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Oscillation criterion for linear equations with coefficients containing powers of natural logarithm

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Abstract
Applying an averaging technique for the adapted Prüfer angle, we obtain an oscillation criterion for linear second order differential equations whose coefficients consist of products of powers of natural logarithm and general (bounded or unbounded) continuous functions. The presented criterion is illustrated by new corollaries and examples. The novelty is caused by the used averaging technique over unbounded intervals.

Keywords Linear differential equation · Oscillation · Averaging technique · Prüfer angle · Logarithm

Mathematics Subject Classification 34C10

1 Introduction

We study the oscillation of linear second order differential equations

$$\big(R(t)x'(t)\big)' + S(t)x(t) = 0, \tag{1.1}$$

where $R > 0$, S are continuous functions on an interval $[T, \infty)$. We recall that Eq. (1.1) is called oscillatory if all its solutions are oscillatory (which means that any solution has zero points in any neighborhood of ∞); and we say that Eq. (1.1) is non-oscillatory

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in the opposite case. Concerning the oscillation theory of Eq. (1.1), see, e.g., [1, 34] with references cited therein.

In the studied equations, it suffices to consider only t large enough because we analyze oscillation properties. Thus, we consider only $t \geq \mathrm{e}$, where e is the base of the natural logarithm. As usual, we put $\mathbb{R}_\mathrm{e} := [\mathrm{e}, \infty)$. Let log denote the natural logarithm and let $p > 0$ be arbitrarily given. In this paper, we concentrate on linear second order differential equations in the form

$$\left(\frac{\log^p t}{r(t)} x'(t)\right)' + \frac{\log^p t}{t^2} s(t)x(t) = 0 \tag{1.2}$$

with continuous coefficients $r : \mathbb{R}_\mathrm{e} \to (0, \infty)$ and $s : \mathbb{R}_\mathrm{e} \to \mathbb{R}$.

In the oscillation theory of linear differential equations, a very useful tool is the combination of the Riccati transformation and the Prüfer angle. This approach enables to describe the oscillation behavior of equations with non-constant coefficients. In addition, very general oscillation criteria can be proved when the combination of the Riccati transformation and the Prüfer angle is followed by a non-trivial averaging technique. We apply such a method to prove an oscillation criterion for Eq. (1.2), where the obtained oscillation criterion can be used through computing averages of treated coefficients.

In this paragraph, we collect an overview of the literature. We begin with strongly relevant papers [10, 12–14, 16, 26–28] (see also [9, 19, 22, 25]). For other relevant results about perturbed differential equations, we refer at least to [5, 7, 21, 23] (see also [18, 33] in the discrete case); and for non-linear equations, to [3, 20, 29, 30]. Concerning the oscillation of corresponding difference equations and dynamic equations on time scales, see, e.g., [2, 6, 15, 31]. More general half-linear discrete equations are treated in [11, 17, 24, 32].

We highlight that our criterion uses intervals of general lengths in the computing averages. This situation is not treated in any previous research paper concerning the studied equations. In this sense, our approach is quite new and the method allows to obtain strong results. More precisely, this study obtained sharper results by using averaging techniques over intervals with variable lengths. Comments on the method are given at appropriate places in the text below. The novelty is demonstrated by three corollaries together with simple examples at the end of this paper.

The basic motivations for the research presented in this paper come from [14, 16], where the following results are proved.

Theorem 1.1 *Let us consider Eq.* (1.2)*, where* $r : \mathbb{R}_\mathrm{e} \to (0, \infty)$*,* $s : \mathbb{R}_\mathrm{e} \to \mathbb{R}$ *are periodic functions with period* $\sigma > 0$.

(A) *If*

$$4\left(\frac{1}{\sigma}\int_{\mathrm{e}}^{\mathrm{e}+\sigma} r(\tau)\,\mathrm{d}\tau\right)\left(\frac{1}{\sigma}\int_{\mathrm{e}}^{\mathrm{e}+\sigma} s(\tau)\,\mathrm{d}\tau\right) > 1,$$

then Eq. (1.2) *is oscillatory.*

(B) *If*

$$4\left(\frac{1}{\sigma}\int_{e}^{e+\sigma} r(\tau)\,\mathrm{d}\tau\right)\left(\frac{1}{\sigma}\int_{e}^{e+\sigma} s(\tau)\,\mathrm{d}\tau\right) < 1,$$

then Eq. (1.2) *is non-oscillatory.*

Theorem 1.2 *Let us consider Eq.* (1.2)*, where* $r : \mathbb{R}_e \to (0,\infty)$*,* $s : \mathbb{R}_e \to \mathbb{R}$ *satisfy*

$$\lim_{t\to\infty} \frac{\int_t^{t+1} r(\tau)\,\mathrm{d}\tau}{\sqrt{t}} = \lim_{t\to\infty} \frac{\int_t^{t+1} |s(\tau)|\,\mathrm{d}\tau}{\sqrt{t}} = 0. \tag{1.3}$$

