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The dissipation inequality in hypoplasticity

Received: 11 October 2010 / Published online: 19 April 2011
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Abstract Standard hypoplasticity is examined with respect to the thermodynamic requirement of non-negative energy dissipation. We introduce a stress energy function and derive a dissipation inequality in terms of the stress-dependent operators of the hypoplastic law. A general form for the non-linear operator is also found, which makes it straightforward to construct thermodynamically consistent hypoplasticity laws. We further examine the subclass of hypoplasticity where the linear term is non-dissipative and construct some examples of hypoplastic laws based on a quadratic stress energy function.

1 Introduction

Hypoplasticity is a constitutive theory in differential form, introduced by Kolymbas [1] by generalisation of Truesdell's *hypoelasticity* [2], and developed to describe the behaviour of granular materials in geomechanics [3]. The main advantages of hypoplasticity over strain-based *elastoplasticity* are simplicity and physical appeal—complex non-linear behaviour can be described by a single equation, with no need to distinguish between elastic and plastic regimes or between loading and unloading [3,4]. Moreover, hypoplasticity has been found to capture the physics of a variety of granular materials [3].

Standard hypoplasticity is isotropic and strain-rate independent, which leads to an evolution equation of the general form [5]

$$\dot{\boldsymbol{\sigma}} = \mathbb{L}(\boldsymbol{\sigma}) : \mathbf{d} - \mathbf{N}(\boldsymbol{\sigma}) \|\mathbf{d}\| + \mathbf{w}\boldsymbol{\sigma} - \boldsymbol{\sigma}\mathbf{w}, \quad (1)$$

where the operators \mathbb{L} and \mathbf{N} are isotropic functions of the Cauchy stress, \mathbf{d} and \mathbf{w} are the symmetric and skew parts of the spatial velocity gradient and $\|\mathbf{d}\| = \sqrt{\text{tr}\mathbf{d}^2}$ is the Euclidean norm of \mathbf{d} . Equation (1) is rate independent because it is homogeneous of degree one in the velocity gradient, and it is frame indifferent owing to the non-objective term¹ $\mathbf{w}\boldsymbol{\sigma} - \boldsymbol{\sigma}\mathbf{w}$.

A shortcoming of the hypoplastic formulation as it stands is that, due to the absence of energy potential, there is no built-in constraint that prevents the energy dissipation from becoming negative. The principle of non-negative dissipation is fundamental in solid mechanics, notably classical elastoplasticity is typically derived via the Coleman–Noll procedure [8], which uses the dissipation inequality as its *starting* point. By contrast, this principle is rarely even mentioned in the dynamics of non-Newtonian fluids. The reason might be that in fluids dissipative processes predominate so that negative dissipation becomes an unlikely event.

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¹ Note that the form (1) incorporates not only the Jaumann rate, but every objective stress rate that is isotropic, homogeneous of degree one, and depends on stress and velocity gradient only [6,7].

An additional point is that fluids are naturally described by *differential* constitutive laws, which are less straightforward to derive from an energy potential. The approach of hypoplasticity is similar to that of visco-elastic fluids, even though the subject matter is actually a solid. Presumably, negative dissipation has not been much of a concern in hypoplasticity, because those models have been so constructed as to always have some dissipation. However, a wider use of differential models for solids will necessitate the addition of non-negative dissipation to the list of mandatory constitutive principles.

The issue has been resolved for certain special cases, in particular *hypoelasticity* (the case obtained by dropping the non-linear second term of Eq. 1) [9,10]. Other restricted classes of hypoplasticity have been treated within the framework of the Müller-Liu entropy principle [11–13]. However, the literature provides no general (sufficient and necessary) condition on the operators $\mathbb{L}(\boldsymbol{\sigma})$ and $\mathbf{N}(\boldsymbol{\sigma})$, to guarantee non-negative energy dissipation of the standard hypoplastic law (1). It is the objective of this contribution to derive such a condition and devise a systematic route for the construction of hypoplastic laws that are thermodynamically consistent.

