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On the number of lattice points in thin sectors

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Abstract
On the circle of radius R centred at the origin, consider a "thin" sector about the fixed line $y = \alpha x$ with edges given by the lines $y = (\alpha \pm \epsilon)x$, where $\epsilon = \epsilon_R \to 0$ as $R \to \infty$. We establish an asymptotic count for $S_\alpha(\epsilon, R)$, the number of integer lattice points lying in such a sector. Our results depend both on the decay rate of ϵ and on the rationality/irrationality type of α. In particular, we demonstrate that if α is Diophantine, then $S_\alpha(\epsilon, R)$ is asymptotic to the area of the sector, so long as $\epsilon R^t \to \infty$ for some $t < 2$.

Keywords Diophantine · Lattice points · Sectors

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1 Introduction

The *Gauss circle problem* is the problem of determining how many integer lattice points lie inside a circle, centred at the origin, with radius $R \to \infty$. This classical problem dates back to Gauss, who employed a simple geometric argument to show that the number of such lattice points is equal to the area of the circle, up to an error term of size $E(R) \leq 2\sqrt{2}\pi R$. In 1906, Sierpiński [8] improved the bound on the error term to $E(R) = O(R^{2/3})$, and further incremental improvements have been subsequently made throughout the years. The current state-of-the-art bound, due to Bourgain and Watt [1], is that $E(R) = O\left(R^{t+\varepsilon}\right)$ for any $\varepsilon > 0$, where $t = 517/824 \approx 0.6274$. It is famously conjectured that $E(R) = O(R^{1/2+\varepsilon})$, for any $\varepsilon > 0$.

A natural related problem is to determine the number of lattice points $S(R)$ inside a *sector* $\text{Sect}(R)$ of a circle with radius $R \to \infty$. For sectors with *fixed* open angle,

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Gauss's argument can be easily extended to show that

$$S(R) = \text{Area}(\text{Sect}(R)) + E(R),$$

where $E(R) = O(R)$. Nowak [7] (who, more generally, considered sectors in domains of the form $\{x^n + y^n \leq R^n : x, y \geq 0\}$ for any $n \geq 2$) showed that the error term can be improved when the slopes of the sector's two respective edges are either rational or irrational of *finite type* (see Definition 1.2). Specifically, when both slopes are *Diophantine* (i.e. of type $\eta = 1 + \varepsilon$ for any $\varepsilon > 0$), we have $E(R) = O(R^{2/3-\delta})$ for a certain (small) $\delta > 0$. Under a suitable assumption on the irrationality type of the edges' slopes, these results were further extended by Kuba [4] tso segments of even more general domains. An additional closely related problem, dating back to the work of Hardy and Littlewood [2, 3], concerns the number of lattice points in right-angled triangles. An asymptotic formula for this count − which plays an important role in the proofs of [4] and [7] − is obtained by applying Koksma's inequality together with standard discrepancy estimates (see, e.g. [5, Theorem 3.2, p. 123 and Theorem 5.1, p. 143]).

In this paper we are interested in counting the number of lattice points, $S_\alpha(\epsilon, R)$, lying within a sector whose open angle *shrinks* as $R \to \infty$. More explicitly, we consider a sector $\text{Sect}_{\alpha,\epsilon}(R)$ about the fixed line $y = \alpha x$ with edges given by the lines $y = (\alpha \pm \epsilon) x$, where now $\epsilon = \epsilon_R \to 0$ as $R \to \infty$. Our main goal is to establish an asymptotic formula for $S_\alpha(\epsilon, R)$ rather than to optimize the relevant error term. In contrast to the case of fixed sectors, our results depend only on the rationality/irrationality type of α, and *not* on the rationality/irrationality type of $\alpha \pm \epsilon$, the slopes of the two edges. For this reason, the results of [4] and [7] are not applicable for our problem, and our argument proceeds in quite a different direction.

If $\epsilon \to 0$ at a rate slower than $1/R$, then upon applying a geometric argument similar to that used in the Gauss circle problem, we find that $S_\alpha(\epsilon, R) \sim \text{Area}\left(\text{Sect}_{\alpha,\epsilon}(R)\right)$ (see Theorem 1.1 below). To obtain an asymptotic count for more quickly shrinking sectors, we must apply an alternative method. First, we approximate $S_\alpha(\epsilon, R)$ by $\Delta_\alpha(\epsilon, R)$, the number of lattice point lying within a thin triangle whose two long edges lie along the lines $y = (\alpha \pm \epsilon) x$. We then fix a rational number $p/q \in \mathbb{Q}$ that well-approximates α, and compute $\Delta_\alpha(\epsilon, R)$ by summing the contributions from lattice points lying on a discrete collection of lines, each of which has rational slope p/q.

When $\alpha \in \mathbb{Q}$ is rational, we obtain an asymptotic for $S_\alpha(\epsilon, R)$, regardless of how fast our sectors shrink. This is due to the fact that, in such a case, all the lattice points in $\text{Sect}_{\alpha,\epsilon}(R)$ lie precisely on the line $y = \alpha x$ once $\text{Sect}_{\alpha,\epsilon}(R)$ is sufficiently thin. If α is irrational of finite type η, we obtain an asymptotic for $S_\alpha(\epsilon, R)$ under the assumption that ϵ decays at a rate slower than $1/R^{1+1/\eta}$ (Theorem 1.3 below). Specifically, when $\alpha \in \mathbb{R}$ is Diophantine, we obtain an asymptotic for $S_\alpha(\epsilon, R)$ under the assumption that $\epsilon \to 0$ at a rate slower than $1/R^t$ for some $t < 2$.

