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k–Generalized Lucas numbers, perfect powers and the problem of Pillai

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Abstract
For an integer $k \geq 2$, let $L^{(k)}$ be the k–generalized Lucas sequence which starts with $0, \ldots, 2, 1$ (a total of k terms) and for which each term afterwards is the sum of the k preceding terms. In this paper we assume that an integer c can be represented in at least two ways as the difference between a k–generalized Lucas number and a power of b, then using the theory of nonzero linear forms in logarithms of algebraic numbers, we bound all possible solutions on this representation of c in terms of b. Finally, combination our general result and some known reduction procedures based on the continued fraction algorithm, we find all the integers c and their representations for $b \in [2, 10]$, this argument can be generalized to any $b > 10$.

Keywords Diophantine equations · Lucas sequence · Pillai's Problem

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1 Introduction

Let $k \geqslant 2$ be a fixed integer. We consider the linear recurrence sequence $G^{(k)} := (G_n^{(k)})_{n\geq 2-k}$ of order k, defined as

$$G_n^{(k)} = G_{n-1}^{(k)} + G_{n-2}^{(k)} + \cdots + G_{n-k}^{(k)} \quad \text{for all } n \geq 2,$$

with the initial conditions

$$G_{-(k-2)}^{(k)} = G_{-(k-3)}^{(k)} = \cdots = G_{-1}^{(k)} = 0, G_0^{(k)} = a \quad \text{and} \quad G_1^{(k)} = b.$$

Observe that if $a = 0$ and $b = 1$, then $G^{(k)}$ is nothing that just the k–generalized Fibonacci sequence or for simplicity, the k–Fibonacci sequence $F^{(k)} := (F_n^{(k)})_{n\geq 2-k}$. In this case, if we choose $k = 2$ we obtain the classical Fibonacci sequence $(F_n)_n$.

On the other hand, if $a = 2$ and $b = 1$ then $G^{(k)}$ is known as the k–generalized Lucas sequence $L^{(k)} := (L_n^{(k)})_{n\geq 2-k}$. In the case of $k = 2$ we obtain the usual Lucas sequence

$$(L_n)_{n\geq 0} := \{2, 1, 3, 4, 7, 11, 18, 29, 47, 76, 123, 199, 322, 521, 843, 1364, \ldots\}.$$

Furthermore, it has been proved in [28] that the only powers of 2 in $L^{(k)}$ are

$$L_0^{(k)} = 2, \quad L_1^{(k)} = 1 = 2^0, \quad L_3^{(2)} = 4 = 2^2, \quad L_7^{(3)} = 64 = 2^6. \tag{1}$$

The above sequences are among the several generalizations of the Fibonacci numbers which have been studied in literature.

Recall the problem of Pillai which states that for each fixed integer $c \geq 1$, the Diophantine equation

$$a^x - b^y = c, \qquad \min\{x, y\} \geq 2, \tag{2}$$

has only a finite number of positive solutions $\{a, b, x, y\}$ [24, 25, 29]. This problem is still open; however, the case $c = 1$, is the conjecture of Catalan and was proved by Mihăilescu [26]. The general problem of Pillai is difficult to solve and this has motivated the consideration of special cases of this problem. In the past years, several special cases of the problem of Pillai have been studied. See, for example, [6–8, 10, 11, 13, 16, 21, 22].

Here we look at a similar problem for the terms of the k–Lucas sequence, namely

$$L_n^{(k)} - b^m = c \quad \text{with} \quad \min\{n, m\} \geq 2, \quad b \geq 2 \quad \text{and} \quad c \in \mathbb{Z}.$$

For $b \geq 2$ fixed integer, we are interested in knowing how many solutions (c, k, n, m) exist for the above equation, under the non-unitary condition of the multiplicity at least for c. For this purpose we study the equation

$$L_n^{(k)} - b^m = L_{n_1}^{(k)} - b^{m_1} \quad (= c), \tag{3}$$

with[1] $n > n_1 \geq 2$ and $m > m_1 \geq 2$.

When b is fixed we bound the solutions for Eq. (3) and present an algorithm that can be generalized to find all its solutions for any known b, in particular we use it to obtain all solutions in the cases $b \in [2, 10]$. We present our main results below.

Theorem 1 *Let $b \geq 2$ be a fixed integer. The solution (k, n, m, n_1, m_1) of the Diophantine equation (3) with $n > n_1 \geq 2, m > m_1 \geq 2$ and $k \geq 2$, satisfies the following.*

(i) If $n \leq k$, then $n = \max\{n, m, n_1, m_1\}$ and Eq. (3) takes the form

$$3 \cdot 2^{n-2} - b^m = 3 \cdot 2^{n_1-2} - b^{m_1} \quad (= c) \quad \textit{for all} \quad k \geq 2.$$

Moreover, if b is a power of 2, there are no solutions for $b > 4$ but there are solutions for $b \in \{2, 4\}$ of the form

$$(b, n, m, n_1, m_1) \in \{(2, m+1, m, m, m-2), (4, 2m+1, m, 2m, m-1)\},$$

with $m \geq 3$. In another case, let p be the largest odd prime divisor of b, then

$$n < 9.13 \times 10^{13} p (\log p)(\log b)^4.$$

(ii) If $n \geq k+1$, then

$$k < 1.4 \times 10^{44} (\log b)^6 \quad \textit{and} \quad m - 2 < n < 1.02 \times 10^{545} (\log b)^{79}.$$

As a consequence to Theorem 1, we find all the solutions for $b \in [2, 10]$ in Corollary 2. The argument of the proof of this numerical result can be extended to find any solution for fixed b.

Corollary 2 *Let $b \in [2, 10]$. The solution (c, k, n, m, n_1, m_1) of Eq. (3) with $n > n_1 \geq 2, m > m_1 \geq 2$ and $k \geq 2$, satisfies the following.*

(i) For $n \leq k$ and $b \notin \{2, 4\}$ (the solutions for these cases are in Theorem 1) there are only solutions

$$3 \cdot 2^{5-2} - 3^3 = 3 \cdot 2^{3-2} - 3^2 = -3, \quad 3 \cdot 2^{7-2} - 3^4 = 3 \cdot 2^{5-2} - 3^2 = 15,$$
$$3 \cdot 2^{10-2} - 3^6 = 3 \cdot 2^{6-2} - 3^2 = 39, \quad 3 \cdot 2^{8-2} - 6^3 = 3 \cdot 2^{4-2} - 6^2 = -24.$$

(ii) For $n \geq k+1$ we obtain for each b the following.

[1] Note that since the sequences $(L_n^{(k)})_{n \geq 2}$ and $(b^m)_{m \geq 2}$ are positive increasing, then $n > n_1$ if and only if $m > m_1$ in the Diophantine equation (3) and without loss of generality we can assume either one.

(a) *If* $b = 2$, *there are only the solutions*

$$
\begin{aligned}
&L_{11}^{(2)} - 2^8 = L_4^{(2)} - 2^6 = -57,\\
&L_6^{(3)} - 2^6 = L_2^{(3)} - 2^5 = -29,\\
&L_5^{(3)} - 2^5 = L_2^{(3)} - 2^4 = -13,\\
&L_8^{(3)} - 2^7 = L_5^{(4)} - 2^5 = L_3^{(4)} - 2^4 = L_3^{(3)} - 2^4 = -10,\\
&L_{10}^{(2)} - 2^7 = L_5^{(2)} - 2^4 = L_2^{(2)} - 2^3 = -5,\\
&L_4^{(2)} - 2^3 = L_2^{(2)} - 2^2 = -1,\\
&L_4^{(3)} - 2^3 = L_3^{(3)} - 2^2 = 2,\\
&L_6^{(3)} - 2^5 = L_5^{(3)} - 2^4 = L_5^{(2)} - 2^3 = L_4^{(2)} - 2^2 = 3,\\
&L_{14}^{(4)} - 2^{13} = L_4^{(4)} - 2^2 = 8,\\
&L_{11}^{(5)} - 2^{10} = L_9^{(5)} - 2^4 = 336.
\end{aligned}
$$

(b) *If* $b = 3$,

$$
\begin{aligned}
&L_7^{(3)} - 3^4 = L_4^{(3)} - 3^3 = -17,\\
&L_9^{(2)} - 3^4 = L_3^{(2)} - 3^2 = -5,\\
&L_7^{(2)} - 3^3 = L_5^{(2)} - 3^2 = 2,\\
&L_{16}^{(2)} - 3^7 = L_8^{(2)} - 3^3 = L_7^{(2)} - 3^2 = 20,\\
&L_8^{(3)} - 3^4 = L_7^{(3)} - 3^3 = 37,\\
&L_{14}^{(2)} - 3^6 = L_{10}^{(2)} - 3^2 = 114,\\
&L_{12}^{(6)} - 3^7 = L_{10}^{(6)} - 3^3 = 709.
\end{aligned}
$$

(c) *For* $b \in \{5, 6, 7, 10\}$ *there are no solutions and the cases* $b \in \{4, 8, 9\}$ *are powers of* 2 *or* 3 *so they are already included.*

Remark 3 In Diophantine equation (3), we have assumed $n > n_1 \geq 2$ and $m > m_1 \geq 2$ preserving the essence of the original problem (2), nevertheless, this could be removed (replacing 2 by 0) and slight adjustments to the arguments presented here would still work.

Let us give a brief overview of the strategy used for proving our results. In the proof of Theorem 1, we distinguished two cases according to $n \leq k$ and $n \geq k + 1$. The case $n \leq k$ was treated by a combination of the theory of nonzero linear forms in logarithms of real algebraic numbers with some elementary arguments on the p–adic valuations of certain Lucas sequences. For the case $n \geq k + 1$, the theory of nonzero linear forms in logarithms is used several times again, to obtain explicit upper bounds for the unknowns (k, n, m, n_1, m_1) depending only on b. The proof of Corollary 2 depends on a combination of Theorem 1 and some known reduction procedures based on the continued fraction algorithm. The computation needed for the proof of Theorem 1 and Corollary 2 was done with the Mathematica software.

2 Preliminary results

In this section, we first recall some general properties of the k–generalized Lucas sequence.

2.1 k–Generalized Lucas numbers

It is known that the characteristic polynomial of the k–generalized Lucas numbers $L^{(k)}$, namely

$$\Psi_k(x) := x^k - x^{k-1} - \cdots - x - 1,$$

is irreducible over $\mathbb{Q}[x]$ and has just one root outside the unit circle. Let $\alpha := \alpha(k)$ denote that single root, which is located between $2\left(1 - 2^{-k}\right)$ and 2 (see [14]). This is called the dominant root of $L^{(k)}$. To simplify notation, in our application we shall omit the dependence on k of α. We shall use $\alpha^{(1)}, \ldots, \alpha^{(k)}$ for all roots of $\Psi_k(x)$ with the convention that $\alpha^{(1)} := \alpha$.

The following appears in [1], responding to a conjecture proposed in [19].

Lemma 1 *Let $\alpha^{(j)} = \rho_j e^{i\theta_j}$ with $\theta_j \in [0, 2\pi)$ for $j = 1, \ldots, k$ be all the roots of $\Psi_k(x)$. Then for every $h \in \{0, 1, \ldots, k-1\}$, there exists j such that*

$$\left|\theta_j - \frac{2\pi h}{k}\right| < \frac{\pi}{k}.$$

We now consider for an integer $k \geq 2$, the function

$$f_k(z) = \frac{z-1}{2 + (k+1)(z-2)} \qquad \text{for} \qquad z \in \mathbb{C}, \quad z \neq 2k/(k+1). \tag{4}$$

In the following lemma, we give some properties of the sequence $L^{(k)}$ which will be used in the proof of Theorem 1. The items of the following lemma was proved by Bravo, Gómez and Luca in [4, 5, 18].

Lemma 2 *Let $k \geq 2$, α be the dominant root of $L^{(k)}$, and consider the function $f_k(z)$ defined in (4). Then,*

(a) If $2 \leq n \leq k$, then $L_n^{(k)} = 3 \cdot 2^{n-2}$.
(b) $\alpha^{n-1} \leq L_n^{(k)} \leq 2\alpha^n$ for all $n \geq 1$.
(c) $L^{(k)}$ satisfies the following formula

$$L_n^{(k)} = \sum_{i=1}^{k} (2\alpha^{(i)} - 1) f_k(\alpha^{(i)}) {\alpha^{(i)}}^{n-1}.$$

(d)

$$\left|L_n^{(k)} - (2\alpha - 1) f_k(\alpha)\alpha^{n-1}\right| < \frac{3}{2} \quad \textit{holds for all } n \geqslant 2-k. \tag{5}$$

(e) The inequalities

$$\frac{1}{2} < f_k(\alpha) < \frac{3}{4} \quad \textit{and} \quad |f_k(\alpha^{(i)})| < 1, \quad 2 \le i \le k$$

hold. In particular, the number $f_k(\alpha)$ is not an algebraic integer.

(f) $L_n^{(k)} = 2F_{n+1}^{(k)} - F_n^{(k)}$.

Next comes another necessary lemma for our work.

Lemma 3 *Let $k \ge 2$, $c \in (0, 1)$ and $n < 2^{ck}$. Then it is satisfied that*

(i) For all $n \ge 2$,

$$L_n^{(k)} = 3 \cdot 2^{n-2}(1 + \zeta_n'), \quad \text{with } |\zeta_n'| < \begin{cases} 4/2^{(1-c)k}; & \textit{if } c \le 0.693, \\ 8.1/2^{(1-c)k}; & \textit{otherwise.} \end{cases}$$

(ii) For all $n \ge k+2$,

$$L_n^{(k)} = 3 \cdot 2^{n-2}\left(1 - \frac{n-k+4/3}{2^{k+1}} + \zeta_n''\right), \quad \text{with } |\zeta_n''| < 8/2^{2(1-c)k}.$$

Proof Howard and Cooper proved in [9] that for all $k \ge 2$, $r \ge k+2$ and $\ell := \lfloor (r+k)/(k+1) \rfloor$ it is satisfied that

$$F_r^{(k)} = 2^{r-2} + \sum_{j=1}^{\ell-1} C_{r,j}\, 2^{r-(k+1)j-2},$$

where $C_{r,j} := (-1)^j \left[\binom{r-jk}{j} - \binom{r-jk-2}{j-2}\right]$. Therefore for $k+2 \le r$ we can write

$$F_r^{(k)} = 2^{r-2}\left(1 - \frac{r-k}{2^{k+1}} + \zeta_r\right), \tag{6}$$

where

$$|\zeta_r| \le \sum_{j=2}^{\ell-1} \frac{|C_{r,j}|}{2^{(k+1)j}} < \sum_{j\ge 2} \frac{2\, r^j}{2^{(k+1)j}\, j!} < \frac{2r^2}{2^{2k+2}} \sum_{j \ge 2} \frac{(r/2^{k+1})^{j-2}}{(j-2)!} < \frac{2r^2}{2^{2k+2}} e^{r/2^{k+1}}. \tag{7}$$

We now prove each case using Lemma 2 and identity (6).