2 Dissipation inequality

In the absence of thermal effects, the second law of thermodynamics takes the form of the isothermal Clausius–Planck inequality:

$$\boldsymbol{\sigma} : \mathbf{d} - \psi \text{tr} \mathbf{d} - \dot{\psi} \geq 0, \quad (2)$$

where ψ is the free energy per unit volume. To enforce the inequality (2), it is necessary to introduce a free energy function. Since the constitutive Eq. (1) uses a single state variable, $\boldsymbol{\sigma}$, to carry all necessary information about the loading history [5], the free energy must be a function of $\boldsymbol{\sigma}$ only and, in order to be objective, may only depend on scalar invariants. Therefore, ψ must have the following form:

$$\psi = \psi(i_1, i_2, i_3); \quad i_1 = \text{tr} \boldsymbol{\sigma}, \quad i_2 = \text{tr} \boldsymbol{\sigma}^2, \quad i_3 = \text{tr} \boldsymbol{\sigma}^3. \quad (3)$$

The stress energy function $\psi(i_1, i_2, i_3)$ will also be required to be positive definite.

Owing to the symmetry of $(d\psi/d\boldsymbol{\sigma}) \boldsymbol{\sigma}$ (cf. Eq. (12)) and the skew symmetry of \mathbf{w} , we have

$$d\psi/d\boldsymbol{\sigma} : \{\mathbf{w}\boldsymbol{\sigma} - \boldsymbol{\sigma}\mathbf{w}\} = 0. \quad (4)$$

Then, the rate of change of free energy becomes

$$\dot{\psi} = \frac{d\psi}{d\boldsymbol{\sigma}} : [\mathbb{L} : \mathbf{d} - \mathbf{N} \|\mathbf{d}\|]. \quad (5)$$

Introducing (5) into the dissipation inequality (2) yields

$$d\psi/d\boldsymbol{\sigma} : \mathbf{N} \|\mathbf{d}\| + \boldsymbol{\zeta} : \mathbf{d} \geq 0, \quad (6)$$

where

$$\boldsymbol{\zeta} = \boldsymbol{\sigma} - \psi \mathbf{I} - d\psi/d\boldsymbol{\sigma} : \mathbb{L}. \quad (7)$$

In the non-trivial case $\mathbf{d} \neq \mathbf{0}$, the inequality (6) reduces to the rate-independent form

$$d\psi/d\boldsymbol{\sigma} : \mathbf{N} + \boldsymbol{\zeta} : \hat{\mathbf{d}} \geq 0, \quad (8)$$

where $\hat{\mathbf{d}} = \mathbf{d} / \|\mathbf{d}\|$ is the normalised strain-rate tensor. Since the product of any two normalised tensors is bounded by $-1 \leq \hat{\boldsymbol{\zeta}} : \hat{\mathbf{d}} \leq 1$, it follows that

$$\max_{\hat{\mathbf{d}}} (\boldsymbol{\zeta} : \hat{\mathbf{d}}) = \|\boldsymbol{\zeta}\|. \quad (9)$$

As (8) must hold for arbitrary $\hat{\mathbf{d}}$, we may thus eliminate the strain rate $\hat{\mathbf{d}}$ in (8) and obtain a condition on the stress functions ψ , \mathbb{L} and \mathbf{N} :

$$d\psi/d\boldsymbol{\sigma} : \mathbf{N} - \|\boldsymbol{\zeta}\| \geq 0 \quad \forall \boldsymbol{\sigma}, \quad (10)$$

defining the complete class of hypoplasticity with non-negative dissipation.

3 General form for \mathbf{N}

Since \mathbf{N} is an isotropic function of $\boldsymbol{\sigma}$, it may be expressed as

$$\mathbf{N} = a_0 \mathbf{I} + a_1 \boldsymbol{\sigma} + a_2 \boldsymbol{\sigma}^2, \quad (11)$$

where a_0, a_1, a_2 are functions of the stress invariants. Using the chain rule,

$$d\psi/d\boldsymbol{\sigma} = \mathbf{I} \partial\psi/\partial i_1 + 2\boldsymbol{\sigma} \partial\psi/\partial i_2 + 3\boldsymbol{\sigma}^2 \partial\psi/\partial i_3, \quad (12)$$

and (11) along with the Cayley–Hamilton theorem, the dissipation inequality (10) may be rewritten as

$$a_0 \theta_0 + a_1 \theta_1 + a_2 \theta_2 - \|\boldsymbol{\zeta}\| \geq 0, \quad (13)$$

where

$$\theta_0 = 3\partial\psi/\partial i_1 + 2i_1 \partial\psi/\partial i_2 + 3i_2 \partial\psi/\partial i_3, \quad (14)$$

$$\theta_1 = i_1 \partial\psi/\partial i_1 + 2i_2 \partial\psi/\partial i_2 + 3i_3 \partial\psi/\partial i_3, \quad (15)$$

$$\theta_2 = i_2 \partial\psi/\partial i_1 + 2i_3 \partial\psi/\partial i_2 + \frac{1}{2} (i_1^4 + 3i_2^2 - 6i_1^2 i_2 + 8i_1 i_3) \partial\psi/\partial i_3. \quad (16)$$

