_{21} :

$$\vec{v}_{21} / \tau + 2\vec{\omega} \times \vec{v}_{21} = \varepsilon \vec{b}. \quad (2.28)$$

The solution of (2.28) can be found as follows. Take the vector product of (2.28) and $\vec{\omega}$, eliminate the vector product $\vec{\omega} \times \vec{v}_{21}$ in that equation with the help of (2.28), use the identity $\vec{\omega} \times (\vec{\omega} \times \vec{v}_{21}) = (\vec{\omega} \cdot \vec{v}_{21}) \vec{\omega} - \omega^2 \vec{v}_{21}$, and eliminate the scalar product $\vec{\omega} \cdot \vec{v}_{21}$ with the help of the equation obtained from the scalar product of (2.28) and $\vec{\omega}$. The result is

$$\vec{v}_{21} = \frac{\varepsilon \tau}{1 + (2\tau\omega)^2} [\vec{b} + 2\tau(\vec{b} \times \vec{\omega}) + 4\tau^2(\vec{g} \cdot \vec{\omega})\vec{\omega}]. \quad (2.29)$$

(2.29) is a generalized drift-flux relation in explicit form. It is much more convenient for applications than the implicit version (2.21) already given in [58]. If the Coriolis force can be neglected, i.e., for $\tau\omega \ll 1$, (2.29) reduces to

$$\vec{v}_{21} = \varepsilon \tau \vec{b}. \quad (2.30)$$

In this case, it is not necessary that $\varepsilon \ll 1$, as (2.27) was used only for simplifying the Coriolis term in (2.21).

2.4 Complete set of basic equations

With the help of (2.4)–(2.6), \vec{v}_{21} may be expressed in terms of the dependent variables ρ_m , \vec{v}_m and \vec{j} . Then (2.29) becomes

$$\vec{v}_m - \vec{j} = \frac{\tau}{1 + (2\tau\omega)^2} \frac{(\rho_m - \rho_1)(\rho_2 - \rho_m)}{\rho_1 \rho_m} [\vec{b} + 2\tau(\vec{b} \times \vec{\omega}) + 4\tau^2(\vec{g} \cdot \vec{\omega})\vec{\omega}]. \quad (2.31)$$

The complete set of basic equations for the unknowns ρ_m , \vec{v}_m and \vec{j} now consists of the continuity Eqs. (2.1) and (2.3), the drift-flux relation (2.31), and the condition of irrotational body force, (2.15). Since only two components of the vector equation (2.15) are independent of each other, the set of equations comprises 7 scalar equations for 7 scalar unknowns. Which equation of the three components of (2.15) is dropped will depend on the particular problem and, perhaps, on the preferences of the researcher.

There are, however, certain special solutions of the set of basic equations that are associated with a non-uniqueness of the solution for \vec{j} , i.e., the bulk flow. This problem will be addressed below; cf. Sect. 3.2.

3 Kinematic waves

3.1 Generalized kinematic-wave equation

The set of equations given above might be convenient for numerical solutions, but for analytical solutions and a better understanding of the settling process, a further analysis is desirable. The basic idea is to eliminate \vec{v}_m from the set of unknowns. This can be accomplished either by rather cumbersome manipulations with the equations given above or—more directly—by considering the mass balance of the particle phase, i.e.,

$$\frac{\partial \alpha}{\partial t} + \nabla \cdot \vec{j}_2 = 0 \tag{3.1}$$

instead of the mass balance of the mixture, (2.3). With

$$\vec{j}_2 = \alpha \vec{j} + \alpha(1 - \alpha)\vec{v}_{21}, \tag{3.2}$$

\vec{v}_{21} according to the drift-flux relation (2.29), $(\vec{b} \times \vec{\omega}) \cdot \nabla \alpha = 0$ according to (2.17), $\nabla \cdot \vec{r} = 2$ according to the definition of \vec{r} , and $\nabla \cdot \vec{j} = 0$ according to the continuity EQ. (2.1), the mass balance (3.1) gives