(i) By item (a) of Lemma 2 we have that the result is trivial for $2 \le n \le k$. If $n = k+1$, it follows from item (g) of Lemma 2 that

$$L_{k+1}^{(k)} = 2F_{k+2}^{(k)} - F_{k+1}^{(k)} = 2^{k+1} - 2^{k-1} - 2 = 3 \cdot 2^{k-1}(1 - 2^{-k+2}/3)$$

with $|\zeta'_{k+1}| := 2^{-k+2}/3 < 4/2^{(1-c)k}$, so the result is true in this case. When $n \ge k+2$, by identity (6) and item (g) of Lemma 2 we have that

$$\begin{aligned} L_n^{(k)} &= 2F_{n+1}^{(k)} - F_n^{(k)} \\ &= 2^n \left(1 - \frac{n+1-k}{2^{k+1}} + \zeta_{n+1}\right) - 2^{n-2}\left(1 - \frac{n-k}{2^{k+1}} + \zeta_n\right) \\ &= 3 \cdot 2^{n-2}\left(1 - \frac{3n+4-3k}{3 \cdot 2^{k+1}} + 4\zeta_{n+1}/3 - \zeta_n/3\right) \end{aligned} \tag{8}$$

and since $n < 2^{ck}$, we get by inequality (7) that

$$\begin{aligned} |\zeta'_n| &:= \frac{3n+4-3k}{3 \cdot 2^{k+1}} + 4|\zeta_{n+1}|/3 + |\zeta_n|/3 \\ &< \left(\frac{1}{2} + \frac{8e^{1/2^{(1-c)k}}}{3 \cdot 2^{(1-c)k}} + \frac{e^{1/2^{(1-c)k+1}}}{3 \cdot 2^{(1-c)k+1}}\right)/2^{(1-c)k} \\ &< \begin{cases} 4/2^{(1-c)k}, & \text{if } c \le 0.693, \\ 8.1/2^{(1-c)k}, & \text{otherwise.} \end{cases} \end{aligned}$$

(ii) Since $n \ge k+2$, we have by identity (8) and inequality (7) for $r \in \{n, n+1\}$ that

$$L_n^{(k)} = 3 \cdot 2^{n-2}\left(1 - \frac{n-k+4/3}{2^{k+1}} + 4\zeta_{n+1}/3 - \zeta_n/3\right)$$

where

$$\begin{aligned} |\zeta''_n| &:= 4|\zeta_{n+1}|/3 + |\zeta_n|/3 < \left(8e^{1/2^{(1-c)k}}/3 + e^{1/2^{(1-c)k+1}}/6\right)/2^{2(1-c)k} \\ &< 8/2^{2(1-c)k}. \end{aligned}$$

□

2.2 Notations and terminology from algebraic number theory

We begin by recalling some basic notions from algebraic number theory.

Let η be an algebraic number of degree d with minimal primitive polynomial over the integers

$$a_0x^d + a_1x^{d-1} + \cdots + a_d = a_0\prod_{i=1}^{d}(x - \eta^{(i)}),$$

where the leading coefficient a_0 is positive and the $\eta^{(i)}$'s are the conjugates of η. Then the *logarithmic height* of η is given by

$$h(\eta) := \frac{1}{d}\left(\log a_0 + \sum_{i=1}^{d} \log\left(\max\{|\eta^{(i)}|, 1\}\right)\right).$$

In particular, if $\eta = p/q$ is a rational number with $\gcd(p, q) = 1$ and $q > 0$, then $h(\eta) = \log \max\{|p|, q\}$. The following are some of the properties of the logarithmic height function $h(\cdot)$, which will be used in the next sections of this paper:

$$\begin{aligned} h(\eta \pm \gamma) &\le h(\eta) + h(\gamma) + \log 2, \\ h(\eta\gamma^{\pm 1}) &\le h(\eta) + h(\gamma), \\ h(\eta^s) &= |s| h(\eta) \qquad (s \in \mathbb{Z}). \end{aligned} \tag{9}$$

On the other hand, it can be proved that

$$h(\alpha) = \log \alpha/k \quad \text{and} \quad h(f_k(\alpha)) < 2\log k, \quad \text{for all } k \ge 2. \tag{10}$$

The logarithmic height of $f_k(\alpha)$ satisfies $h(f_k(\alpha)) < 2\log k$. See [17, 18] for details when $k \ge 3$ and the case $k = 2$ is easily verified computationally.

2.3 Linear forms in logarithms and continued fractions

In order to prove our main result Theorem 1, we need to use several times a Baker–type lower bound for a nonzero linear form in logarithms of algebraic numbers. There are many such in the literature like that of Baker and Wüstholz from [3]. We use the following result by Matveev [25], which is one of our main tools in this paper.

Theorem 4 *Let $\gamma_1, \ldots, \gamma_t$ be positive real algebraic numbers in a real algebraic number field $\mathbb{K}$ of degree D, $b_1, \ldots, b_t$ be nonzero integers with $B \ge \max\{|b_1|, \ldots, |b_t|\}$, and assume that*

$$\Lambda := \gamma_1^{b_1} \cdots \gamma_t^{b_t} - 1 \tag{11}$$

is nonzero. Then

$$\log|\Lambda| > -1.4 \times 30^{t+3} \times t^{4.5} \times D^2(1 + \log D)(1 + \log B)A_1 \cdots A_t,$$

and

$$A_i \ge \max\{Dh(\gamma_i), |\log \gamma_i|, 0.16\}, \qquad \text{for all} \qquad i = 1, \ldots, t.$$

During the course of our calculations, we get some upper bounds on our variables which are too large, thus we need to reduce them. To do so, we use some results from the theory of continued fractions. Specifically, for a nonhomogeneous linear form in two integer variables, we use a slight variation of a result due to Dujella and Pethő (see [15], Lemma 5a), which itself is a generalization of a result of Baker and Davenport [2].

For a real number X, we write $||X|| := \min\{|X-n| : n \in \mathbb{Z}\}$ for the distance from X to the nearest integer.

Lemma 4 (Dujella, Pethő) *Let M be a positive integer, p/q be a convergent of the continued fraction of the irrational number τ such that $q > 6M$, and A, B, μ be some real numbers with $A > 0$ and $B > 1$. Let further $\varepsilon := ||\mu q|| - M||\tau q||$. If $\varepsilon > 0$, then there is no solution to the inequality*

$$0 < |u\tau - v + \mu| < AB^{-w},$$

in positive integers u, v and w with

$$u \leq M \quad and \quad w \geq \frac{\log(Aq/\varepsilon)}{\log B}.$$

The above lemma cannot be applied when $\mu = 0$ (since then $\varepsilon < 0$). In this case, we use the following criterion of Legendre.

Lemma 5 *(Legendre) Let τ be real number and x, y integers such that*

$$\left|\tau - \frac{x}{y}\right| < \frac{1}{2y^2}. \tag{12}$$

Then $x/y = p_j/q_j$ is a convergent of τ. Furthermore, if $[a_0, a_1, a_2, \ldots]$ is the continued fraction of τ, then

$$\left|\tau - \frac{x}{y}\right| \geq \frac{1}{(a_{j+1}+2)y^2}. \tag{13}$$

For the use of the above two lemmas we will sometimes use the well-known inequality

$$|\log(1+x)| \leq 2|x| \quad \text{if} \quad |x| \leq 1/2, \quad \text{for} \quad x \in \mathbb{R}. \tag{14}$$

Finally, the following lemma is also useful. It is Lemma 7 in [20].

Lemma 6 (Gúzman, Luca) *If $s \geqslant 1$, $T > (4s^2)^s$ and $T > x/(\log x)^s$, then*

$$x < 2^s T(\log T)^s.$$

3 Case $n \leq k$

In this section we prove the first items of Theorem 1 and Corollary 2. When $n \leq k$, equality (3) becomes

$$3 \cdot 2^{n-2} - 3 \cdot 2^{n_1-2} = b^m - b^{m_1} \tag{15}$$

with $n > n_1 \geq 2$ and $m > m_1 \geq 2$. To bound the possible solutions of the above equation we present the following lemma.

Lemma 7 *Let $a, \ell, r \in \mathbb{Z}^+$, with $a \geq 3$ odd.*

(i) If $\ell \geq \nu_2(a-1)$ and $2^\ell \mid a^r - 1$, then $2^\ell \mid r(a^2-1)/2$.
(ii) If p is the greatest prime divisor of a and $a^\ell \mid 2^r - 1$. Then $a^\ell | r a^{p-1}$.

Proof We consider each case.

(i) Since $\nu_2(a^r - 1) \geq \ell \geq \nu_2(a-1)$, we have that

$$\nu_2\left(\sum_{j=0}^{r-1} a^j\right) = \nu_2\left(\frac{a^r-1}{a-1}\right) \geq 1.$$

Moreover, that a is odd implies

$$r \equiv \sum_{j=0}^{r-1} a^j \equiv 0 \pmod 2.$$

Now we note that $u_r := (a^r - 1)/(a-1)$ is a sequence of Lucas with characteristic polynomial $x^2 - (a+1)x + a$ whose discriminant is $\Delta = (a-1)^2$. Therefore $2 \nmid a$, $2 \mid \Delta$ and $2 \mid r$, then from Theorem 1.5 in [29] it follows that

$$\nu_2(u_r) = \nu_2(r) + \nu_2(u_2) - 1$$

and this implies that

$$\ell \leq \nu_2(a^r - 1) = \nu_2\left(r(a^2-1)/2\right),$$

i.e. $2^\ell \mid r(a^2-1)/2$.

(ii) First we assume that $a = p^\gamma$ for some $\gamma \in \mathbb{Z}^+$. We know that the sequence $u_r := 2^r - 1$ is a sequence of Lucas with characteristic polynomial $x^2 - 3x + 2$ whose discriminant is $\Delta = 1$ and $p \nmid 2$. Then by Corollary 1.6 in [29] and as $p^{\gamma\ell} \mid 2^r - 1$, we obtain that

$$\begin{aligned} 0 < \gamma\ell \leq \nu_p(u_r) &= \nu_p(r) + \nu_p(u_{\tau(p)}) \\ &\leq \nu_p(r) + \log_p(u_{\tau(p)}) < \nu_p(r) + \tau(p)\log_p 2 \\ &< \nu_p(r) + p - 1 = \nu_p(rp^{p-1}), \end{aligned}$$

where $\tau(p)$ is the multiplicative order of 2 modulo p. That is,

$$p^{\gamma\ell} \mid rp^{p-1} \tag{16}$$

and in particular $a^{\ell} \mid ra^{p-1}$. In general, if $a = p_1^{\gamma_1} \cdots p_s^{\gamma_s}$ with $p_1 < p_2 < \cdots < p_s$ primes and $\{\gamma_j\}_{1 \le j \le s} \subset \mathbb{Z}^+$, we obtain from conclusion (16) that

$$p_1^{\gamma_1 \ell} \mid r p_1^{p_1 - 1}, \ldots, p_s^{\gamma_s \ell} \mid r p_s^{p_s - 1},$$

then $a^{\ell} \mid ra^{p_s - 1}$.

□

On the other hand, we see that

$$\begin{aligned} 3 \cdot 2^{n-2} > 3 \cdot 2^{n-2} - 3 \cdot 2^{n_1 - 2} &= b^m - b^{m_1} \ge b^m - b^{m-1} \\ &= b^{m-3} b^2 (b-1) \ge 4b^{m-3} \end{aligned}$$

implies $(n-2)\log 2 > (m-3)\log 2$, therefore $n > m-1$ and $n \ge m$. Then

$$\max\{n_1, m_1, n, m\} = n.$$

Equation (15) can be factored as

$$3 \cdot 2^{n_1 - 2}(2^{n-n_1} - 1) = b^{m_1}(b^{m-m_1} - 1). \tag{17}$$

We consider the following cases over b.

3.1 Case *b* power of 2

Let $b = 2^t$ for some $t \in \mathbb{Z}^+$. By identity (17) and since $m - m_1 \ge 1$, $n - n_1 \ge 1$, we have that $b^{m_1} = 2^{n_1 - 2}$. Then, by identity (15)

$$3 \cdot 2^{n-2} - 2^{tm} = 3 \cdot 2^{n-2} - b^m = 2^{n_1 - 1},$$

so

$$0 < 2^{n_1 - n + 1}/3 = 1 - 2^{tm - n + 2}/3,$$

where $3 > 2^{tm-n+2}$ and $n - tm \ge 1$. Therefore

$$1/3 = 1 - 2^{-1+2}/3 \le 1 - 2^{-(n-tm)+2}/3 = 2^{n_1 - n + 1}/3$$

implies $n - n_1 = 1$. Now identity (17) becomes

$$2^{t(m-m_1)} = b^{m-m_1} = 2^2$$

and it follows that $t(m - m_1) = 2$. The above combined with Eq. (17) leads to

$$3 \cdot 2^{n-3} = 3 \cdot 2^{n-2}(1 - 2^{n_1 - n}) = b^m(1 - 2^{t(m_1 - m)}) = 3 \cdot 2^{tm-2},$$

i.e. $n = tm + 1$. In conclusion, there are infinitely many solutions of the form

$$(b, n, m, n_1, m_1) \in \left\{(2, m+1, m, m, m-2)\,, \left(2^2, 2m+1, m, 2m, m-1\right)\right\}$$

for Eq. (15).

3.2 Case *b* with odd prime divisor

Let p be the greatest odd prime divisor of b. First we bound n_1 and m_1 in terms of b and n.