With no loss of generality, we may now replace $a_0 \theta_0, a_1 \theta_1, a_2 \theta_2$ in (13) by another set of stress functions η_0, η_1, η_2 and w_0, w_1, w_2 , such that

$$a_0 \theta_0 = w_0 \|\boldsymbol{\zeta}\| + \eta_0, \quad a_1 \theta_1 = w_1 \|\boldsymbol{\zeta}\| + \eta_1, \quad a_2 \theta_2 = w_2 \|\boldsymbol{\zeta}\| + \eta_2, \quad (17)$$

whereupon (13) reduces to

$$\eta_0 + \eta_1 + \eta_2 \geq 0, \quad w_0 + w_1 + w_2 \geq 1. \quad (18)$$

We thus find that \mathbf{N} may be generally expressed as

$$\mathbf{N} = \alpha_0 \mathbf{I} + \alpha_1 \hat{\boldsymbol{\sigma}} + \alpha_2 \hat{\boldsymbol{\sigma}}^2, \quad (19)$$

where $\hat{\boldsymbol{\sigma}} = \boldsymbol{\sigma}/\sqrt{i_2}$ and $\alpha_0 = a_0, \alpha_1 = a_1 \sqrt{i_2}, \alpha_2 = a_2 i_2$ are functions of the stress invariants,

$$\alpha_k = \begin{cases} \frac{i_2^{k/2}}{\theta_k} (w_k \|\boldsymbol{\zeta}\| + \eta_k) & \text{if } \theta_k \neq 0 \\ \lim_{\sigma \rightarrow 0} \frac{i_2^{k/2}}{\theta_k} (w_k \|\boldsymbol{\zeta}\| + \eta_k) & \text{if } \theta_k = 0. \end{cases} \quad (20)$$

The existence of the limit requires the additional conditions on η_k and w_k

$$\begin{aligned} \eta_0, \quad w_0 \|\boldsymbol{\zeta}\| &= O(\theta_0), \\ \eta_1 \sqrt{i_2}, \quad w_1 \|\boldsymbol{\zeta}\| \sqrt{i_2} &= O(\theta_1), \\ \eta_2 i_2, \quad w_2 \|\boldsymbol{\zeta}\| i_2 &= O(\theta_2). \end{aligned} \quad (21)$$

The form (19) together with (20), subject to conditions (18) and (21), expresses explicitly the complete set of non-linear operators \mathbf{N} that are defined for all stress states and satisfy the dissipation inequality for any given choice of ψ and \mathbb{L} .

This result suggests a straightforward procedure to construct hypoplastic laws: first postulate the linear operator \mathbb{L} and a stress energy function ψ , and thereafter construct \mathbf{N} according to (19) and (20) subject to the inequalities (18) and limit conditions (21). This seems to be the only route that allows an unlimited degree of complexity of the resulting constitutive law and, in particular, is feasible for any given \mathbb{L} . Alternative routes would be to postulate \mathbf{N} and ψ and construct \mathbb{L} , or to specify the forms of all three functions, and then satisfy the dissipation inequality by means of suitable free functions. These routes may be practicable in some cases where \mathbf{N} and ψ are particularly simple; but not in general. The idea of postulating \mathbb{L} and \mathbf{N} is also inadvisable, because the task then becomes to prove the existence of a stress energy function, which will be very difficult in general. For this reason, it also appears difficult to test existing hypoplastic laws with respect to the dissipation inequality (10).

4 A subclass of hypoplasticity

The particular case

$$\boldsymbol{\zeta} = \mathbf{0} \quad (22)$$

defines a physically appealing subclass of hypoplastic models, in that the linear operator \mathbb{L} is non-dissipative, and any dissipation is due to \mathbf{N} :

$$\frac{d\psi}{d\boldsymbol{\sigma}} : \mathbb{L} = \boldsymbol{\sigma} - \psi \mathbf{I}, \quad (23)$$

$$\frac{d\psi}{d\boldsymbol{\sigma}} : \mathbf{N} \geq 0. \quad (24)$$

Since the conditions for \mathbb{L} and \mathbf{N} are now separated, (23) and (24), the operators may be constructed independently. Another feature of this subclass is to allow for an elastic, i.e. zero-dissipation-, regime. The limiting case of $\mathbf{N} = \mathbf{0}$ is admissible here and corresponds to potential *hypoeasticity*.