The behaviour of lattice points in even faster shrinking sectors about irrational slopes is a more subtle question. If ϵ decays at a rate $1/R^{1+1/\eta}$ or faster, the above method fails to produce an asymptotic count for $S_\alpha(\epsilon, R)$. However, if ϵ shrinks *sufficiently* quickly, then the count once again becomes much simpler. Specifically, when ϵ decays faster than $1/R^{1+\eta}$, we may apply an elementary argument to prove that for sufficiently large R, $\mathrm{Sect}_{\alpha,\epsilon}(R)$ contains no lattice points whatsoever (Proposition 1.7). A related question concerns the distribution of lattice points in a *randomly* chosen sector of width $\epsilon \asymp 1/R^2$. This interesting question has been addressed by Marklof and Strömbergsson [6], who successfully applied tools from homogeneous dynamics to prove the existence of a (non-Poissonian) limiting distribution for the number of lattice points in such sectors.

1.1 Notation

Fix $\alpha \in \mathbb{R}$, and consider the interval $I_\epsilon(\alpha) := (\alpha - \epsilon, \alpha + \epsilon)$, for some $\epsilon > 0$. Let

$$\mathrm{Sect}_{\alpha,\epsilon}(R) := \{(x, y) \in \mathbb{R}_{>0} \times \mathbb{R} : x^2 + y^2 \le R^2,\ y/x \in I_\epsilon(\alpha)\}$$

denote the sector of radius R with edges given by the lines $y = (\alpha \pm \epsilon)\,x$, which has an open angle of size

$$\theta := \tan^{-1}(\alpha + \epsilon) - \tan^{-1}(\alpha - \epsilon).$$

In what follows, we view $\epsilon = \epsilon_R$ as a function of R. Our main interest will be in *thin* sectors, i.e. when $\epsilon \to 0$ as $R \to \infty$. Taylor expanding about α, we find that as $\epsilon \to 0$, the area of $\mathrm{Sect}_{\alpha,\epsilon}(R)$ is equal to

$$\mathrm{Area}(\mathrm{Sect}_{\alpha,\epsilon}(R)) = \frac{R^2}{2}\left(\tan^{-1}(\alpha + \epsilon) - \tan^{-1}(\alpha - \epsilon)\right) = \frac{\epsilon R^2}{1+\alpha^2} + O\left(\epsilon^3 R^2\right).$$

Let

$$\begin{aligned} S_\alpha(\epsilon, R) &:= \#\{\mathbb{Z}^2 \cap \mathrm{Sect}_{\alpha,\epsilon}(R)\} \\ &= \#\{(m, n) \in \mathbb{Z}_{>0} \times \mathbb{Z} : m^2 + n^2 \le R^2,\ |n - \alpha m| < m\epsilon\} \end{aligned}$$

count the number of integer lattice points in $\mathrm{Sect}_{\alpha,\epsilon}(R)$.

We are interested in the value of $S_\alpha(\epsilon, R)$ in the limit as $R \to \infty$. For example, we may consider the case $\epsilon := R^{-\lambda}$ for some fixed $\lambda \ge 0$. We then classify our sectors based on the decay rate of ϵ.

Remark Our results may be easily extended to more general sectors about the line $y = \alpha x$. In particular, we note that Theorem 1.3 continues to hold when counting lattice points in any sector of the form

$$\mathrm{Sect}_{\alpha,\epsilon_1,\epsilon_2}(R) := \{(x, y) \in \mathbb{R}_{>0} \times \mathbb{R} : x^2 + y^2 \le R^2,\ y/x \in (\alpha - \epsilon_1, \alpha + \epsilon_2)\},$$

where, say, $\epsilon_1 \asymp \epsilon_2 \asymp \epsilon$. Consequently, one may alternatively consider a sector centred about the angle $\Phi := \tan^{-1}(\alpha)$ with radius R and open angle $\theta \asymp \epsilon$; and express the resulting lattice point count in terms of the properties of $\tan\Phi$ and the decay rate of θ

without any alterations to Theorem 1.3. Nonetheless, we have chosen to formulate our results in terms of slopes, rather than angles, in order to simplify the exposition, and because our analysis naturally depends upon the Diophantine properties of the slope α.

1.2 "Slowly" shrinking sectors

Suppose first that ϵ is either fixed or decays slower than $1/R$, in the sense that $\epsilon R \to \infty$ in the limit as $R \to \infty$ (e.g. $0 \leq \lambda < 1$). Upon refining the elementary geometric argument of the $O(R)$ bound for the error term in the Gauss circle problem, we obtain the following result, which yields an asymptotic count for the number of lattice points in such slowly shrinking sectors:

Theorem 1.1 *Fix $\alpha \in \mathbb{R}$, and assume that $\epsilon R \to \infty$ as $R \to \infty$. Then*

$$S_\alpha(\epsilon, R) = \text{Area}(\text{Sect}_{\alpha,\epsilon}(R)) + O(R). \tag{1.1}$$

1.3 "Quickly" shrinking sectors

In our investigation of more quickly shrinking sectors, our results depend heavily upon the rationality/irrationality type of α, defined as follows:

Definition 1.2 We say that an irrational $\alpha \in \mathbb{R}$ is of finite **type** η, if there exists a constant $c = c(\alpha) > 0$ such that

$$\left|\alpha - \frac{p}{q}\right| > \frac{c}{q^{1+\eta}}$$

for all integers pairs $(p, q) \in \mathbb{Z} \times \mathbb{Z}_{>0}$.

Note that for irrational $\alpha \in \mathbb{R}$ of type η we necessarily have $\eta \geq 1$ by Dirichlet's theorem. We say that α is **Diophantine** if $\alpha \in \mathbb{R}$ is irrational of type $\eta = 1 + \varepsilon$ for every $\varepsilon > 0$. It is well-known that almost all $\alpha \in \mathbb{R}$ are Diophantine (Khinchin's theorem), and every algebraic number is Diophantine (Roth's theorem).

1.3.1 Irrational slopes

For irrational $\alpha \in \mathbb{R}$, our main result is as follows:

Theorem 1.3 *Let $\alpha \in \mathbb{R}$ be irrational of finite type η, and assume that $\epsilon \to 0$, as well as that $\epsilon R^{1+1/\eta} \to \infty$ as $R \to \infty$. Then*

$$S_\alpha(\epsilon, R) = \text{Area}(\text{Sect}_{\alpha,\epsilon}(R)) + O\left(\epsilon^{\frac{1}{1+\eta}} R + R^2\epsilon^2\right),$$

in the limit as $R \to \infty$.