- If b be odd. By Eq. (17) we have

$$2^{n_1-2} \mid b^{m-m_1} - 1 \quad \text{and} \quad b^{m_1-1} \mid 2^{n-n_1} - 1.$$

  Then by Lemma 7 we get that

$$n_1 \leq \max\left\{\nu_2(b-1), \log_2\left((m-m_1)(b^2-1)/2\right)\right\} + 2 < \log_2\left(2n(b^2-1)\right) \text{ and}$$
$$m_1 \leq \log_b(n-n_1) + p < p + \log_b n.$$

- If $b = 2^t b'$ with $t \geq 1$ and $b' > 1$ odd. Then from Eq. (17) it follows that $2^{tm_1} = 2^{n_1-2}$ and

$$3(2^{n-n_1} - 1) = b'^{m_1}(b^{m-m_1} - 1).$$

  So $b'^{m_1-1} \mid 2^{n-n_1} - 1$ and by Lemma 7 we obtain

$$\frac{n_1-2}{\nu_2(b)} = m_1 \leq \log_{b'}(n-n_1) + p < p + \log_{b/2^{\nu_2(b)}} n.$$

In any case we conclude that

$$n_1 < \left(p + \log_2\left(2nb^2\right)\right)\log_2 b \qquad \text{and} \qquad m_1 < p + \log_{b/2^{\nu_2(b)}} n. \tag{18}$$

Now, we see that from identity (15), we have

$$|\Lambda_0| := \left|\frac{b^m}{3\cdot 2^{n-2}} - 1\right| \leq \frac{3\cdot 2^{n_1-2} + b^{m_1}}{3\cdot 2^{n-2}}. \tag{19}$$

If $b^m = 3\cdot 2^{n-2}$, we get that $2 < m \leq \nu_3(b^m) = 1$ and this is absurd. Therefore $|\Lambda_0| \neq 0$ and we use Theorem 4 with the parameters $\mathbb{K} = \mathbb{Q}$, $D = 1$,

$$(\gamma_1, \gamma_2, \gamma_3) = (3, 2, b)\,, \qquad (b_1, b_2, b_3) = (-1, -n+2, m),$$

$B = n$, $A_1 = \log 3$, $A_2 = \log 2$, $A_3 = \log b$. So

$$-\log|\Lambda_0| < 2.7 \times 10^{11}(\log n)(\log b). \tag{20}$$

Finally we combine inequalities (18), (19) and (20) to obtain

$$\begin{aligned} n &< 2 + \log_2\left(3 \cdot 2^{n_1-2} + b^{m_1}\right) - \log_2|\Lambda_0| - \log_2 3 \\ &< \max\{n_1, m_1\}\log_2 b + 3.9 \times 10^{11}(\log n)(\log b) \\ &< \left(p + \log_2\left(2nb^2\right)\right)(\log_2 b)^2 + 3.9 \times 10^{11}(\log n)(\log b) \\ &< 5.63 \times 10^{11} \cdot p(\log_2 b)^3 \log n, \end{aligned}$$

then we use the Lemma 6 and we get

$$n < 9.13 \times 10^{13} p(\log p)(\log b)^4.$$

Now we summarize what we obtained in the following lemma.

Lemma 8 *Given $b \geq 2$, the solution (n, m, n_1, m_1) of Eq. (15) satisfies that*

$$n = \max\{n, m, n_1, m_1\}.$$

Moreover, if b is a power of two, there are no solutions for $b > 4$ but there are solutions for $b \in \{2, 4\}$ of the form

$$(b, n, m, n_1, m_1) \in \{(2, m+1, m, m, m-2), (4, 2m+1, m, 2m, m-1)\}$$

with $m \geq 3$. In another case, let p be the largest odd prime divisor of b, then

$$n < 9.13 \times 10^{13} p(\log p)(\log b)^4.$$

This completed the proof of the first part of Theorem 1.

3.3 Proof of the first part of Corollary 2

By Lemma 8 we know that

$$\max\{n, m, n_1, m_1\} = n < 3.5 \times 10^{16} \quad \text{for} \quad b \in \{3, 5, 6, 7, 9, 10\}, \tag{21}$$

in fact for $b \in \{2, 4\}$ the solutions are given and for $b = 8$ there are no solutions. Now, by bounds (18) and (21) we have

$$2 \leq n_1 \leq 231 \qquad \text{and} \qquad 2 \leq m_1 \leq 41.$$

So if we assume for a moment that $n \geq 233$, by inequality (19) we will have that $|\Lambda_0| < 1/2$, then we apply inequality (14) and arrive at that

$$\frac{9.96 \times 10^{69}}{2^n} \geq \begin{cases} \left|m \log_2 b - (n-2) + \log_2(1/3)\right|; & \text{if } b \in \{5,7,9,10\}, \\ \left|(m-1)\log_2 3 - (n-2)\right|; & \text{if } b = 3, \\ \left|(m-1)\log_2 6 - (n-3)\right|; & \text{if } b = 6. \end{cases}$$

So, for $b \in \{5, 7, 9, 10\}$ we use Lemma 4 on the above inequality with the parameters

$$A := 9.96 \times 10^{69}, \quad B := 2, \quad \mu := \log_2(1/3), \quad \tau := \log_2 b, \quad v := n-2, \quad u := m,$$

$M := 3.5 \times 10^{16}$ and we obtain

$$\begin{aligned} &\varepsilon \in (0.27, 0.425), \quad q \in (3.32 \times 10^{17}, 5.62 \times 10^{18}), \\ &n \leq 296 \quad \text{for all} \quad b \in \{5, 7, 9, 10\}. \end{aligned}$$

For $b \in \{3, 6\}$ we use Lemma 5 with values

$$\tau := \log_2 b, \qquad y := m-1, \qquad x := \begin{cases} n-2; & \text{if } \; b = 3, \\ n-3; & \text{if } \; b = 6 \end{cases}$$

and assuming before without loss of generality that

$$\log_2\left(2 \times 9.96 \times 10^{69}(m-1)\right) \leq \left\lceil \log_2\left(2 \times 9.96 \times 10^{69}(3.5 \times 10^{16} - 1)\right)\right\rceil \\ = 289 < n$$

for the hypothesis to be fulfilled, then we get

$$\frac{1}{(\max_{0 \leq j \leq 33}\{a_{j+1}\} + 1)(m-1)^2} < \left|\log_2 b - \frac{x}{m-1}\right| < \frac{9.96 \times 10^{69}}{2^n(m-1)} < \frac{1}{2(m-1)^2}$$

where

$$j = 32, \quad q_j := 130441933147714940, \quad \max_{0 \leq j \leq 33}\{a_{j+1}\} = 55$$

and therefore

$$\frac{1}{56(3.5 \times 10^{16} - 1)} < \frac{9.96 \times 10^{69}}{2^n}$$

i.e. $n \leq 293$. In conclusion

$$m \leq n \leq 296, \quad 2 \leq n_1 \leq 75 \quad \text{and} \quad 2 \leq m_1 \leq 12 \quad \text{for all} \quad b \in \{3, 5, 6, 7, 9, 10\}. \tag{22}$$

Finally a brief computational verification with the help of mathematica for Eq. (15) on Bounds (22)

c	b	n	m	n_1	m_1
−3	3	5	3	3	2
15	3	7	4	5	2
39	3	10	6	6	2
−24	6	8	3	4	2

are the only solutions for this equation with $b \in \{3, 5, 6, 7, 9, 10\}$. Note that k takes arbitrary values greater than n in each case.

This concludes the proof of the first part of Corollary 2.

4 Case $n \geq k + 1$

4.1 Bounding n in terms of m and k

We also have that $m > m_1 \geq 2$, $n > n_1 \geq 2$ and $k \geq 2$.

So, from Lemma 2(b) and (3), we have

$$\begin{aligned} \alpha^{n-3} &\leq L_{n-2}^{(k)} \leq L_n^{(k)} - L_{n_1}^{(k)} = b^m - b^{m_1} < b^m, \text{ and} \\ 2\alpha^n &\geq L_n^{(k)} > L_n^{(k)} - L_{n_1}^{(k)} = b^m - b^{m_1} \geq b^{m-1}, \end{aligned} \tag{23}$$

leading to

$$\left(\frac{\log b}{\log \alpha}\right)(m-1) - \frac{\log 2}{\log \alpha} < n < \left(\frac{\log b}{\log \alpha}\right) m + 3. \tag{24}$$

We note that the above inequality (24) in particular implies that $m < n + 2$. By Lemma 2(d) and (3), we get

$$\begin{aligned} &\left|(2\alpha-1) f_k(\alpha)\alpha^{n-1} - b^m\right| = \left|((2\alpha-1) f_k(\alpha)\alpha^{n-1} - L_n^{(k)}) + (L_{n_1}^{(k)} - b^{m_1})\right| \\ &= \Big|((2\alpha-1) f_k(\alpha)\alpha^{n-1} - L_n^{(k)}) + (L_{n_1}^{(k)} - (2\alpha-1) f_k(\alpha)\alpha^{n_1-1}) \\ &\quad + ((2\alpha-1) f_k(\alpha)\alpha^{n_1-1} - b^{m_1})\Big| \\ &< \frac{3}{2} + \frac{3}{2} + 9\alpha^{n_1}/8 + b^{m_1} \\ &< 5\max\{\alpha^{n_1}, b^{m_1}\}. \end{aligned}$$

In the above, we have also used the fact that $1/2 < f_k(\alpha) < 3/4$ (see Lemma 2(*e*)). Dividing through by b^m, we get

$$\left|(2\alpha - 1) f_k(\alpha)\alpha^{n-1}b^{-m} - 1\right| < 5 \max\left\{\frac{\alpha^{n_1}}{b^m}, b^{m_1-m}\right\}$$
$$< 5 \max\{\alpha^{n_1-n+3}, b^{m_1-m}\}, \tag{25}$$

where for the right–most inequality in (25) we used (23).

For the left–hand side of (25) above, we apply Theorem 4 with the data: $t := 3$ and

$$\gamma_1 := (2\alpha - 1) f_k(\alpha), \quad \gamma_2 := \alpha, \quad \gamma_3 := b, \quad b_1 := 1, \quad b_2 := n - 1, \quad b_3 := -m.$$

We begin by noticing that the three numbers $\gamma_1, \gamma_2, \gamma_3$ are positive real numbers and belong to the field $\mathbb{K} := \mathbb{Q}(\alpha)$, so we can take $D := [\mathbb{K} : \mathbb{Q}] = k$. Put

$$\Lambda := (2\alpha - 1) f_k(\alpha)\alpha^{n-1}b^{-m} - 1.$$

To see that $\Lambda \neq 0$, observe that imposing $\Lambda = 0$, leads to $b^m = (2\alpha - 1) f_k(\alpha)\alpha^{n-1}$. Conjugating the above relation by some automorphism of the Galois group of the decomposition field of $\Psi_k(x)$ over $\mathbb{Q}$ and then taking absolute values, we get that for any $i \geq 2$, we have

$$b^m = \left|2\alpha^{(i)} - 1\right| \left|f_k(\alpha^{(i)})\right| \left|\alpha^{(i)}\right|^{n-1}.$$

But the above relation is not possible since its left-hand side is greater than 8, while its right-hand side is smaller than 3. Thus, $\Lambda \neq 0$.

Since $h(\gamma_2) = (\log \alpha)/k < (\log 2)/k$ and $h(\gamma_3) = \log b$, it follows that we can take $A_2 := \log 2$ and $A_3 := k \log b$. Furthermore, by properties (9) and bounds (10), we obtain that

$$h(\gamma_1) \leq h(\alpha) + h(f_k(\alpha)) + \log 4 < (\log 2)/k + 2 \log k$$
$$7 \quad + \log 4 < 5 \log k \quad \text{for all } k \geq 2,$$

so we can take $A_1 := 5k \log k$. Finally, since $\max\{1, n - 1, m\} \leq n + 1$, we take $B := n + 1$. Then, the left–hand side of (25) is bounded below, by Theorem 4, as

$$\log |\Lambda| > -1.4 \times 30^6 \times 3^{4.5} \times k^4(1 + \log k)(1 + \log(n + 1))(5 \log k)(\log 2)(\log b).$$

Comparing with (25), we get

$$\min\{(n - n_1) \log \alpha, (m - m_1) \log b\} < 2.73 \times 10^{12}k^4(\log k)^2(\log n)(\log b). \tag{26}$$

Now the argument is split into two cases.

Case 1. $\min\{(n-n_1)\log\alpha, (m-m_1)\log b\} = (n-n_1)\log\alpha$.
In this case, we rewrite (3) as

$$\left|(2\alpha-1)f_k(\alpha)\alpha^{n-1} - (2\alpha-1)f_k(\alpha)\alpha^{n_1-1} - b^m\right| = \left|((2\alpha-1)f_k(\alpha)\alpha^{n-1} - L_n^{(k)}) + (L_{n_1}^{(k)} - (2\alpha-1)f_k(\alpha)\alpha^{n_1-1}) - b^{m_1}\right| < \frac{3}{2} + \frac{3}{2} + b^{m_1} \le b^{m_1+2}.$$

Dividing through by b^m gives

$$\left|(2\alpha-1)f_k(\alpha)(\alpha^{n-n_1}-1)\alpha^{n_1-1}b^{-m} - 1\right| < b^{m_1-m+2}. \tag{27}$$

Now we put

$$\Lambda_1 := (2\alpha-1)f_k(\alpha)(\alpha^{n-n_1}-1)\alpha^{n_1-1}b^{-m} - 1.$$

We apply again Theorem 4 with the following data

$$t := 3, \quad \gamma_1 := (2\alpha-1)f_k(\alpha)(\alpha^{n-n_1}-1), \quad \gamma_2 := \alpha, \quad \gamma_3 := b,$$
$$b_1 := 1, \quad b_2 := n_1 - 1, \quad b_3 := -m.$$

As before, we begin by noticing that the three numbers $\gamma_1, \gamma_2, \gamma_3$ belong to the field $\mathbb{K} := \mathbb{Q}(\alpha)$, so we can take $D := [\mathbb{K} : \mathbb{Q}] = k$. To see why $\Lambda_1 \neq 0$, note that otherwise, we would get the relation $(2\alpha-1)f_k(\alpha)(\alpha^{n-n_1}-1) = b^m\alpha^{1-n_1}$. Conjugating this last equation with any automorphism σ of the Galois group of $\Psi_k(x)$ over $\mathbb{Q}$ such that $\sigma(\alpha) = \alpha^{(i)}$ for some $i \geq 2$ and then taking absolute values, we arrive at the equality

$$6 > |(2\alpha^{(i)}-1)f_k(\alpha^{(i)})((\alpha^{(i)})^{n-n_1}-1)| = |b^m(\alpha^{(i)})^{1-n_1}| > 8$$

because $b \geq 2$, $m > m_1 \geq 2$ and $n_1 \geq 2$, but this is an absurd.

Since

$$h(\gamma_1) \leq h((2\alpha-1)f_k(\alpha)) + h(\alpha^{n-n_1}-1) < 5\log k + (n-n_1)\frac{\log\alpha}{k} + \log 2,$$

it follows that

$$\begin{aligned} kh(\gamma_1) &< 6k\log k + (n-n_1)\log\alpha \\ &< 6k\log k + 2.73 \times 10^{12}k^4(\log k)^2(\log n)(\log b). \end{aligned}$$

So, we can take $A_1 := 2.731 \times 10^{12}k^4(\log k)^2(\log n)(\log b)$. Further, as before, we take $A_2 := \log 2$ and $A_3 := k\log b$. Finally, by recalling that $m \leq n+1$, we can take $B := n+1$.