The condition for the linear operator (23) may be rewritten as

$$(\psi_1 \mathbf{I} + 2\psi_2 \boldsymbol{\sigma} + 3\psi_3 \boldsymbol{\sigma}^2) : \mathbb{L} : \mathbf{d} - \boldsymbol{\sigma} : \mathbf{d} + \psi \text{tr} \mathbf{d} = \mathbf{0}, \quad (25)$$

where $\psi_k = \partial\psi/\partial i_k$. Based on the representation theorems for second-order tensors, the general form for the terms $\mathbb{L}(\boldsymbol{\sigma}) : \mathbf{d}$ and $\mathbf{N}(\boldsymbol{\sigma})$ is obtained as, adopting the notation by Lanier et al. [5],

$$\begin{aligned} \mathbb{L}(\boldsymbol{\sigma}) = & \mathbf{I} \otimes [b_0 \mathbf{I} + c_0 \boldsymbol{\sigma} + d_0 \boldsymbol{\sigma}^2] \\ & + \boldsymbol{\sigma} \otimes [b_3 \mathbf{I} + c_3 \boldsymbol{\sigma} + d_3 \boldsymbol{\sigma}^2] \\ & + \boldsymbol{\sigma}^2 \otimes [b_4 \mathbf{I} + c_4 \boldsymbol{\sigma} + d_4 \boldsymbol{\sigma}^2] \\ & + g_1 \mathbf{I}^{4s} + g_5 (\mathbf{I} \bar{\otimes} \boldsymbol{\sigma} + \boldsymbol{\sigma} \bar{\otimes} \mathbf{I}) + g_6 (\mathbf{I} \bar{\otimes} \boldsymbol{\sigma}^2 + \boldsymbol{\sigma}^2 \bar{\otimes} \mathbf{I}), \end{aligned} \quad (26)$$

where, in general, all the 12 coefficients are functions of the stress invariants i_1, i_2, i_3 . Now introducing the general form of \mathbb{L} , Eq. (26), and using the Cayley–Hamilton identity,

$$\boldsymbol{\sigma}^3 = \left(\frac{1}{3} i_3 + \frac{1}{6} i_1^3 - \frac{1}{2} i_1 i_2 \right) \mathbf{I} - \frac{1}{2} (i_1^2 - i_2) \boldsymbol{\sigma} + i_1 \boldsymbol{\sigma}^2, \quad (27)$$

we obtain

$$\begin{aligned} 0 = & [b_0 \text{tr} \mathbf{d} + c_0 \boldsymbol{\sigma} : \mathbf{d} + d_0 \boldsymbol{\sigma}^2 : \mathbf{d}] (3\psi_1 + 2\psi_2 i_1 + 3\psi_3 i_2) \\ & + [b_3 \text{tr} \mathbf{d} + c_3 \boldsymbol{\sigma} : \mathbf{d} + d_3 \boldsymbol{\sigma}^2 : \mathbf{d}] (\psi_1 i_1 + 2\psi_2 i_2 + 3\psi_3 i_3) \\ & + [b_4 \text{tr} \mathbf{d} + c_4 \boldsymbol{\sigma} : \mathbf{d} + d_4 \boldsymbol{\sigma}^2 : \mathbf{d}] \left(\psi_1 i_2 + 2\psi_2 i_3 + 3\psi_3 \left(\frac{1}{6} i_1^4 + \frac{1}{2} i_2^2 - i_1^2 i_2 + \frac{4}{3} i_1 i_3 \right) \right) \\ & + \psi_1 g_1^{(1)} \text{tr} \mathbf{d} + 2 \left(\psi_2 g_1^{(1)} + \psi_1 g_5^{(0)} \right) \boldsymbol{\sigma} : \mathbf{d} \\ & + \left(3\psi_3 g_1^{(1)} + 4\psi_2 g_5^{(0)} + 2\psi_1 g_6^{(-1)} \right) \boldsymbol{\sigma}^2 : \mathbf{d} \\ & + \left(6\psi_3 g_5^{(0)} + 4\psi_2 g_6^{(-1)} \right) \left[\left(\frac{1}{3} i_3 + \frac{1}{6} i_1^3 - \frac{1}{2} i_1 i_2 \right) \mathbf{I} - \frac{1}{2} (i_1^2 - i_2) \boldsymbol{\sigma} + i_1 \boldsymbol{\sigma}^2 \right] : \mathbf{d} \\ & + 3\psi_3 g_6^{(-1)} 2 \left[\left(\frac{1}{6} i_1^4 + \frac{1}{3} i_1 i_3 - \frac{1}{2} i_1^2 i_2 \right) \mathbf{I} + \frac{1}{3} (i_3 - i_1^3) \boldsymbol{\sigma} + \frac{1}{2} (i_1^2 + i_2) \boldsymbol{\sigma}^2 \right] : \mathbf{d} \\ & - \boldsymbol{\sigma} : \mathbf{d} + \psi \text{tr} \mathbf{d}. \end{aligned} \quad (28)$$

