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Constraints on the applicability range of Hill's criterion: strong orthotropy or transverse isotropy

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Abstract Commonly used orthotropic Hill's criterion of plastic flow initiation (Hill in Proc R Soc Lond A 193:281–297, 1948) suffers from some constraints and inconsistencies, which are of two different origins. Firstly, in case of high orthotropy degree, the quadratic form corresponding to Hill's criterion may change type from convex and closed elliptic to concave and open hyperbolic in the deviatoric stress space (Ottosen and Ristinmaa in The Mechanics of Constitutive Modeling, Elsevier, Amsterdam, 2005). Secondly, application of classical Hill's criterion to transversely isotropic materials shows a discrepancy between Hill's limit curves in the transverse isotropy plane and the Huber-von Mises prediction for isotropic materials (Huber 1904; von Mises 1913). The basic result of the present paper is to propose the new transversely isotropic von Mises–Hu–Marin's-type criterion of hexagonal symmetry that is free from both constraints. The new enhanced Hu–Marin's-type limit surface represents an elliptic cylinder, the axis of which is proportional to stress/strength, in contrast to Hill's-type limit surface possessing the hydrostatic axis. Hence, this condition does not exhibit the deviatoricity property, which is a price for coincidence with the Huber–von Mises condition in the transverse isotropy plane, but with cylindricity ensured for an arbitrarily high orthotropy degree. The hybrid-type transversely isotropic Hu–Marin's criterion of mixed symmetry based on additional biaxial bulge test, capable of fitting experimental findings for some complex composites, is also proposed. Application of this criterion has been verified for a unidirectional SiC/Ti composite examined by Herakovich (Thermal stresses V, Lastran Corp. Publ. Division, pp 1–142, 1999).

1 Introduction

A general tensorial polynomial anisotropic plastic flow or failure criterion was first proposed by Goldenblat and Kopnov [6] and later by Sayir [30]. It is based on a concept of common invariants of the stress tensor σ and of the structural tensors of plastic or failure anisotropy $\mathbb{\Pi}$, e.g.: $\Pi_{ij}\sigma_{ij}$ (2nd rank), $\Pi_{ijkl}\sigma_{ij}\sigma_{kl}$ (4th rank), $\Pi_{ijklmn}\sigma_{ij}\sigma_{kl}\sigma_{mn}$ (6th rank), etc. Structural tensors of plastic/failure anisotropy $\Pi_{ij}^{\mathrm{p/f}}$ 2nd rank, $\Pi_{ijkl}^{\mathrm{p/f}}$ 4th rank and $\Pi_{ijklmn}^{\mathrm{p/f}}$ 6th rank, different for plasticity (p) or failure (f) initiations, are satisfactory to describe basic transformation modes of limit surfaces due to plastic or damage hardening processes, namely isotropic change of size of limit surface, its translation and rotation, as well as distortion due to a curvature change (cf. Kowalsky et al. [17]). The basic postulates of material stability: in a Drucker's sense for ductile materials (cf.

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Drucker [3]), or the Hessian matrix be positive definite $[_{\tan}\mathbb{E}]_{mn}\varepsilon_m\varepsilon_n > 0$ in a Sylvester's sense, for brittle materials (cf. Kuna-Ciskał and Skrzypek [18]) imply a restriction, which allows the yield or failure initiation surfaces to be always closed and convex surfaces in the stress space.

In case of ductile materials (metals, alloys, intermetallics), second-rank tensors Π_{ij} of plastic anisotropy are usually neglected since the hydrostatic stress does not influence the yield initiation criterion. Additionally, the strength differential effect due to a different plastic behavior in uniaxial tension or compression is negligible ($k_t \approx k_c$). On the other hand, in case of brittle materials (concrete, ceramic materials, rocks, composite materials, etc.), where initiation of failure or damage manifests mostly or prior to other dissipative phenomena, the first stress invariant plays an important role, such that the strength differential effect is essential ($k_t \neq k_c$), so the first (linear) term $\Pi_{ij}\sigma_{ij}$ cannot be omitted (e.g., criterion of Tsai–Wu [36]). Moreover, the third term (cubic) $\Pi_{ijklmn}\sigma_{ij}\sigma_{kl}\sigma_{mn}$, which describes limit surface distortion, can play an essential role if consecutive hardening phenomena due to advanced plasticity and damage or other microstructure changes responses occur (e.g., Kowalsky et al. [17]). However, when only initiation of plastic or failure mechanisms is considered, this term is also consequently omitted.

Hence, in what follows, we shall reduce a class of the limit surface to the forms independent of both the first and the third common invariants, but preserving the most general representation for the second common invariant, according to von Mises [24]. In such a case, the fourth-rank tensor of plastic anisotropy Π_{ijkl} is in general defined by 21 anisotropy modules (but 18 of them independent), since the anisotropy 6×6 matrix $[\mathbb{II}]_{ij}$ can completely be populated. Further reduction in the number of parameters to 15 will be achieved, when the invariance of the general von Mises quadratic form with respect to the change in hydrostatic stress is assumed. In such a way, the general tensorial von Mises criterion will be reduced to the deviatoric form defined by 15 anisotropy modules. A choice of 15 anisotropy modules considered as independent is, in general, not unique (cf. Szczepiński [34]). However, the 15-parameter von Mises deviatoric criterion is sensitive to the change in the sign of shear stresses, which may be considered as questionable (cf., e.g., Malinin and Rżysko [22]). The simplest way to avoid a doubtful physical explanation for existence of terms linear for shear stresses τ_{ij}, the reduction of the 15-parameter von Mises equation to the 9-parameter orthotropic criterion can be done. This form does not satisfy the deviatoric property, but when the constraint of independence of the hydrostatic stress is consistently applied, it is easily reduced to the deviatoric form, known as Hill's criterion, with only 6 independent moduli of orthotropy (cf. Hill [8]).

Classical Hill's criterion, despite obvious advantages and common technical applications, is limited, however, by some constraints of applicability, which are the main topic of the present paper.

The first limitation of the applicability range of the classical Hill's criterion is established through the inequality bounding values of the engineering orthotropy constants k_x, k_y and k_z in order to avoid ellipticity loss of the limit surface in the stress space (see, e.g., Ottosen and Ristinmaa [27]). Such limit bounds put upon the orthotropy yield limits usually hold in case if the degree of material orthotropy is moderate. It is shown, for example if the material ensures the transverse isotropy symmetry (z orthotropy axis), that the orthotropy degree bounded by the inequality $k_z/k_x > 0.5$ guarantees ellipticity of the limit surface to be saved. However, if the orthotropy bound is violated, the Hill criterion becomes useless, when the degeneration of the cylindrical (elliptic) surface into two concave hyperbolic cylinders occurs, what is not admissible in light of Drucker's or Sylvester's stability postulates. It will be shown that in a case of high orthotropy degree (observed for the majority of the long-fiber- reinforced composites, for instant: boron/Al, SiC/Ti, glass/ epoxy, graphite/epoxy, etc., e.g., Herakovich and Aboudi [7], Sun and Vaidya [33], and others), a concept other than Hill's is proposed. This other approach requires abandonment of the deviatoric form Hill's criterion and suggests the formulation of a new limit criterion based on the 9-parameter von Mises condition, but enhanced by the Hu–Marin's-type auxiliary conditions (cf. Hu and Marin [12], Skrzypek and Ganczarski [31]). For calibration of this criterion, despite classical uniaxial tensile and shear tests, the biaxial constraints have to be postulated. It will be demonstrated that, even in a case of arbitrarily strong orthotropy (for instance, $k_{\max}/k_{\min} \approx 9$, if brass Ł62 is tested), the property of ellipticity is saved.

The second limitation of the applicability range of Hill's criterion arises when the description of the orthotropic material that exhibits transverse isotropy property is considered. It will be shown that, if the reduction of the 6-parameter Hill's criterion to the transverse isotropy symmetry is performed, the 4-parameter form that satisfies the tetragonal symmetry class is furnished (cf., e.g., Sun and Vaidya [33], who considered two symmetry classes of the tetragonal or hexagonal configurations of long-fiber-reinforced elastic composites). In such a case, moduli k_x, k_y, k_z and k_{xy} are considered as independent, which makes it impossible to obtain transition of the classical Hill's criterion to the Huber–von Mises isotropic yield condition in the plane of transverse isotropy. To avoid this discrepancy, instead of the deviatoric transversely isotropic Hill's criterion exhibiting tetragonal symmetry, the new Hu–Marin's-based transversely isotropic approach exhibiting hexagonal symmetry is proposed. It enables us to achieve coincidence with the Huber–von Mises condition in the transverse isotropy plane, preserving cylindricity regardless of the magnitude of the orthotropy degree.

Finally, it will be demonstrated that for some composite materials it is necessary to further modify the 3-parameter Hu–Marin's-type criterion to the 4-parameter hybrid-type criterion taking advantage of the bulge test that differs essentially from both the Hu–Marin's hexagonal-type criterion and the Huber–von Mises criterion in the isotropy plane. Bulge tests have been performed and described, e.g., by Jackson et al [14] with equipment used by Lankford et al [19]. This new criterion is capable of properly describing the SiC/Ti long-fiber-reinforced composite examined by Herakovich and Aboudi [7].

2 Von Mises deviatoric yield criterion

In a most general case of both elastic and plastic material anisotropy, extension of the isotropic plastic yield initiation criterion to the anisotropic flow by the use of common invariants of the stress tensor and of the structural tensors of plastic anisotropy (cf. Hill [8], Sayir [30], Betten [2], Życzkowski [37]) can be shown in a general fashion,

$$f^{\mathrm{p}} = f^{\mathrm{p}}\left(\Pi, \Pi_{ij}\sigma_{ij}, \sigma_{ij}\Pi_{ijkl}\sigma_{kl}, \sigma_{ij}\Pi_{ijklmn}\sigma_{kl}\sigma_{mn}, \ldots\right) = 0, \tag{1}$$

where Einstein's summation holds.

In such a case, initiation of plastic flow is governed by the structural tensors of plastic anisotropy of even-ranks: $\boldsymbol{\Pi}^{(0)} = \Pi, \boldsymbol{\Pi}^{(2)} = \Pi_{ij}, \boldsymbol{\Pi}^{(4)} = \Pi_{ijkl}, \boldsymbol{\Pi}^{(6)} = \Pi_{ijklmn}, \ldots$, etc., instead of the scalar constants k_i as it is known for isotropic materials. Equation (1) owns a general representation, but its practical identification is limited by a large number of required material tests, and, additionally, because the components of the structural tensors are temperature dependent, which makes identification much more complicated (cf., e.g., Herakovich and Aboudi [7], Tamma and Avila [35]). Hence, a general form (1) is usually more specified and limited for engineering needs.

In a particular case, Goldenblat and Kopnov [6], and later Sayir [30] proposed a polynomial representation for Eq. (1), which controls initiation of anisotropic plastic flow or failure in a material by the tensorial polynomial anisotropic criterion

$$\left(\Pi_{ij}\sigma_{ij}\right)^{\alpha} + \left(\sigma_{ij}\Pi_{ijkl}\sigma_{kl}\right)^{\beta} + \left(\sigma_{ij}\Pi_{ijklmn}\sigma_{kl}\sigma_{mn}\right)^{\gamma} + \cdots - 1 = 0. \tag{2}$$

The even-rank structural anisotropy tensors $\Pi_{ij}, \Pi_{ijkl}, \Pi_{ijklmn}, \ldots$, in Eq. (2), are normalized by the common constant Π, $\alpha, \beta, \gamma \ldots$, etc., are arbitrary exponents of a polynomial representation. Assuming, further, $\alpha = 1, \beta = 1/2, \gamma = 1/3$, and limiting an infinite form (2) to the equation that contains only three common invariants of the stress and structural anisotropy tensors of appropriate ranks, we arrive at the simpler form, which satisfies the homogeneity of three polynomial components, known as the Goldenblat and Kopnov criterion (cf. Goldenblat and Kopnov [6]),

$$\Pi_{ij}\sigma_{ij} + \left(\sigma_{ij}\Pi_{ijkl}\sigma_{kl}\right)^{1/2} + \left(\sigma_{ij}\Pi_{ijklmn}\sigma_{kl}\sigma_{mn}\right)^{1/3} - 1 = 0. \tag{3}$$

Equation (3), when limited only to three common invariants of the stress tensor σ and structural anisotropy tensors of even orders: 2nd Π_{ij}, 4th Π_{ijkl} and 6th Π_{ijklmn}, is not the most general one, in the meaning of the representation theorems, which determine the most general irreducible representation of the scalar and tensor functions that satisfy the invariance with respect to change in coordinates and material symmetry properties (cf., e.g., Spencer [32], Rymarz [29], Rogers [28]). However, second-, fourth- and sixth-order structural anisotropy tensors, which are used in (3), are found satisfactory for describing fundamental transformation modes of limit surfaces caused by plastic or damage processes, namely isotropic change of size, kinematic translation and rotation, as well as surface distortion (cf. Kowalsky et al. [17], Betten [2]).