We then get that

$$\log|\Lambda_1| > -1.4\times 30^6\times 3^{4.5}\times k^3(1+\log k)(1+\log(n+1)) \\ \times(2.731\times 10^{12}k^4(\log k)^2(\log n)(\log b))(\log 2)(\log b),$$

which yields

$$\log|\Lambda_1| > -1.491\times 10^{24}k^7(\log k)^3(\log n)^2(\log b)^2.$$

Comparing this with inequality (27), we obtain that

$$(m-m_1)\log b < 1.5\times 10^{24}k^7(\log k)^3(\log n)^2(\log b)^2. \tag{28}$$

Case 2. $\min\{(n-n_1)\log\alpha, (m-m_1)\log b\} = (m-m_1)\log b$.
In this case, we write (3) as

$$\begin{aligned}\left|(2\alpha-1)f_k(\alpha)\alpha^{n-1}-b^m+b^{m_1}\right| &= \Big|((2\alpha-1)f_k(\alpha)\alpha^{n-1}-L_n^{(k)})+(L_{n_1}^{(k)} \\ &\quad -(2\alpha-1)f_k(\alpha)\alpha^{n_1-1})+(2\alpha-1)f_k(\alpha)\alpha^{n_1-1}\Big| \\ &< \frac{3}{2}+\frac{3}{2}+9\alpha^{n_1}/8 < 2.4\alpha^{n_1},\end{aligned}$$

so, by inequality (23), $\alpha^{n-3} < b^m - b^{m_1}$ and we obtain that

$$\begin{aligned}\left|(2\alpha-1)f_k(\alpha)(b^{m-m_1}-1)^{-1}\alpha^{n-1}b^{-m_1}-1\right| &< \frac{2.4\alpha^{n_1}}{b^m-b^{m_1}} \\ &\le 2.4\alpha^{n_1-n+3} < \alpha^{n_1-n+5}.\end{aligned} \tag{29}$$

The above inequality (29) suggests once again studying a lower bound for the absolute value of

$$\Lambda_2 := (2\alpha-1)f_k(\alpha)(b^{m-m_1}-1)^{-1}\alpha^{n-1}b^{-m_1}-1.$$

We again apply Matveev's theorem with the following data

$$t:=3,\quad \gamma_1:=(2\alpha-1)f_k(\alpha)(b^{m-m_1}-1)^{-1},\quad \gamma_2:=\alpha,\quad \gamma_3:=b,$$
$$b_1:=1,\quad b_2:=n-1,\quad b_3:=-m_1.$$

We can again take $B:=n+1$ and $\mathbb{K}:=\mathbb{Q}(\alpha)$, so that $D:=k$. We also note that, if $\Lambda_2=0$, then we would get to the relation $(2\alpha-1)f_k(\alpha)\alpha^{n-1}=b^{m_1}(b^{m-m_1}-1)$. With a similar argument to $\Lambda_1\neq 0$, we arrive at

$$3 > |(2\alpha^{(i)}-1)f_k(\alpha^{(i)})(\alpha^{(i)})^{n-1}| = |b^{m_1}(b^{m-m_1}-1)| \ge 4$$

since $b \geq 2$, $m_1 \geq 2$ and $m - m_1 \geq 1$, but this is a contradiction. Then, $\Lambda_2 \neq 0$.

Now, we note that

$$h(\gamma_1) \leq h((2\alpha - 1) f_k(\alpha)) + h(b^{m-m_1} - 1) < 5 \log k + (m - m_1) \log b + \log 2.$$

Thus,

$$kh(\gamma_1) < 6k \log k + (m - m_1)k \log b < 2.731 \times 10^{12} k^5 (\log k)^2 (\log n)(\log b),$$

and so we can take $A_1 := 2.731 \times 10^{12} k^5 (\log k)^2 (\log n)(\log b)$. As before, we take $A_2 := \log 2$ and $A_3 := k \log b$. It then follows from Matveev's theorem, after some calculations, that

$$\log |\Lambda_2| > -1.491 \times 10^{24} k^8 (\log k)^3 (\log n)^2 (\log b)^2.$$

From this and inequality (29), we obtain that

$$(n - n_1) \log \alpha < 1.5 \times 10^{24} k^8 (\log k)^3 (\log n)^2 (\log b)^2. \tag{30}$$

In both Case 1 and Case 2 and from inequations (26), (28) and (30), we have

$$\begin{aligned} (m - m_1) \log b &< 1.5 \times 10^{24} k^7 (\log k)^3 (\log n)^2 (\log b)^2 \quad \text{and} \\ (n - n_1) \log \alpha &< 1.5 \times 10^{24} k^8 (\log k)^3 (\log n)^2 (\log b)^2. \end{aligned} \tag{31}$$

We now finally rewrite Eq. (3) as

$$\begin{aligned} &\left|(2\alpha - 1) f_k(\alpha)\alpha^{n-1} - (2\alpha - 1) f_k(\alpha)\alpha^{n_1-1} - b^m + b^{m_1}\right| \\ &= \Big|((2\alpha - 1) f_k(\alpha)\alpha^{n-1} - L_n^{(k)}) \\ &\quad + (L_{n_1}^{(k)} - (2\alpha - 1) f_k(\alpha)\alpha^{n_1-1})\Big| < 3. \end{aligned}$$

We divide through both sides by $b^m - b^{m_1}$ getting

$$\left|\frac{(2\alpha - 1) f_k(\alpha)(\alpha^{n-n_1} - 1)}{b^{m-m_1} - 1}\alpha^{n_1-1} b^{-m_1} - 1\right| < \frac{3}{b^m - b^{m_1}} \leq \frac{6}{b^m}. \tag{32}$$

To find a lower–bound on the left–hand side of (32) above, we again apply Theorem 4 with the data

$$\begin{aligned} &t := 3, \quad \gamma_1 := \frac{(2\alpha - 1) f_k(\alpha)(\alpha^{n-n_1} - 1)}{b^{m-m_1} - 1}, \quad \gamma_2 := \alpha, \\ &\gamma_3 := b, \quad b_1 := 1, \quad b_2 := n_1 - 1, \quad b_3 := -m_1. \end{aligned}$$

We also take $B := n + 1$ and we take $\mathbb{K} := \mathbb{Q}(\alpha)$ with $D := k$. From the properties of the logarithmic height function, we have that

$$\begin{aligned} kh(\gamma_1) &\le k\left(h((2\alpha - 1)f_k(\alpha)) + h(\alpha^{n-n_1} - 1) + h(b^{m-m_1} - 1)\right) \\ &< 5k\log k + (n - n_1)\log\alpha + k(m - m_1)\log b + 2k\log 2 \\ &< 3.01 \times 10^{24}k^8(\log k)^3(\log n)^2(\log b)^2, \end{aligned}$$

where in the above chain of inequalities we used the bounds (31). So we can take $A_1 := 3.01 \times 10^{24}k^8(\log k)^3(\log n)^2(\log b)^2$, and certainly as before we take $A_2 := \log 2$ and $A_3 := k\log b$. We need to show that if we put

$$\Lambda_3 := \frac{(2\alpha - 1)f_k(\alpha)(\alpha^{n-n_1} - 1)}{b^{m-m_1} - 1}\alpha^{n_1-1}b^{-m_1} - 1,$$

then $\Lambda_3 \neq 0$. To see why $\Lambda_3 \neq 0$, note that otherwise, we would get the relation

$$(2\alpha - 1)f_k(\alpha)\alpha^{n_1-1}(\alpha^{n-n_1} - 1) = b^{m_1}(b^{m-m_1} - 1).$$

Conjugating this equation with the automorphism σ of the Galois group of $\Psi_k(x)$ over $\mathbb{Q}$ such that $\sigma(\alpha) = \alpha^{(i)}$, for some $i \ge 2$ and the argument of $\alpha^{(i)}$ is in $[0, \pi/4)$ for all $k \ge 12$ (which is possible using the Lemma 1 with $h = 1$) we obtain

$$|(2\alpha^{(i)} - 1)f_k(\alpha^{(i)})(\alpha^{(i)})^{n_1-1}((\alpha^{(i)})^{n-n_1} - 1)| = |b^{m_1}(b^{m-m_1} - 1)|,$$

where

$$2 > |f_k(\alpha^{(i)})(\alpha^{(i)})^{n_1-1}((\alpha^{(i)})^{n-n_1} - 1)|$$

and for the law of cosines we obtain

$$|2\alpha^{(i)} - 1| = \sqrt{|2\alpha^{(i)}|^2 + 1 - 2|2\alpha^{(i)}|\cos\theta} < \sqrt{5 - 4\cos(\pi/4)} < 1.5,$$

in conclusion

$$3 > |(2\alpha^{(i)} - 1)f_k(\alpha^{(i)})(\alpha^{(i)})^{n_1-1}((\alpha^{(i)})^{n-n_1} - 1)| = |b^{m_1}(b^{m-m_1} - 1)| > 4$$

since $m - m_1 \ge 1$, $m_1 \ge 2$ and $b \ge 2$, but this is not true. For $k \le 11$, after making an appropriate choice i, an exact calculation of the value absolutes of $2\alpha^{(i)} - 1$, $f_k(\alpha^{(i)})$ and $\alpha^{(i)}$, show that $\Lambda_1 \neq 0$.

Then Theorem 4 gives

$$\begin{aligned} \log|\Lambda_3| &> -1.4 \times 30^6 \times 3^{4.5}k^{11}(1 + \log k)(1 + \log(n + 1)) \\ &\quad \times \left(3.01 \times 10^{24}(\log k)^3(\log n)^2(\log b)^2\right)(\log 2)(\log b), \end{aligned}$$

which together with inequalities (23) and (32) gives

$$(n-3)\log\alpha < m\log b < 1.645\times 10^{36}k^{11}(\log k)^4(\log n)^3(\log b)^3,$$

i.e.

$$\frac{n}{(\log n)^3} < 3.421\times 10^{36}k^{11}(\log k)^4(\log b)^3. \tag{33}$$

We apply Lemma 6 with the data $s=3,\ x=n,\ T=3.421\times 10^{36}k^{11}(\log k)^4(\log b)^3$. Thus,

$$\begin{aligned} n &< 2^3\cdot\left(3.421\times 10^{36}k^{11}(\log k)^4(\log b)^3\right)(13\log k+120\log b)^3\\ &< 1.94\times 10^{44}k^{11}(\log k)^7(\log b)^6. \end{aligned}$$

We then record what we have proved so far as a lemma.

Lemma 9 *If (n, m, n_1, m_1, k) is a solution in positive integers to Eq. (3) with $n > n_1 \geq 2$, $m > m_1 \geq 2$ and $k \geq 2$, we then have that*

$$n < 1.94\times 10^{44}k^{11}(\log k)^7(\log b)^6.$$

4.2 An absolute upper bound for *k* and *n* on *b*

First we assume $2^{0.49k} \leq n$. Then

$$\begin{aligned} k < 3\log n &< 3\left(\log\left(k^{11}(\log k)^7\right)+\log\left(1.94\times 10^{44}(\log b)^6\right)\right)\\ &< 3\,(14\log k+144\log b)\\ &< 684(\log b)(\log k). \end{aligned}$$

Hence, by Lemma 6 with $s=1,\ \ x=k,\ \ T=684(\log b)$, we have

$$k < 2(684(\log b))(\log(684(\log b))) < 1.3\times 10^4(\log b)^2 \quad \text{for all} \quad b\geq 2.$$

In another case $n_1 < n < 2^{0.49k}$ and we assume without loss of generality that $k \geq 200$. By Lemma 3 with $c = 0.49$ and $r \in \{n_1, n\}$ we get

$$L_r^{(k)} = 3\cdot 2^{r-2}\left(1+\zeta_r'\right), \qquad |\zeta_r'| < 4/2^{0.51k} \quad \text{for all} \quad r\geq 2; \tag{34}$$

$$L_r^{(k)} = 3\cdot 2^{r-2}\left(1-\frac{r-k+4/3}{2^{k+1}}+\zeta_r''\right), \quad |\zeta_r''| < 8/2^{1.02k} \quad \text{for all} \quad r\geq k+2. \tag{35}$$

We need distinguing the following cases.

4.2.1 The case $3 \cdot 2^{n-2} \neq b^{m_1}\left(b^{m-m_1}-1\right)$

Using (34) we can write (3) as

$$3 \cdot 2^{n-2}\left(1+\zeta_n'\right) - b^m = 3 \cdot 2^{n_1-2}\left(1+\zeta_{n_1}'\right) - b^{m_1}. \tag{36}$$

Then

$$\begin{aligned}\left|3 \cdot 2^{n-2} - b^m\right| &\leq 3 \cdot 2^{n-2}|\zeta_n'| + 3 \cdot 2^{n_1-2}\left(1+|\zeta_{n_1}'|\right) + b^{m_1}\\ &< \frac{3 \cdot 2^n}{2^{0.51k}}\left(1+2^{n_1-n}\right) + 3 \cdot 2^{n_1-2} + b^{m_1}.\end{aligned}$$

Dividing throught by $3 \cdot 2^{n-2}$ and considering that $2^{n+1} > 2\alpha^n > b^{m-1}$ (according to 23), we conclude

$$\begin{aligned}\left|3^{-1} \cdot 2^{-(n-2)} b^m - 1\right| &< \frac{6}{2^{0.51k}} + \frac{3b}{b^{m-m_1}} + \frac{1}{2^{n-n_1}}\\ &< 7b \max\left\{2^{-0.51k}, b^{m_1-m}, 2^{n_1-n}\right\}.\end{aligned} \tag{37}$$

Now we put

$$\Lambda_4 := 3^{-1} \cdot 2^{-(n-2)} b^m - 1$$

and see that it is nonzero analogously to how we obtained that $\Lambda_0 \neq 0$ in (19). We apply Theorem 4 on Λ_4 with the datas

$$t := 3, \quad \gamma_1 := 3, \quad \gamma_2 := 2, \quad \gamma_3 := b, \quad b_1 := -1, \quad b_2 := -(n-2), \quad b_3 := m.$$

We note that $\gamma_1, \gamma_2, \gamma_3$ belong to $\mathbb{K} := \mathbb{Q}$, so $D := 1$. Now, $A_1 := \log 3$, $A_2 := \log 2$ and $A_3 := \log b$. Thus,

$$\log|\Lambda_4| > -2.4 \times 10^{11}(\log b)(\log n).$$

This inequality together with inequality (37), lead to

$$\nabla := \min\left\{0.51k, (m-m_1)\log_2 b, n-n_1\right\} < 3.5 \times 10^{11}(\log b)(\log n). \tag{38}$$

If $\nabla = 0.51k$, we get

$$k < 6.9 \times 10^{11}(\log b)(\log n). \tag{39}$$

If $\nabla = (m-m_1)\log_2 b$, then $m - m_1 < 2.43 \times 10^{11} \log n$. We rewriting equality (36) to obtain

$$\left|3 \cdot 2^{n-2} - b^{m_1}\left(b^{m-m_1}-1\right)\right| \leq \frac{3 \cdot 2^n}{2^{0.51k}}\left(1+2^{n_1-n}\right) + 3 \cdot 2^{n_1-2}.$$

Then

$$\left|3^{-1}2^{-(n-2)}b^{m_1}\left(b^{m-m_1}-1\right)-1\right| < \frac{6}{2^{0.51k}}+\frac{1}{2^{n-n_1}}$$
$$< 7\max\left\{2^{n_1-n}, 2^{-0.51k}\right\}. \qquad (40)$$