Since this equation must hold for arbitrary $\boldsymbol{\sigma}$, it splits into three separate conditions:

$$\psi_1 A_{11} + \psi_2 A_{12} + \psi_3 A_{13} = -\psi, \quad (29)$$

$$\psi_1 A_{21} + \psi_2 A_{22} + \psi_3 A_{23} = 1, \quad (30)$$

$$\psi_1 A_{31} + \psi_2 A_{32} + \psi_3 A_{33} = 0, \quad (31)$$

with

$$A_{11} = 3b_0 + b_3i_1 + b_4i_2 + g_1 + 2g_6 \left(\frac{2}{3}i_3 + \frac{1}{3}i_1^3 - i_1i_2 \right), \quad (32)$$

$$A_{12} = 2(b_0i_1 + b_3i_2 + b_4i_3), \quad (33)$$

$$A_{13} = 3b_0i_2 + 3b_3i_3 + b_4 \left(\frac{1}{2}i_1^4 + \frac{3}{2}i_2^2 - 3i_1^2i_2 + 4i_1i_3 \right) + g_5(2i_3 + i_1^3 - 3i_1i_2) + g_6(i_1^4 + 2i_1i_3 - 3i_1^2i_2), \quad (34)$$

$$A_{21} = 3c_0 + c_3i_1 + c_4i_2 + 2g_5, \quad (35)$$

$$A_{22} = 2(c_0i_1 + c_3i_2 + c_4i_3 + g_1 - g_6(i_1^2 - i_2)), \quad (36)$$

$$A_{23} = 3c_0i_2 + 3c_3i_3 + c_4 \left(\frac{1}{2}i_1^4 + \frac{3}{2}i_2^2 - 3i_1^2i_2 + 4i_1i_3 \right) + 2g_6(i_3 - i_1^3) - 3g_5(i_1^2 - i_2), \quad (37)$$

$$A_{31} = 3d_0 + d_3i_1 + d_4i_2 + 2g_6, \quad (38)$$

$$A_{32} = 2(d_0i_1 + d_3i_2 + d_4i_3 + 2g_5 + 2g_6i_1), \quad (39)$$

$$A_{33} = 3d_0i_2 + 3d_3i_3 + d_4 \left(\frac{1}{2}i_1^4 + \frac{3}{2}i_2^2 - 3i_1^2i_2 + 4i_1i_3 \right) + 3g_1 + 6g_5i_1 + 3g_6(i_1^2 + i_2). \quad (40)$$

Clearly, the conditions (29)–(31) allow considerable freedom in the choice of the thirteen functions ψ and $d_0 \dots g_6$, all of which may be functions of the stress invariants i_1, i_2, i_3 . By direct inspection, the first two conditions (29) and (30) show that at least one of b_0, b_3, b_4, g_1, g_5 and g_6 and at least one of c_0, c_3, c_4, g_1, g_5 and g_6 must be non-zero.

A possible route for constructing $\mathbb{L}(\boldsymbol{\sigma})$ subject to the above condition is to start from a given form, which must be a special case of (26), and reduce the condition (29)–(31) accordingly. This yields three partial differential equations in $\psi(i_1, i_2, i_3)$. These equations must then be solved analytically, since ψ_1, ψ_2 and ψ_3 are needed for the construction of $\mathbf{N}(\boldsymbol{\sigma})$. This solution is, however, difficult to obtain in general. A much more straightforward procedure is to start from a postulated free energy potential $\psi = \psi(i_1, i_2, i_3)$, introduce ψ and its derivatives ψ_1, ψ_2 and ψ_3 , into Eqs. (29)–(31) and simplify. One may then select a set of suitable terms from Eq. (26), such that Eqs. (29)–(31) have at least one solution for the chosen stress functions. Finally, select a suitable solution.