Let $X, Y, \sigma > 0$*. If* $4XY > 1$ *and if the inequalities*

$$\frac{1}{\sigma}\int_{t}^{t+\sigma} r(\tau)\,\mathrm{d}\tau \geq X, \qquad \frac{1}{\sigma}\int_{t}^{t+\sigma} s(\tau)\,\mathrm{d}\tau \geq Y$$

are valid for all large t*, then Eq.* (1.2) *is oscillatory.*

Using Theorems 1.1 and 1.2, we can simply describe the aim of this paper. Our goal is to significantly enlarge the set of oscillatory equations in the studied form, i.e., to modify Theorem 1.2. Theorem 1.1 shows that the oscillation criterion mentioned in Theorem 1.2 cannot be improved by a modification of the inequality $4XY > 1$. Thus, via an averaging technique over intervals of general lengths, we prove a modification of Theorem 1.2 which covers coefficients without average values over intervals of finite lengths.

This paper is organized as follows. In the upcoming section, we recall the Riccati method together with the Prüfer angle. In Sect. 3, we describe the used averaging technique which is the main tool. In addition, we prove auxiliary results in Sect. 3. Section 4 presents our main result with its corollaries and examples.

2 Riccati transformation and Prüfer angle

In this section, we describe the used combination of the standard Riccati transformation and the adapted Prüfer angle. For arbitrary continuous functions $R : \mathbb{R}_e \to (0,\infty)$ and $S : \mathbb{R}_e \to \mathbb{R}$, let us consider Eq. (1.1). For a non-trivial solution x of Eq. (1.1), applying the Riccati transformation

$$w(t) = R(t)\frac{x'(t)}{x(t)}, \qquad x(t) \neq 0,$$

we obtain the Riccati equation

$$w'(t) + S(t) + R^{-1}(t)w^2(t) = 0. \tag{2.1}$$

In addition, using the substitution

$$v(t) = \frac{t}{\log^p t} w(t),$$

from Eq. (2.1), we obtain

$$v'(t) = \frac{\log t - p}{t \log t} v(t) - \frac{t}{\log^p t} S(t) - R^{-1}(t) \frac{\log^p t}{t} v^2(t).$$

Finally, for a non-trivial solution x of Eq. (1.1), we apply the adapted Prüfer transformation

$$x(t) = \rho(t) \sin \varphi(t), \qquad x'(t) = \rho(t) R^{-1}(t) \frac{\log^p t}{t} \cos \varphi(t)$$

which yields the equation (of the adapted Prüfer angle)

$$\varphi'(t) = \frac{\log^p t}{t R(t)} \cos^2 \varphi(t) - \frac{\log t - p}{t \log t} \cos \varphi(t) \sin \varphi(t) + \frac{t S(t)}{\log^p t} \sin^2 \varphi(t). \tag{2.2}$$

For the derivation of Eq. (2.2), we refer to [14].

3 Averaging function of Prüfer angle

We consider two auxiliary functions f, g. Let a continuously differentiable function $f : \mathbb{R}_e \to (0, \infty)$ and a continuous function $g : \mathbb{R}_e \to [1, \infty)$ satisfy

$$\lim_{t\to\infty} f'(t) g(t) = 0, \qquad \lim_{t\to\infty} \frac{f(t) g^2(t)}{t} = 0. \tag{3.1}$$

Taking into account that $g(t) \geq 1$ for all $t \in \mathbb{R}_e$, from the second limit in (3.1), one can see that

$$\lim_{t\to\infty} \frac{f(t) g(t)}{t} = 0, \qquad \text{i.e.,} \qquad \lim_{t\to\infty} \frac{t}{f(t) g(t)} = \infty. \tag{3.2}$$