Goldenblat and Kopnov's Eq. (3) is quite general, too, because of a large number of material tests required for its calibration. Hence, for some engineering applications, its further reduction is performed. It is based on the behavior of two basic classes of structural materials depending on the dominant dissipative phenomena responsible for the termination of pure elastic behavior: ductile or brittle. It is observed that plastic yield initiation occurs in the majority of ductile metallic materials, where the hydrostatic stress does not affect plastic yield initiation. In such materials, the strength differential effect is usually negligible, because limit stresses in tension and compression are comparable, $k_{\mathrm{t}} \approx k_{\mathrm{c}}$. By contrast, in most of brittle materials (concrete, rock-like, ceramics), the hydrostatic stress does have the essential effect on the initiation of failure or damage. That is why in brittle materials also the strength differential effect may be important, $k_t \neq k_c$. Following the above reasoning, the first linear term $\Pi_{ij}\sigma_{ij}$ in the Goldenblat–Kopnov's Eq. (3) when applied to initiation of plastic yield in ductile materials can be ignored, whereas when used for the appearance of initial failure in brittle material, the stress sign may play an essential role, such that the linear invariant cannot be omitted. Moreover, the third term in Eq. (3), $\sigma_{ij}\Pi_{ijklmn}\sigma_{kl}\sigma_{mn}$, which is dependent of the third stress invariant, is basically responsible for curvature change in limit surface (distortion), which occurs mainly due to material hardening effect, such that its omitting for the initiation of both dissipative phenomena is usually acceptable.

Consequently, limiting ourselves to plastic yield initiation in ductile materials, a consecutive reduction of the Goldenblat and Kopnov criterion (3) to the form dependent only on the fourth-rank common invariant $\sigma_{ij}\Pi_{ijkl}\sigma_{kl}$ holds, as it was proposed by the von Mises criterion for anisotropic yield initiation (cf. von Mises [24]),

$$\sigma_{ij}\Pi_{ijkl}\sigma_{kl} - 1 = 0. \tag{4}$$

The structural fourth-rank tensor of plastic anisotropy in Eq. (4) must be symmetric: $\Pi_{ijkl} = \Pi_{klij} = \Pi_{jikl} = \Pi_{ijlk}$, if stress tensor symmetry is assumed. Hence, in case if no other symmetry properties are implied, the von Mises plastic anisotropy tensor is defined by 21 modules. However, due to its invariance of the tensorial transformation rule, the number of independent anisotropy modules is reduced to 18. Finally, the general von Mises anisotropic criterion can be furnished as

$$\begin{aligned}
&\Pi_{xxxx}\sigma_x^2 + \Pi_{yyyy}\sigma_y^2 + \Pi_{zzzz}\sigma_z^2 + 2\Pi_{xxyy}\sigma_x\sigma_y + 2\Pi_{yyzz}\sigma_y\sigma_z \\
&\quad + 2\Pi_{zzxx}\sigma_z\sigma_x + 4\Pi_{xxyz}\sigma_x\tau_{yz} + 4\Pi_{xxzx}\sigma_x\tau_{zx} + 4\Pi_{xxxy}\sigma_x\tau_{xy} \\
&\quad + 4\Pi_{yyyz}\sigma_y\tau_{yz} + 4\Pi_{yyzx}\sigma_y\tau_{zx} + 4\Pi_{yyxy}\sigma_y\tau_{xy} + 4\Pi_{zzyz}\sigma_z\tau_{yz} \\
&\quad + 4\Pi_{zzzx}\sigma_z\tau_{zx} + 4\Pi_{zzxy}\sigma_z\tau_{xy} + 8\Pi_{xyyz}\tau_{xy}\tau_{yz} + 8\Pi_{yzzx}\tau_{yz}\tau_{zx} \\
&\quad + 8\Pi_{zxxy}\tau_{zx}\tau_{xy} + 4\Pi_{yzyz}\tau_{yz}^2 + 4\Pi_{zxzx}\tau_{zx}^2 + 4\Pi_{xyxy}\tau_{xy}^2 - 1 = 0
\end{aligned} \tag{5}$$

where Π_{ijkl} denote 21 tensorial coordinates of the von Mises plastic anisotropy tensor.

When the more convenient Voigt's vector-matrix notation is used, the form equivalent to (4) is obtained,

$$\{\sigma\}^{\mathrm{T}}\,[\mathbb{II}]\,\{\sigma\} - 1 = 0. \tag{6}$$

The von Mises 6×6 matrix of plastic anisotropy, being a symmetric and fully populated matrix representation of the fourth-rank anisotropy tensor Π_{ijkl}, is furnished as follows:

$$[\mathbb{II}] = \left[\begin{array}{ccc|ccc}
\Pi_{11} & \Pi_{12} & \Pi_{13} & \Pi_{14} & \Pi_{15} & \Pi_{16} \\
 & \Pi_{22} & \Pi_{23} & \Pi_{24} & \Pi_{25} & \Pi_{26} \\
 & & \Pi_{33} & \Pi_{34} & \Pi_{35} & \Pi_{36} \\
\hline
 & & & \Pi_{44} & \Pi_{45} & \Pi_{46} \\
 & & & & \Pi_{55} & \Pi_{56} \\
 & & & & & \Pi_{66}
\end{array}\right] \tag{7}$$

if engineering vectorial representation of the stress tensor is chosen as

$$\{\sigma\} = \{\sigma_1, \sigma_2, \sigma_3, \sigma_4, \sigma_5, \sigma_6\}^{\mathrm{T}} = \left\{\sigma_x, \sigma_y, \sigma_z, \tau_{yz}, \tau_{zx}, \tau_{xy}\right\}^{\mathrm{T}}. \tag{8}$$

When matrix coordinates Π_{ij} are consistently defined by the tensorial coordinates Π_{ijkl}, we arrive at the full 21-parameter equation equivalent to (5),

$$\begin{aligned}
&\Pi_{11}\sigma_x^2 + \Pi_{22}\sigma_y^2 + \Pi_{33}\sigma_z^2 + 2(\Pi_{12}\sigma_x\sigma_y + \Pi_{23}\sigma_y\sigma_z + \Pi_{31}\sigma_z\sigma_x \\
&\quad + \Pi_{14}\sigma_x\tau_{yz} + \Pi_{15}\sigma_x\tau_{zx} + \Pi_{16}\sigma_x\tau_{xy} + \Pi_{24}\sigma_y\tau_{yz} + \Pi_{25}\sigma_y\tau_{zx} \\
&\quad + \Pi_{26}\sigma_y\tau_{xy} + \Pi_{34}\sigma_z\tau_{yz} + \Pi_{35}\sigma_z\tau_{zx} + \Pi_{36}\sigma_z\tau_{xy} + \Pi_{45}\tau_{yz}\tau_{zx} \\
&\quad + \Pi_{46}\tau_{xy}\tau_{yz} + \Pi_{56}\tau_{zx}\tau_{xy}) + \Pi_{44}\tau_{yz}^2 + \Pi_{55}\tau_{zx}^2 + \Pi_{66}\tau_{xy}^2 = 1.
\end{aligned} \tag{9}$$

The von Mises equation in tensorial representation (4) or the vector-matrix notation (6) depends on both the stress deviator $\boldsymbol{s}$ and the stress axiator $\sigma_{\mathrm{h}}\mathbf{1}$, when stress decomposition $\sigma = \boldsymbol{s} + \sigma_{\mathrm{h}}\mathbf{1}$ is applied, namely

$$\{\boldsymbol{s}\}^{\mathrm{T}}\,[\mathbb{II}]\,\{\boldsymbol{s}\} + \left(2\,\{\boldsymbol{s}\}^{\mathrm{T}} + \sigma_{\mathrm{h}}\,\{\mathbf{1}\}^{\mathrm{T}}\right)\left([\mathbb{II}]\,\{\mathbf{1}\}\,\sigma_{\mathrm{h}}\right) - 1 = 0. \tag{10}$$

The tensorial von Mises Eq. (10) can further be reduced to the deviatoric form independent of the hydrostatic pressure as follows:

$$\{\boldsymbol{s}\}^{\mathrm{T}}\left[\mathbb{II}\right]\{\boldsymbol{s}\}-1=0 \tag{11}$$

if the constraint

$$\left[\mathbb{II}\right]\{\mathbf{1}\}=0 \tag{12}$$

is consistently applied. The constraint (12) guarantees the deviatoric von Mises Eq. (11) be represented in the reduced 6-dimensional stress space by a cylindrical surface defined by 15 independent anisotropy modules, when 6 constraints are satisfied,

$$\begin{array}{ll}\Pi_{11}+\Pi_{12}+\Pi_{13}=0, & \Pi_{14}+\Pi_{24}+\Pi_{34}=0,\\ \Pi_{12}+\Pi_{22}+\Pi_{23}=0, & \Pi_{15}+\Pi_{25}+\Pi_{35}=0,\\ \Pi_{13}+\Pi_{23}+\Pi_{33}=0, & \Pi_{16}+\Pi_{26}+\Pi_{36}=0.\end{array} \tag{13}$$

However, the final matrix representation (7) with (13) employed depends on a choice of independent elements. Two of such representations are of special importance.