The non-linear operator may be constructed almost independently (provided that $\psi(i_1, i_2, i_3)$ is given), by application of the general form (20) with $\|\boldsymbol{\zeta}\| = 0$:

$$\alpha_0 = \eta_0/\theta_0, \quad \alpha_1 = \eta_1\sqrt{i_2}/\theta_1, \quad \alpha_2 = \eta_2i_2/\theta_2, \quad (41)$$

where η_k is subject to the restrictions (18) and (21) and the special treatment (Eq. (20)) when the θ -functions vanish. The absence of $\|\boldsymbol{\zeta}\|$ allows an alternative form which avoids the division by functions that may become zero. If we express the functions a_0, a_1 and a_2 in terms of a new set of stress functions q_0, q_1 and q_2 , such that

$$a_0 = q_0\theta_0(\theta_1\theta_2)^2, \quad (42)$$

$$a_1 = q_1\theta_1(\theta_0\theta_2)^2, \quad (43)$$

$$a_2 = q_2\theta_2(\theta_0\theta_1)^2, \quad (44)$$

the inequality (13) becomes

$$q_0 + q_1 + q_2 \geq 0. \quad (45)$$

All these constructs are, however, more or less awkward and, as exemplified in the next Section, the simplest results may be more easily obtained by directly considering inequality (13):

$$a_0(3\psi_1 + 2i_1\psi_2 + 3i_2\psi_3) + a_1(i_1\psi_1 + 2i_2\psi_2 + 3i_3\psi_3) + a_2 \left(i_2\psi_1 + 2i_3\psi_2 + \frac{1}{2}(i_1^4 + 3i_2^2 - 6i_1^2i_2 + 8i_1i_3)\psi_3 \right) \geq 0. \quad (46)$$

It is sometimes possible to test for (22), by introducing the linear operator into Eqs. (29)–(31) and checking for the existence of a solution for ψ (i_1, i_2, i_3). In this way, it is straightforward to show that the popular hypoplastic laws of Kolymbas [3] and Wu [3] do *not* belong to this class. This means that they do not satisfy the condition (10) for every parameter choice (e.g., setting the parameters such that $\mathbf{N} = \mathbf{0}$ will make them thermodynamically inconsistent). It remains an open question, however, under what precise conditions non-negative dissipation can be guaranteed for those models. It appears that these models at least *nearly* satisfy (22): The author has experimented with integrating these models with $\mathbf{N} = \mathbf{0}$ over various strain paths and upon returning to the original stress state the total work done was small but significant, and sometimes negative. It should be noted that the *hypoelastic* laws obtained within this subclass (by dropping the \mathbf{N} -term) are thoroughly strain path dependent, yet non-dissipative—exactly the amount of work done will be returned upon unloading to its original stress state. A residual strain will remain, however, unless the loading path is self-retracing.

5 Example

We now provide a straightforward example of the construction of a thermodynamically consistent hypoplastic model. The example belongs to the subclass defined by (22) and is based on a simple quadratic stress energy function:

$$\psi = \alpha i_1^2 + \beta i_2, \quad (47)$$

where α and β are constants. Since $0 \leq i_1^2/i_2 \leq 3$, the positive definiteness of ψ , requires that

$$\beta \geq 0 \quad \text{and} \quad 3\alpha + \beta \geq 0. \quad (48)$$

To construct the linear operator, we introduce Eq. (47) into (29)–(31):

$$2\alpha i_1 A_{11} + \beta A_{12} + \alpha i_1^2 + \beta i_2 = 0, \quad (49)$$

$$2\alpha i_1 A_{21} + \beta A_{22} - 1 = 0, \quad (50)$$

$$2\alpha i_1 A_{31} + \beta A_{32} = 0. \quad (51)$$

Introducing the expressions for A_{ij} from (32)–(40) and collecting terms according to their powers in the stress invariants gives three equations:

$$\begin{aligned} & (2(3b_0 + g_1)\alpha + 2b_0\beta) i_1 + (2b_3 + 1)\beta i_2 + (2b_3 + 1)\alpha i_1^2 + 2b_4\alpha i_1 i_2 \\ & + \frac{4}{3}g_6\alpha i_1^4 - 4g_6\alpha i_1^2 i_2 + 2b_4\beta i_3 + \frac{8}{3}g_6\alpha i_1 i_3 = 0, \end{aligned} \quad (52)$$

$$\begin{aligned} & (6c_0\alpha + 4g_5\alpha + 2c_0\beta) i_1 + 2(c_3 + g_6)\beta i_2 + 2(c_3\alpha - g_6\beta) i_1^2 + 2c_4\alpha i_1 i_2 + 2c_4\beta i_3 \\ & = 1 - 2\beta g_1, \end{aligned} \quad (53)$$

$$2(3d_0\alpha + 2g_6\alpha + d_0\beta + 2g_6\beta) i_1 + 2d_3\alpha i_1^2 + 2d_4\alpha i_1 i_2 + 2d_3\beta i_2 + 2d_4\beta i_3 = -4\beta g_5. \quad (54)$$