$$\frac{\partial \alpha}{\partial t} + \left[\vec{j} + f_1'(\alpha)\vec{b} + f_2'(\alpha)(\vec{g} \cdot \vec{\omega})\vec{\omega} \right] \cdot \nabla \alpha = -2\omega^2 f_1(\alpha), \tag{3.3}$$

with

$$f_1(\alpha) = [\varepsilon\alpha(1 - \alpha)\tau] / [1 + (2\tau\omega)^2]; \quad f_2(\alpha) = 4\tau^2 f_1(\alpha) \tag{3.4}$$

and $\tau = \tau(\alpha)$ according to (2.23). If $\tau\omega \ll 1$, i.e., in the case of negligibly small Coriolis force, the third term in the brackets on the left-hand side of (3.3) and the term $(2\tau\omega)^2$ in (3.4) can be dropped to simplify the equations.

The inhomogeneous partial differential Eq. (3.3) is of the general form that was discussed in [58] without providing details. According to (3.3), the concentration field is coupled with the bulk flow via the term $\vec{j} \cdot \nabla \alpha$. Since $\nabla \alpha$ is parallel to the body force \vec{b} as indicated by (2.18), (3.3) can be written as

$$\frac{\partial \alpha}{\partial t} + w \frac{\partial \alpha}{\partial s} = -2\omega^2 f_1(\alpha), \tag{3.5}$$

with

$$w = j_b + f_1'(\alpha)|\vec{b}| + f_2'(\alpha)(\vec{g} \cdot \vec{\omega})^2 / |\vec{b}|, \tag{3.6}$$

where j_b is the component of \vec{j} in the direction of \vec{b} . (3.5) is a generalized kinematic-wave equation. It shows that the settling process is governed by one-dimensional kinematic waves that propagate along the lines of the body force \vec{b} with velocity w . The waves are embedded in a bulk flow \vec{j} that may be one-, two- or three-dimensional. Since \vec{b} does not contain the Coriolis force, cf. (2.14), the direction of the propagation of the kinematic waves is independent of the Coriolis force, whereas the third term on the right-hand side of (3.6) gives a contribution of the Coriolis force to the wave propagation velocity.

3.2 Complete set of kinematic-wave equations

The generalized kinematic-wave Eq. (3.5) is to be supplemented by (2.1) and (2.17) to obtain a complete set of equations for the unknowns α and \vec{j} . Since only two components of the vector Eq. (2.17) are independent of each other, the set of equations consists of 4 scalar equations for 4 scalar unknowns. There is, however, a problem with certain solutions, as previous investigations, e.g., [13], have shown. In particular, if α is assumed to be a function of time only, (2.17) is trivially satisfied and the term containing \vec{j} in (3.3) vanishes. This allows to integrate (3.3) to obtain α independent of \vec{j} , and only the scalar continuity Eq. (2.1) remains for determining the unknown vector field \vec{j} . To deal with that problem, an approach has been proposed in [13] for the case of

gravity settling that may be generalized as follows. Instead of directly using (2.17), the derivative of (3.5) in any direction normal to the body force \vec{b} is taken. Observing (2.19), one obtains

$$\frac{\partial w}{\partial n} \frac{\partial \alpha}{\partial s} = 0. \quad (3.7)$$

In view of (3.7), two cases have to be distinguished. First, it is assumed that α varies in space, i.e., $\partial\alpha/\partial s \neq 0$. It follows from (3.7) that w has to satisfy the relation