The conditions $\epsilon \to 0$ and $\epsilon R^{1+1/\eta} \to \infty$ (e.g. $0 < \lambda < 1 + 1/\eta$) consequently guarantee the asymptotic

$$S_\alpha(\epsilon, R) \sim \mathrm{Area}(\mathrm{Sect}_{\alpha,\epsilon}(R)). \tag{1.2}$$

In particular, if α is Diophantine, then (1.2) holds whenever $\epsilon \to 0$ and $\epsilon R^t \to \infty$ for some $t < 2$ (e.g. $0 < \lambda < 2$). Note furthermore that $\mathrm{Area}(\mathrm{Sect}_{\alpha,\epsilon}(R))$ grows if and only if $\epsilon R^2 \to \infty$, and thus our results in such a case are essentially optimal (and "strictly" so whenever $\alpha \in \mathbb{R}$ is a badly approximable irrational, i.e. irrational of type $\eta = 1$).

Note that Theorem 1.3 gives a better error term than (1.1) whenever $\epsilon = o(R^{-1/2})$ and $\epsilon R \to \infty$ (e.g. $1/2 < \lambda < 1$). Upon comparing the error terms in Theorem 1.1 and Theorem 1.3 we obtain the following corollary:

Corollary 1.4 *Let $\alpha \in \mathbb{R}$ be irrational of finite type η, and let $\epsilon = R^{-\lambda}$. Then in the limit as $R \to \infty$,*

$$S_\alpha(\epsilon, R) = \begin{cases} \mathrm{Area}(\mathrm{Sect}_{\alpha,\epsilon}(R)) + O(R), & 0 \le \lambda < \frac{1}{2} \\ \mathrm{Area}(\mathrm{Sect}_{\alpha,\epsilon}(R)) + O\left(R^{2-2\lambda}\right), & \frac{1}{2} \le \lambda < \frac{1+\eta}{1+2\eta} \\ \mathrm{Area}(\mathrm{Sect}_{\alpha,\epsilon}(R)) + O\left(R^{1-\frac{\lambda}{1+\eta}}\right), & \frac{1+\eta}{1+2\eta} \le \lambda < 1 + \frac{1}{\eta}. \end{cases} \tag{1.3}$$

In particular, when $\alpha \in \mathbb{R}$ is Diophantine, Corollary 1.4 yields

$$S_\alpha(\epsilon, R) = \begin{cases} \mathrm{Area}(\mathrm{Sect}_{\alpha,\epsilon}(R)) + O(R), & 0 \le \lambda < \frac{1}{2} \\ \mathrm{Area}(\mathrm{Sect}_{\alpha,\epsilon}(R)) + O\left(R^{2-2\lambda}\right), & \frac{1}{2} \le \lambda < \frac{2}{3} \\ \mathrm{Area}(\mathrm{Sect}_{\alpha,\epsilon}(R)) + O_\delta\left(R^{1-\lambda/2+\delta}\right), & \frac{2}{3} \le \lambda < 2. \end{cases} \tag{1.4}$$

1.3.2 Rational slopes

For rational $\alpha \in \mathbb{Q}$, we obtain the following result:

Theorem 1.5 *Fix $\alpha = p/q \in \mathbb{Q}$, where $q > 0$ and $(p, q) = 1$. Then in the limit as $R \to \infty$, we have*

$$S_\alpha(\epsilon, R) = \frac{\epsilon q^2 R^2}{p^2 + q^2} + \frac{1}{q^2 \epsilon} \left\{ \frac{\epsilon q^2 R}{\sqrt{p^2+q^2}} \right\} \left(1 - \left\{ \frac{\epsilon q^2 R}{\sqrt{p^2+q^2}} \right\} \right) + O\left(1 + (R\epsilon)^2\right),$$

where $\{x\} := x - \lfloor x \rfloor$ denotes the fractional part of x.

When $\epsilon = o(R^{-1})$ (e.g. $\lambda > 1$), Theorem 1.5 simplifies to

$$S_\alpha(\epsilon, R) = \frac{R}{\sqrt{p^2+q^2}} + O(1).$$

In this case, $S_\alpha(\epsilon, R)$ is no longer asymptotic to $\text{Area}(\text{Sect}_{\alpha,\epsilon}(R))$, and the only points that contribute to $S_\alpha(\epsilon, R)$ are those which lie precisely on the line $y = \alpha x$.

When $\epsilon \to 0$ and $\epsilon R \to \infty$ (e.g. $0 < \lambda < 1$), Theorem 1.5 yields

$$S_\alpha(\epsilon, R) = \text{Area}(\text{Sect}_{\alpha,\epsilon}(R)) + \beta/\epsilon + O(\epsilon^2 R^2), \tag{1.5}$$

where

$$\beta := \frac{1}{q^2}\left\{\frac{\epsilon q^2 R}{\sqrt{p^2+q^2}}\right\}\left(1 - \left\{\frac{\epsilon q^2 R}{\sqrt{p^2+q^2}}\right\}\right)$$

is a bounded function of R. In particular, as in the case of irrational slopes, if $\epsilon = o(R^{-1/2})$ and $\epsilon R \to \infty$ (e.g. $1/2 < \lambda < 1$), then (1.5) yields a more precise count than (1.1). The following corollary summarizes the above analysis in the case $\epsilon = R^{-\lambda}$:

Corollary 1.6 *Let $\alpha = p/q \in \mathbb{Q}$, where $q > 0$ and $(p, q) = 1$, and let $\epsilon = R^{-\lambda}$. Then in the limit as $R \to \infty$, we have*

$$S_\alpha(\epsilon, R) = \begin{cases} \text{Area}(\text{Sect}_{\alpha,\epsilon}(R)) + O(R), & 0 \le \lambda \le \frac{1}{2} \\ \text{Area}(\text{Sect}_{\alpha,\epsilon}(R)) + O\left(R^{2-2\lambda}\right), & \frac{1}{2} < \lambda \le \frac{2}{3} \\ \text{Area}(\text{Sect}_{\alpha,\epsilon}(R)) + \beta R^{\lambda} + O\left(R^{2-2\lambda}\right), & \frac{2}{3} < \lambda < 1 \\ \frac{R}{\sqrt{p^2+q^2}} + O(1), & 1 < \lambda. \end{cases} \tag{1.6}$$