In the first case, the elements of matrix (7) considered as independent are the following: $\Pi_{12}, \Pi_{13}, \Pi_{23}$; $\Pi_{15}, \Pi_{16}, \Pi_{24}, \Pi_{26}, \Pi_{34}, \Pi_{35}$ and $\Pi_{44}, \Pi_{55}, \Pi_{66}$; $\Pi_{45}, \Pi_{46}, \Pi_{56}$, such that the following first representation for the deviatoric von Mises matrix is furnished:

$$\left[{}_{\mathrm{dev}}\mathbb{II}\right]=\left[\begin{array}{ccc|ccc}-\Pi_{12}-\Pi_{13} & \Pi_{12} & \Pi_{13} & -\Pi_{24}-\Pi_{34} & \Pi_{15} & \Pi_{16}\\ & -\Pi_{12}-\Pi_{23} & \Pi_{23} & \Pi_{24} & -\Pi_{15}-\Pi_{35} & \Pi_{26}\\ & & -\Pi_{13}-\Pi_{23} & \Pi_{34} & \Pi_{35} & -\Pi_{16}-\Pi_{26}\\ \hline & & & \Pi_{44} & \Pi_{45} & \Pi_{46}\\ & & & & \Pi_{55} & \Pi_{56}\\ & & & & & \Pi_{66}\end{array}\right] \tag{14}$$

if constraints (13) are applied as follows:

$$\begin{array}{ll}\Pi_{11}=-\Pi_{12}-\Pi_{13}, & \Pi_{14}=-\Pi_{24}-\Pi_{34},\\ \Pi_{22}=-\Pi_{12}-\Pi_{23}, & \Pi_{25}=-\Pi_{15}-\Pi_{35},\\ \Pi_{33}=-\Pi_{13}-\Pi_{23}, & \Pi_{36}=-\Pi_{16}-\Pi_{26}.\end{array} \tag{15}$$

In the second case, the elements of matrix (7) chosen as independent are the following: $\Pi_{11}, \Pi_{22}, \Pi_{33}$; $\Pi_{15}, \Pi_{16}, \Pi_{24}, \Pi_{26}, \Pi_{34}, \Pi_{35}$ and $\Pi_{44}, \Pi_{55}, \Pi_{66}$; $\Pi_{45}, \Pi_{46}, \Pi_{56}$; hence, we arrive at the second representation of the deviatoric von Mises matrix as follows:

$$\left[{}_{\mathrm{dev}}\mathbb{II}\right]=\left[\begin{array}{ccc|ccc}\Pi_{11} & \frac{1}{2}(\Pi_{33}-\Pi_{11}-\Pi_{22}) & \frac{1}{2}(\Pi_{22}-\Pi_{11}-\Pi_{33}) & -\Pi_{24}-\Pi_{34} & \Pi_{15} & \Pi_{16}\\ & \Pi_{22} & \frac{1}{2}(\Pi_{11}-\Pi_{22}-\Pi_{33}) & \Pi_{24} & -\Pi_{15}-\Pi_{35} & \Pi_{26}\\ & & \Pi_{33} & \Pi_{34} & \Pi_{35} & -\Pi_{16}-\Pi_{26}\\ \hline & & & \Pi_{44} & \Pi_{45} & \Pi_{46}\\ & & & & \Pi_{55} & \Pi_{56}\\ & & & & & \Pi_{66}\end{array}\right] \tag{16}$$

if, instead of (15), an other substitution is used:

$$\begin{array}{ll}\Pi_{12}=\frac{1}{2}(\Pi_{33}-\Pi_{11}-\Pi_{22}), & \Pi_{14}=-\Pi_{24}-\Pi_{34},\\ \Pi_{13}=\frac{1}{2}(\Pi_{22}-\Pi_{11}-\Pi_{33}), & \Pi_{25}=-\Pi_{15}-\Pi_{35},\\ \Pi_{23}=\frac{1}{2}(\Pi_{11}-\Pi_{22}-\Pi_{33}), & \Pi_{36}=-\Pi_{16}-\Pi_{26}.\end{array} \tag{17}$$

A choice of 15 elements in the von Mises matrix (7) considered as independent is not a unique procedure and can result in the different deviatoric von Mises equation forms. In particular, when a more convenient representation (14) is substituted for $\left[{}_{\mathrm{dew}}\mathbb{II}\right]$ in (11), we arrive at the following von Mises equation expressed in the deviatoric stress space:

$$\begin{aligned}&-\Pi_{12}\left(s_x-s_y\right)^2-\Pi_{13}\left(s_x-s_z\right)^2-\Pi_{23}\left(s_y-s_z\right)^2+\underline{2\left\{\tau_{yz}\left[\Pi_{24}\left(s_y-s_x\right)\right.\right.}\\&\underline{\left.+\Pi_{34}\left(s_z-s_x\right)\right]+\tau_{zx}\left[\Pi_{15}\left(s_x-s_y\right)+\Pi_{35}\left(s_z-s_y\right)\right]+\tau_{xy}\left[\Pi_{16}\left(s_x-s_z\right)\right.}\\&\underline{\left.\left.+\Pi_{26}\left(s_y-s_z\right)\right]+\Pi_{45}\tau_{yz}\tau_{zx}+\Pi_{46}\tau_{xy}\tau_{yz}+\Pi_{56}\tau_{zx}\tau_{xy}\right\}}+\Pi_{44}\tau_{yz}^2\\&+\Pi_{55}\tau_{zx}^2+\Pi_{66}\tau_{xy}^2=1.\end{aligned} \tag{18}$$

It is visible that the above equation owns the clear deviatoric structure; hence, when the tensorial stress space is used instead of the deviatoric one, the analogous equivalent to (18) representation of the deviatoric von Mises equation is also true in terms of stress components (cf. Szczepiński [34]),

$$\begin{aligned} &-\Pi_{12}\left(\sigma_x-\sigma_y\right)^2-\Pi_{13}\left(\sigma_x-\sigma_z\right)^2-\Pi_{23}\left(\sigma_y-\sigma_z\right)^2 \\ &\quad+2\underline{\left\{\tau_{yz}\left[\Pi_{24}\left(\sigma_y-\sigma_x\right)+\Pi_{34}\left(\sigma_z-\sigma_x\right)\right]+\tau_{zx}\left[\Pi_{15}\left(\sigma_x-\sigma_y\right)\right.\right.} \\ &\quad+\underline{\left.\Pi_{35}\left(\sigma_z-\sigma_y\right)\right]+\tau_{xy}\left[\Pi_{16}\left(\sigma_x-\sigma_z\right)+\Pi_{26}\left(\sigma_y-\sigma_z\right)\right]+\Pi_{45}\tau_{yz}\tau_{zx}} \\ &\quad+\underline{\left.\Pi_{46}\tau_{xy}\tau_{yz}+\Pi_{56}\tau_{zx}\tau_{xy}\right\}}+\Pi_{44}\tau_{yz}^2+\Pi_{55}\tau_{zx}^2+\Pi_{66}\tau_{xy}^2=1. \end{aligned} \tag{19}$$

Note that Eqs. (18) and (19) are defined by 15 elements Π_{ij}. However, the underlined terms are sensitive to a change in sign of shear stresses, e.g., $\tau_{yz}(\sigma_y-\sigma_x)$, etc., which is physically questionable and, finally, such terms are consequently omitted is some cases (cf., e.g., Malinin and Rżysko [22]). Nevertheless, the full representation (19) might occur useful when the von Mises–Tsai–Wu extension to the brittle-like material is sought for (cf. Tsai and Wu [36]).

3 Advantages of classical Hill's criterion

The general form of the 21-parameter anisotropic von Mises criterion (9) does involve none of material symmetry properties. In a particular case if plastic orthotropy is assumed for the initial yield criterion (6), when represented in principal orthotropy axes, the 9-parameter orthotropic von Mises matrix (17) takes the form

$$[{}_{\text{ortho}}\mathbb{\Pi}]=\left[\begin{array}{ccc|ccc} \Pi_{11} & \Pi_{12} & \Pi_{13} & 0 & 0 & 0 \\ & \Pi_{22} & \Pi_{23} & 0 & 0 & 0 \\ & & \Pi_{33} & 0 & 0 & 0 \\ \hline & & & \Pi_{44} & 0 & 0 \\ & & & & \Pi_{55} & 0 \\ & & & & & \Pi_{66} \end{array}\right]. \tag{20}$$

In such a case, the general anisotropic von Mises Eq. (9) is reduced to the narrower 9-parameter orthotropic von Mises criterion,

$$\begin{aligned} &\Pi_{11}\sigma_x^2+\Pi_{22}\sigma_y^2+\Pi_{33}\sigma_z^2+2(\Pi_{12}\sigma_x\sigma_y+\Pi_{23}\sigma_y\sigma_z+\Pi_{31}\sigma_z\sigma_x) \\ &\quad+\Pi_{44}\tau_{yz}^2+\Pi_{55}\tau_{zx}^2+\Pi_{66}\tau_{xy}^2=1. \end{aligned} \tag{21}$$

When the Voigt notation is used, the 9-parameter orthotropic von Mises criterion takes the form

$$\{\sigma\}^{\mathrm{T}}[{}_{\text{ortho}}\mathbb{\Pi}]\{\sigma\}-1=0 \tag{22}$$

that involves definition (20). Note that Eq. (22) is, in general, dependent on the hydrostatic stress. If independence of the hydrostatic stress σ_{h} is assumed,

$$[{}_{\text{ortho}}\mathbb{\Pi}]\{\mathbf{1}\}=0, \tag{23}$$

three constraints must additionally be fulfilled (see the first three equations of (13)), which results in the reduced 6-parameter deviatoric representation of the von Mises equation, known as Hill's criterion,

$$\{\boldsymbol{s}\}^{\mathrm{T}}[\mathbb{\Pi}^{\mathrm{H}}]\{\boldsymbol{s}\}-1=0. \tag{24}$$

Applying the general representations for deviatoric von Mises matrices (14) or (16) to the considered orthotropic symmetry case, we arrive at the following Hill's matrices:

$$[\mathbb{\Pi}^{\mathrm{H}}]=\left[\begin{array}{ccc|ccc} -\Pi_{12}-\Pi_{13} & \Pi_{12} & \Pi_{13} & & & \\ & -\Pi_{12}-\Pi_{23} & \Pi_{23} & & & \\ & & -\Pi_{13}-\Pi_{23} & & & \\ \hline & & & \Pi_{44} & & \\ & & & & \Pi_{55} & \\ & & & & & \Pi_{66} \end{array}\right] \tag{25}$$

or

$$[\mathbb{I}\mathbb{I}^{\mathrm{H}}] = \left[\begin{array}{ccc|ccc} \Pi_{11} & \frac{1}{2}(\Pi_{33}-\Pi_{11}-\Pi_{22}) & \frac{1}{2}(\Pi_{22}-\Pi_{11}-\Pi_{33}) & & & \\ & \Pi_{22} & \frac{1}{2}(\Pi_{11}-\Pi_{22}-\Pi_{33}) & & & \\ & & \Pi_{33} & & & \\ \hline & & & \Pi_{44} & & \\ & & & & \Pi_{55} & \\ & & & & & \Pi_{66} \end{array}\right]. \tag{26}$$

When the engineering notation is used, corresponding representations of the Hill's criterion are

$$-\left[\Pi_{23}\left(\sigma_y-\sigma_z\right)^2+\Pi_{13}\left(\sigma_z-\sigma_x\right)^2+\Pi_{12}\left(\sigma_x-\sigma_y\right)^2\right] +\Pi_{44}\tau_{yz}^2+\Pi_{55}\tau_{zx}^2+\Pi_{66}\tau_{xy}^2=1 \tag{27}$$

or

$$\Pi_{11}\sigma_x^2+\Pi_{22}\sigma_y^2+\Pi_{33}\sigma_z^2+(\Pi_{33}-\Pi_{11}-\Pi_{22})\,\sigma_x\sigma_y +(\Pi_{22}-\Pi_{11}-\Pi_{33})\,\sigma_x\sigma_z+(\Pi_{11}-\Pi_{22}-\Pi_{33})\,\sigma_y\sigma_z +\Pi_{44}\tau_{yz}^2+\Pi_{55}\tau_{zx}^2+\Pi_{66}\tau_{xy}^2=1. \tag{28}$$

Both representations (27) and (28) describe the same Hill's limit surface, but applying two different choices of six independent elements of the Hill matrices (25) or (26). In order to calibrate Hill's criterion in the form (27) or (28), three tests of uniaxial tension $\sigma_x = k_x, \sigma_y = k_y, \sigma_z = k_z$ and three tests of pure shear $\tau_{xy} = k_{xy}, \tau_{yz} = k_{yz}, \tau_{zx} = k_{zx}$, in directions and planes of material orthotropy (Fig. 1), must be performed.