$$\frac{\partial w}{\partial n} = 0, \quad (3.8)$$

i.e., the wave velocity may vary in the direction of the body force \vec{b} , but not normal to it. Writing (3.8) for two main normals $n^{(1)}$ and $n^{(2)}$, gives two auxiliary conditions that, together with the kinematic-wave Eq. (3.5) and the continuity Eq. (2.1), provide a complete set of equations for the unknowns α and \vec{J} . The second case is less straightforward to deal with. If α is a function of time only, say $\alpha = \bar{\alpha}(t)$, one obtains $\partial\alpha/\partial s = 0$, and (3.7) is satisfied trivially. In this case, a solution that is stable with respect to small perturbations of $\bar{\alpha}(t)$ is sought. Thus, if $\alpha = \bar{\alpha}(t) + \gamma\alpha'(t, s)$ with γ as a small parameter, it is required that $w = \bar{w} + \gamma w'$ as $\gamma \rightarrow 0$. Taking into account that $\partial\bar{\alpha}/\partial s = 0$, whereas $\partial\alpha'/\partial s \neq 0$, (3.7) gives $\partial\bar{w}/\partial n = 0$, which is in accord with (3.8). Thus, the condition (3.8) is applied in both cases.

As mentioned above, the Coriolis force affects only the magnitude of the propagation velocity of the kinematic waves, not the direction of the propagation. However, the drift velocity vector contains a term orthogonal to \vec{b} , cf. (2.29). Via the boundary conditions, as given below, the Coriolis force may then give rise to a bulk and particle flow normal to the direction of the wave propagation.

4 Boundary conditions

In previous work, various boundary conditions have been applied as required by particular applications. A review, together with a few new results, was presented in [58]. In what follows, a more systematic account of the boundary conditions for two- and three-dimensional flow will be given. With regard to details, the reader will be referred to previous publications.

The orientation of a wall with respect to the body force is characterized by the scalar product $(\vec{b} \cdot \vec{n})$, where \vec{n} is the unit normal vector, pointing from the wall into the suspension. The special case $(\vec{b} \cdot \vec{n}) = 0$, i.e., the wall parallel to the body force, has been considered extensively in the past; cf. the Introduction. With viscosity being neglected for the bulk flow of the mixture, such a wall does not affect the flow. This special case is of little interest here. Another special case is a wall normal to the body force, i.e., $(\vec{b} \cdot \vec{n}) = \pm|\vec{b}|$. Typical examples are the top and the bottom, respectively, of a vessel in gravity settling, or the inner and outer walls, respectively, of a centrifuge with negligible gravity force. Those walls, too, are part of practically all classical applications of kinematic-wave theory to separation processes. Two types of walls that are of particular interest for the present investigation remain.

4.1 Wall type I: $0 < (\vec{b} \cdot \vec{n}) < |\vec{b}|$

The body force points from the wall into the suspension. This leads to a layer of clear liquid at the wall. Provided $\varepsilon > 0$, cf. the discussion following (2.22), the mass density of the suspension is larger than the mass density of the liquid. Thus, buoyancy causes the clear liquid in the wall layer to flow in the direction opposite to the body force, giving rise to enhanced separation of particles and liquid (Boycott effect). If the Reynolds number of the bulk flow is sufficiently large, the wall layer of clear liquid is very thin, and the flow velocity in the wall layer is rather large. For details, in particular for an analysis of the flow in the wall layer, see [13] and the references given in the survey article [58]. Experimental verification can be found in [46,47] for gravity settling and in [14] for centrifugal separation. It may be worth mentioning that in a low Reynolds number regime, where the kinematic-wave theory is not applicable, entrainment of particles into the wall layer has to be considered as a possibility [69].

To keep the wall layer of clear liquid free from particles, the normal component of the volume flux of particles, $j_2^{(n)}$, has to vanish at the interface between liquid and suspension. Since the wall layer is very thin, the boundary condition for the bulk flow can be prescribed at the wall rather than at the interface. This gives the boundary condition

$$\rho_1 j^{(n)} - \rho_m v_m^{(n)} = 0 \text{ at the wall (type I)} \quad (4.1)$$

or

$$j^{(n)} + (1 - \alpha)v_{21}^{(n)} = 0 \text{ at the wall (type I),} \quad (4.2)$$

where the superscript⁽ⁿ⁾ refers to the component in the direction of the wall normal vector \vec{n} .

4.2 Wall type II: $-\vec{b} \cdot \vec{n} < 0$

The body force in the suspension points toward the wall, and sediment forms at the wall. Then two possibilities are to be considered [59].