Finally, we consider the case $\epsilon \asymp R^{-1}$ (e.g. $\epsilon R = c$, for some $c \in \mathbb{R}_{>0}$). Then Theorem 1.5 yields

$$S_\alpha(\epsilon, R) = \gamma R + O(1),$$

where

$$\gamma := \frac{\epsilon q^2 R}{p^2+q^2} + \frac{1}{\epsilon q^2 R}\left\{\frac{\epsilon q^2 R}{\sqrt{p^2+q^2}}\right\}\left(1 - \left\{\frac{\epsilon q^2 R}{\sqrt{p^2+q^2}}\right\}\right).$$

In particular, whenever $\epsilon < \frac{\sqrt{p^2+q^2}}{q^2 R}$, the only points which contribute to $S_\alpha(\epsilon, R)$ are those which lie precisely on the line $y = \alpha x$, and we find that

$$\gamma = \frac{1}{\sqrt{p^2+q^2}}.$$

We moreover note that $S_\alpha(\epsilon, R)$ is asymptotic to $\text{Area}(\text{Sect}_{\alpha,\epsilon}(R))$ if and only if $\gamma = \frac{\epsilon q^2 R}{p^2+q^2}$, i.e. if and only if ϵ is an integer multiple of $\frac{\sqrt{p^2+q^2}}{q^2 R}$.

1.4 "Very quickly" shrinking sectors

While in the range $R^{-1-\eta} \ll \epsilon \ll R^{-(1+1/\eta)}$ we are unable to obtain an asymptotic formula for $S_\alpha(\epsilon, R)$, for sectors that shrink even more quickly the situation becomes rather trivial. Specifically, whenever $\epsilon = o(R^{-1-\eta})$ (e.g. $\lambda > 1+\eta$), we show that $S_\alpha(\epsilon, R) = 0$ for sufficiently large R:

Proposition 1.7 *Let $\alpha \in \mathbb{R}$ be irrational of finite type η, and suppose that $\epsilon = o(R^{-1-\eta})$. Then there exists $R_0 > 0$ such that for all $R > R_0$,*

$$S_\alpha(\epsilon, R) = 0.$$

In particular, if α is a Diophantine irrational, then for sufficiently large R, $S_\alpha(\epsilon, R) = 0$ whenever $\epsilon = o(R^{-t})$ for some $t > 2$ (e.g. $\lambda > 2$).

1.5 Structure of paper

The remainder of this paper is structured as follows. In Sect. 2 we apply a simple geometric argument to compute $S_\alpha(\epsilon, R)$ in the case that $\epsilon \to 0$ at a rate slower than $1/R$. In Sect. 3 we approximate $S_\alpha(\epsilon, R)$ by $\Delta_\alpha(\epsilon, R)$, i.e. by the number of lattice points in a triangle whose two long edges lie along the lines $y = (\alpha \pm \epsilon)x$. In Sect. 4 we then apply this approximation to compute $S_\alpha(\epsilon, R)$ when $\alpha \in \mathbb{R}$ is irrational of finite type; and in Sect. 5 we address the case when $\alpha \in \mathbb{Q}$ is rational. Finally, in Sect. 6, we address the case in which $\mathrm{Sect}_{\alpha,\epsilon}(R)$ shrinks "very quickly", i.e. when $\epsilon \to 0$ at a rate faster than $1/R^{1+\eta}$.

2 Lattice points in slowly shrinking sectors

In this section we provide a proof of Theorem 1.1, namely a count for $S_\alpha(\epsilon, R)$ when $\epsilon R \to \infty$ as $R \to \infty$. The proof is an easy adaptation of the elementary geometric argument applied in the classical Gauss circle problem. As evidenced by the proof, this argument remains valid for slowly shrinking sectors.

Proof of Theorem 1.1 For each $z \in \mathbb{Z}^2 \cap \mathrm{Sect}_{\alpha,\epsilon}(R)$, let $\Box_z$ denote a square-box of unit area, centred at the point z. Then

$$S_\alpha(\epsilon, R) = \mathrm{Area}\left(\bigcup_{z \in \mathbb{Z}^2 \cap \mathrm{Sect}_{\alpha,\epsilon}(R)} \Box_z\right),$$

i.e. $S_\alpha(\epsilon, R)$ is equal to the area formed by the union of such boxes. Note, moreover, that if $w \in \Box_z$ for some $z \in \mathbb{Z}^2 \cap \mathrm{Sect}_{\alpha,\epsilon}(R)$, then

$$\mathrm{dist}(w, \mathrm{Sect}_{\alpha,\epsilon}(R)) \le \sqrt{2}/2,$$

i.e. the distance between w and $\mathrm{Sect}_{\alpha,\epsilon}(R)$ is bounded by $\sqrt{2}/2$. We therefore define a wider sector, $\mathrm{Sect}^{+}_{\alpha,\epsilon}(R')$, with the same open angle and direction as $\mathrm{Sect}_{\alpha,\epsilon}(R)$, but extended by a distance of $\sqrt{2}/2$ on all sides, so that

$$\bigcup_{z\in\mathbb{Z}^2\cap\mathrm{Sect}_{\alpha,\epsilon}(R)} \square_z \subseteq \mathrm{Sect}^{+}_{\alpha,\epsilon}(R').$$

To construct $\mathrm{Sect}^{+}_{\alpha,\epsilon}(R')$ explicitly, we expand $\mathrm{Sect}_{\alpha,\epsilon}(R)$ by drawing parallel lines distanced $d = \sqrt{2}/2$ away from each of its two respective straight edges. Let x denote the distance between their point of intersection and the origin. Note that

$$x \cdot \sin\frac{\theta}{2} = \frac{\sqrt{2}}{2},$$

from which we obtain

$$x = \frac{\sqrt{2}}{2\cdot\sin\frac{\theta}{2}} = \frac{\sqrt{2}}{\theta + O(\theta^3)} = \frac{\sqrt{2}}{\theta} + O(\theta).$$