These tests allow to express 6 modules of material orthotropy in Eqs. (27) and (28) in terms of 3 independent plastic tension limits k_x, k_y, k_z (in directions of orthotropy), and 3 independent plastic shear limits k_{yz}, k_{zx}, k_{xy} (in planes of material orthotropy). Hence,

$$\begin{aligned} -\Pi_{23} &= \tfrac{1}{2}\left(\tfrac{1}{k_y^2}+\tfrac{1}{k_z^2}-\tfrac{1}{k_x^2}\right), \quad \Pi_{44}=\tfrac{1}{k_{yz}^2}, \\ -\Pi_{13} &= \tfrac{1}{2}\left(\tfrac{1}{k_z^2}+\tfrac{1}{k_x^2}-\tfrac{1}{k_y^2}\right), \quad \Pi_{55}=\tfrac{1}{k_{zx}^2}, \\ -\Pi_{12} &= \tfrac{1}{2}\left(\tfrac{1}{k_x^2}+\tfrac{1}{k_y^2}-\tfrac{1}{k_z^2}\right), \quad \Pi_{66}=\tfrac{1}{k_{xy}^2} \end{aligned} \tag{29}$$

such that orthotropic Hill's conditions equivalent to (27) or (28) can be furnished,

$$\frac{1}{2}\left(\frac{1}{k_y^2}+\frac{1}{k_z^2}-\frac{1}{k_x^2}\right)\left(\sigma_y-\sigma_z\right)^2+\frac{1}{2}\left(\frac{1}{k_z^2}+\frac{1}{k_x^2}-\frac{1}{k_y^2}\right)\left(\sigma_z-\sigma_x\right)^2 +\frac{1}{2}\left(\frac{1}{k_x^2}+\frac{1}{k_y^2}-\frac{1}{k_z^2}\right)\left(\sigma_x-\sigma_y\right)^2+\left(\frac{\tau_{yz}}{k_{yz}}\right)^2+\left(\frac{\tau_{zx}}{k_{zx}}\right)^2+\left(\frac{\tau_{xy}}{k_{xy}}\right)^2=1 \tag{30}$$

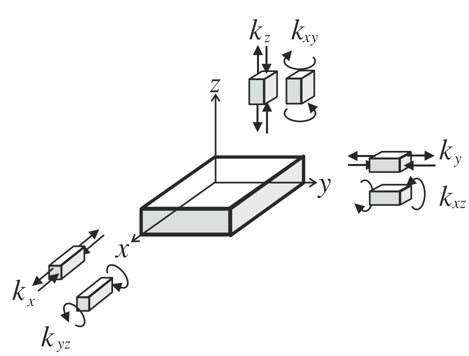


Fig. 1 Six tests for Hill's criterion calibration

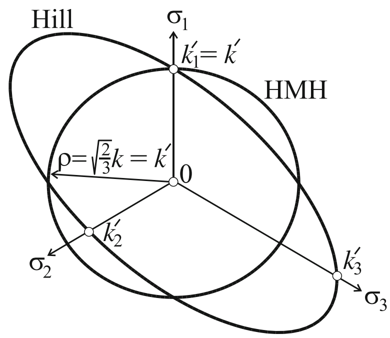


Fig. 2 Comparison of the Huber–von Mises and the Hill criteria in deviatoric plane applying the Haigh–Westergaard coordinates $\rho(\theta)$ ($k_1 = k$, $k_2 = 0.8k$, $k_3 = 1.5k$)

or

$$\left(\frac{\sigma_x}{k_x}\right)^2 + \left(\frac{\sigma_y}{k_y}\right)^2 + \left(\frac{\sigma_z}{k_z}\right)^2 - \left(\frac{1}{k_x^2} + \frac{1}{k_y^2} - \frac{1}{k_z^2}\right)\sigma_x\sigma_y - \left(\frac{1}{k_y^2} + \frac{1}{k_z^2} - \frac{1}{k_x^2}\right)\sigma_y\sigma_z - \left(\frac{1}{k_z^2} + \frac{1}{k_x^2} - \frac{1}{k_y^2}\right)\sigma_z\sigma_x + \left(\frac{\tau_{yz}}{k_{yz}}\right)^2 + \left(\frac{\tau_{zx}}{k_{zx}}\right)^2 + \left(\frac{\tau_{xy}}{k_{xy}}\right)^2 = 1. \tag{31}$$

Note that under a particular plane stress condition, e.g., in the x, y plane, when $\sigma_z = \tau_{zx} = \tau_{yz} = 0$, both formulas (30) and (31) reduce to the 4-parameter orthotropic Hill's condition,

$$\frac{\sigma_x^2}{k_x^2} + \frac{\sigma_y^2}{k_y^2} - \left(\frac{1}{k_x^2} + \frac{1}{k_y^2} - \frac{1}{k_z^2}\right)\sigma_x\sigma_y + \frac{\tau_{xy}^2}{k_{xy}^2} = 1, \tag{32}$$

where initiation of plastic flow in the x, y plane is controlled not only by the in-plane limits k_x, k_y and k_{xy}, but also by the out-of-plane limit k_z, which may physically be considered as doubtful. Let us mention also that in the space of 3 principal stresses, due to the deviatoric property, Hill's criterion (30) or (31) represents a cylindrical surface with elliptic identical cross sections, the axis of which coincides with the hydrostatic axis. To illustrate this property, it is convenient to use the Haigh–Westergaard coordinates (cf. Ganczarski and Lenczowski [4])

$$\begin{Bmatrix} \sigma_1 \\ \sigma_2 \\ \sigma_3 \end{Bmatrix} = \frac{\xi}{\sqrt{3}} \begin{Bmatrix} 1 \\ 1 \\ 1 \end{Bmatrix} + \sqrt{\frac{2}{3}}\rho(\theta) \begin{Bmatrix} \cos\theta \\ \cos(\theta - \frac{2\pi}{3}) \\ \cos(\theta + \frac{2\pi}{3}) \end{Bmatrix} \tag{33}$$

to obtain Hill's criterion in the reduced three-dimensional space of principal stresses,

$$\rho(\theta) = \left[\frac{2}{\left(\frac{1}{k_2^2} + \frac{1}{k_3^2} - \frac{1}{k_1^2}\right)\sin^2\left(\theta + \frac{\pi}{3}\right) + \left(\frac{1}{k_3^2} + \frac{1}{k_1^2} - \frac{1}{k_2^2}\right)\sin^2\left(\theta - \frac{\pi}{3}\right) + \left(\frac{1}{k_1^2} + \frac{1}{k_2^2} - \frac{1}{k_3^2}\right)\sin^2\theta}\right]^{1/2}, \tag{34}$$

as shown in Fig. 2.

4 Hill's criterion versus Hu–Marin's concept in case of strong orthotropy

The applicability range of Hill's orthotropic criterion (30) or (31), to properly describe initiation of plastic yield in some engineering materials that exhibit strong orthotropy degree, is bounded by a possible ellipticity loss of the limit surface. To illustrate this restriction, we consider two types of true materials for which

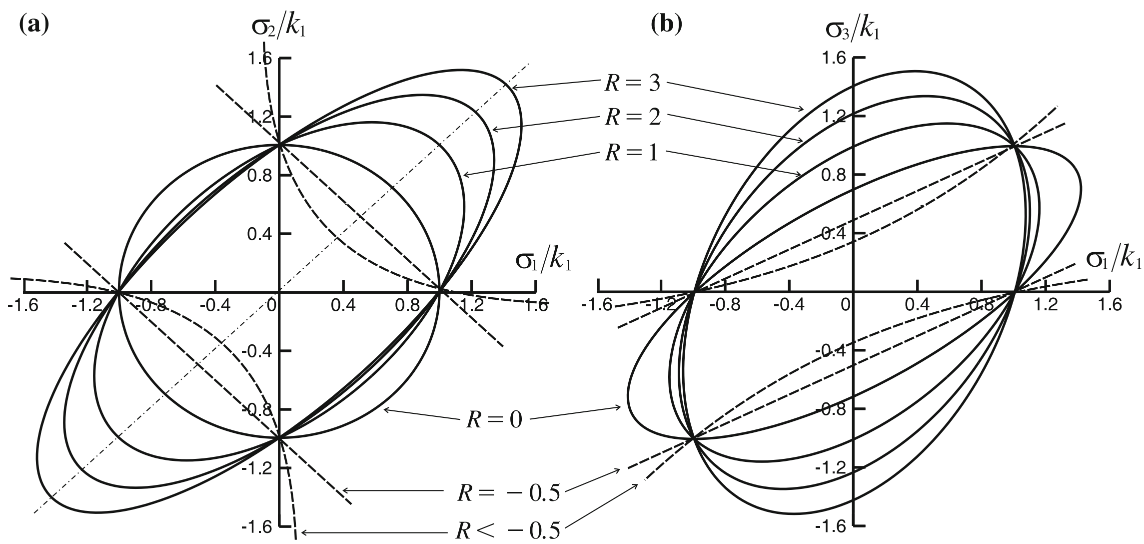


Fig. 3 Degeneration of the Hill limit surface with the magnitude of the Hosford and Backhofen parameter R: **a** transverse isotropy plane, **b** orthotropy plane

Table 1 Mechanical properties of orthotropic OTCz titanium alloy after Malinin and Rżysko [22]

Yield limits	k_1 (MPa)	k_2 (MPa)	k_3 (MPa)
	490	520	800

the classical Hill criterion occurs to be: either useful, if the material orthotropy degree is not very high such that the ellipticity property of the limit surface is preserved, or useless, if the orthotropy degree is as high as the described limit surface no longer holds the ellipticity requirement property. In other words, a physically inadmissible degeneration of the single convex and simply connected elliptical limit surface into two concave hyperbolic surfaces occurs.

The following inequality bounds the range of applicability for Hill's criterion (cf., e.g., Ottosen and Ristinmaa [27]);

$$\frac{2}{k_1^2 k_2^2} + \frac{2}{k_2^2 k_3^2} + \frac{2}{k_3^2 k_1^2} > \frac{1}{k_1^4} + \frac{1}{k_2^4} + \frac{1}{k_3^4}. \tag{35}$$

For simplicity, a coincidence of the principal stress axes with the material orthotropy axes is assumed in (35). In the narrower case of transverse isotropy $k_1 = k_2$, condition (35) reduces to the simple form

$$\frac{1}{k_3^2}\left(\frac{4}{k_1^2} - \frac{1}{k_3^2}\right) > 0. \tag{36}$$

Substitution of the dimensionless parameter $R = 2(k_3/k_1)^2 - 1$, after Hosford and Backhofen [11], leads to the simplified restriction

$$R > -0.5. \tag{37}$$

If the above inequalities (36), (37) do not hold, elliptic cross sections of the limit surface degenerate to two hyperbolic branches, and the lack of convexity occurs. To illustrate this limitation, the yield curves in two planes: the transverse isotropy (σ_1, σ_2) and the orthotropy plane (σ_1, σ_3) for various R-values are sketched in Fig. 3a, b, respectively. It is observed that when R, starting from $R = 3$, approaches the limit $R = -0.5$, the curves change from closed ellipses to two parallel lines, whereas for $R < -0.5$ concave hyperbolas appear.

As example of orthotropic engineering material for which the classical Hill's criterion can correctly predict the limit surface, consider first the OTCz Titanium Alloy, the mechanical orthotropic properties of which are given in Table 1 (cf. Malinin and Rżysko [22]).