If the coefficients are all *constant*, each one of the above terms must vanish for all σ . This yields a unique solution:

$$\begin{aligned} b_0 &= -\frac{\alpha}{2\beta(3\alpha + \beta)}, \quad b_3 = -1/2, \quad g_1 = \frac{1}{2\beta} \quad \text{and} \\ b_4 &= c_0 = c_3 = c_4 = d_0 = d_3 = d_4 = g_5 = g_6 = 0, \end{aligned} \quad (55)$$

whence the linear operator becomes

$$\mathbb{L}(\sigma) = \lambda \mathbf{I} \otimes \mathbf{I} + 2\mu \mathbf{I}^{4s} - \frac{1}{2} \sigma \otimes \mathbf{I}. \quad (56)$$

We see that (56) provides a *hypoelastic* form of Hooke's law with the various coefficients identified as

$$g_1 = 2\mu, \quad b_0 = \lambda \quad \text{and} \quad \alpha = \frac{1}{18K} - \frac{1}{12\mu}, \quad \beta = \frac{1}{4\mu}, \quad (57)$$

where λ and μ are the Lamé constants, and μ and K are the shear and bulk moduli, respectively. While constant coefficients will serve the purpose of our example, considerably more complicated responses could be described by allowing the coefficients to be functions of the stress invariants.

Next, consider the non-linear operator. Introducing the energy function (47) into (46), the dissipation inequality reads

$$(3\alpha + \beta) i_1 a_0 + (\alpha i_1^2 + \beta i_2) a_1 + (\alpha i_1 i_2 + \beta i_3) a_2 \geq 0. \quad (58)$$

Apart from the trivial case $\mathbf{N} = \mathbf{0}$ (*hypoelasticity*), the simplest will be to introduce constant coefficients a_0, a_1, a_2 . With this restriction and the relations $0 \leq i_1^2 \leq 3i_2$ and $0 \leq i_3^{2/3} \leq i_2$, inequality (58) immediately leads to

$$a_1 \geq 0 \quad \text{and} \quad a_2 = 0. \quad (59)$$

The remaining inequality may be rewritten as

$$(3\alpha + \beta) |a_0| |i_1| \leq a_1 (\alpha i_1^2 + \beta i_2), \quad (60)$$

since the parenthetic expressions are both non-negative. Then, using $|i_1| \leq \sqrt{3i_2}$, we obtain

$$(3\alpha + \beta) |a_0| \sqrt{3i_2} \leq a_1 (3\alpha + \beta) i_2, \quad (61)$$

which demands that $a_0 = 0$. Thus, constant coefficients necessarily lead to the form

$$\mathbf{N} = a\boldsymbol{\sigma}, \quad (62)$$

where a is a positive constant. An admissible generalisation of (62) is to make a a (positive) function of the stress invariants. For instance, $a_1 = a_1(i_1, i_2, i_3)$ will allow for a more or less pronounced elastic regime, and even a distinct yield surface. One may further retain a_0 in order to separate isochoric plasticity from volumetric: $a_0 = (a'_0 - \frac{1}{3}a_1) i_1$ leads to

$$\mathbf{N} = a'_0 i_1 \mathbf{I} + a_1 \left(\boldsymbol{\sigma} - \frac{1}{3} i_1 \mathbf{I} \right), \quad (63)$$

where a_1 and a'_0 are any positive functions of the stress invariants.

Introducing the operators (56) and (62) into (1) gives the constitutive equation

$$\dot{\boldsymbol{\sigma}} = \lambda \mathbf{I} \text{tr} \mathbf{d} + 2\mu \mathbf{d} - \frac{1}{2} \boldsymbol{\sigma} \text{tr} \mathbf{d} - a'_0 i_1 \mathbf{I} \|\mathbf{d}\| - a_1 \left(\boldsymbol{\sigma} - \frac{1}{3} i_1 \mathbf{I} \right) \|\mathbf{d}\| + \mathbf{w} \boldsymbol{\sigma} - \boldsymbol{\sigma} \mathbf{w}. \quad (64)$$