Ia: Sediment deposited at the wall

In general, the pressure gradient is too small to give rise to a volume flux of liquid in the sediment that would be comparable to the volume flux in the bulk of the suspension. Darcy's law can be applied to provide estimates, if required. Thus, the bulk flow has to satisfy the boundary condition

$$j^{(n_s)} = 0 \text{ at the surface of the sediment,} \quad (4.3)$$

where the superscript^(n_s) refers to the component normal to the surface of the sediment. In case the concentration in the bulk flow is small, the sediment layer will be thin in comparison with the characteristic length of the vessel or centrifuge, and the boundary condition can be prescribed at the wall in a first approximation.

Solutions for gravity settling with boundary condition (4.3) for particles being deposited at the sidewalls have been given in [13, 59]. Concerning centrifugal settling, work is under progress [70].

Ib: Sediment moves along the wall

Driven by the tangential component of the body force, the sediment may move along the wall. A quantitative description of the motion of a sediment at a wall is quite difficult [71–76], and there are still many open questions. The following problems may be of relevance, but are beyond the scope of the present investigation: resuspension of particles [77, 78] and related processes [79]; shear-induced migration of particles in concentrated suspensions [73–75]; not a sharp interface [80] and shear-induced corrugation of interfaces [81]; wall slip (depending on surface properties and particle material) [82–89]; stick-slip events [90]. For the present investigation, it is only of importance how the motion of the sediment affects the bulk flow, as this is what controls the settling time and the concentration in the suspension during the settling process. The details of the flow of the sediment are of lesser interest.

Applying the kinematic-wave theory to the bulk flow of the suspension requires that the thickness of the sediment layer at the wall, δ , is much smaller than the characteristic length of the vessel or centrifuge, L ; i.e., $\delta/L \ll 1$. In addition, a small thickness of the sediment layer leads to an essential simplification of the equations that govern the transport of the sediment. According to the mass (volume) balance of the moving sediment, the total volume flux into the sediment layer, i.e., $j^{(n)}$, has to be balanced by the rate of change of the volume flow of sediment in tangential direction and the rate of change of the layer thickness, $\partial\delta/\partial t$. Since the characteristic time scale of the process is $L/|\vec{j}|$, the order of magnitude of the rate of change of the layer thickness is $\partial\delta/\partial t \sim |\vec{j}|\delta/L$. Thus, $\partial\delta/\partial t$ is negligible in the mass balance of the moving sediment layer and the process can be considered as quasi-steady, if $\delta/L \ll 1$.

To determine the conditions for the validity of the assumption $\delta/L \ll 1$, the following estimates of the orders of magnitude are provided. With $j^{(n)} \sim |\vec{j}|$, the volume balance of the moving sediment for quasi-steady flow shows that

$$\delta/L \sim |\vec{j}|/\vec{j}_c, \quad (4.4)$$

with \bar{j}_c as a characteristic averaged volume flux of the sediment layer. In the layer, the body force is to be balanced by inertial terms, i.e.,

$$\bar{j}_c^2/L \sim |\bar{b}|, \quad (4.5)$$

and/or by viscosity effects, i.e.,

$$\mu_c \bar{j}_c / \rho_c \delta^2 \sim |\bar{b}|, \quad (4.6)$$

where μ_c and ρ_c are the viscosity and the mass density, respectively, of the concentrated suspension of mean particle concentration $\bar{\alpha}_c$. Buoyancy forces due to the hydrostatic pressure gradient in the bulk of the suspension are discarded on the ground that they do not affect the order of magnitude of \bar{j}_c , as long as the particle concentration in the bulk is not very close to the concentration in the sediment. With $|j| \sim \bar{v}_{21} \sim \varepsilon \tau |\bar{b}|$, cf. (2.29), (4.4) with (4.5) and (4.6) gives

$$\delta/L \sim \varepsilon \tau (|\bar{b}|/L)^{1/2} \quad (4.7)$$

and

$$\delta/L \sim (\varepsilon \tau \mu_c / \rho_c L^2)^{1/3}, \quad (4.8)$$

respectively. The larger one of these two expressions for δ/L must be much smaller than 1 in order to justify the assumption $\delta/L \ll 1$.