We now apply Theorem 4 on

$$\Lambda_5 := 3^{-1}\cdot 2^{-(n-2)}\left(b^{m-m_1}-1\right)b^{m_1}-1$$

which is nonzero because $3\cdot 2^{n-2}\neq b^{m_1}\left(b^{m-m_1}-1\right)$. Set the datas

$$t := 3,\quad \gamma_1 := 3^{-1}\left(b^{m-m_1}-1\right),\quad \gamma_2 := 2,\quad \gamma_3 := b,$$
$$b_1 := 1,\quad b_2 := -(n-2),\quad b_3 := m_1.$$

As before we take $\mathbb{K} := \mathbb{Q}$, $D := 1$ and $B = n+1$. Now, $A_1 := 4(m-m_1)(\log b)$, $A_2 := \log 2$ and $A_3 := \log b$. Thus,

$$\log|\Lambda_5| > -8.74\times 10^{11}(m-m_1)(\log b)^2\log n$$
$$> -2.13\times 10^{23}(\log b)^2(\log n)^2.$$

This inequality together with inequality (40), lead to

$$k < 6.1\times 10^{23}(\log b)^2(\log n)^2 \qquad (41)$$

or

$$n-n_1 < 3.1\times 10^{23}(\log b)^2(\log n)^2. \qquad (42)$$

If $\nabla = n-n_1$, then $n-n_1 < 3.5\times 10^{11}(\log b)(\log n)$. We rewrite equality (36) as

$$\left|3\cdot 2^{n_1-2}\left(2^{n-n_1}-1\right)-b^m\right| \le \frac{3\cdot 2^n}{2^{0.51k}}\left(1+2^{n_1-n}\right)+b^{m_1}.$$

Hence,

$$\left|3\cdot 2^{n_1-2}\left(2^{n-n_1}-1\right)b^{-m}-1\right| < \frac{6}{2^{0.51k}}\frac{2^n}{b^m}+\frac{1}{b^{m-m_1}}$$
$$< \frac{6\alpha^3}{2^{0.51k}}\left(\frac{2}{\alpha}\right)^n+\frac{1}{b^{m-m_1}}$$
$$< 64\max\left\{2^{-0.51k}, b^{m_1-m}\right\}. \qquad (43)$$

Here we have used (23). Furthermore, given that $\alpha > 2(1-2^{-k})$ and $n < 2^{0.49k}$, we conclude

$$\left(\frac{2}{\alpha}\right)^n < \left(\frac{1}{1-2^{-k}}\right)^n = \left(1+\frac{2^{-k}}{1-2^{-k}}\right)^n < \left(1+\frac{2^{-0.51k+1}}{2^{0.49k}}\right)^{2^{0.49k}}$$
$$< e^{2^{-0.51k+1}} < 1.3 \quad \text{for all } k \geq 6,$$

since $(1+z/\ell)^\ell$ converges increasingly to e^z for all $z \in \mathbb{R}^+$ when $\ell \to \infty$.

Now we make

$$\Lambda_6 := 3 \cdot 2^{n_1-2}\left(2^{n-n_1}-1\right) b^{-m} - 1.$$

If $\Lambda_6 = 0$ we obtain that $3 \cdot 2^{n_1-2}\left(2^{n-n_1}-1\right) = b^m$ so $3^{m-1} \mid 2^{n-n_1}-1$ and by item (ii) of Lemma 7 we conclude that $3^{m-3} \mid n-n_1$, therefore (see (24))

$$\begin{aligned}(n-3)(\log\alpha)/\log b < m \leq 3+\log_3(n-n_1) &< 3+\log_3(3.5\times 10^{11}(\log b)(\log n))\\ &< 3+13\log n+0.4\log b\\ &< 24(\log b)(\log n)\end{aligned}$$

and using Lemma 6 it follows that

$$n < 2(57(\log b)^2)(\log(57(\log b)^2)) < 548(\log b)^3. \tag{44}$$

In another case $\Lambda_6 \neq 0$ and we apply similarly Theorem 4 as for the last Λ's, with the datas: Λ_6, $t = 3$,

$$\begin{aligned}&\gamma_1 := 3\left(2^{n-n_1}-1\right), \quad \gamma_2 := 2, \quad \gamma_3 := b,\\ &b_1 := 1, \quad b_2 := n_1-2, \quad b_3 := -m\end{aligned}$$

which leads to

$$\begin{aligned}\log|\Lambda_6| &> -5.46\times 10^{11}(n-n_1)(\log b)(\log n)\\ &> -1.911\times 10^{23}(\log b)^2(\log n)^2.\end{aligned}$$

Comparing this inequality with inequality (43), we arrive at

$$k < 3.75\times 10^{23}(\log b)^2(\log n)^2 \tag{45}$$

or

$$m-m_1 < 1.912\times 10^{23}(\log b)(\log n)^2. \tag{46}$$

Hence, from (39), (41), (42), (44) (45) and (46) we conclude that

$$n < 548(\log b)^3 \tag{47}$$

or

$$k < 6.1 \times 10^{23}(\log b)^2(\log n)^2 \tag{48}$$

or

$$n - n_1 < 3.1 \times 10^{23}(\log b)^2(\log n)^2 \ \text{ and } \ m - m_1 < 1.912 \times 10^{23}(\log b)(\log n)^2. \tag{49}$$

Assume that (49) is fulfilled. Returning to equality (36) and rearranging it as

$$\left|3 \cdot 2^{n_1-2}\left(2^{n-n_1}-1\right) - b^{m_1}\left(b^{m-m_1}-1\right)\right| < \frac{3 \cdot 2^{n_1}\left(2^{n-n_1}+1\right)}{2^{0.51k}}$$

we get

$$\left|3^{-1} \cdot 2^{-(n_1-2)}\left(2^{n-n_1}-1\right)^{-1} b^{m_1}\left(b^{m-m_1}-1\right) - 1\right| < 12 \cdot 2^{-0.51k}. \tag{50}$$

Finally, we want to apply Theorem 4 to

$$\Lambda_7 := 3^{-1} \cdot 2^{-(n_1-2)}\left(2^{n-n_1}-1\right)^{-1} b^{m_1}\left(b^{m-m_1}-1\right) - 1.$$

Note that if $\Lambda_7 = 0$, then

$$3 \cdot 2^{n-2} - 3 \cdot 2^{n_1-2} = b^m - b^{m_1}. \tag{51}$$

We consider the following cases over n_1.

- First, we assume that $n_1 \leq k+1$. If in addition $n = k+1 > n_1$, then combining Eq. (3) with items (a) and (g) of Lemma 2 together with identity (51), we obtain

  $$3 \cdot 2^{k-1} - 2 - b^m = 3 \cdot 2^{n_1-2} - b^{m_1} = 3 \cdot 2^{k-1} - b^m$$

  i.e. $-2 = 0$ which is absurd. Otherwise $n_1 \leq k+1 < n$ and we use a similar argument adding identity (35) to obtain

  $$3 \cdot 2^{n-2}\left(-\frac{n-k+4/3}{2^{k+1}} + \zeta_n''\right) + 3 \cdot 2^{n_1-2} = \begin{cases} 3 \cdot 2^{n_1-2} - 2; & \text{if } n_1 = k+1, \\ 3 \cdot 2^{n_1-2}; & \text{if } n_1 \leq k. \end{cases}$$

Therefore

$$\frac{24}{2^{1.02k}} > 3|\zeta_n''| = \frac{1}{2^{k+1}} \begin{cases} \left|3(n-k)+4-2^{-n+3}\right|; & \text{if } n_1 = k+1, \\ |3(n-k)+4|; & \text{if } n_1 \leq k \end{cases}$$
$$> \frac{6}{2^{k+1}}$$

but the above can only occur when $k \leq 149$ which contradicts the assumption of the beginning.

- Now we assume that $n_1 \geq k+2$ and since $n > n_1$ we can use the same argument of the previous item but using identity (35) for n and n_1. So

$$3 \cdot 2^{n-2}\left(-\frac{n-k+4/3}{2^{k+1}} + \zeta_n''\right) = 3 \cdot 2^{n_1-2}\left(-\frac{n_1-k+4/3}{2^{k+1}} + \zeta_{n_1}''\right)$$

implies that

$$\frac{1}{2^{k+1}}\left|2^{n_1-n}(n_1-k+4/3)-(n-k+4/3)\right| < \frac{12}{2^{1.02k}}$$

i.e.

$$\frac{2.5}{2^{k+1}} \leq \frac{1}{2^{k+1}}\left|(n-n_1)+(1-2^{n_1-n})(n_1-k+4/3)\right| < \frac{12}{2^{1.02k}}$$

and this only occurs when $k \leq 163$, so we get a contradiction.

In conclusion $\Lambda_7 \neq 0$ and continuing with our apply of Theorem 4, as for the last Λ's, we take $t = 3$ and

$$\gamma_1 := 3^{-1}\left(2^{n-n_1}-1\right)^{-1}\left(b^{m-m_1}-1\right), \quad \gamma_2 := 2, \quad \gamma_3 := b,$$
$$b_1 := 1, \quad b_2 := -(n_1-2), \quad b_3 := m_1.$$

Further

$$\begin{aligned} A_3 := h(\gamma_3) &\leq h(3) + h(2^{n-n_1}-1) + h(b^{m-m_1}-1) \\ &< (n-n_1)\log 2 + (m-m_1)\log b + \log 12 \\ &< 2.15 \times 10^{23}(\log b)^2(\log n)^2 + 1.912 \times 10^{23}(\log b)^2(\log n)^2 + \log 12 \\ &< 4.1 \times 10^{23}(\log b)^2(\log n)^2. \end{aligned}$$

Hence

$$\log|\Lambda_7| > -8.14 \times 10^{34}(\log b)^3(\log n)^3.$$

Therefore inequality (50), together with the previous one, leads to

$$k < 2.303 \times 10^{35}(\log b)^3(\log n)^3 \tag{52}$$

which includes inequality (48). So, by Lemma 9

$$\begin{aligned} k &< 2.303\times 10^{35}(\log b)^3\left(\log\left(1.94\times 10^{44}k^{11}(\log k)^7\right)+6\log(\log b)\right)^3\\ &< 2.303\times 10^{35}(\log b)^3\,(35\log k+6\log(\log b))^3\\ &< 2.303\times 10^{35}(\log b)^3\,(41\max\{\log k,\log(\log b)\})^3 \end{aligned}$$

therefore

$$k < 9.45\times 10^{36}(\log b)^3(\log k)^3.$$

Finally, using Lemma 6 with $(x,s) := (k,3)$ and $T := 9.45\times 10^{36}(\log b)^3$, we conclude that

$$\begin{aligned} k &< 2^3\cdot\left(9.45\times 10^{36}(\log b)^3\right)\left(\log\left(9.45\times 10^{36}(\log b)^3\right)\right)^3\\ &< 1.4\times 10^{44}(\log b)^6. \end{aligned}$$

Sustituying this inequality in Lemma 9, we have

$$n < 1.02\times 10^{545}(\log b)^{79}$$

which includes inequality (47).

4.2.2 The case $3\cdot 2^{n-2} = b^{m_1}\left(b^{m-m_1}-1\right)$

In this case from Eq. (3) we know that $3\cdot 2^{n-2} = L_n^{(k)} - L_{n_1}^{(k)}$, so combining this with Items (a) and (g) of Lemma 2 and the identity (35), it follows that

$$0 = \begin{cases} -2-3\cdot 2^{n_1-2}; & \text{if } n_1\le k \text{ and } n=k+1,\\ 3\cdot 2^{n-2}\left(-\dfrac{n-k+4/3}{2^{k+1}}+\zeta_n''\right)-3\cdot 2^{n_1-2}; & \text{if } n_1\le k \text{ and } n\ge k+2,\\ 3\cdot 2^{n-2}\left(-\dfrac{n-k+4/3}{2^{k+1}}+\zeta_n''\right)-(3\cdot 2^{k-1}-2); & \text{if } n>n_1=k+1 \end{cases}$$

or

$$0 = 3\cdot 2^{n-2}\left(-\frac{n-k+4/3}{2^{k+1}}+\zeta_n''\right)-3\cdot 2^{n_1-2}\left(1-\frac{n_1-k+4/3}{2^{k+1}}+\zeta_{n_1}''\right)$$

if $n > n_1 \geq k+2$. Then we conclude that the case $n_1 \leq k$ and $n = k+1$ is an absurd and the others imply

$$\frac{8}{2^{1.02k}} > \frac{1}{2^{k+1}} \begin{cases} \left|n - k + 4/3 + 2^{n_1-n+k+1}\right|; & \text{if } n_1 \leq k \text{ and } n \geq k+2, \\ \left|n - k + 4/3 + 2^{k-n+3}(3 \cdot 2^{k-1} - 2)/3\right|; & \text{if } n > n_1 = k+1 \end{cases}$$
$$> \frac{3}{2^{k+1}}$$

or

$$\begin{aligned} \frac{12}{2^{1.02k}} &> \frac{1}{2^{k+1}} \left|(n-k+4/3) + 2^{n_1-n}\left(2^{k+1} - (n_1 - k + 4/3)\right)\right| \\ &= \frac{1}{2^{k+1}} \left|(1-2^{n_1-n})(n-k+4/3) + 2^{n_1-n}(n-n_1) + 2^{k+n_1-n+1}\right| \\ &> \frac{1.5}{2^{k+1}} \end{aligned}$$

if $n > n_1 \geq k+2$. In any case we obtain a contradiction since we had assumed $k \geq 200$ from the beginning.

In resume, we have the follow result.

Lemma 10 *Let $n > n_1 \geq 2$ and $m > m_1 \geq 2$, be solutions of (3) with $n \geq k+1$, then*

$$k < 1.4 \times 10^{44}(\log b)^6 \quad \text{and} \quad m - 1 \leq n < 1.02 \times 10^{545}(\log b)^{79}.$$

This completed the proof of the second part of Theorem 1.

4.3 Proof of the second part of Corollary 2

In this section $b \in [2, 10]$ then it is only enough to study the values of $b \in \{2, 3, 5, 6, 7, 10\}$, since 4, 8 and 9 are powers of 2 or 3.

4.3.1 Case $k > 625$ and $b \in \{2, 3, 5, 6, 7, 10\}$.

Combining inequalities (33) and (52) we obtain computationally that

$b =$	2	3	5	6	7	10
$n <$	1.6×10^{539}	4.4×10^{546}	6.4×10^{552}	3.5×10^{554}	7.4×10^{555}	3.9×10^{558}

(53)

Also, by Lemma 9 we know that $n < 2^{0.49k}$ because we have assumed $k > 625$.

In Sect. 4.2.2 we saw that if $k+1 \leq n < 2^{0.49k}$ and $3 \cdot 2^{n-2} = b^{m_1}\left(b^{m-m_1} - 1\right)$ then Eq. (3) has no solution, so in this section we will always assume that $3 \cdot 2^{n-2} \neq b^{m_1}\left(b^{m-m_1} - 1\right)$.