We therefore set the radius of our desired sector, $\mathrm{Sect}^{+}_{\alpha,\epsilon}(R')$, to be equal to

$$R' = R + \frac{\sqrt{2}}{2\cdot\sin\frac{\theta}{2}} + \frac{\sqrt{2}}{2} = R + \frac{\sqrt{2}}{\theta} + O(1),$$

which yields

$$\begin{aligned}\mathrm{Area}\left(\mathrm{Sect}^{+}_{\alpha,\epsilon}(R')\right) &= \frac{\theta}{2}\cdot(R')^2 = \frac{\theta}{2}\cdot\left(R + \frac{\sqrt{2}}{\theta} + O(1)\right)^2 \\ &= \mathrm{Area}(\mathrm{Sect}_{\alpha,\epsilon}(R)) + O\left(R\right),\end{aligned}$$

upon noting that $\theta \asymp \epsilon$, so that $\theta^{-1} = o(R)$. Thus

$$\begin{aligned}S_\alpha(\epsilon, R) &= \mathrm{Area}\left(\bigcup_{z\in\mathbb{Z}^2\cap\mathrm{Sect}_{\alpha,\epsilon}(R)} \square_z\right) \le \mathrm{Area}\left(\mathrm{Sect}^{+}_{\alpha,\epsilon}(R')\right) \\ &= \mathrm{Area}(\mathrm{Sect}_{\alpha,\epsilon}(R)) + O\left(R\right). \end{aligned} \tag{2.1}$$

To obtain a lower bound for $S_\alpha(\epsilon, R)$, we similarly construct a sector, denoted by $\mathrm{Sect}^{-}_{\alpha,\epsilon}(R'')$, with the same open angle and direction as $\mathrm{Sect}_{\alpha,\epsilon}(R)$, but now *shrunk* by a distance of $\sqrt{2}/2$ on all sides, of radius

$$R'' = R - \frac{\sqrt{2}}{2\cdot\sin\frac{\theta}{2}} - \frac{\sqrt{2}}{2},$$

which we note is clearly possible since $\theta^{-1} = o(R)$. Any point $w \in \mathrm{Sect}^{-}_{\alpha,\epsilon}(R'')$ is within a distance of at most $\sqrt{2}/2$ from some lattice point z, which, by construction, must lie in $\mathrm{Sect}_{\alpha,\epsilon}(R)$. It follows that

$$\mathrm{Sect}^{-}_{\alpha,\epsilon}(R'') \subseteq \left(\bigcup_{z\in\mathbb{Z}^2\cap\mathrm{Sect}_{\alpha,\epsilon}(R)} \square_z \right).$$

Using a similar analysis to that above, we find that

$$\mathrm{Area}\left(\mathrm{Sect}^{-}_{\alpha,\epsilon}(R'')\right) = \frac{\theta}{2} \cdot R^2 + O\left(R\right),$$

and therefore

$$S_\alpha(\epsilon, R) \geq \mathrm{Area}\left(\mathrm{Sect}^{-}_{\alpha,\epsilon}(R'')\right) = \mathrm{Area}(\mathrm{Sect}_{\alpha,\epsilon}(R)) + O\left(R\right). \tag{2.2}$$

Combining (2.1) and (2.2) we conclude that

$$S_\alpha(\epsilon, R) = \mathrm{Area}(\mathrm{Sect}_{\alpha,\epsilon}(R)) + O\left(R\right), \tag{2.3}$$

as desired. □

3 Approximating sectors by triangles

In this section we approximate $S_\alpha(\epsilon, R)$ by considering lattice points in a triangle, namely the summation

$$\Delta_\alpha(\epsilon, R) := \sum_{1\leq m\leq \frac{R}{\sqrt{1+\alpha^2}}} \#\{n \in \mathbb{Z} : m(\alpha - \epsilon) < n < m(\alpha + \epsilon)\}$$

We have the following lemma:

Lemma 3.1 *Assume that* $\epsilon \to 0$. *Then*

$$S_\alpha(\epsilon, R) = \Delta_\alpha(\epsilon, R) + O\left(1 + (R\epsilon)^2\right)$$

In particular, if $\epsilon = O(R^{-1})$, *then*

$$S_\alpha(\epsilon, R) = \Delta_\alpha(\epsilon, R) + O(1).$$

Proof Assume $\alpha > 0$, as the proof for the cases $\alpha = 0$ and $\alpha < 0$ follow similarly. Suppose $(m, n) \in S_\alpha(\epsilon, R)$. Then $m^2 + n^2 \leq R^2$ and $n > m(\alpha - \epsilon) > 0$ (which holds for sufficiently small ϵ) together imply

$$m^2\left(1+(\alpha-\epsilon)^2)\right) \leq R^2,$$

i.e. that

$$m \leq \frac{R}{\sqrt{1+(\alpha-\epsilon)^2}}.$$

We may therefore write

$$S_\alpha(\epsilon, R) = S_\alpha^1(\epsilon, R) - S_\alpha^2(\epsilon, R),$$

where

$$S_\alpha^1(\epsilon, R) := \sum_{1\leq m\leq \frac{R}{\sqrt{1+(\alpha-\epsilon)^2}}} \#\{n : m(\alpha-\epsilon) < n < m(\alpha+\epsilon)\}$$

and

$$S_\alpha^2(\epsilon, R) := \sum_{1\leq m\leq \frac{R}{\sqrt{1+(\alpha-\epsilon)^2}}} \#\{n : m(\alpha-\epsilon) < n < m(\alpha+\epsilon), m^2+n^2 > R^2\}.$$