Provided the layer of moving sediment is very thin, the bulk flow is affected only via boundary conditions that are associated with the entrainment from the bulk flow into the wall layer. Leaving the details of the sediment transport to future investigations, only an approximate formulation of the boundary conditions is given here. Thus, following [59], it is assumed that the cross-sectional mean value of the particle concentration in the wall layer, $\bar{\alpha}_c$, is a constant that is known either from experiments or an analysis of the wall layer motion. This assumption resembles the common assumption of a known particle concentration in the sediment at the bottom of the vessel or the outer wall of the centrifuge.

Furthermore, it is quite natural to assume that the drift flux (or relative velocity) in the sediment moving along the wall is negligible. As in IIa, Darcy's law can be applied to provide estimates, if required. Then the ratio of the particle and liquid flow rates, respectively, in the moving sediment layer is $\bar{\alpha}_c / (1 - \bar{\alpha}_c)$, leading to the boundary condition

$$j_2^{(n)} / j_1^{(n)} = \bar{\alpha}_c / (1 - \bar{\alpha}_c) \text{ at the wall (type II, sliding sediment),} \quad (4.9)$$

or, with (3.2),

$$\frac{\alpha [j^{(n)} + (1 - \alpha) v_{21}^{(n)}]}{(1 - \alpha) [j^{(n)} - \alpha v_{21}^{(n)}]} = \frac{\bar{\alpha}_c}{1 - \bar{\alpha}_c} \text{ at the wall (type II, sliding sediment).} \quad (4.10)$$

The boundary condition (4.9) was already given in [59], where the effect of sediment motion on kinematic-wave propagation was investigated, apparently for the first time, for the case of gravity settling.

5 Initial conditions and instabilities

It has been noted first in [13] for the particular case of gravity settling that the initial condition for the total flux \bar{j} cannot be prescribed independently of the initial particle concentration, α_0 . In the present, more general formulation, it ought to be taken into account that (3.8) is independent of time, and, therefore, has to be satisfied also in the initial state. Substituting for w according to (3.6) gives a condition for j_b in terms of α_0 .

At high Reynolds numbers, inertial effects may cause various deviations from the initial conditions that would be proper for a kinematic-wave theory. Those deviations may—or may not—have noticeable effects on the separation process. As a particularly striking phenomenon, an initial vortex was observed in [16]. However, for applying the kinematic-wave theory to problems with more than one spatial coordinate, it is reassuring that experiments have confirmed the theoretical predictions for gravity settling [24, 46, 47] as well as for centrifugal separation [14–16, 18].

At low Reynolds numbers, instabilities that lead to fluctuations have been observed [91, 92], in particular in sedimentation of polydisperse suspensions [93] and in a stratified ambient [94]. An analysis of those phenomena in the framework of kinematic-wave theory seems to be lacking, but is beyond the scope of the present paper.

6 Kinematic shock relations

According to (3.6) the propagation velocity of the kinematic waves depends on the particle concentration α . It is well known [2, 3] that this may lead to wave steepening up to a discontinuity in the concentration, which is then called a kinematic shock wave. Kinematic shock waves separate the suspension, on one side, from the clear liquid on top of the vessel or at the inner wall of the centrifuge and, on the other side, from the sediment at the bottom of the vessel or at the outer wall of the centrifuge. Under certain conditions, kinematic shock waves can also occur within the suspension [6, 8, 13].

Since kinematic shock waves are a classical topic, we refrain here from reviewing the details. It is supposed to suffice to discuss a few points that are of particular interest for the separation processes with more than one spatial coordinate.