Note that for the OTCz Titanium Alloy, yield limits in the plane of weak orthotropy 1,2 differ not so much, but the 3 axis is the dominant orthotropy axis. As consequence, in the plane of weak orthotropy 1,2, Hill's

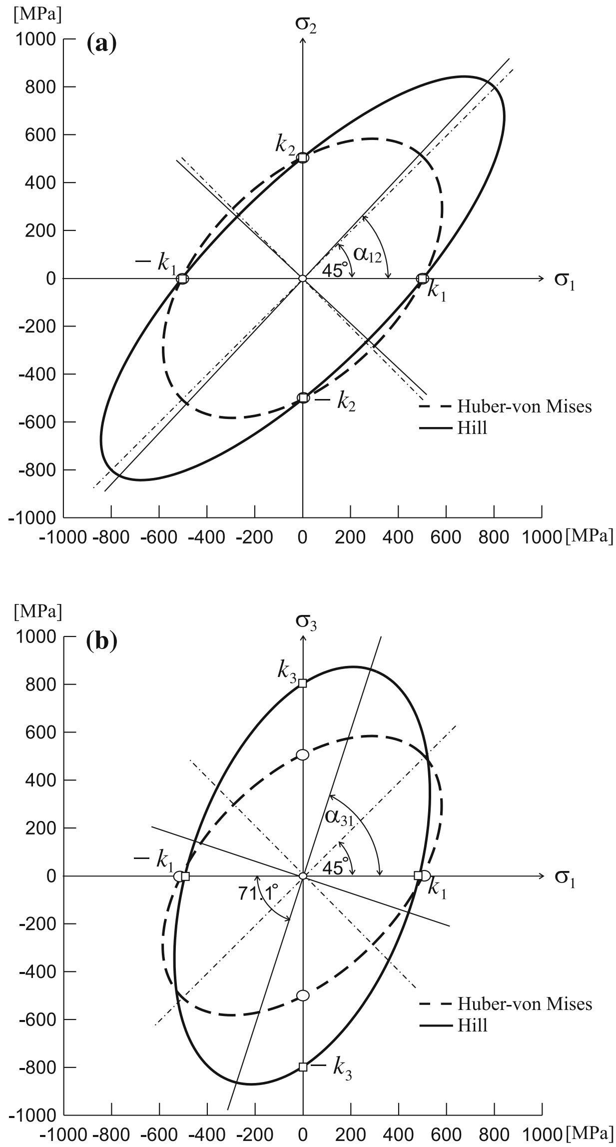


Fig. 4 Hill's deviatoric initial yield conditions versus Huber–von Mises isotropic approximation for the OTCz titanium alloy (cf. Table 1, $k_1 = 490\,\text{MPa}$, $k_2 = 520\,\text{MPa}$, $k_3 = 800\,\text{MPa}$: **a** the plane of "weak" orthotropy (σ_1, σ_2), **b** the plane of "strong" orthotropy (σ_1, σ_3)

ellipse is slightly rotated toward 2-axis ($\alpha_{12} \approx 45°$), in contrast to the plane of strong orthotropy 1,3, where the rotation of the Hill ellipse is significant ($\alpha_{13} \approx 71°$), as shown in Fig. 4a, b, respectively.

In the general case of strong orthotropy, when the ellipticity condition (35) does not hold, the deviatoric Hill's criterion (30) or (31) becomes useless. Hence, in order to describe a physically admissible close and convex limit surface, the more general 9-parameter orthotropic von Mises Eq. (20) must be recalled. In a narrower case of principal stress axes coinciding with principal orthotropy axes Eq. (20) reads as

$$\Pi_{11}\sigma_1^2 + \Pi_{22}\sigma_2^2 + \Pi_{33}\sigma_3^2 + 2(\Pi_{12}\sigma_1\sigma_2 + \Pi_{23}\sigma_2\sigma_3 + \Pi_{31}\sigma_3\sigma_1) = 1. \tag{38}$$

Condition (38) is defined by 6 material parameters only, because $\tau_{23} \equiv \tau_{31} \equiv \tau_{12} \equiv 0$; hence, its calibration requires 6 conditions:

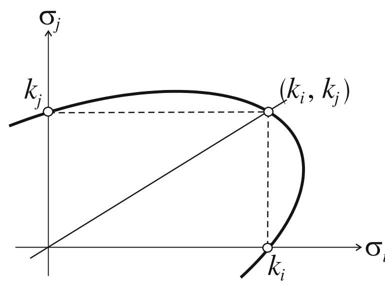


Fig. 5 Graphical illustration of auxiliary conditions of coincidence (40)

3 tests of uniaxial tension according orthotropy axes

$$\begin{aligned} \sigma_1 = k_1 \ \sigma_2 = 0 \ \sigma_3 = 0 &\longrightarrow \Pi_{11} = \tfrac{1}{k_1^2}, \\ \sigma_2 = k_2 \ \sigma_1 = 0 \ \sigma_3 = 0 &\longrightarrow \Pi_{22} = \tfrac{1}{k_2^2}, \\ \sigma_3 = k_3 \ \sigma_1 = 0 \ \sigma_2 = 0 &\longrightarrow \Pi_{33} = \tfrac{1}{k_3^2} \end{aligned} \tag{39}$$

and 3 auxiliary conditions of coincidence of pairs of yield limits k_i and k_j for biaxial stress states (k_i, k_j) cf. Fig. 5

$$\begin{aligned} \sigma_1 = k_1 \ \sigma_2 = k_2 \ \sigma_3 = 0 &\longrightarrow \Pi_{12} = -\tfrac{1}{2k_1k_2}, \\ \sigma_1 = k_1 \ \sigma_3 = k_3 \ \sigma_2 = 0 &\longrightarrow \Pi_{13} = -\tfrac{1}{2k_1k_3}, \\ \sigma_2 = k_2 \ \sigma_3 = k_3 \ \sigma_1 = 0 &\longrightarrow \Pi_{23} = -\tfrac{1}{2k_2k_3}. \end{aligned} \tag{40}$$

Calibration of (38), performed with conditions (39) and (40), leads to the three-axial extension of the Hu–Marin-type criterion (cf. Hu–Marin [12], Ganczarski and Skrzypek [5], Skrzypek and Ganczarski [31])

$$\left(\frac{\sigma_1}{k_1}\right)^2 - \frac{\sigma_1\sigma_2}{k_1k_2} + \left(\frac{\sigma_2}{k_2}\right)^2 - \frac{\sigma_2\sigma_3}{k_2k_3} + \left(\frac{\sigma_3}{k_3}\right)^2 - \frac{\sigma_1\sigma_3}{k_1k_3} = 1. \tag{41}$$

The enhanced Hu–Marin's criterion (41) is free from the Hill's deficiency even in case of arbitrarily strong orthotropy degree, since it never violates the Drucker stability postulate, which is not guaranteed by the Hill's-type equations. The Hu–Marin's-type Eq. (41) can easily be presented in the "pseudo–deviatoric" form

$$\frac{1}{2}\left(\frac{\sigma_1}{k_1} - \frac{\sigma_2}{k_2}\right)^2 + \frac{1}{2}\left(\frac{\sigma_2}{k_2} - \frac{\sigma_3}{k_3}\right)^2 + \frac{1}{2}\left(\frac{\sigma_3}{k_3} - \frac{\sigma_1}{k_1}\right)^2 = 1. \tag{42}$$

Three orthotropy limit yield points k_1, k_2 and k_3 establish the proportional stress/strength axis of the cylindrical Hu–Marin's surface. Note that this proportional stress/strength axis, which determines a position of the limit surface axis in the principal stress space, is different from the hydrostatic axis, but the condition of equal ratios $\sigma_i/k_i = \alpha$ holds at all points belonging to this axis.

In a particular case of plane stress state $\sigma_3 = 0$, the three-parameter enhanced Hu–Marin's Eq. (41) is reduced to a two-parameter one, as proposed by Hu–Marin [12],

$$\left(\frac{\sigma_1}{k_1}\right)^2 - \frac{\sigma_1\sigma_2}{k_1k_2} + \left(\frac{\sigma_2}{k_2}\right)^2 = 1. \tag{43}$$

Comparison of the 2-parameter Hu–Marin's plane stress Eq. (43) with 3-parameter plane stress Hill's Eq. (32) written for principal axes leads to the form

$$\left(\frac{\sigma_1}{k_1}\right)^2 - \left(\frac{1}{k_1^2} + \frac{1}{k_2^2} - \frac{1}{k_3^2}\right)\sigma_1\sigma_2 + \left(\frac{\sigma_2}{k_2}\right)^2 = 1, \tag{44}$$

which becomes identical to the Hu–Marin's Eq. (32) only when following constraint holds:

$$\frac{1}{k_3^2} = \frac{1}{k_1^2} + \frac{1}{k_2^2} - \frac{1}{k_1k_2}, \tag{45}$$

which is usually not true.

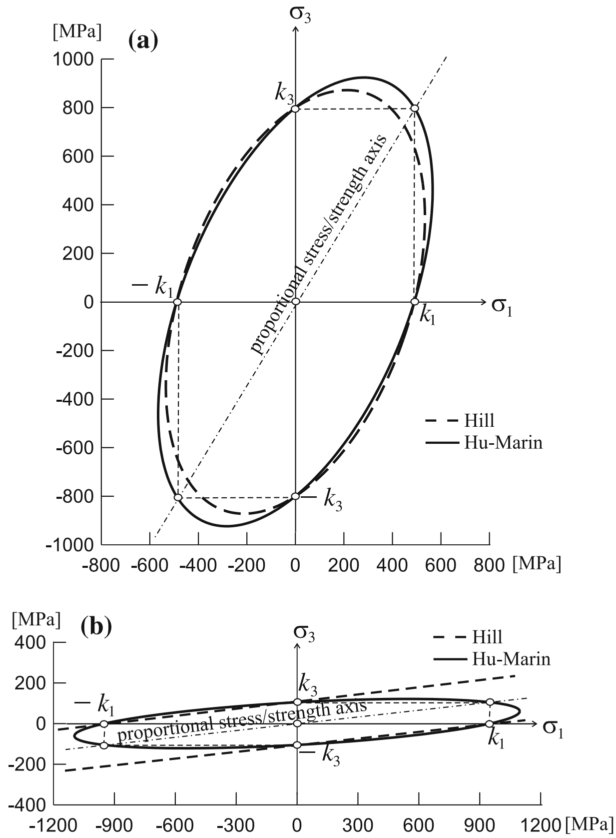


Fig. 6 Comparison of the Hill and the Hu–Marin plastic yield criteria for two orthotropic materials of different orthotropy degrees: **a** "weak" orthotropy in case of OTCz titanium alloy ($k_1 = 490$ MPa, $k_2 = 520$ MPa, $k_3 = 800$ MPa), **b** "strong" orthotropy in case of Ł62 brass ($k_1 = 105$ MPa, $k_2 = 120$ MPa, $k_3 = 950$ MPa)

In order to illustrate the suitability of the Hu–Marin's orthotropic Eq. (41), when compared to certain limitations of the Hill's deviatoric Eqs. (30) or (31), two engineering materials characterized by different degrees of orthotropy: OTCz Titanium Alloy ("weak" orthotropy) and Ł62 brass ("strong" orthotropy) are studied. The results are presented in Fig. 6a, b on the planes σ_1, σ_3 (for OTCz Titanium Alloy and for Ł62 brass). In case of "weak" orthotropy, both the Hill's and Hu–Marin's ellipses differ not so much, and both concepts are recommended (Fig. 6a). However, in case of "strong" orthotropy, when the inequality (35) is not satisfied, following the Hill's concept, two concave hyperbolic cylinders are formed by opening of the elliptic cylinder toward the proportional stress/strength axis (Fig. 6b). On the other hand, the Hu–Marin's surface saves the ellipticity property, regardless of the magnitude of the orthotropy degree considered. It is possible due to three additional constraints (40) satisfied for the pairs of orthotropy yield limits (k_1, k_2), (k_2, k_3) and (k_3, k_1). But, it should be pointed out that the Hu–Marin's cylindrical surface does not satisfy the condition of deviatoricity; hence, this condition is dependent on the hydrostatic stress.

5 Transverse isotropy case—tetragonal Hill's versus hexagonal von Mises' formulations

Classical orthotropic Hill's Eqs. (27) and (28), which are expressed in terms of six independent plastic yield limits $k_x, k_y, k_z, k_{yz}, k_{zx}$ and k_{xy}, are often too general for engineering applications. Orthotropic structural materials usually exhibit the transversely isotropic symmetry, basically due to either fabrication process or microstructure texture, as often observed in many long-parallel- fiber-reinforced composites. In particular, if in the elastic range the transversely isotropic symmetry group holds, it is expected that, also for the plastic yield initiation criterion, such narrower symmetry is true.

In what follows, a distinction between two symmetry classes of the transverse isotropy—tetragonal or hexagonal has to be done. Such distinction is known, e.g., from definitions of a representative unit cell used in homogenization methods for composite materials (cf., e.g., Berryman [1], Sun and Vaidya [33], etc.).