Therefore, we use inequalities (14) and (37) to obtain[2]

$$\begin{aligned}\left|(n-2)\log_b 2 - m + \log_b 3\right| &< 14b\max\left\{2^{-0.51k}, b^{m_1-m}, 2^{n_1-n}\right\}/\log b\\ &\le \frac{14b}{\log b}\times 2^{-\nabla},\end{aligned} \tag{54}$$

where ∇ is already defined in inequality (38). When $b=2$, we have

$$0.41 < \left\|\log_2 3\right\| \le \left|n-m-2+\log_2 3\right| < 41\times 2^{-\nabla}$$

and therefore $\nabla < 6.644$ but we had assumed that $\nabla \ge 9$ which is an absurd, so it only remains that

$$\nabla < 9. \tag{55}$$

If $b \in \{3, 6\}$, let $c_3 := 2$ and $c_6 := 3$, so we obtain

$$\left|(n-c_b)\log_b 2 - (m-1)\right| < \frac{14b}{\log b}\times 2^{-\nabla} \tag{56}$$

and using bounds (53) we assume for a moment that

$$\nabla > \begin{cases}1823; & \text{if } b=3,\\ 1849; & \text{if } b=6\end{cases} > \log_2\left(\frac{14b(2(n-c_b))}{\log b}\right).$$

Then by Lemma 5 it follows that

$$\frac{1}{(a_b+2)(n-c_b)^2} < \left|\log_b 2 - \frac{m-1}{n-c_b}\right| < \frac{14b\times 2^{-\nabla}}{(n-c_b)\log b} < \frac{1}{2(n-c_b)^2}$$

where

b	t_b	q_{t_b+1}	$a_b := \max_{0\le\ell\le t_b+1}\{a_\ell\}$
3	1048	4.86×10^{546}	3308
6	1070	6.49×10^{554}	3308

(57)

Therefore

$$\nabla < \begin{cases}1832; & \text{if } b=3,\\ 1859; & \text{if } b=6.\end{cases} \tag{58}$$

[2] Assuming for a moment that $\nabla \ge 9$.

In another case $b \in \{5, 7, 10\}$, We use Lemma 4, with the parameters

$$u := n - 2, \qquad \tau := \log_b 2, \qquad \mu := \log_b 3, \qquad v := m,$$
$$A := \frac{14b}{\log b}, \qquad B := 2, \qquad w := \nabla \tag{59}$$

and M as in Table (53). Thus we obtain

b	q_b	ε_b	$\log_B\left(\frac{Aq_b}{\varepsilon_b}\right)$
5	1.0481×10^{554}	0.34419	1847
7	6.0928×10^{557}	0.12152	1861
10	2.0504×10^{560}	0.11046	1870

By the above table and inequalities (55) and (58), we arrive at

$b =$	2	3	5	6	7	10
$\nabla \leq$	9	1832	1847	1859	1861	1870

(60)

Case 1. $\nabla = 0.51k$. Then

$b =$	2	3	5	6	7	10
$k \leq$	17	3592	3621	3645	3649	3666

(61)

Case 2. $\nabla = (m - m_1)\log_2 b$. From inequalities (14) and (40), we know that[3]

$$|\Gamma_5| < \frac{14}{\log b} \times 2^{-\min\{n-n_1, 0.51k\}},$$

where

$$\Gamma_5 := \begin{cases} n - m_1 - 2 + \log_2\left(3/\left(2^{m-m_1} - 1\right)\right); & \text{if } (m - m_1, b) = (m - m_1, 2) \neq (2, 2), \\ n - m_1 - 2; & \text{if } (m - m_1, b) = (2, 2), \\ (n-3)\log_3 2 - (m_1 - 1); & \text{if } (m - m_1, b) = (1, 3), \\ (n-5)\log_3 2 - (m_1 - 1); & \text{if } (m - m_1, b) = (2, 3), \\ (n-5)\log_5 2 - m_1; & \text{if } (m - m_1, b) = (2, 5), \\ (n-3)\log_7 2 - m_1; & \text{if } (m - m_1, b) = (1, 7), \\ (n-6)\log_7 2 - m_1; & \text{if } (m - m_1, b) = (2, 7), \\ (n-2)\log_b 2 - m_1 + \log_b\left(3/\left(b^{m-m_1} - 1\right)\right); & \text{otherwise.} \end{cases}$$

If $b = 2$ and $m - m_1 \neq 2$ we obtain

$$0.22 < \left\|\log_2\left(3/\left(2^{m-m_1} - 1\right)\right)\right\| < \frac{14}{\log 2} \times 2^{-\min\{n-n_1, 0.51k\}}.$$

[3] Here we assume for a moment that $\min\{n - n_1, 0.51k\} \geq 6$.

If $b = 2$ and $m - m_1 = 2$, we get

$$1 \leq |n - m| = |n - m_1 - 2| < \frac{14}{\log 2} \times 2^{-\min\{n-n_1, 0.51k\}}$$

since $3 \cdot 2^{n-2} \neq b^{m_1}\left(b^{m-m_1} - 1\right) = 3 \cdot 2^{m-2}$ implies $n \neq m$. Therefore

$$\min\{n - n_1, 0.51k\} < 6.521 \quad \text{for} \quad b = 2. \tag{62}$$

If $(m - m_1, b) \in \{(1,3), (2,3), (2,5), (1,7), (2,7)\}$, we use Lemma 5 with $\tau = \log_b 2$, M as in Table (53) and $y < M$ on

$$\left|\log_b 2 - \frac{x}{y}\right| < \frac{14}{y \log b} \times 2^{-\min\{n-n_1, 0.51k\}}, \tag{63}$$

where $x/y \in \{(m_1-1)/(n-3), (m_1-1)/(n-5), m_1/(n-5), m_1/(n-3), m_1/(n-6)\}$. Then we assume for a moment that

$$\min\{n - n_1, 0.51k\} > \begin{cases} 1821; & \text{if } b = 3, \\ 1841; & \text{if } b = 5, \\ 1851; & \text{if } b = 7 \end{cases} = \left\lceil \log_2\left(\frac{28M}{\log b}\right)\right\rceil$$

and obtain

$$\frac{1}{(a_{\ell_b} + 2)y^2} < \left|\log_b 2 - \frac{x}{y}\right| < \frac{14}{y \log b} \times 2^{-\min\{n-n_1, 0.51k\}} < \frac{1}{2y^2},$$

given that

b	t_b	q_{t_b+1}	$a_b := \max_{0 \leq \ell \leq t_b+1}\{a_\ell\}$
3	1048	4.86×10^{546}	3308
5	1094	9.31×10^{552}	5393
7	1107	9.07×10^{555}	2038

(64)

So

$b =$	3	5	7
$\min\{n - n_1, 0.51k\} <$	1831.27	1851.9	1860.4

(65)

In another case

$$(m - m_1, b) \notin \{(m - m_1, 2), (1,3), (2,3), (2,5), (1,7), (2,7)\}$$

and we use Lemma 4 with the parameters

$$u := n-2, \qquad \tau := \log_b 2, \qquad \mu := \log_b\left(3/\left(b^{m-m_1}-1\right)\right), \qquad v := m_1,$$
$$A := \frac{14}{\log b}, \qquad B := 2, \qquad w := \min\{n-n_1, 0.51k\}, \tag{66}$$

M as in Table (53) and $m-m_1$ between 1 and the integer part of bounds (60) divided by $\log_2 b$. Then we obtain

b	$q_b \in$	$\varepsilon_b \in$	$\log_B\left(\frac{Aq_b}{\varepsilon_b}\right) \in$
3	$[2.8272 \times 10^{547}, 5.6237 \times 10^{548}]$	$[0.000066516, 0.44354]$	$[1823, 1836]$
5	$[1.0481 \times 10^{554}, 3.3307 \times 10^{554}]$	$[0.00010719, 0.47347]$	$[1844, 1856]$
6	$[6.2120 \times 10^{555}, 1.0338 \times 10^{557}]$	$[0.0010799, 0.47241]$	$[1850, 1863]$
7	$[6.0928 \times 10^{557}, 6.1836 \times 10^{557}]$	$[0.00045702, 0.49322]$	$[1856, 1866]$
10	$[2.0504 \times 10^{560}, 2.1576 \times 10^{560}]$	$[0.00093374, 0.48994]$	$[1864, 1873]$

Thus, the above table together with inequalities (62) and (65), we obtain

$b =$	2	3	5	6	7	10
$\min\{n-n_1, 0.51k\} \leq$	6	1836	1856	1863	1866	1873

(67)

Case 3. $\nabla = n - n_1$. From inequalities (14) and (43), we know that[4]

$$|\Gamma_6| < \frac{128}{\log b} \times 2^{-\min\{0.51k, (m-m_1)\log_2 b\}}, \tag{68}$$

where

$$\Gamma_6 := \begin{cases} (n_1-2)\log_3 2 - (m-1); & \text{if } \ (m-m_1, b) = (1,3), \\ (n_1-2)\log_3 2 - (m-2); & \text{if } \ (m-m_1, b) = (2,3), \\ (n_1-3)\log_6 2 - (m-1); & \text{if } \ (m-m_1, b) = (1,6), \\ (n_1-4)\log_6 2 - (m-2); & \text{if } \ (m-m_1, b) = (2,6), \\ (n_1-2)\log_b 2 - m + \log_b\left(3\left(2^{n-n_1}-1\right)\right); & \text{otherwise.} \end{cases}$$

If $b = 2$, by bounds (60) we have $n - n_1 < 9$ and

$$0.169925 < \left\|\log_2\left(3\left(2^{n-n_1}-1\right)\right)\right\| < |\Gamma_6| < \frac{128}{\log 2} \times 2^{-\min\{0.51k, (m-m_1)\log_2 2\}},$$

so

$$\min\left\{0.51k, (m-m_1)\log_2 2\right\} < 10.1. \tag{69}$$

[4] Here we assume for a moment that $\min\left\{0.51k, (m-m_1)\log_2 b\right\} \geq 8$.

If $(m - m_1, b) \in \{(1,3), (2,3), (1,6), (2,6)\}$, we can use Lemma 5 on inequality (68) and we will get the results in (57), therefore we get

$$\frac{1}{(a_b+2)y^2} < \left|\log_b 2 - \frac{x}{y}\right| < \frac{128}{y\log b} \times 2^{-\min\{0.51k,(m-m_1)\log_2 b\}} < \frac{1}{2y^2} \quad (70)$$

with

$$x/y \in \{(m-1)/(n_1-2), (m-2)/(n_1-2), (m-1)/(n_1-3), (m-2)/(n_1-4)\},$$

assuming only for a moment

$$\min\left\{0.51k, (m-m_1)\log_2 b\right\} > \left\lceil \log_2\left(\frac{256M}{\log b}\right)\right\rceil = \begin{cases} 1824; & \text{if } b=3,\\ 1850; & \text{if } b=6\end{cases}$$

and taking $M > y$ as in Table (53). Then

$$\min\left\{0.51k, (m-m_1)\log_2 b\right\} < \begin{cases} 1834.47; & \text{if } b=3,\\ 1860.01; & \text{if } b=6.\end{cases} \quad (71)$$

If $(m-m_1, b) \notin \{(m-m_1, 2), (1,3), (2,3), (1,6), (2,6)\}$, we use Lemma 4 with the parameters

$$u := n_1 - 2, \quad \tau := \log_b 2, \quad \mu := \log_b\left(3\left(2^{n-n_1}-1\right)\right), \quad v := m,$$
$$A := \frac{128}{\log b}, \quad B := 2, \quad w := \min\left\{0.51k, (m-m_1)\log_2 b\right\}, \quad (72)$$

M as in Table (53) and $n - n_1$ between 1 and the values in Table (60). Then

b	$q_b \in$	$\varepsilon_b \in$	$\log_B\left(\frac{Aq_b}{\varepsilon_b}\right) \in$
3	$[2.8272\times 10^{547}, 2.0935\times 10^{549}]$	$[0.000064074, 0.44458]$	$[1826, 1840]$
5	$[1.0481\times 10^{554}, 1.2275\times 10^{555}]$	$[0.00026953, 0.47260]$	$[1847, 1858]$
6	$[6.2120\times 10^{555}, 1.0338\times 10^{557}]$	$[0.00021938, 0.48645]$	$[1853, 1867]$
7	$[6.0928\times 10^{557}, 1.2276\times 10^{558}]$	$[0.00025027, 0.49381]$	$[1859, 1870]$
10	$[2.0504\times 10^{560}, 2.1576\times 10^{560}]$	$[0.00031011, 0.48998]$	$[1868, 1878]$

Therefore, the above table and inequalities (69) and (71), imply that

$b =$	2	3	5	6	7	10
$\min\left\{0.51k, (m-m_1)\log_2 b\right\} \leq$	10	1840	1858	1867	1870	1878

(73)

Summarizing, for the bounds (60), (61), (67) and (73), we obtain

$$
\begin{array}{|c|c|c|c|c|c|c|}
\hline
b = & 2 & 3 & 5 & 6 & 7 & 10 \\
\hline
k \le & 19 & 3607 & 3643 & 3660 & 3666 & 3682 \\
\hline
\end{array}
\tag{74}
$$

or

$$
\begin{array}{|c|c|c|c|c|c|c|}
\hline
b = & 2 & 3 & 5 & 6 & 7 & 10 \\
\hline
n - n_1 \le & 9 & 1836 & 1856 & 1863 & 1866 & 1873 \\
\hline
m - m_1 \le & 10 & 1160 & 800 & 722 & 666 & 565 \\
\hline
\end{array}
\tag{75}
$$

If we assume that bounds (75) are satisfied, then using inequality (50), we obtain that

$$|\Gamma_7| < \frac{24}{\log b} \times 2^{-0.51k},$$

where

$$\Gamma_7 := \begin{cases} (n_1 - 3)\log_3 2 - (m_1 - 1); & \text{if} \quad (b, n - n_1, m - m_1) = (3, 1, 1), \\ (n_1 - 5)\log_3 2 - (m_1 - 1); & \text{if} \quad (b, n - n_1, m - m_1) = (3, 1, 2), \\ (n_1 - 3)\log_3 2 - (m_1 - 2); & \text{if} \quad (b, n - n_1, m - m_1) = (3, 2, 1), \\ (n_1 - 5)\log_3 2 - (m_1 - 2); & \text{if} \quad (b, n - n_1, m - m_1) = (3, 2, 2), \\ (n_1 - 6)\log_3 2 - (m_1 - 2); & \text{if} \quad (b, n - n_1, m - m_1) = (3, 4, 4), \\ (n_1 - 5)\log_5 2 - m_1; & \text{if} \quad (b, n - n_1, m - m_1) = (5, 1, 2), \\ (n_1 - 4)\log_6 2 - (m_1 - 2); & \text{if} \quad (b, n - n_1, m - m_1) = (6, 4, 1), \\ (n_1 - 3)\log_7 2 - m_1; & \text{if} \quad (b, n - n_1, m - m_1) = (7, 1, 1), \\ (n_1 - 6)\log_7 2 - m_1; & \text{if} \quad (b, n - n_1, m - m_1) = (7, 1, 2), \\ (n_1 - 3)\log_7 2 - (m_1 - 1); & \text{if} \quad (b, n - n_1, m - m_1) = (7, 3, 1), \\ (n_1 - 6)\log_7 2 - (m_1 - 1); & \text{if} \quad (b, n - n_1, m - m_1) = (7, 3, 2), \\ (n_1 - 2)\log_b 2 - m_1 + \log_b\left(\frac{3(2^{n-n_1}-1)}{b^{m-m_1}-1}\right); & \text{otherwise.} \end{cases}$$