First, conservation of momentum flux across the kinematic shock, with inertial effects and friction in the bulk flow being neglected, requires that the pressure is continuous at the shock. Since ρ_m is discontinuous at the shock, it follows from (2.7) that the shock front is orthogonal to the body force [13, 58]. The latter is given by (2.14), as the Coriolis force term has to be dropped in the momentum equation of the mixture.

Secondly, the propagation speed W of the kinematic shock wave is to be determined, in the classical way, from the continuity of the normal component of the total volume flux, $j^{(n)}$, together with the conservation of particulate volume, requiring $(\alpha W - j_2^{(n)})$ to be continuous across the shock. For \vec{j}_2 , (3.2) is available. The normal component of the relative velocity can be written as $v_{21}^{(n)} = (\vec{v}_{21} \cdot \vec{b})/|\vec{b}|$, with \vec{b} according to (2.14). Inserting (2.29) for the relative velocity eventually gives

$$W = j^{(n)} + \frac{1}{\hat{\alpha} - \alpha} \{ [f_1(\hat{\alpha}) - f_1(\alpha)] |\vec{b}| + [f_2(\hat{\alpha}) - f_2(\alpha)] (\vec{g} \cdot \vec{\omega})^2 / |\vec{b}| \}, \quad (6.1)$$

where α and $\hat{\alpha}$ are the values of the particle concentration in front and behind the shock wave, respectively, with the functions f_1 and f_2 as given in (3.4). Note that in the limit of very weak shocks, i.e., $(\hat{\alpha} - \alpha) \rightarrow 0$, (6.1) for the shock velocity reduces to (3.6) for the wave velocity, as required by the theory of kinematic waves.

In the classical kinematic-wave analyses of settling processes with only one spatial coordinate, e.g., [1, 6, 8], the shock relation (6.1) is considerably simplified for the kinematic shock waves that separate the suspension from the clear liquid and the sediment, respectively. First, the total volume flux \vec{j} vanishes. Secondly, both the clear liquid and the sediment are at rest. However, this is not always true for more than one spatial coordinate. In case of the Boycott effect, clear liquid flows along the sidewall into the bulk of the clear liquid. Similarly, when sediment slides along a sidewall, particles are transported into the sediment at the bottom of the vessel or at the outer wall of the centrifuge. In both cases, mass conservation requires that there is a non-vanishing total volume flux at certain kinematic shock waves; cf. [13] and [14–16, 18] for the Boycott effect in gravity and centrifugal settling, respectively, whereas the effects of sediment transport are taken into account in [59, 60, 70]. As was already observed in [13], the kinematic-wave analysis is not applicable within the region of clear liquid at the top of the vessel when the Boycott effect takes place. This might cast doubts on the validity of the results of the kinematic-wave analysis, which are, however, verified by a variety of experiments, e.g., [24, 46, 47] for gravity settling and [14–16, 18] for centrifugal settling. The case of sediment transport along the sidewalls is even more challenging for the analysis. In particular, it is not a priori clear how the mass balance ought to be satisfied at the kinematic shock that separates the sediment at the bottom of the vessel or at the outer wall of the centrifuge from the bulk of the suspension. In [59] it was assumed that the particles are at rest at the surface of the sediment at the bottom of the vessel. However, it may be more consistent with the basic assumptions of the kinematic-wave theory to follow the well-established analysis of the Boycott effect [13] and assume that the total bulk flow \vec{j} vanishes in the sediment at the bottom of the vessel or at the outer wall of the centrifuge. Certainly, theoretical investigations that are more detailed and supported by experiments are desirable, though for small initial particle concentrations the particles sliding along the sidewalls have only a small effect on the motion of the sediment at the bottom of the vessel or at the outer wall of the centrifuge.

In general, a kinematic shock wave propagates with a speed that is between the wave velocities in front and behind the shock wave, respectively. However, in a kinematic-wave analysis of centrifuges, wave fronts were found that run ahead of the shock wave instead of into it [6, 8]. In order not to contradict stability requirements, the wave fronts, in this case, emerge from the shock wave with equal propagation velocity.