Let us first estimate the size of $S_\alpha^2(\epsilon, R)$. Note that if $m^2+n^2 > R^2$ and $m(\alpha-\epsilon) < n < m(\alpha+\epsilon)$, then $m^2(1+(\alpha+\epsilon)^2) > R^2$, and therefore $m > R/\sqrt{1+(\alpha+\epsilon)^2}$. Moreover, since the length of the interval $(m(\alpha-\epsilon), m(\alpha+\epsilon))$ is $2m\epsilon \leq 2R\epsilon$, we find that, for any $m \in \mathbb{N}$, there exist at most $O(1+R\epsilon)$ integers $n \in \mathbb{Z}$ such that

$$m(\alpha-\epsilon) < n < m(\alpha+\epsilon).$$

Thus

$$\begin{aligned} S_\alpha^2(\epsilon, R) &\ll \sum_{\frac{R}{\sqrt{1+(\alpha+\epsilon)^2}} < m \leq \frac{R}{\sqrt{1+(\alpha-\epsilon)^2}}} (1+R\epsilon) \\ &\leq (1+R\epsilon)\left(1+\frac{R}{\sqrt{1+(\alpha-\epsilon)^2}} - \frac{R}{\sqrt{1+(\alpha+\epsilon)^2}}\right). \end{aligned}$$

Note furthermore that

$$\sqrt{1+(\alpha\pm\epsilon)^2} = \sqrt{1+\alpha^2}(1+O(\epsilon)).$$

It follows that

$$\frac{R}{\sqrt{1+(\alpha-\epsilon)^2}} - \frac{R}{\sqrt{1+(\alpha+\epsilon)^2}} = \frac{R}{\sqrt{1+\alpha^2}}(1+O(\epsilon)) - \frac{R}{\sqrt{1+\alpha^2}}(1+O(\epsilon))$$
$$= O(R\epsilon).$$

Hence

$$S_\alpha^2(\epsilon, R) \ll (1+R\epsilon)^2 \ll 1+(R\epsilon)^2,$$

from which we obtain that

$$S_\alpha(\epsilon, R) = S_\alpha^1(\epsilon, R) + O\left(1+(R\epsilon)^2\right).$$

Next, we wish to show that

$$S_\alpha^1(\epsilon, R) = \Delta_\alpha(\epsilon, R) + O\left(1+(R\epsilon)^2\right).$$

Indeed, note that

$$S_\alpha^1(\epsilon, R) - \Delta_\alpha(\epsilon, R) = \sum_{\frac{R}{\sqrt{1+\alpha^2}} < m \le \frac{R}{\sqrt{1+(\alpha-\epsilon)^2}}} \#\{n : m(\alpha-\epsilon) < n < m(\alpha+\epsilon))\}, \tag{3.1}$$

and that each summand in (3.1) is $O(1+R\epsilon)$. It follows that

$$S_\alpha^1(\epsilon, R) - \Delta_\alpha(\epsilon, R) \ll (1+R\epsilon)\cdot\left(1+\frac{R}{\sqrt{1+(\alpha-\epsilon)^2}} - \frac{R}{\sqrt{1+\alpha^2}}\right)$$
$$\ll (1+R\epsilon)^2 \ll 1+(R\epsilon)^2,$$

as desired. □

4 Sectors about irrational slopes

In this section we provide a proof of Theorem 1.3, namely a count for $S_\alpha(\epsilon, R)$ when $\alpha \in \mathbb{R}$ is irrational of finite type.

Let $\alpha \in \mathbb{R}$ be irrational. For any rational $p/q \in \mathbb{Q}$, we define $\delta := \alpha - p/q$. For the purposes of this proof, we will moreover assume that $|\delta| < \epsilon/2$, which, in particular, implies that $\delta - \epsilon < 0$ and $\epsilon + \delta > 0$. We then write

$$\Delta_\alpha(\epsilon, R) = \{(m,n) \in \mathbb{Z}^2 : |n/m - \alpha| < \epsilon, \quad 1 \le m \le R/\sqrt{1+\alpha^2}\}$$
$$= \{(m,n) \in \mathbb{Z}^2 : -\epsilon+\delta < n/m - p/q < \epsilon+\delta, \quad 1 \le m \le R/\sqrt{1+\alpha^2}\}$$
$$= \{(m,n) \in \mathbb{Z}^2 : mq(\delta-\epsilon) < nq - mp < (\epsilon+\delta)mq, \quad 1 \le m \le R/\sqrt{1+\alpha^2}\}.$$

Let $d = nq - mp$, so that

$$(\delta - \epsilon)mq < d < (\epsilon + \delta)mq.$$

Together with the conditions on m, this implies that

$$\frac{(\delta - \epsilon)qR}{\sqrt{1+\alpha^2}} \leq d \leq \frac{(\epsilon + \delta)qR}{\sqrt{1+\alpha^2}}.$$

When $d > 0$, the condition on m is equivalent to

$$\frac{d}{(\delta + \epsilon)q} < m \leq \frac{R}{\sqrt{1+\alpha^2}},$$

while when $d < 0$, the condition is then

$$\frac{d}{(\delta - \epsilon)q} < m \leq \frac{R}{\sqrt{1+\alpha^2}}.$$

Partitioning with respect to d, we then write

$$\Delta_\alpha(\epsilon, R) = \Delta_\alpha^+(\epsilon, R) + \Delta_\alpha^0(\epsilon, R) + \Delta_\alpha^-(\epsilon, R), \tag{4.1}$$

with

$$\begin{aligned}
\Delta_\alpha^+(\epsilon, R) &:= \sum_{0<d\leq \frac{(\epsilon+\delta)Rq}{\sqrt{1+\alpha^2}}} \sum_{\substack{\frac{d}{(\epsilon+\delta)q} < m \leq \frac{R}{\sqrt{1+\alpha^2}} \\ m \equiv -d\bar{p}\ (q)}} 1 \\
\Delta_\alpha^-(\epsilon, R) &:= \sum_{\frac{(\delta-\epsilon)Rq}{\sqrt{1+\alpha^2}} \leq d < 0} \sum_{\substack{\frac{d}{q(\delta-\epsilon)} < m \leq \frac{R}{\sqrt{1+\alpha^2}} \\ m \equiv -d\bar{p}\ (q)}} 1, \\
\Delta_\alpha^0(\epsilon, R) &:= \sum_{\substack{1 \leq m \leq \frac{R}{\sqrt{1+\alpha^2}} \\ m \equiv 0\ (q)}} 1,
\end{aligned}$$