We consider Eq. (1.2), where continuous functions $r : \mathbb{R}_e \to (0, \infty)$ and $s : \mathbb{R}_e \to \mathbb{R}$ satisfy

$$\tilde{r} := \limsup_{t\to\infty} \frac{\int_t^{t+f(t)} r(\tau)\, d\tau}{f(t) g(t)} < \infty, \qquad \tilde{s} := \limsup_{t\to\infty} \frac{\int_t^{t+f(t)} |s(\tau)|\, d\tau}{f(t) g(t)} < \infty. \tag{3.3}$$

Thus, we deal with Eq. (1.1) for $R(t) = \log^p t/r(t)$, $S(t) = s(t)\log^p t/t^2$, $t \in \mathbb{R}_e$, i.e., for Eq. (1.2), Eq. (2.2) takes the form

$$\varphi'(t) = \frac{1}{t}\left(r(t)\cos^2\varphi(t) - \frac{\log t - p}{\log t}\cos\varphi(t)\sin\varphi(t) + s(t)\sin^2\varphi(t)\right). \tag{3.4}$$

Let φ be an arbitrary solution of Eq. (3.4) on $\mathbb{R}_e$. For the auxiliary function $f : \mathbb{R}_e \to (0, \infty)$, we define the averaging function $\varphi_f : \mathbb{R}_e \to \mathbb{R}$ as

$$\varphi_f(t) := \frac{1}{f(t)}\int_t^{t+f(t)} \varphi(\tau)\,\mathrm{d}\tau, \qquad t \in \mathbb{R}_e. \tag{3.5}$$

Concerning the averaging function φ_f, we prove the following properties.

Lemma 3.1 *Let $\varphi : \mathbb{R}_e \to \mathbb{R}$ be a solution of Eq. (3.4). The inequality*

$$\limsup_{t\to\infty} \frac{t}{f(t)g(t)}\left|\varphi(\tau) - \varphi_f(t)\right| < \infty \tag{3.6}$$

holds uniformly for $\tau \in [t, t + f(t)]$.

Proof From (3.5), we have

$$\left|\varphi(\tau) - \varphi_f(t)\right| \le \max_{\varrho_1,\varrho_2\in[0,f(t)]} \left|\varphi(t+\varrho_1) - \varphi(t+\varrho_2)\right|, \quad \tau \in [t, t+f(t)],\ t \in \mathbb{R}_e.$$

Considering (3.3), we obtain

$$\begin{aligned}
&\limsup_{t\to\infty} \frac{t}{f(t)g(t)}\left|\varphi(\tau) - \varphi_f(t)\right| \\
&\quad \le \limsup_{t\to\infty} \frac{t}{f(t)g(t)} \max_{\varrho_1,\varrho_2\in[0,f(t)]} \left|\varphi(t+\varrho_1) - \varphi(t+\varrho_2)\right| \\
&\quad = \limsup_{t\to\infty} \frac{t}{f(t)g(t)} \max_{\varrho_1,\varrho_2\in[0,f(t)]} \left|\int_{t+\varrho_2}^{t+\varrho_1} \varphi'(\sigma)\,\mathrm{d}\sigma\right| \\
&\quad \le \limsup_{t\to\infty} \frac{t}{f(t)g(t)} \int_t^{t+f(t)} \left|\varphi'(\sigma)\right|\mathrm{d}\sigma \\
&\quad = \limsup_{t\to\infty} \frac{t}{f(t)g(t)} \int_t^{t+f(t)} \left|\frac{1}{\sigma}\left(r(\sigma)\cos^2\varphi(\sigma)\right.\right. \\
&\qquad \left.\left. - \frac{\log\sigma - p}{\log\sigma}\cos\varphi(\sigma)\sin\varphi(\sigma) + s(\sigma)\sin^2\varphi(\sigma)\right)\right| \mathrm{d}\sigma
\end{aligned}$$

$$
\begin{aligned}
&\leq \limsup_{t\to\infty} \frac{1}{f(t)g(t)} \int\limits_t^{t+f(t)} \left(r(\sigma)+1+p+|s(\sigma)|\right) \mathrm{d}\sigma \\
&\leq \limsup_{t\to\infty} \frac{\int_t^{t+f(t)} r(\sigma)\,\mathrm{d}\sigma}{f(t)g(t)} + \limsup_{t\to\infty} \frac{1+p}{g(t)} + \limsup_{t\to\infty} \frac{\int_t^{t+f(t)} |s(\sigma)|\,\mathrm{d}\sigma}{f(t)g(t)} \\
&\leq \tilde{r} + 1 + p + \tilde{s} < \infty,
\end{aligned}
$$

where $\tau \in [t, t+f(t)]$. □

Remark 1 From (3.2) and from (3.6) in Lemma 3.1, we have

$$
\lim_{t\to\infty} \left(\varphi(t) - \varphi_f(t)\right) = 0
$$

for any solution $\varphi : \mathbb{R}_\mathrm{e} \to \mathbb{R}$ of Eq. (3.4) and φ_f defined in (3.5).