If $b = 2$, we take $n - n_1$ and $m - m_1$ between 1 and bounds (75) omitting the case

$$n - n_1 = 1 \quad \text{and} \quad m - m_1 = 2 = n_1 - m_1$$

because in this case $\Lambda_7 = 0$ and before we had proved that Eq. (3) has no solution if this occurs. Thus we obtain

$$0.00282 < \left\| \log_2\left(\frac{3(2^{n-n_1} - 1)}{2^{m-m_1} - 1}\right) \right\| < \frac{24}{\log 2} \times 2^{-0.51k}$$

so

$$k \le 26 \quad \text{for} \quad b = 2. \tag{76}$$

If $(b, n-n_1, m-m_1)$ belongs to the set

$$\mathcal{I} := \{(3,1,1),(3,1,2),(3,2,1),(3,2,2),(3,4,4),(5,1,2),\\ (6,4,1),(7,1,1),(7,1,2),(7,3,1),(7,3,2)\},$$

we use Lemma 5 on

$$\left|\log_b 2 - \frac{x}{y}\right| < \frac{24}{y\log b} \times 2^{-0.51k}, \tag{77}$$

with $x \in \{n_1-3, n_1-4, n_1-5, n_1-6\}$, $y \in \{m_1, m_1-1, m_1-2\}$ according to the corresponding order and $M > y$ as in Table (53). Then assuming for a moment that

$$k > \left\lceil \left(\log_2\left(\frac{48M}{\log b}\right)\right)/0.51\right\rceil = \begin{cases} 3572; & \text{if } b=3,\\ 3611; & \text{if } b=5,\\ 3622; & \text{if } b=6,\\ 3630; & \text{if } b=7,\end{cases}$$

we obtain

$$\frac{1}{(a_b+2)y^2} < \frac{24}{y\log b} \times 2^{-0.51k} < \frac{1}{2y^2}$$

where a_b is the same as in data list (57) and (64). Therefore

$b =$	3	5	6	7
$k \le$	3592	3632	3642	3649

(78)

Then, we use Lemma 4 with the parameters

$$u := n_1 - 2, \qquad \tau := \log_b 2, \qquad \mu := \log_b\left(\frac{3(2^{n-n_1}-1)}{b^{m-m_1}-1}\right), \qquad v := m_1,$$
$$A := \frac{24}{\log b}, \qquad B := 2, \qquad w := 0.51k, \tag{79}$$

M as in Table (53) and $n-n_1$ and $m-m_1$ from 1 to bounds (75). So

b	$q_b \in$	$\varepsilon_b \in$	$\log_B\left(\frac{Aq_b}{\varepsilon_b}\right) \in$
3	$[2.83\times10^{547}, 8.94\times10^{549}]$	$[1.06\times10^{-7}, 0.445]$	$[1825, 1847]$
5	$[1.05\times10^{554}, 3.76\times10^{556}]$	$[3.64\times10^{-7}, 0.485]$	$[1846, 1868]$
6	$[6.21\times10^{555}, 8.49\times10^{558}]$	$[6.76\times10^{-7}, 0.487]$	$[1852, 1872]$
7	$[6.09\times10^{557}, 2.69\times10^{560}]$	$[5.12\times10^{-8}, 0.494]$	$[1858, 1881]$
10	$[2.05\times10^{560}, 3.58\times10^{561}]$	$[5.50\times10^{-7}, 0.491]$	$[1866, 1886]$

Finally, the above table and bounds (74), (76) and (78), imply that

$b=$	2	3	5	6	7	10
$k\le$	26	3621	3662	3670	3688	3698

(80)

Now, we had assumed that $k > 625$ for all $b \in \{2,3,5,6,7,10\}$ then for $b = 2$ we get a contradiction and conclude that there are no solutions to Eq. (3) when $k > 625$ and $b = 2$.

On the other hand for the values of $b \in \{3,5,6,7,10\}$ we will do one more cycle of reduction, where bounds (53) have reduced them using inequality (33) with bounds (80) as follows.

$b=$	3	5	6	7	10
$n<$	2.26×10^{86}	8.232×10^{86}	1.171×10^{87}	1.594×10^{87}	2.7463×10^{87}

(81)

We note that for some previous reductions we use Lemma 5 for $b \in \{3,5,6,7\}$, then under the appropriate hypothesis we will always obtain that

$$\frac{1}{(a_b+2)y^2} < \left|\log_b 2 - \frac{x}{y}\right| < \frac{1}{2y^2}$$

for some $x, y \in \mathbb{Z}$, where $0 < y < M$, M is as in Table (81) and

b	t_b	q_{t_b+1}	$a_b := \max_{0\le\ell\le t_b+1}\{a_\ell\}$
3	184	4.08×10^{86}	55
5	177	8.8×10^{86}	5393
6	185	4.105×10^{87}	55
7	170	4.01×10^{88}	197

(82)

Then we will use in all these cases the lower bound

$$y\left|\log_b 2 - \frac{x}{y}\right| > \frac{1}{(a_b+2)M} = \begin{cases} 7.76277\times10^{-89}; & \text{if } b=3,\\ 2.25166\times10^{-91}; & \text{if } b=5,\\ 1.49819\times10^{-89}; & \text{if } b=6,\\ 3.15253\times10^{-90}; & \text{if } b=7. \end{cases} \tag{83}$$

We start by realizing the second cycle of reduction to ∇ using inequality (54). Thus, reduction (58) becomes using the lower bounds (83) and after another iteration

$$\nabla < \begin{cases} 298; & \text{if } b=3,\\ 301; & \text{if } b=6. \end{cases} \tag{84}$$

For the other cases we use Lemma 4 with parameters (59) except that M is taken from Table (81), so we get

b	q_b	ε_b	$\log_B\left(\frac{Aq_b}{\varepsilon_b}\right)$
5	1.1380×10^{88}	0.0977842	301
7	4.0119×10^{88}	0.419435	301
10	6.7270×10^{88}	0.175735	303

and obtain

$$\nabla \leq 303 \quad \text{for all} \quad b \in \{3, 5, 6, 7, 10\}. \tag{85}$$

As in the first reduction cycle, we consider each case over ∇.

If $\nabla = 0.51k$, then

$$k \leq 594 \qquad \text{for all} \qquad b \in \{3, 5, 6, 7, 10\}. \tag{86}$$

If $\nabla = (m - m_1)\log_2 b \leq 303$ then by inequalities (63) and (83) we obtain

$b =$	3	5	7
$\min\{n - n_1, 0.51k\} <$	297	305	301

(87)

for $(m - m_1, b) \in \{(1, 3), (2, 3), (2, 5), (1, 7), (2, 7)\}$. In another case we use Lemma 4 with parameters (66) but with M as in Table (81) and get

b	$q_b \in$	$\varepsilon_b \in$	$\log_B\left(\frac{Aq_b}{\varepsilon_b}\right) \in$
3	$[2.5174 \times 10^{87}, 1.2996 \times 10^{88}]$	$[0.0003591, 0.481619]$	$[295, 307]$
5	$[1.1380 \times 10^{88}, 4.7020 \times 10^{88}]$	$[0.00545658, 0.475573]$	$[296, 303]$
6	$[2.1195 \times 10^{88}, 7.0356 \times 10^{89}]$	$[0.000840818, 0.496361]$	$[297, 306]$
7	$[4.0119 \times 10^{88}, 1.2080 \times 10^{89}]$	$[0.0197481, 0.479736]$	$[298, 303]$
10	$[6.7270 \times 10^{88}, 6.7270 \times 10^{88}]$	$[0.00242457, 0.488085]$	$[298, 306]$

So

$$k \leq 602 \quad \text{or} \quad n - n_1 \leq 307 \qquad \text{for all} \qquad b \in \{3, 5, 6, 7, 10\}. \tag{88}$$

If $\nabla = n - n_1 \leq 303$ then similar to the previous case we obtain from inequalities (70) and (83) that

$$\min\left\{0.51k, (m - m_1)\log_2 b\right\} < \begin{cases} 300; & \text{if } b = 3, \\ 302; & \text{if } b = 6. \end{cases} \tag{89}$$

for $(m - m_1, b) \in \{(1,3), (2,3), (1,6), (2,6)\}$. In another case we use Lemma 4 with parameters (72) and obtain

b	$q_b \in$	$\varepsilon_b \in$	$\log_B\left(\frac{Aq_b}{\varepsilon_b}\right) \in$
3	$[2.5174 \times 10^{87}, 1.2996 \times 10^{88}]$	$[0.000208392, 0.493232]$	$[298, 311]$
5	$[1.1380 \times 10^{88}, 4.7020 \times 10^{88}]$	$[0.00180693, 0.480547]$	$[299, 309]$
6	$[2.1195 \times 10^{88}, 7.0356 \times 10^{89}]$	$[0.00125949, 0.496717]$	$[300, 310]$
7	$[4.0119 \times 10^{88}, 1.2080 \times 10^{89}]$	$[0.00155886, 0.485693]$	$[301, 309]$
10	$[6.7270 \times 10^{88}, 6.2171 \times 10^{89}]$	$[0.00599169, 0.493197]$	$[301, 309]$

Therefore

$$k \le 609 \quad \text{or} \quad m - m_1 \le \lfloor 311/\log_2 b \rfloor \qquad \text{for all} \qquad b \in \{3, 5, 6, 7, 10\}. \tag{90}$$

In conclusion, by bounds (85), (86), (88) and (90) it follows that

$$k \le 609 \qquad \text{for all} \qquad b \in \{3, 5, 6, 7, 10\} \tag{91}$$

or

$$n - n_1 \le 307 \quad \text{and} \quad m - m_1 \le \lfloor 311/\log_2 b \rfloor \quad \text{for all} \quad b \in \{3, 5, 6, 7, 10\}. \tag{92}$$

Using inequalities (77) and (83) we obtain that

$b =$	3	5	6	7
$k \le$	582	598	585	590

(93)

for $(b, n - n_1, m - m_1) \in \mathcal{I}$. In another case we use Lemma 4 with data (79) and obtain

b	$q_b \in$	$\varepsilon_b \in$	$\log_B\left(\frac{Aq_b}{\varepsilon_b}\right) \in$
3	$[2.52 \times 10^{87}, 8.76 \times 10^{89}]$	$[9.01 \times 10^{-6}, 0.499]$	$[296, 319]$
5	$[1.14 \times 10^{88}, 4.35 \times 10^{89}]$	$[0.0000125, 0.498]$	$[298, 313]$
6	$[2.12 \times 10^{88}, 3.56 \times 10^{90}]$	$[0.0000201, 0.498]$	$[299, 318]$
7	$[4.01 \times 10^{88}, 6.84 \times 10^{89}]$	$[0.0000121, 0.495]$	$[299, 315]$
10	$[6.73 \times 10^{88}, 6.22 \times 10^{89}]$	$[0.0000326, 0.499]$	$[300, 314]$

Finally, comparing bounds (91), (93) and from the above table we conclude that

$$k \le 625 \qquad \text{for all} \qquad b \in \{3, 5, 6, 7, 10\},$$

but this is a contradiction since from the beginning we assumed $k > 625$. Therefore, there are no solutions to Eq. (3) when $k > 625$ for all $b \in \{3, 5, 6, 7, 10\}$.

4.3.2 Case $2 \le k \le 625$ and $b \in \{2, 3, 5, 6, 7, 10\}$

By Eq. (33) we obtain

$b =$	2	3	5	6	7	10
$n <$	6.15×10^{76}	2.51×10^{77}	8.04×10^{77}	1.12×10^{78}	1.435×10^{78}	2.4×10^{78}

(94)

First, we use the inequality (25) to see that

$\Lambda < 1/2$ assuming for a moment that $\nabla_1 := \min\{(n - n_1 - 3)\log_b \alpha, m - m_1\} \ge 7$.

Then we use the inequality (14) on Λ in (25) and dividing by $\log b$ on both sides, we obtain

$$|\Gamma| = \left|(n-1)\log_b \alpha - m + \log_b((2\alpha - 1) f_k(\alpha))\right| < \frac{10}{\log b} \times b^{-\nabla_1}. \tag{95}$$

If $k = 2$,

$$|\Gamma|/n = |\log_b \alpha - m/n| < \frac{10}{n \log b} \times b^{-\nabla_1},$$

then we use Lemma 5 with $y := n < M$, M as in Table (94) and assuming that

$$\left\lceil \log_b \left(\frac{20M}{\log b} \right) \right\rceil < \nabla_1.$$

So, we obtain

$$\frac{1}{(a_b + 2)n^2} \le |\Gamma|/n < \frac{10}{n \log b} \times b^{-\nabla_1} < \frac{1}{2n^2} \tag{96}$$

where

b	t_b	q_{t_b+1}	$a_b := \max_{0 \le \ell \le t_b+1}\{a_\ell\}$
2	151	1.15154×10^{77}	880
3	145	3.6664×10^{77}	871
5	160	1.34411×10^{79}	59
6	161	3.84468×10^{78}	347
7	154	7.75164×10^{78}	94
10	155	4.71288×10^{78}	770

(97)

and therefore

$b =$	2	3	5	6	7	10
$\nabla_1 \le$	269	171	116	105	96	82

(98)

Now we apply the Lemma 4 on inequality (95) with $k > 2$ and the parameters

$$u := n-1, \qquad \tau := \log_b \alpha, \qquad \mu := \log_b\left((2\alpha-1)f_k(\alpha)\right), \qquad v := m,$$
$$A := 10/\log b, \qquad B := b, \qquad w := \nabla_1$$

and M as in Table (94). Then we obtain

b	$q_b^{(k)} \in$	$\varepsilon_b^{(k)} \in$	$\log_B\left(Aq_b^{(k)}/\varepsilon_b^{(k)}\right) \in$
2	$[3.7152 \times 10^{77}, 1.9302 \times 10^{188}]$	$[3.5105 \times 10^{-7}, 0.49840]$	[262, 631]
3	$[1.5099 \times 10^{78}, 3.5935 \times 10^{131}]$	$[5.5922 \times 10^{-55}, 0.49244]$	[166, 391]
5	$[4.8352 \times 10^{78}, 9.4465 \times 10^{81}]$	[0.00064540, 0.49196]	[114, 119]
6	$[6.7455 \times 10^{78}, 1.3659 \times 10^{132}]$	$[4.1106 \times 10^{-55}, 0.49170]$	[102, 239]
7	$[8.6331 \times 10^{78}, 1.6014 \times 10^{81}]$	[0.00017225, 0.49714]	[94, 99]
10	$[1.4470 \times 10^{79}, 1.5036 \times 10^{82}]$	[0.0028794, 0.49459]	[80, 83]

From the above and bounds (98), we arrive at

$b =$	2	3	5	6	7	10
$\nabla_1 \leq U_b^{(1)} :=$	631	391	119	239	99	83

(99)

Now, we consider each possibility of ∇_1.