where $\bar{p}$ denotes the inverse of p modulo q. Upon recalling that

$$\sum_{0<d\leq x} d = \frac{\lfloor x \rfloor (\lfloor x \rfloor + 1)}{2} = \frac{1}{2}(x + O(1))^2,$$

we see that

$$\begin{aligned}\Delta_\alpha^+(\epsilon, R) &= \sum_{0<d\le\frac{(\epsilon+\delta)Rq}{\sqrt{1+\alpha^2}}} \left(\frac{1}{q}\left(\frac{R}{\sqrt{1+\alpha^2}} - \frac{d}{(\epsilon+\delta)q}\right) + O(1)\right)\\ &= \frac{R}{q\sqrt{1+\alpha^2}}\left(\frac{(\epsilon+\delta)Rq}{\sqrt{1+\alpha^2}} + O(1)\right)\\ &\quad - \frac{1}{2(\epsilon+\delta)q^2}\left(\frac{(\epsilon+\delta)Rq}{\sqrt{1+\alpha^2}} + O(1)\right)^2 + O(\epsilon q R)\\ &= \frac{1}{2}\frac{R^2(\epsilon+\delta)}{1+\alpha^2} + O\left(\frac{R}{q} + \frac{1}{\epsilon q^2} + \epsilon q R\right).\end{aligned}$$

Similarly, we compute

$$\begin{aligned}\Delta_\alpha^-(\epsilon, R) &= \sum_{\frac{(\delta-\epsilon)Rq}{\sqrt{1+\alpha^2}}\le d<0} \left(\frac{1}{q}\left(\frac{R}{\sqrt{1+\alpha^2}} - \frac{d}{(\delta-\epsilon)q}\right) + O(1)\right)\\ &= \sum_{0<d\le\frac{(\epsilon-\delta)Rq}{\sqrt{1+\alpha^2}}} \left(\frac{1}{q}\left(\frac{R}{\sqrt{1+\alpha^2}} - \frac{d}{(\epsilon-\delta)q}\right) + O(1)\right)\\ &= \frac{1}{2}\frac{R^2(\epsilon-\delta)}{1+\alpha^2} + O\left(\frac{R}{q} + \frac{1}{\epsilon q^2} + \epsilon q R\right).\end{aligned}$$

Finally, we note that

$$\Delta_\alpha^0(\epsilon, R) = \frac{R}{q\sqrt{1+\alpha^2}} + O(1). \tag{4.2}$$

It then follows from (4.1) that

$$\Delta_\alpha(\epsilon, R) = \frac{\epsilon R^2}{(1+\alpha^2)} + O\left(\frac{R}{q} + \frac{1}{\epsilon q^2} + \epsilon q R + 1\right). \tag{4.3}$$

4.1 Choosing an appropriate convergent

Suppose $\alpha \in \mathbb{R}$ is irrational of finite type η, and let $\{p_i/q_i\}_{i=1}^{\infty}$ denote the sequence of convergents to the continued fraction of α. Upon choosing an appropriate pair $\{p_i/q_i\}$, we are able to proceed with a proof of Theorem 1.3:

Proof of Theorem 1.3 For any $X := X(R)$, there exists a unique i such that $q_i \le X < q_{i+1}$. There moreover exists a $c = c(\alpha) > 0$ such that

$$\frac{c}{q_i^{1+\eta}} < \left|\alpha - \frac{p_i}{q_i}\right| < \frac{1}{q_i q_{i+1}}. \tag{4.4}$$

Hence

$$X < q_{i+1} < \frac{1}{c} \cdot q_i^{\eta},$$

which further implies that

$$X^{\frac{1}{\eta}} < q_{i+1}^{\frac{1}{\eta}} < c^{-\frac{1}{\eta}} \cdot q_i.$$

In other words, there exists a constant $C > 0$ such that

$$X^{\frac{1}{\eta}} < C \cdot q_i.$$

Let $p = p_i$ and $q = q_i$. By (4.4), it follows that

$$|\delta| = \left|\alpha - \frac{p_i}{q_i}\right| < \frac{1}{q_i q_{i+1}} < \frac{1}{q_i X} < C \cdot \frac{1}{X^{1+1/\eta}}. \tag{4.5}$$

To ensure that $|\delta| < \epsilon/2$, we choose X such that

$$C \cdot X^{-(1+1/\eta)} \leq \frac{\epsilon}{2}, \tag{4.6}$$

namely, we subject X to the restriction

$$\epsilon^{-\frac{1}{1+1/\eta}} \ll X. \tag{4.7}$$

To *optimize* our error term, we seek a choice of X, subject to the restriction (4.7), which minimizes the value of

$$O\left(\frac{R}{q} + \frac{1}{\epsilon q^2} + \epsilon q R + 1\right).$$

Note first that by (4.5) and (4.6),

$$\frac{1}{qX} \leq C \cdot \frac{1}{X^{1+1/\eta}} \leq \frac{\epsilon}{2}, \tag{4.8}$$

which in turn implies that

$$\frac{1}{q^2 X^2} \leq \frac{\epsilon^2}{4},$$

and therefore that

$$\frac{1}{\epsilon q^2} \leq \frac{\epsilon X^2}{4}. \tag{4.9}$$

Similarly, since (4.8) implies $q^{-1} \leq \epsilon X/2$, we find that

$$\frac{R}{q} = O\left(\epsilon X R\right). \tag{4.10}$$

Next, since $q \leq X$, it follows that

$$\epsilon q R = O\left(\epsilon X R\right), \tag{4.11}$$

and finally it similarly follows from (4.8) that

$$1 \leq \frac{X}{q} = O\left(\epsilon X^2\right). \tag{4.12}$$

By (4.9), (4.10), (4.11), and (4.12), we have that

$$O\left(\frac{R}{q} + \frac{1}{\epsilon q^2} + \epsilon q R + 1\right) = O\left(\epsilon X^2 + \epsilon X R\right).$$

