Acta Mechanica 169, 225–226 (2004)
DOI 10.1007/s00707-004-0086-1

Acta Mechanica
Printed in Austria

*Erratum*

# On stresses conjugate to Eulerian strains (Acta Mech. 165, 87–98, 2003)

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Received December 30, 2003
Published online: April 15, 2004 

The author is grateful to Dr. M. Itskov (University of Bayreuth, Germany) who identified and communicated an error in the above titled article. The error is described and corrected below.

The stress conjugate to the left Cauchy-Green (Finger) strain $\mathbf{B}$ is denoted as $\boldsymbol{\Upsilon}$ and is reported in Eq. (35) as $\boldsymbol{\Upsilon} = \frac{\boldsymbol{\tau}\mathbf{B}^{-1}+\mathbf{B}^{-1}\boldsymbol{\tau}}{4}$. The correct answer, shown below, is that $\boldsymbol{\Upsilon}$ is the solution of $\boldsymbol{\tau} = \mathbf{B}\boldsymbol{\Upsilon} + \boldsymbol{\Upsilon}\mathbf{B}$. To sketch the proof of this result, using the notation of the article, we may write the Jaumann flux of $\mathbf{B}$ as

$$\overset{\circ}{\mathbf{B}} = \mathbf{BD} + \mathbf{DB}$$

from which

$$VEC\,\overset{\circ}{\mathbf{B}} = \mathbf{B} \oplus \mathbf{B}VEC(\mathbf{D})$$

and finally

$$VEC(\mathbf{D}) = (\mathbf{B} \oplus \mathbf{B})^{-1} VEC\,\overset{\circ}{\mathbf{B}}. \tag{1}$$

Note that, since $\mathbf{B}$ is positive definite, $\mathbf{B} \oplus \mathbf{B}$ and $(\mathbf{B} \oplus \mathbf{B})^{-1}$ are positive definite.

Next

$$trace(\boldsymbol{\tau}\mathbf{D}) = VEC^T(\boldsymbol{\tau})VEC(\mathbf{D}) = VEC^T(\boldsymbol{\tau})(\mathbf{B} \oplus \mathbf{B})^{-1} VEC\,\overset{\circ}{\mathbf{B}}. \tag{2}$$

If we introduce the conjugate stress $\boldsymbol{\Upsilon}$ such that $VEC^T(\boldsymbol{\Upsilon}) = VEC^T(\boldsymbol{\tau})(\mathbf{B} \oplus \mathbf{B})^{-1}$, it follows that $trace(\boldsymbol{\Upsilon}\overset{\circ}{\mathbf{B}}) = VEC^T(\boldsymbol{\Upsilon})VEC\,\overset{\circ}{\mathbf{B}} = trace(\boldsymbol{\tau}\mathbf{D})$. Accordingly, the conjugate stress satisfies $(\mathbf{B} \otimes \boldsymbol{I} + \boldsymbol{I} \otimes \mathbf{B})VEC(\boldsymbol{\Upsilon}) = VEC(\boldsymbol{\tau})$, implying $\boldsymbol{\tau} = \mathbf{B}\boldsymbol{\Upsilon} + \boldsymbol{\Upsilon}\mathbf{B}$. It is readily seen that $\Upsilon$ must be symmetric.

Numerical Evaluation of $\boldsymbol{\Upsilon}$ involves little more than determining the eigenvalues and eigenvectors of the positive definite tensor $\mathbf{B}$. Note that $\boldsymbol{\Upsilon} = IVEC((\mathbf{B} \oplus \mathbf{B})^{-1}\boldsymbol{\tau})$. The evaluation of $(\mathbf{B} \oplus \mathbf{B})^{-1}$ is facilitated by the fact that its eigenvalues are sums of the eigenvalues of $\mathbf{B}^{-1}$, and its eigenvectors are *Kronecker* products of the eigenvectors of $\mathbf{B}$. Suppose that $\beta_i$ is the $i^{th}$ eigenvalue of $\mathbf{B}$, and $\zeta_i$ is the corresponding eigenvector. The matrix $\mathbf{Q} = [\,\zeta_1 \quad \zeta_2 \quad \zeta_3\,]$ serves to diagonalize $\mathbf{B} : \mathbf{Q}^T\mathbf{B}\mathbf{Q} = \boldsymbol{\Delta}_b$, $\boldsymbol{\Delta}_b = diag(\beta_i)$. Now,

$$(\mathbf{B} \oplus \mathbf{B})^{-1} = [\mathbf{B} \otimes \boldsymbol{I} + \boldsymbol{I} \otimes \mathbf{B}]^{-1} = [\mathbf{Q}^T\boldsymbol{\Delta}_b\mathbf{Q} \otimes \mathbf{Q}^T\mathbf{Q} + \mathbf{Q}^T\mathbf{Q} \otimes \mathbf{Q}^T\boldsymbol{\Delta}_b\mathbf{Q}]^{-1}. \tag{3}$$

Next,

$$\mathbf{Q}^T\boldsymbol{\Delta}_b\mathbf{Q}\otimes\mathbf{Q}^T\mathbf{Q}=\left(\mathbf{Q}^T\otimes\mathbf{Q}^T\right)(\boldsymbol{\Delta}_b\mathbf{Q})\otimes\mathbf{Q}=\left(\mathbf{Q}^T\otimes\mathbf{Q}^T\right)(\boldsymbol{\Delta}_b\mathbf{Q})\otimes(\boldsymbol{I}\mathbf{Q})=\mathbf{Q}^T\otimes\mathbf{Q}^T\boldsymbol{\Delta}_b\otimes\boldsymbol{I}\mathbf{Q}\otimes\mathbf{Q}.$$

Similarly, $\mathbf{Q}^T\mathbf{Q}\otimes\mathbf{Q}^T\boldsymbol{\Delta}_b\mathbf{Q}=\mathbf{Q}^T\otimes\mathbf{Q}^T\boldsymbol{I}\otimes\boldsymbol{\Delta}_b\mathbf{Q}\otimes\mathbf{Q}$.

It follows that

$$\begin{aligned}(\mathbf{B}\oplus\mathbf{B})^{-1}&=[\mathbf{Q}^T\otimes\mathbf{Q}^T(\boldsymbol{\Delta}_b\otimes\boldsymbol{I}+\boldsymbol{I}\otimes\boldsymbol{\Delta}_b)\mathbf{Q}\otimes\mathbf{Q}]^{-1}\\&=\mathbf{Q}^T\otimes\mathbf{Q}^T(\boldsymbol{\Delta}_b\otimes\boldsymbol{I}+\boldsymbol{I}\otimes\boldsymbol{\Delta}_b)^{-1}\mathbf{Q}\otimes\mathbf{Q}.\end{aligned}\tag{4}$$

Also

$$(\boldsymbol{\Delta}_b\otimes\boldsymbol{I}+\boldsymbol{I}\otimes\boldsymbol{\Delta}_b)^{-1}=diag\left(\frac{1}{\beta_1}+\frac{1}{\beta_1},\frac{1}{\beta_1}+\frac{1}{\beta_2},\frac{1}{\beta_1}+\frac{1}{\beta_3},\frac{1}{\beta_2}+\frac{1}{\beta_1},\ldots,\frac{1}{\beta_3}+\frac{1}{\beta_3}\right).\tag{5}$$

Finally, the second line in Eq. (40) in the article quotes the erroneous expression for $\boldsymbol{\Upsilon}$ and should be deleted.

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