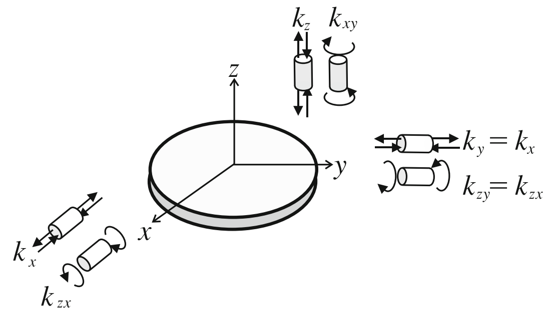


Fig. 7 Four independent tests for transversely isotropic Hill's criterion calibration

Assume that z-axis is the orthotropy axis, whereas x, y is the transversely isotropic plane. Applying Eqs. (27) or (28) with calibration (29) and $k_x = k_y \neq k_z$, $k_{zx} = k_{zy} \neq k_{xy}$, the number of independent limits in Hill's equation reduces to 4, for instance: two axial yield limits k_x and k_z, and two shear yield limits k_{zx} and k_{xy} (see Fig. 7). In this way, the following is furnished:

$$\begin{aligned} &-\Pi_{13} = -\Pi_{23} = \tfrac{1}{2k_z^2}, \quad \Pi_{44} = \Pi_{55} = \tfrac{1}{k_{zx}^2}, \\ &-\Pi_{12} = \tfrac{1}{2}\left(\tfrac{2}{k_x^2} - \tfrac{1}{k_z^2}\right), \quad \Pi_{66} = \tfrac{1}{2k_{xy}^2}. \end{aligned} \tag{46}$$

Substitution of (46) into (25) and (26) yields the transversely isotropic Hill's matrices

$$[{}_{\text{tris}}\mathbb{\Pi}^{\text{H}}] = \left[\begin{array}{ccc|ccc} -\Pi_{12} - \Pi_{13} & \Pi_{12} & \Pi_{13} & & & \\ & -\Pi_{12} - \Pi_{23} & \Pi_{13} & & & \\ & & -2\Pi_{13} & & & \\ \hline & & & \Pi_{44} & & \\ & & & & \Pi_{44} & \\ & & & & & \Pi_{66} \end{array}\right], \tag{47}$$

or

$$[{}_{\text{tris}}\mathbb{\Pi}^{\text{H}}] = \left[\begin{array}{ccc|ccc} \Pi_{11} & \tfrac{1}{2}(\Pi_{33} - 2\Pi_{11}) & -\tfrac{1}{2}\Pi_{33} & & & \\ & \Pi_{11} & -\tfrac{1}{2}\Pi_{33} & & & \\ & & \Pi_{33} & & & \\ \hline & & & \Pi_{44} & & \\ & & & & \Pi_{44} & \\ & & & & & \Pi_{66} \end{array}\right]. \tag{48}$$

The transversely isotropic 4-parameter Hill's criteria corresponding to orthotropic Hill's criteria (30) and (31) take the following representations:

$$\frac{(\sigma_y - \sigma_z)^2 + (\sigma_z - \sigma_x)^2}{2k_z^2} + \underline{\left(\frac{1}{k_x^2} - \frac{1}{2k_z^2}\right)}\left(\sigma_x - \sigma_y\right)^2 + \frac{\tau_{yz}^2 + \tau_{zx}^2}{k_{zx}^2} + \frac{\tau_{xy}^2}{k_{xy}^2} = 1 \tag{49}$$

or equivalently

$$\frac{\sigma_x^2 + \sigma_y^2}{k_x^2} + \frac{\sigma_z^2}{k_z^2} - \underline{\left(\frac{2}{k_x^2} - \frac{1}{k_z^2}\right)}\sigma_x\sigma_y - \frac{\sigma_y\sigma_z + \sigma_z\sigma_x}{k_z^2} + \frac{\tau_{yz}^2 + \tau_{zx}^2}{k_{zx}^2} + \frac{\tau_{xy}^2}{k_{xy}^2} = 1. \tag{50}$$

Both forms involve 4 plastic limits k_x, k_z, k_{zx} and k_{xy} as independent parameters. The underlined factor in (50) includes not only k_x but also k_z. The other explicitly deviatoric form (49) also exhibits a similar feature. Analogously to the general orthotropy (30) or (31), the plastic state in the transverse isotropy plane x, y is controlled not only by the tensile yield limit in this plane k_x, but also by the out-of-plane tensile yield limit k_z. Finally, the transversely isotropic Hill's criteria (49) and (50) have to be classified as the tetragonal symmetry formulation. In case of plane stress state in the transverse isotropy plane (x, y), $\sigma_x, \sigma_y, \tau_{xy} \neq 0$, Eqs. (49) and (50) reduce to (32) with $k_x = k_y$,

$$\frac{\sigma_x^2+\sigma_y^2}{k_x^2} - \left(\frac{2}{k_x^2} - \frac{1}{k_z^2}\right)\sigma_x\sigma_y + \frac{\tau_{xy}^2}{k_{xy}^2} = 1. \tag{51}$$

The above form simply means that the commonly used "transversely isotropic Hill's criterion" does not coincide in the "transverse isotropy plane" with the isotropic Huber–von Mises's equation

$$\frac{\sigma_x^2+\sigma_y^2}{k_x^2} - \frac{\sigma_x\sigma_y}{k_x^2} + 3\frac{\tau_{xy}^2}{k_{xy}^2} = 1. \tag{52}$$

In other words, when the new transversely isotropic yield criterion, that is free from inconsistencies between (51) and (52), is sought for, the material parameter preceding term $\sigma_x\sigma_y$ must be equal to $\Pi_{33} - 2\Pi_{11} = 1/k_x^2$ and not depend on k_z, and simultaneously, the material parameter $\Pi_{66} = 3/k_x^2$ must depend on k_x only.

In order to derive the transversely isotropic yield criterion that assures coincidence with the Huber–von Mises's criterion in the isotropy plane, the new transversely isotropic hexagonal Hu–Marin's criterion will be postulated. To obtain this criterion, the general orthotropic von Mises Eq. (21), which is not deviatoric, can be calibrated in the way analogous to that presented in (39) and (40). Namely, when the constraints of transverse isotropy are used, we apply three tests (tensile tests in the isotropy x and the orthotropy z axes and shear test in the orthotropy plane zx),

$$\begin{aligned}
&\sigma_x = k_x \quad \sigma_y = \sigma_z = \tau_{xy} = \tau_{yz} = \tau_{zx} = 0 \longrightarrow \Pi_{11} = \Pi_{22} = \tfrac{1}{k_x^2},\\
&\sigma_z = k_z \quad \sigma_x = \sigma_y = \tau_{xy} = \tau_{yz} = \tau_{zx} = 0 \longrightarrow \Pi_{33} = \tfrac{1}{k_z^2},\\
&\tau_{zx} = k_{zx}\ \sigma_x = \sigma_y = \sigma_z = \tau_{xy} = \tau_{yz} = 0 \ \longrightarrow \Pi_{44} = \Pi_{55} = \tfrac{1}{k_{zx}^2},
\end{aligned} \tag{53}$$

as well as three additional coincidence conditions between appropriate pairs of yield limits in biaxial states,

$$\sigma_x = k_x \quad \sigma_y = k_x \quad \sigma_z = \tau_{xy} = \tau_{yz} = \tau_{zx} = 0 \longrightarrow \Pi_{12} = -\tfrac{1}{2k_x^2}, \tag{51.1.1–4}$$

$$\sigma_x = k_x \quad \sigma_z = k_z \quad \sigma_y = \tau_{xy} = \tau_{yz} = \tau_{zx} = 0 \longrightarrow \Pi_{13} = -\tfrac{1}{2k_xk_z}, \tag{51.2.1–4}$$

$$\sigma_x = k_x \quad \tau_{xy} = \tfrac{k_x}{\sqrt{3}}\ \sigma_y = \sigma_z = \tau_{yz} = \tau_{zx} = 0 \ \longrightarrow \Pi_{66} = \tfrac{3}{k_x^2}. \tag{51.3.1–4}$$

Introduction of (53) and (54) to von Mises' Eq. (21) leads to the hexagonal Hu–Marin's criterion as follows:

$$\frac{\sigma_x^2+\sigma_y^2}{k_x^2} - \underline{\frac{\sigma_x\sigma_y}{k_x^2}} + \frac{\sigma_z^2}{k_z^2} - \frac{\sigma_y\sigma_z+\sigma_z\sigma_x}{k_zk_x} + \frac{\tau_{yz}^2+\tau_{zx}^2}{k_{zx}^2} + 3\frac{\tau_{xy}^2}{k_x^2} = 1, \tag{55}$$

or

$$\frac{1}{2}\left(\frac{\sigma_x-\sigma_y}{k_x}\right)^2 + \frac{1}{2}\left(\frac{\sigma_y}{k_x} - \frac{\sigma_z}{k_z}\right)^2 + \frac{1}{2}\left(\frac{\sigma_z}{k_z} - \frac{\sigma_x}{k_x}\right)^2 + \frac{\tau_{yz}^2+\tau_{zx}^2}{k_{zx}^2} + 3\frac{\tau_{xy}^2}{k_x^2} = 1. \tag{56}$$

Note that the above conditions corresponding to generalized Hu–Marin's Eqs. (41) and (42), with $k_1 = k_2$, were enhanced by the additional shear terms and are referring to optional directions x, y, z. Equations (55) and (56) reduce to the Huber–von Mises Eq. (52) in case of plane stress state in the transverse isotropy plane (x, y), which means that this new criterion can finally be recognized as the transversely isotropic hexagonal symmetry von Mises-based criterion.

Transversely isotropic conditions–tetragonal Hill's (49) or (50) and hexagonal Hu–Marin's (55) or (56), are examined for given orthotropy degrees $R = 2(k_z/k_x)^2 - 1 = 2$, $k_{xy}/k_x = 0.8$, $k_{(xy)}/k_x = 0.9$ and $k_{zx}/k_x = 0.8$, for the following stress states: biaxial normal stresses (σ_x, σ_y) and combined normal with shear stresses (σ_x, τ_{xy}) in the transverse isotropy plane (see Fig. 8a, b), as well as biaxial normal stresses (σ_x, σ_z) and combined normal with shear stresses (σ_x, τ_{zx}) in the orthotropy plane (see Fig. 9a, b). It is worth to mention that the transversely isotropic Hill's condition of tetragonal symmetry (49) or (50) comprises 4 independent plastic yield limits: k_x, k_z, k_{zx} and k_{xy}, because the shear yield limit in isotropy plane k_{xy} is considered as independent. On the contrary, the transversely isotropic enhanced Hu–Marin's-type condition, the symmetry class of which is hexagonal, is defined by 3 independent yield limits only: k_x, k_z and k_{zx}, since shear yield limit