Equation (64) is linear elastic in the limit of small loads and yields increasingly towards higher load levels. The isochoric plastic flow behaviour is controlled by the stress function a_1 and the volumetric plasticity by a'_0 . Equation (64) was integrated here by means of the simple quadratic algorithm

$$\boldsymbol{\sigma}_{n+1} = \boldsymbol{\sigma}_{n-1} + 2\dot{\boldsymbol{\sigma}}_n \Delta t, \quad (65)$$

where Δt is the timestep. Figure 1 shows the response of Eq. (64) to cyclic isochoric shear deformation. The case $a_1 = 0$ is elastic, and thus unloading retraces loading (3 cycles are plotted for each case). When $a_1 \neq 0$ the response exhibits a dissipative loop, which becomes stationary within two cycles. Figure 2 shows the corresponding volumetric response. The case $a'_0 = 0$ is elastic and reversible, while $a'_0 \neq 0$ produces a dissipative loop. Because a_1 and a'_0 are constants here, the isochoric response is a pure shear stress and the volumetric response (Fig. 2) is a purely spherical stress. Moreover, the isochoric response (Fig. 1) is independent of a'_0 , and the volumetric response is independent of a_1 . Cross-coupling between these two modes may, however, be introduced by means of a suitable stress dependence of a'_0 and a_1 . This simple model displays a fairly standard yield behaviour. It may be noted that the quadratic stress energy function used here is one of the simplest possible, yet it permits a fair variety of responses, with the linear elastic response as a baseline.

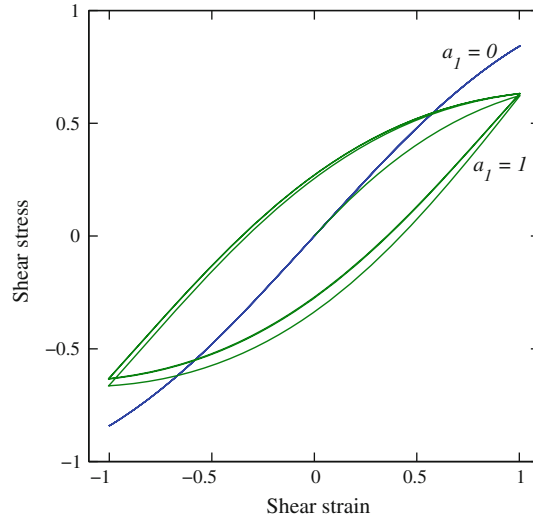


Fig. 1 Equation (64) in reciprocating shear, with $\lambda = \mu = 1$. Three cycles are plotted for each of the cases, $a_1 = 0$ (*thick line*) and $a_1 = 1$ (*thin line*) [beginning at $(0, 0)$]

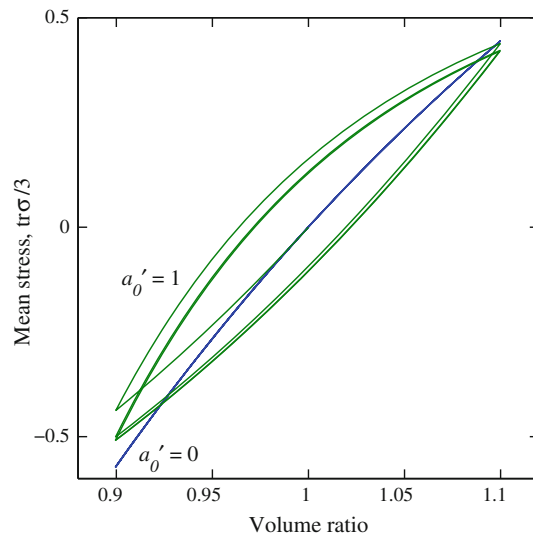


Fig. 2 Equation (64) in cyclic dilation, with $\lambda = \mu = 1$. Three cycles are plotted for each case, $a'_0 = 0$ (*thick line*) and $a'_0 = 1$ (*thin line*) [beginning at $(1, 0)$]

6 Conclusions

A dissipation inequality for hypoplasticity was derived by introducing a stress energy function. Owing to the rate independence of hypoplasticity, the dissipation inequality reduces to a simple condition on the hypoplastic operators and stress energy function (10). This condition defines the complete class of hypoplastic laws (as defined by Eq. (1)) that are thermodynamically consistent. It was further shown that the non-linear operator of the hypoplastic law can be written in a general form, provided that the linear and non-linear operators of the hypoplastic law, along with the stress energy function, are given. A subclass of hypoplasticity was defined, in which the linear operator is *hypoelastic*. This allows a non-dissipative regime and simplifies the construction of the linear and non-linear operators. The example provided shows how a simple hypoplastic law may be constructed with a linear elastic response in the limit of small loads.

Acknowledgments This work was carried out under a grant from the Swedish Research Council (VR).

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