Case 1. $\nabla_1 = (n-n_1-3)\log_b \alpha \leq U_b^{(1)}$. Here, we use the inequality (27) to see that

$$\Lambda_1 < 1/2 \quad \text{for} \quad m - m_1 \geq 3.$$

Then we use the inequality (14) on Λ_1 in (27) and dividing by $\log b$ on both sides, we obtain

$$|\Gamma_1| < \frac{2b^2}{\log b} \times b^{-(m-m_1)}. \tag{100}$$

where

$$\Gamma_1 := \begin{cases} (n_1-1)\log_b \alpha - m; & \text{if } (k, n-n_1) = (2,1), \\ (n_1+1)\log_b \alpha - m; & \text{if } (k, n-n_1) = (2,2), \\ (n_1+1)\log_b \alpha - (m-1); & \text{if } (k, n-n_1, b) = (2,3,2), \\ (n_1+3)\log_b \alpha - (m-2); & \text{if } (k, n-n_1, b) = (2,6,2), \\ (n_1-1)\log_b \alpha - m + \log_b\left((2\alpha-1)f_k(\alpha)(\alpha^{n-n_1}-1)\right); & \text{otherwise.} \end{cases}$$

First let us consider the cases

$$(k, n-n_1, b) \in \{(2,1,b), (2,2,b), (2,3,2), (2,6,2)\},$$

for them we use Lemma 5 on the inequality (see inequality (100))

$$|\Gamma_1|/y = \left|\tau - \frac{x}{y}\right| < \frac{2b^2}{y\log b} \times b^{-(m-m_1)},$$

with $M > y - 3$ as before,

$$\tau := \log_b \alpha \qquad \text{and} \qquad \frac{x}{y} \in \left\{\frac{m}{n_1 \pm 1}, \frac{m-1}{n_1+1}, \frac{m-2}{n_1+3}\right\}.$$

We assume for a moment that

$$\left\lceil \log_b\left(\frac{4b^2(M+3)}{\log b}\right)\right\rceil < m - m_1$$

and therefore

$$\frac{1}{(a_b+2)y^2} \le |\Gamma_1|/y < \frac{2b^2}{y\log b} \times b^{-(m-m_1)} < \frac{1}{2y^2}.$$

Then we note that the above inequality leads us to obtain the same values from Table (97) since we again use Lemma 5 with $\tau := \log_2 \alpha$ and the upper bound for y given by $M + 3$ also satisfies those results. So

$b =$	2	3	5	6	7	10
$m - m_1 \le$	268	170	116	105	96	83

(101)

Now, we apply the Lemma 4 on inequality (100) with $(k, n - n_1, b)$ in the set

$$\left([2, 625] \times \left[1, \left\lfloor 3 + \frac{U_b^{(1)}}{\log_b \alpha}\right\rfloor\right] \times \{2, 3, 5, 6, 7, 10\}\right) \setminus \{(2, 1, b), (2, 2, b), (2, 3, 2), (2, 6, 2)\}$$

and the parameters

$$u := n_1 - 1, \quad \tau := \log_b \alpha, \quad \mu := \log_b \left((2\alpha - 1) f_k(\alpha)(\alpha^{n-n_1} - 1)\right),$$
$$v := m, \quad A := \frac{2b^2}{\log b}, \quad B := b, \quad w := m - m_1,$$

with M is as in Table (94). Then we obtain

b	$q_b \in$	$\varepsilon_b \in$	$\log_B\left(\frac{Aq_b}{\varepsilon_b}\right) \in$
2	$[3.72 \times 10^{77}, 1.93 \times 10^{188}]$	$[1.72 \times 10^{-59}, 0.5]$	[263, 638]
3	$[1.51 \times 10^{78}, 1.46 \times 10^{132}]$	$[4.01 \times 10^{-57}, 0.5]$	[168, 397]
5	$[4.84 \times 10^{78}, 9.45 \times 10^{81}]$	$[2.17 \times 10^{-7}, 0.499]$	[116, 126]
6	$[6.73 \times 10^{78}, 2.84 \times 10^{132}]$	$[3.91 \times 10^{-57}, 0.5]$	[104, 245]
7	$[8.63 \times 10^{78}, 3.04 \times 10^{81}]$	$[6.74 \times 10^{-6}, 0.5]$	[96, 104]
10	$[1.45 \times 10^{79}, 1.50 \times 10^{82}]$	$[1.94 \times 10^{-6}, 0.5]$	[82, 88]

and comparing the bounds in the above table and bounds (101), we conclude that

$b =$	2	3	5	6	7	10
$m - m_1 \leq$	638	397	126	245	104	88

(102)

Case 2. $\nabla_1 = m - m_1 \leq U_b^{(1)}$. We use the inequality (29) to see that

$$\Lambda_2 < 1/2 \quad \text{for} \quad n - n_1 \geq 6.$$

Then we use the inequality (14) on Λ_2 in (29) and dividing by $\log b$ on both sides, we obtain

$$|\Gamma_2| < \frac{2.4\alpha^3}{\log b} \times \alpha^{-(n-n_1)} \tag{103}$$

where

$$\Gamma_2 := \begin{cases} n \log_b \alpha - m_1; & \text{if } (k, m - m_1, b) = (2, 1, 2), \\ (n-1) \log_b \alpha - m_1 + \log_b \left(\frac{(2\alpha-1) f_k(\alpha)}{b^{m-m_1} - 1}\right); & \text{otherwise.} \end{cases}$$

We consider the case $(k, m - m_1, b) = (2, 1, 2)$ and use Lemma 5 assuming that

$$\left\lceil \log_\alpha \left(\frac{2.4M\alpha^3}{\log b}\right) \right\rceil < n - n_1$$

where M is as in Table (94). This implies that

$$\frac{1}{(a_2+2)n^2} \leq |\Gamma_2| / n < \frac{2.4\alpha^3}{n \log b} \times \alpha^{-(n-n_1)} < \frac{1}{2n^2}$$

and therefore, from Table (97) (which is obtained with the same value of τ and M), it follows that

$b =$	2	3	5	6	7	10
$n - n_1 \leq$	387	389	385	389	386	391

(104)

Next, we apply Lemma 4 on inequality (103) with

$$(k, m - m_1, b) \in \left([2, 625] \times [1, U_b^{(1)}] \times \{2, 3, 5, 6, 7, 10\}\right) \setminus \{(2, 1, 2)\}$$

and the parameters

$$u := n - 1, \quad \tau := \log_b \alpha, \quad \mu := \log_b \left((2\alpha - 1) f_k(\alpha)(b^{m-m_1} - 1)^{-1}\right),$$
$$v := m_1, \quad A := 2.4\alpha^3/\log b, \quad B := \alpha, \quad w := n - n_1$$

and M as before. Then we obtain

b	$q_b \in$	$\varepsilon_b \in$	$\log_B\left(\frac{Aq_b}{\varepsilon_b}\right) \in$
2	$[3.72 \times 10^{77}, 1.93 \times 10^{188}]$	$[2.94 \times 10^{-59}, 0.5]$	$[264, 919]$
3	$[1.51 \times 10^{78}, 7.61 \times 10^{131}]$	$[8.15 \times 10^{-56}, 0.499]$	$[265, 899]$
5	$[4.84 \times 10^{78}, 7.22 \times 10^{131}]$	$[2.69 \times 10^{-55}, 0.5]$	$[266, 623]$
6	$[6.73 \times 10^{78}, 9.97 \times 10^{131}]$	$[3.03 \times 10^{-56}, 0.5]$	$[267, 896]$
7	$[8.63 \times 10^{78}, 6.54 \times 10^{131}]$	$[8.86 \times 10^{-56}, 0.5]$	$[267, 624]$
10	$[1.45 \times 10^{79}, 1.50 \times 10^{82}]$	$[5.57 \times 10^{-8}, 0.499]$	$[268, 403]$

Therefore the above table and bounds (104) imply that

$b =$	2	3	5	6	7	10
$n - n_1 \leq$	919	899	623	896	624	403

(105)

By bounds (99), (102) and (105), we conclude

$b =$	2	3	5	6	7	10
$n - n_1 \leq$	919	899	623	896	624	403
$m - m_1 \leq$	638	397	126	245	104	88

(106)

Finally we will use inequality (32) to bound m. We assume for a moment that $m \geq 6$, so that $\Lambda_3 < 1/2$. Then we use inequality (14) on Λ_3 in (32) and dividing by $\log b$ on both sides, we obtain

$$|\Gamma_3| := \left|(n_1 - 1)\log_b \alpha - m_1 + \log_b \left(\frac{(2\alpha - 1) f_k(\alpha)(\alpha^{n-n_1} - 1)}{b^{m-m_1} - 1}\right)\right| < \frac{12 \times b^{-m}}{\log b}. \tag{107}$$

In fact we note that

$$\Gamma_3 := \begin{cases} (n_1-1)\log_2\alpha - m_1; & \text{if } (b,k,n-n_1,m-m_1)=(2,2,1,1),\\ (n_1+1)\log_2\alpha - m_1; & \text{if } (b,k,n-n_1,m-m_1)=(2,2,2,1),\\ (n_1+1)\log_2\alpha - (m_1-1); & \text{if } (b,k,n-n_1,m-m_1)=(2,2,3,1),\\ (n_1+3)\log_2\alpha - (m_1-2); & \text{if } (b,k,n-n_1,m-m_1)=(2,2,6,1),\\ (n_1+1)\log_3\alpha - m_1; & \text{if } (b,k,n-n_1,m-m_1)=(3,2,3,1),\\ (n_1+3)\log_5\alpha - m_1; & \text{if } (b,k,n-n_1,m-m_1)=(5,2,6,1). \end{cases}$$

Therefore we use Lemma 5 in these cases assuming that

$$\left\lceil \log_b\left(\frac{24M}{\log b}\right)\right\rceil < m$$

only for a moment and taking $x \in \{m_1, m_1-1, m_1-2\}$, $y \in \{n_1-1, n_1+1, n_1+3\}$ with $y < M+3$ where M is as in Table (94). Then

$$\frac{1}{(a_b+2)y^2} < |\Gamma_3|/y < \frac{12\times b^{-m}}{y\log b} < \frac{1}{2y^2}$$

and the values of a_b in Table (97) are still valid for these cases, in conclusion

$$m \le \begin{cases} 268; & \text{if } b=2,\\ 170; & \text{if } b=3,\\ 115; & \text{if } b=5. \end{cases} \tag{108}$$

In another case $(b,k,n-n_1,m-m_1)$ is not in the set

$$\{(2,2,1,1),(2,2,2,1),(2,2,3,1),(2,2,6,1),(3,2,3,1),(5,2,6,1)\}$$

and we apply Lemma 4 on inequality (107) with the parameters

$$u := n_1-1,\quad \tau := \log_b\alpha,\quad \mu := \log_b\left(\frac{(2\alpha-1)f_k(\alpha)(\alpha^{n-n_1}-1)}{b^{m-m_1}-1}\right),$$
$$v := m_1,\quad A := 12/\log b,\quad B := b,\quad w := m$$

and M as before. Thus we obtain

b	$q_b \in$	$\varepsilon_b \in$	$\log_B\left(\frac{Aq_b}{\varepsilon_b}\right) \in$
2	$[3.72\times10^{77}, 1.93\times10^{188}]$	$[4\times10^{-62}, 0.5]$	$[262, 833]$
3	$[1.51\times10^{78}, 3.86\times10^{133}]$	$[2.10\times10^{-58}, 0.4]$	$[166, 403]$
5	$[4.84\times10^{78}, 9.86\times10^{132}]$	$[2.68\times10^{-56}, 0.4]$	$[114, 271]$
6	$[6.73\times10^{78}, 3.64\times10^{134}]$	$[1.2\times10^{-57}, 0.5]$	$[102, 247]$
7	$[8.63\times10^{78}, 2.60\times10^{132}]$	$[3.26\times10^{-57}, 0.4]$	$[94, 224]$
10	$[1.45\times10^{79}, 5.02\times10^{132}]$	$[1.16\times10^{-55}, 0.4]$	$[80, 188]$

Comparing bounds (108) and those of the previous table we arrive at

$b=$	2	3	5	6	7	10
m	833	403	271	247	224	188

Moreover, by inequality (24) we have that $n \leq \lfloor 3 + m \log_\alpha b \rfloor$ and $m \leq n + 1$, then by the above bounds we have that

$b=$	2	3	5	6	7	10
$n \leq U_b :=$	1202	923	909	922	908	902

(109)

Finally with the help of Mathematica we computationally search all solutions for Eq. (3) with parameters

$$k \in [2, 625], \qquad n \in [k+1, U_b], \qquad n_1 \in [2, n-1],$$
$$m_1 \in [2, n] \qquad \text{and} \qquad m \in [m_1 + 1, n + 1].$$

Thus, we obtain that for $b \in \{5, 6, 7, 10\}$ there are no solutions to Eq. (3). For $b = 2$ there are solutions

k	4	4	2	2	2	2	2	2	5	3	3	3	3	3
n	5	14	4	5	5	10	10	11	11	4	5	6	6	8
n_1	3	4	2	4	2	5	2	4	9	3	2	5	2	3
m	5	13	3	3	4	7	7	8	10	3	5	5	6	7
m_1	4	2	2	2	3	4	3	6	4	2	4	4	5	4
c	−10	8	−1	3	−5	−5	−5	−57	336	2	−13	3	−29	−10

and for $b = 3$,

k	6	2	2	2	2	2	2	3	3
n	12	7	8	9	14	16	16	7	8
n_1	10	5	7	3	10	8	7	4	7
m	7	3	3	4	6	7	7	4	4
m_1	3	2	2	2	2	3	2	3	3
c	709	2	20	−5	114	20	20	−17	37

This ends the proof of the second part of Corollary 2.

This computation was done with the *Mathematica* software at Computer Center Jurgen Tischer in the Department of Mathematics at the Universidad del Valle on 24 parallel Pc's (Intel Xeon E3-1240 v5, 3.5 GHz, 16 Gb of RAM).

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