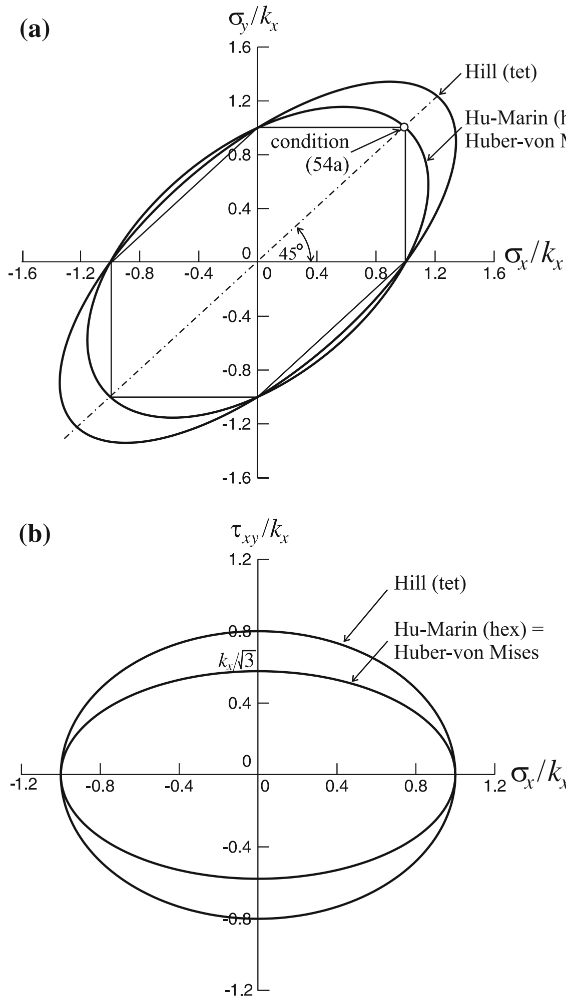


Fig. 8 Comparison of transversely isotropic criteria: Hill's tetragonal (50), Hu–Marin's hexagonal (55) and Huber–von Mises for certain magnitudes of orthotropy ratios: $R = 2$, $k_{zx}/k_x = 0.8$, $k_{(xy)}/k_x = 0.9$ in case of 2D states of stress in the transverse isotropy plane: **a** biaxial normal stresses (σ_x, σ_y) and **b** combined normal with shear stresses (σ_x, τ_{xy})

k_{xy} must agree with the Huber–von Mises criterion in the isotropy plane $k_{xy} = k_x/\sqrt{3}$. Hence, representation of the Hu–Marin's constitutive matrix of plasticity is as follows:

$$[{}_{\text{tris}}\mathbb{I}^{\text{HM}}] = \left[\begin{array}{ccc|ccc} \frac{1}{k_x^2} & -\frac{1}{2k_x^2} & -\frac{1}{2k_x k_z} & & & \\ & \frac{1}{k_x^2} & -\frac{1}{2k_x k_z} & & & \\ & & \frac{1}{k_z^2} & & & \\ \hline & & & \frac{1}{k_{zx}^2} & & \\ & & & & \frac{1}{k_{zx}^2} & \\ & & & & & \frac{3}{k_x^2} \end{array}\right]. \tag{57}$$

Both transversely isotropic criteria: Hill's type of tetragonal symmetry (49) and Hu–Marin's type of hexagonal symmetry (55) describe cylindrical surfaces in the space of principal stresses. However, Hill's-type limit surface represents an elliptical cylinder the axis of which coincides with the hydrostatic axis, in contrast to the enhanced Hu–Marin's-type limit surface that represents an elliptic cylinder the axis of which forms a proportional stress/strength axis, different from the hydrostatic axis. It means that the enhanced Hu–Marin's condition does not satisfies the deviatoricity property, which is a price for the property of coincidence with the Huber–von Mises condition in the isotropy plane, with cylindricity ensured regardless of the magnitude of orthotropy degree.

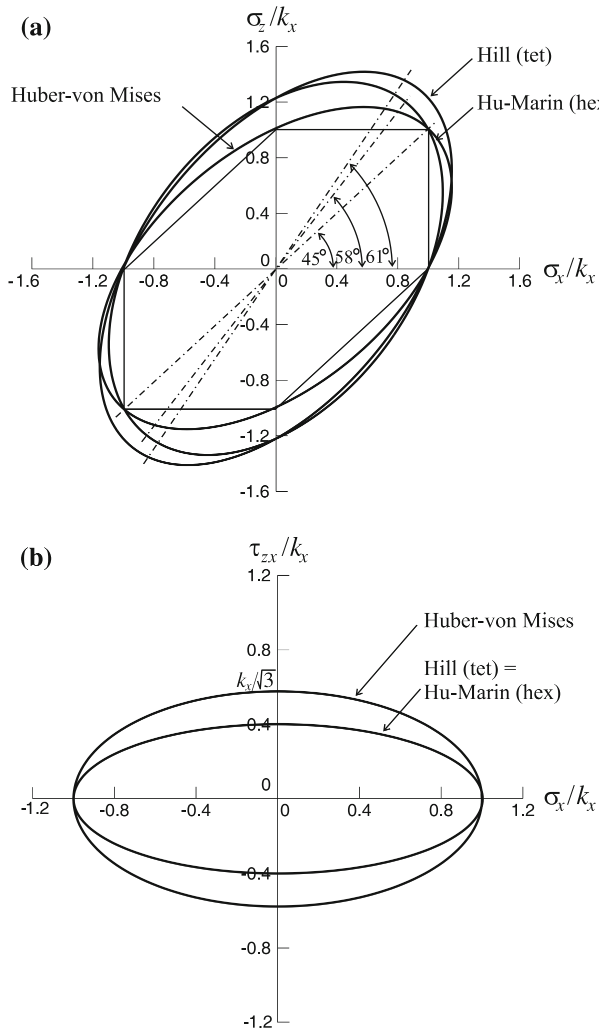


Fig. 9 Comparison of transversely isotropic criteria: Hill's tetragonal (50), Hu–Marin's hexagonal (55) and Huber–von Mises for certain magnitudes of orthotropy ratios: $R = 2$, $k_{zx}/k_x = 0.8$, $k_{(xy)}/k_x = 0.9$, in case of 2D states of stress in the orthotropy plane: **a** biaxial normal stresses (σ_x, σ_z) and **b** combined normal with shear stresses (σ_x, τ_{zx})

6 Hybrid formulation between Hill's and Hu–Marin's yield criteria

A choice of an appropriate transversely isotropic limit criterion, of either tetragonal symmetry (49) or hexagonal symmetry (55), depends on the coincidence with experimental findings for real material. This may often lead to one of two above-considered symmetry classes, but sometimes material limit response is different from both of them. Note that the shape of the limit curves in the transverse isotropy plane is the key to appropriate classification of real transversely isotropic material as exhibiting tetragonal or hexagonal or mixed symmetry properties.

In what follows, a description of the new limit criterion of hybrid symmetry property between tetragonal (49) or (50) and hexagonal (55) or (56) symmetry classes is proposed. The Hu–Marin's-type equation of pure hexagonal symmetry property (55) or (56) comprises three independent material constants k_x, k_z and k_{zx}. However, real engineering materials of hybrid-type nature are characterized by four independent material constants established in four tests: two limits in uniaxial tensions k_x and k_z, shear limit in orthotropy plane k_{zx} (53) and additionally in the biaxial tension test (bulge test) $k_{(xy)}$ instead of condition (54.1.1), namely

$$\sigma_x = \sigma_y = k_{(xy)} \; \sigma_z = \tau_{xy} = \tau_{yz} = \tau_{zx} = 0 \longrightarrow \Pi_{12} = -\frac{1}{2k^2_{(xy)}}. \tag{58}$$

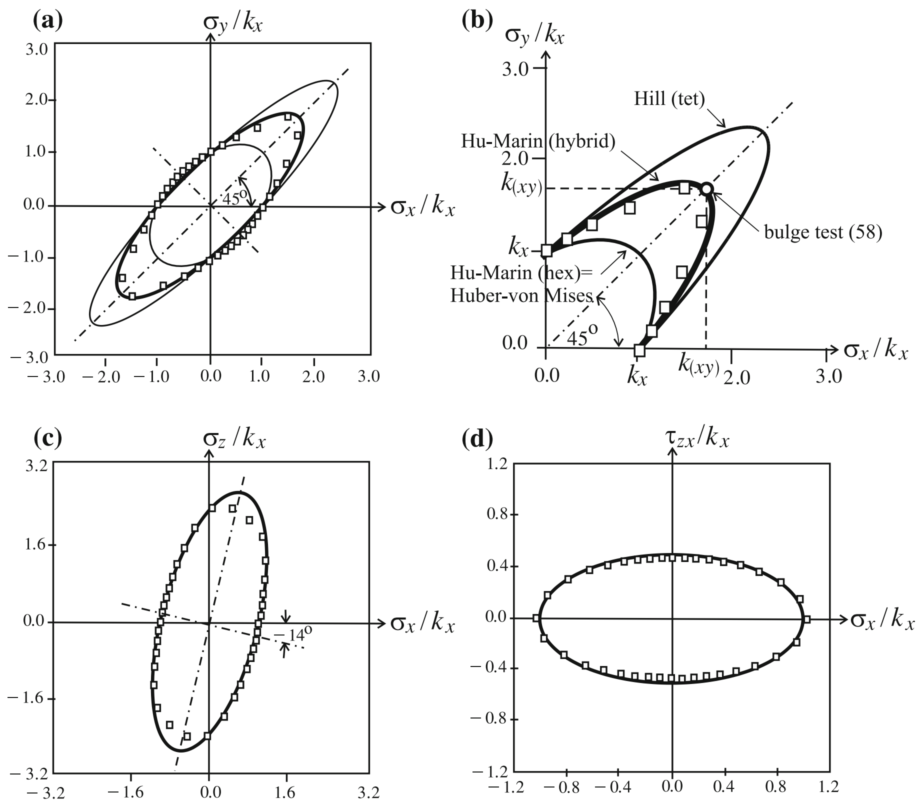


Fig. 10 Fitting of the initial yield surface of a unidirectional SiC/Ti composite according to Herakovich and Aboudi finding (*box symbol*) [7] by the use of transversely isotropic Hu-Marin's hybrid-type criterion (59): **a**, **b** transverse isotropy plane (σ_x, σ_y), **c** orthotropy plane (σ_x, σ_z), **d** orthotropy shear plane (σ_x, τ_{zx})

The above condition leads to the hybrid formulation of the enhanced Hu–Marin's condition,

$$\frac{\sigma_x^2+\sigma_y^2}{k_x^2}-\underline{\frac{\sigma_x\sigma_y}{k_{(xy)}^2}}+\frac{\sigma_z^2}{k_z^2}-\frac{\sigma_y\sigma_z+\sigma_z\sigma_x}{k_zk_x}+\frac{\tau_{yz}^2+\tau_{zx}^2}{k_{zx}^2}+3\frac{\tau_{xy}^2}{k_x^2}=1. \tag{59}$$

Equation (59) differs from the hexagonal form of the Hu–Marin condition (55) in the underlined term, where the fourth independent material constant $k_{(xy)}$ is taken from the bulge test (58), additionally to conditions (54.2,3). Hybrid formulation of the 4-parameter transversely isotropic Hu–Marin's condition (59) is illustrated in Fig. 10a–d by the use of a thick solid line.

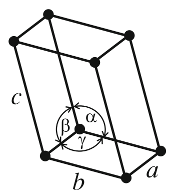
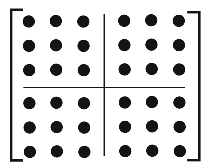
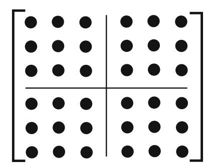
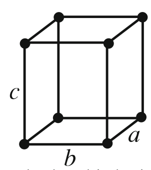
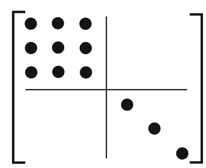
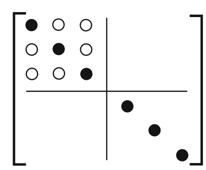
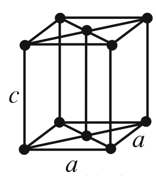
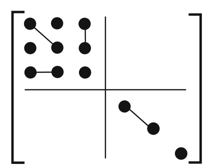
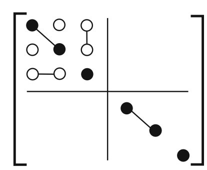
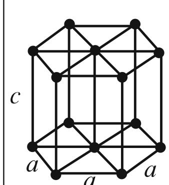
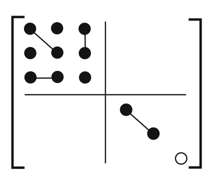
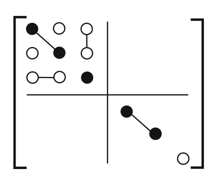
The hybrid-type enhanced Hu–Marin's criterion is capable of capturing the behavior of some long-fiber-reinforced composite materials which in the transverse isotropy plane exhibit a limit response different from both the Hill and the Huber–von Mises materials (cf., e.g., Herakovich and Aboudi [7]).