The appearance of kinematic shock waves as discontinuities is, of course, a consequence of the basic assumptions of the kinematic-wave theory. By accounting for inertial and other effects, an internal structure of kinematic shocks is obtained. In addition, splitting of kinematic shocks and other “strange properties” have been found [4, 11, 95].

Particles: gliding towards the bottom of the tube

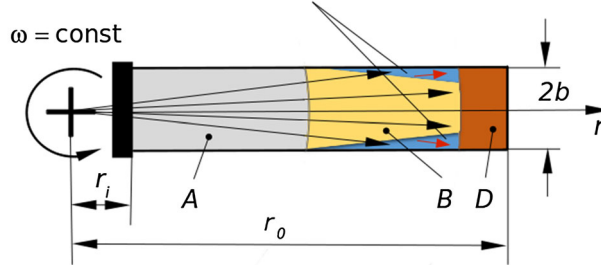


Fig. 1 Concentration zones in a tube centrifuge: A—clear liquid; B—suspension; D—sediment. Radial arrows indicate direction of body force. [60]

7 Application to tube centrifuges

As outlined in the Introduction, dealing with M. Ungarish's criticism of applying the approximation of one-dimensional flow to tube centrifuges [55–57] was one of the motivations for the present work. As a result, means are now available for comparing the different approaches. It is hoped that this may shed some new light on the applicability of the approximations under discussion. The related problem for gravity settling was already considered in [59].

Consider a long tube with constant, rectangular cross section, rotating about an axis that is normal to the tube center line and parallel to two of the four tube walls, addressed as sidewalls in what follows; cf. Figure 1 [60,70]. The kinematic-wave theory presented in the previous sections is applied. Since viscosity in the bulk flow is neglected, the tube walls normal to the sidewalls are of no relevance for the bulk flow, which, therefore, may be considered as two-dimensional. In accordance with Ungarish's work [55,56], the Coriolis force is also neglected; cf., however, the discussion in Sect. 2.3.2.

Of course, it is convenient to introduce cylindrical coordinates r, φ, z , with the z -axis coinciding with the axis of rotation. Vector components are indicated by subscripts. The kinematic-wave Eq. (3.5) then becomes

$$\frac{\partial \alpha}{\partial t} + w \frac{\partial \alpha}{\partial r} = -2\omega^2 f_1(\alpha), \quad (7.1)$$

with

$$w = j_r + \omega^2 r f_1'(\alpha), \quad (7.2)$$

$$f_1(\alpha) = \varepsilon \alpha (1 - \alpha) \tau(\alpha) \quad (7.3)$$

according to (3.6) and (3.4). From (2.19) and (3.8) it follows that neither the particle concentration nor the total bulk flow depend on the azimuthal coordinate, i.e.,

$$\alpha = \alpha(t, r); \quad j_r = j_r(t, r). \quad (7.4)$$

Thus, the continuity Eq. (2.1) can be integrated immediately to obtain

$$j_\varphi = -\varphi \frac{\partial (r j_r)}{\partial r} + c(t, r), \quad (7.5)$$

with $c(t, r)$ as a free function to be determined from the boundary conditions. In case of a symmetric channel, $c(t, r)$ vanishes identically.

The kinematic-wave Eq. (7.1) can be solved by the method of characteristics. Transforming (7.1) from the polar coordinate system (t, r, φ) to the characteristics coordinate system $(\tilde{t}, \zeta, \varphi)$ with

$$t = \tilde{t}; \quad \frac{\partial r}{\partial \tilde{t}} = w, \quad (7.6)$$

one obtains

$$\frac{\partial \alpha}{\partial \tilde{t}} = -2\omega^2 f_1(\alpha) \quad (7.7)$$

and

$$t = \tilde{t} = -\frac{1}{2\omega^2} \int_{\alpha_0}^{\alpha} \frac{d\bar{\alpha}}{f_1(\bar{\alpha})} + \Phi(\zeta, \varphi), \quad (7.8)$$