We thus choose the minimal possible value for X, namely $X \asymp \epsilon^{-\frac{1}{1+1/\eta}}$, which is moreover $o(R)$ by the assumption that $\epsilon R^{1+1/\eta} \to \infty$. In particular,

$$\epsilon X^2 + \epsilon X R \ll \epsilon X R \ll \epsilon^{\frac{1}{1+\eta}} R. \tag{4.13}$$

By (4.3) and (4.13), we conclude that

$$\Delta_\alpha(\epsilon, R) = \frac{\epsilon R^2}{1+\alpha^2} + O\left(\epsilon^{\frac{1}{1+\eta}} R\right). \tag{4.14}$$

Theorem 1.3 now follows directly from (4.14) and Lemma 3.1. □

5 Sectors about rational slopes

In this section we provide a proof of Theorem 1.5, namely a count for $S_\alpha(\epsilon, R)$ when $\alpha \in \mathbb{Q}$. The proof proceeds similarly to that of Theorem 1.3, upon setting $\delta = 0$:

Proof of Theorem 1.5 Recall that $\alpha = p/q$, where $(p, q) = 1$. Note that

$$\begin{aligned}\Delta_\alpha(\epsilon, R) &= \#\left\{(m, n) \in \mathbb{Z}^2 : m(p/q - \epsilon) < n < m(p/q + \epsilon) : 1 \leq m \leq \frac{qR}{\sqrt{p^2+q^2}}\right\} \\ &= \#\left\{(m, n) \in \mathbb{Z}^2 : |nq - mp| < mq\epsilon : 1 \leq m \leq \frac{qR}{\sqrt{p^2+q^2}}\right\}.\end{aligned}$$

Let $d = nq - mp$, and note that $|d| < mq\epsilon$ implies

$$|d| \leq \frac{\epsilon q^2 R}{\sqrt{p^2+q^2}},$$

as well as that

$$\frac{|d|}{q\epsilon} < m.$$

Partitioning with respect to d, we then write

$$\begin{aligned}\Delta_\alpha(\epsilon, R) &= \sum_{|d| \leq \frac{\epsilon q^2 R}{\sqrt{p^2+q^2}}} \# \left\{ (m,n) \in \mathbb{Z}^2 : nq - mp = d, \frac{|d|}{q\epsilon} < m \leq \frac{qR}{\sqrt{p^2+q^2}} \right\} \\ &= \sum_{|d| \leq \frac{\epsilon q^2 R}{\sqrt{p^2+q^2}}} \sum_{\substack{\frac{|d|}{q\epsilon} < m \leq \frac{Rq}{\sqrt{p^2+q^2}} \\ m \equiv -d\bar{p}\ (q)}} 1,\end{aligned}$$

where $\bar{p}$ denotes the inverse of p modulo q (in particular, if $\epsilon q^2 R/\sqrt{p^2+q^2} < 1$, then the only contribution to $\Delta_\alpha(\epsilon, R)$ comes from the term $d = 0$, i.e. points $(m, n) \in \mathbb{Z}^2$ lying precisely on the line $y = \alpha x$). Upon setting

$$A := \frac{R}{\sqrt{p^2+q^2}} \quad \text{and} \quad B := \epsilon q^2,$$

and recalling that

$$\sum_{|d| \leq x} |d| = \lfloor x \rfloor (\lfloor x \rfloor + 1),$$

it follows that

$$\begin{aligned}\Delta_\alpha(\epsilon, R) &= \sum_{|d| \leq AB} \left(A - \frac{|d|}{B} + O(1) \right) \\ &= A(1 + 2\lfloor AB \rfloor) - \frac{1}{B} \lfloor AB \rfloor (\lfloor AB \rfloor + 1) + O(1 + AB).\end{aligned}$$

We furthermore note that

$$A(1 + 2\lfloor AB \rfloor) = 2A^2 B + (1 - 2\{AB\}) A$$

and similarly that

$$\frac{1}{B}\lfloor AB\rfloor(\lfloor AB\rfloor+1)=\left(A-\frac{\{AB\}}{B}\right)(1+AB-\{AB\})$$
$$=A^2B+A(1-2\{AB\})-\frac{\{AB\}}{B}(1-\{AB\}).$$

Combining the above expressions we see that

$$\Delta_\alpha(\epsilon,R)=A^2B+\frac{1}{B}\{AB\}(1-\{AB\})+O(1+AB)$$
$$=\frac{\epsilon q^2R^2}{p^2+q^2}+\frac{1}{q^2\epsilon}\left\{\frac{\epsilon q^2R}{\sqrt{p^2+q^2}}\right\}\left(1-\left\{\frac{\epsilon q^2R}{\sqrt{p^2+q^2}}\right\}\right)+O(1+\epsilon R),$$

and the desired result now follows from Lemma 3.1. □

6 Very quickly shrinking sectors

Finally, in this section we provide a proof of Proposition 1.7, namely that when $\alpha\in\mathbb{R}$ is irrational of finite type η and $\epsilon=o(R^{-1-\eta})$, we find that $S_\alpha(\epsilon,R)=0$ for sufficiently large R:

Proof of Proposition 1.7 Since α is of finite type η, there exists a constant $c=c(\alpha)>0$ such that for all $(p,q)\in\mathbb{Z}\times\mathbb{Z}_{>0}$,

$$\left|\alpha-\frac{p}{q}\right|>\frac{c}{q^{1+\eta}}.$$

Take R_0 sufficiently large such that for any $R>R_0$ we have

$$\epsilon<\frac{c}{R^{1+\eta}}.$$

Then for any $R>R_0$, and any $(p,q)\in\mathbb{Z}\times\mathbb{Z}_{>0}$ with $0<q\le R$, we find that

$$\left|\alpha-\frac{p}{q}\right|>\frac{c}{q^{1+\eta}}\ge\frac{c}{R^{1+\eta}}>\epsilon.$$

It follows that for all $R>R_0$ we have $S_\alpha(\epsilon,R)=0$, as desired. □

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