7 Final remarks and discussion

The main goals of the present paper were to find new formulations of the initial yield criterion, originated basically from von Mises anisotropic yield criterion. The proposed new formulations allow avoiding disadvantages of usually applied Hill's criterion, namely loss of ellipticity of Hill's surface in case of strong orthotropy degree and irreducibility of transversely isotropic Hill's criterion to the Huber–von Mises condition in the isotropy plane.

In order to emphasize new goals of the present paper, a certain analogy between crystal unit cells of space lattices and constitutive matrices of elasticity and initiation of plasticity is presented in Table 2. In the fundamental book by Love [21], the analogy between crystal symmetry classes and groups from one hand and appropriate forms of elastic strain energy function $W=\frac{1}{2}\{\boldsymbol{\varepsilon}\}^{\mathrm{T}}[\mathbb{C}]\{\boldsymbol{\varepsilon}\}$ from the other is demonstrated. In the

Table 2 Analogy between chosen symmetry groups: triclinic, orthorhombic, tetragonal and hexagonal symmetry of Hooke's matrix and plastic yield initiation matrix

Conventional unit cells of space lattices	Chosen constitutive matrix symmetry	
	Elastic Hooke's matrix $2W = \{\varepsilon\}^{\mathrm{T}}[\mathbb{C}]\{\varepsilon\}$	Plastic yield initiation matrix $\{\sigma\}^{\mathrm{T}}[\mathbb{H}]\{\sigma\} = 1$
 triclinic lattice	 Hooke's (21 constants)	 von Mises (21 constants)
 orthorhombic lattice	 orthotropic Hooke's (9 constants)	 deviatorc Hill's (6 constants)
 tetragonal lattice	 transversely isotropic tetragonal Hooke's (6 constants)	 deviatoric transversely isotropic teragonal Hill's (4 constants)
 hexagonal lattice	 transversely isotropic hexagonal Hooke's (5 constants)	 pseudodeviatoric transversely isotropic hexagonal Hu-Marin's (3 constants)

present paper, an extension of the aforementioned analogy also for the symmetry of the constitutive matrix of plastic yield initiation $[\mathbb{H}]$ appearing in the von Mises criterion $\{\sigma\}^{\mathrm{T}}[\mathbb{H}]\{\sigma\} = 1$ was proposed. Unit cells of the four chosen space lattices have been presented following Jastrzebski [15], whereas corresponding constitutive elasticity matrices have schematically been presented applying Nye [26] graphics (symbol • refers to independent element, symbol ∘ refers to dependent element, whereas symbols •–• or ∘–∘ represent pairs of identical matrix elements).

In case of full anisotropy, the complete analogy between the Hooke matrix and the von Mises plasticity matrix holds (21 independent matrix elements in both classes).

However, when narrower symmetry groups are considered: orthotropic, transversely isotropic of tetragonal or hexagonal classes, it is necessary to notice that elastic matrices are usually defined in stress tensor coordinates, whereas plastic constitutive matrices are often defined in the narrower stress deviator coordinates. Reduction in the tensorial space to the deviatoric one is always equivalent to imposing additional constraints (see Eq. (13)); hence, the number of independent elements of plasticity matrix is always lower than the corresponding number of independent elements of elasticity matrix. Namely, it is clear that the 6-element orthotropic deviatoric Hill's matrix corresponds to the 9-element orthotropic Hooke's matrix. Similarly, the 4-element transversely isotropic tetragonal class Hill's matrix corresponds to the 6-element Hooke's matrix, when the independence of Hill's matrix of hydrostatic stress is imposed. Finally, the 3-element transversely isotropic hexagonal class Hu–Marin's matrix corresponds to the 5-element transversely isotropic hexagonal class Hooke matrix. Let us note that pairs of identical matrix elements are arranged in the same way in both matrices of elasticity and plasticity. Nevertheless, some dependent elements in the plasticity matrix (as represented by symbol ∘) correspond to independent elements of the elasticity matrix (sketched by symbol •), but the general population of both matrices remains unchanged.

The commonly used term "transversely isotropic criterion" may be misleading as long as an additional distinction between the tetragonal and the hexagonal symmetry is not introduced. The aforementioned distinction is known from the literature dealing with the prediction of composite behavior in the elastic range and its validation by experiments. For example, Sun and Vaidya [33] examined two types of materials: boron/Al composite and graphite/epoxy composite, and found that some of them exhibit the tetragonal but other the hexagonal symmetry classes. However, even this distinction between tetragonal and hexagonal symmetry classes may occur insufficient to describe some composite materials, for example SiC/Ti unidirectional lamina examined by Herakovich and Aboudi [7]. This is basically caused by residual stresses that appear after cooling-down during the fabrication process. The corners observed at limit curves of the composite result from the intersection of different families of individual limit curves of fiber and matrix material.

The above considerations are limited to the description of the initial yield surface only. Usually, it is assumed that during plastic hardening the initial yield surface possessing certain symmetry is rebuilt in an isotropic way, which is generally not true. This question was discussed, e.g., by Malinin and Rżysko [22], who invoked Mursa's [25] results for OTCz titanium alloy which confirms the assumption of isotropic nature of plastic hardening. However, Hu and Marin's [12] findings for aluminum alloy showed the anisotropic nature of plastic hardening rather than isotropic one.

Nevertheless, the plastic hardening theory is usually taken in an isotropic fashion, e.g., Malinin and Rżysko [22], Ottosen and Ristinmaa [27], Hill [8,9]. Such approach, although commonly used, may be questionable in the light of aforementioned experimental testing, some of which confirms such assumption, cf. Mursa [25] (Titanium Alloy) but the other contradict it, cf. Hu and Marin [12] (aluminum alloy), Kowalewski and Śliwowski [16] (influence of first common invariant).

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References

1. Berryman, J.G.: Bounds and self-consistent estimates for elastic constants of random polycrystals with hexagonal, trigonal, and tetragonal symmetries. J. Mech. Phys. Solids **53**, 2141–2173 (2005)
2. Betten, J.: Applications of tensor functions to the formulation of yield criteria for anisotropic materials. Int. J. Plast. **4**, 29–46 (1988)
3. Drucker, D.C.: A more fundamental approach to plastic stress-strain relations. In: Proceedings of 1st US National Congress on Applied Mechanics, Chicago, pp. 487–491 (1951)
4. Ganczarski, A., Lenczowski, J.: On the convexity of the Goldenblat-Kopnov yield condition. Arch. Mech. **49**, 461–475 (1997)
5. Ganczarski, A., Skrzypek, J.: Modeling of limit surfaces for transversely isotropic composite SCS-6/Ti-15-3 (in Polish). Acta Mech. Et Autom. **5**, 24–30 (2011)
6. Goldenblat, I.I., Kopnov, V.A.: A generalized theory of plastic flow of anisotropic metals (in Russian). Stroit. Mekh. pp. 307–319 (1966)
7. Herakovich, C.T., Aboudi, J.: Thermal effects in composites. In: Hetnarski, R.B. (ed.) Thermal Stresses V, Lastran Corp. Publ. Division, pp. 1–142 (1999)
8. Hill, R.: A theory of the yielding and plastic flow of anisotropic metals. Proc. R. Soc. Lond. A **193**, 281–297 (1948)
9. Hill, R.: The Mathematical Theory of Plasticity. Clarendon Press, Oxford (1950)
10. Hill, R.: Continuum micro-mechanics of elastoplastic polycrystals. J. Mech. Phys. Solids **13**, 89–101 (1965)
11. Hosford, W.F., Backhofen, W.A.: Strength and plasticity of textured metals. In: Backhofen, W.A., Burke, J., Coffin, L., Reed, N., Weisse, V. (eds.) Fundamentals of Deformation Processing, Syracuse Univ. Press, pp. 259–298 (1964)
12. Hu, Z.W., Marin, J.: Anisotropic loading functions for combined stresses in the plastic range. J. Appl. Mech. **22**, 1 (1956)
13. Huber, M.T.: Właściwa praca odkształcenia jako miara wytężenia materiału. Czas. Techn. 22, Lwów; Pisma, vol.II, PWN, Warszawa 1956 (1904)
14. Jackson, L.R., Smith, K.F., Lankford, W.T.: Plastic flow in anisotropic sheet steel. Am. Inst. Mining. Metall. Eng. **2440**, 1–15 (1948)
15. Jastrzebski, Z.D.: The Nature and Properties of Engineering Materials. Wiley, New York (1987)

16. Kowalewski, Z.L., Śliwowski, M.: Effect of cyclic loading on the yield surface evolution of 18G2A low-alloy steel. Int. J. Mech. Sci. **39**, 1, 51–68 (1997)
17. Kowalsky, U.K., Ahrens, H., Dinkler, D.: Distorted yield surfaces—modeling by higher order anisotropic hardening tensors. Comput. Mater. Sci. **16**, 81–88 (1999)
18. Kuna-Ciskał, H., Skrzypek, J.: CDM based modelling of damage and fracture mechanisms in concrete under tension and compression. Eng. Fract. Mech. **71**, 681–698 (2004)
19. Lankford, W.T., Low, J.R., Gensamer, M.: The plastic flow of aluminium alloy sheet under combined loads. Trans. AIME 171, 574; TP 2238, Met. Techn., Aug. 1947 (1947)
20. Lemaitre, J., Chaboche, J.-L.: Méchanique des Matériaux Solides. Dunod Publ., Paris (1985)
21. Love, A.E.H.: A Treatise on the Mathematical Theory of Elasticity. Dover Publ., New York (1944)
22. Malinin, N.N, Rżysko, J.: Mechanics of Materials (in Polish). PWN, Warszawa (1981)
23. von Mises, R.: Mechanik der festen Körper im plastisch-deformablen Zustand, Nachrichten der Gesellschaft der Wissenschaften zu Göttingen (1913)
24. Mises, R.von : Mechanik der plastischen Formänderung von Kristallen. ZAMM **8**, 161–185 (1928)
25. Mursa, K.S.: Examination of orthotropic metal sheets under uniaxial tension (in Russian). Izv. Vys. Ucheb. Zav., Mash. **6** (1972)
26. Nye, J.F.: Physical Properties of Crystals. Their Representations by Tensor and Matrices. Clarendon Press, Oxford (1957)
27. Ottosen, N.S., Ristinmaa, M.: The Mechanics of Constitutive Modeling. Elsevier, Amsterdam (2005)
28. Rogers, T.G.: Yield criteria, flow rules, and hardening in anisotropic plasticity. In: Boehler, J.P. (ed.) Yielding, Damage and Failure of Anisotropic Solids, pp. 53–79. Mech. Eng. Publ., London (1990)
29. Rymarz, Cz.: Continuum Mechanics (in Polish). PWN, Warszawa (1993)
30. Sayir, M.: Zur Fließbedingung der Plastiztätstheorie. Ingenievrarchiv **39**, 414–432 (1970)
31. Skrzypek J., Ganczarski A. (2013) Anisotropic initial yield and failure criteria including temperature effect. In: Hetnarski, R. (ed.), Encyclopedia of Thermal Stresses, Springer Science+Business Media Dordrecht
32. Spencer A.J.M.: Theory of invariants. In: Eringen, C. (ed.) Continuum Physics, Academic Press, New York, pp. 239–353
33. Sun, C.T., Vaidya, R.S.: Prediction of composite properties from a representative volume element. Compos. Sci. Technol. **56**, 171–179 (1996)
34. Szczepiński, W.: On deformation-induced plastic anisotropy of sheet metals. Arch. Mech. **45**, 3–38 (1993)
35. Tamma K.K., Avila A.F. (1999) An integrated micro/macro modelling and computational methodology for high temperature composites. In: Hetnarski R.B. (ed.) Thermal Stresses V, Lastran Corp. Publ. Division, Rochester, NY, pp. 143–256
36. Tsai, S.T., Wu, E.M.: A general theory of strength for anisotropic materials. Int. J. Numer. Methods. Eng. **38**, 2083–2088 (1971)
37. Życzkowski, M.: Anisotropic yield conditions. In: Lemaitre, J. (ed.) Handbook of Materials Behavior Models, pp. 155–165. Academic Press, San Diego (2001)