in agreement with the solution given in [8] for the cylindrical centrifuge. The initial concentration α_0 has been chosen as the lower boundary of the integral. The free function $\Phi(\zeta, \varphi)$ is to be determined from boundary conditions and/or jump conditions at kinematic shocks; cf. Sects. 4 and 6 above. It was already shown in [8] that, depending on the value of the initial concentration, several types of settling processes have to be distinguished. In particular, centered kinematic waves and/or kinematic shock waves within the suspension may occur. In addition, sediment sliding along the sidewalls gives rise to a bulk flow \vec{j} , which affects the solution via the wave velocity w according to (7.2) and via the shock velocity W according to (6.1). The latter effect was studied, apparently for the first time, in [59], yet for the somewhat simpler problem of gravity settling.

As far as other approaches are concerned, it ought to be noted that the bulk flow is taken into account neither in the one-dimensional flow approximation [8] nor in Ungarish's analysis [55, 56]. Nevertheless, both approaches may lead to results that resemble the experimental findings [51–54] under certain conditions. Since the one-dimensional flow approximation [8] is based on cross-sectional averages of the particle concentration, the sliding of particles along the sidewalls may have a favorable effect on the comparison with experimental results, though the bulk concentration is not correctly predicted. Resuspension of the particles after settling at the sidewalls might also contribute to a better agreement of the experiments with the one-dimensional flow approximation, as noted in [56]. Ungarish's analysis [55, 56], however, is restricted to very small values of the initial concentration α_0 . For sufficiently small values of the initial concentration there are only two kinematic shock waves, one of them separating the suspension from the purified liquid, the other one separating the suspension from the sediment at the bottom of the tube. The separation process is terminated when the two kinematic shock waves meet at a certain time t_s . Since, for reasons of mass conservation, the bulk flow cannot vanish at both shocks at the same time, the sliding of particles along the sidewalls will affect the settling process even in this simplest case. However, the effect can be expected to be small for very small initial concentrations. Details of the analysis, which is quite elaborate for geometric reasons, will be provided in [70]. Solutions in closed form can be obtained for a tube rotating sufficiently far from the axis to satisfy the condition $b/r_i \ll 1$. Quantitative comparisons with the one-dimensional approximation [8], on the one hand, and with Ungarish's analysis [55, 56] on the other hand, will also be given in [70].

8 Conclusions

The generalized drift-flux model leads to a complete set of equations that is substantially simpler than the full set of basic equations of two-phase flow. Thus, the criticism expressed in [37, 48–50] does not appear to be justified. Misleading remarks in previous papers on the subject [13, 58] may have been a source of that criticism.

For reasons of self-consistency of the generalized drift-flux model, the Coriolis force must be dropped in the momentum equation of the mixture. Furthermore, accounting for the Coriolis force in the drift-flux relation is justified only for small mass density differences between the liquid and the particle material.

The explicit form of the generalized drift-flux equation given in the present paper allowed deriving a generalized kinematic-wave equation for the particle concentration, with an explicit relation for the wave propagation speed. In accord with previous, more restricted investigations [9, 13–16, 18, 58, 59] it shows that the two-dimensional or three-dimensional settling process can be identified as a quasi-one-dimensional concentration wave that is embedded in a two-dimensional or three-dimensional incompressible flow of the mixture. This allows solving the generalized kinematic-wave equation by the method of characteristics. Depending on the orientation of the sidewalls with respect to the body force, the boundary conditions at the sidewalls account for the appearance of thin layers of clear liquid and of sediment that may be deposited at the wall or be transported along the wall. Mass conservation may then give rise to a bulk flow, which affects the settling process via the kinematic-wave velocity and/or via the kinematic shock relations.

Although it was one of the aims of the present paper to clarify controversial points, some open questions remain subject to further investigations. Among others, a more detailed analysis of the coupling of sediment transport and bulk flow should lead to a better understanding of the rather complex processes. Concerning the particular problem of tube centrifuges, viscosity may affect the bulk flow in long tubes to such an extent that

the basic assumptions of the kinematic-wave theory are violated. An analysis similar to [69] may be promising in that case.

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