Lemma 3.2 *Let* $\varphi : \mathbb{R}_\mathrm{e} \to \mathbb{R}$ *be a solution of Eq.* (3.4). *Then, there exists a continuous function* $\psi : \mathbb{R}_\mathrm{e} \to \mathbb{R}$ *such that* $\lim_{t\to\infty} \psi(t) = 0$ *and that*

$$
\begin{aligned}
\varphi_f'(t) = \frac{1}{t} \Bigg(&\cos^2 \varphi_f(t) \left(\frac{1}{f(t)} \int\limits_t^{t+f(t)} r(\tau)\,\mathrm{d}\tau \right) - \frac{\log t - p}{\log t} \cos \varphi_f(t) \sin \varphi_f(t) \\
&+ \sin^2 \varphi_f(t) \left(\frac{1}{f(t)} \int\limits_t^{t+f(t)} s(\tau)\,\mathrm{d}\tau \right) + \psi(t) \Bigg)
\end{aligned} \tag{3.7}
$$

for all $t >$ e.

Proof For any $t >$ e, it holds

$$
\begin{aligned}
\varphi_f'(t) &= \left(\frac{1}{f(t)} \int\limits_t^{t+f(t)} \varphi(\tau)\,\mathrm{d}\tau \right)' \\
&= -\frac{f'(t)}{f^2(t)} \int\limits_t^{t+f(t)} \varphi(\tau)\,\mathrm{d}\tau + \frac{\left(1+f'(t)\right)\varphi(t+f(t)) - \varphi(t)}{f(t)} \\
&= \frac{1}{f(t)} \int\limits_t^{t+f(t)} \varphi'(\tau)\,\mathrm{d}\tau + \frac{f'(t)}{f(t)} \left(\varphi(t+f(t)) - \frac{1}{f(t)} \int\limits_t^{t+f(t)} \varphi(\tau)\,\mathrm{d}\tau \right).
\end{aligned} \tag{3.8}
$$

Using (3.1) and (3.6) in Lemma 3.1, we have

$$\begin{aligned}
&\limsup_{t\to\infty} t \left|\frac{f'(t)}{f(t)}\right| \left|\varphi(t+f(t)) - \frac{1}{f(t)} \int_t^{t+f(t)} \varphi(\tau)\,\mathrm{d}\tau\right| \\
&= \limsup_{t\to\infty} t \left|\frac{f'(t)}{f(t)}\right| \left|\varphi(t+f(t)) - \varphi_f(t)\right| \\
&\quad = \limsup_{t\to\infty} \left|f'(t)g(t)\right| \frac{t}{f(t)g(t)} \left|\varphi(t+f(t)) - \varphi_f(t)\right| = 0. \qquad (3.9)
\end{aligned}$$

Next, (3.8) and (3.9) give

$$\lim_{t\to\infty} \left| t\,\varphi_f'(t) - \frac{t}{f(t)} \int_t^{t+f(t)} \varphi'(\tau)\,\mathrm{d}\tau \right| = 0. \qquad (3.10)$$

Therefore, it is sufficient to consider

$$\frac{1}{f(t)} \int_t^{t+f(t)} \varphi'(\tau)\,\mathrm{d}\tau.$$

We have

$$\begin{aligned}
&\left|\frac{1}{f(t)} \int_t^{t+f(t)} \frac{1}{\tau} \left(r(\tau)\cos^2\varphi(\tau) - \frac{\log\tau - p}{\log\tau} \cos\varphi(\tau)\sin\varphi(\tau) + s(\tau)\sin^2\varphi(\tau)\right) \mathrm{d}\tau \right.\\
&\quad - \frac{1}{f(t)} \int_t^{t+f(t)} \frac{1}{t} \left(r(\tau)\cos^2\varphi(\tau) - \frac{\log\tau - p}{\log\tau} \cos\varphi(\tau)\sin\varphi(\tau)\right.\\
&\quad \left.\left. + s(\tau)\sin^2\varphi(\tau)\right) \mathrm{d}\tau \right|\\
&\le \frac{1}{f(t)} \int_t^{t+f(t)} \left(\frac{1}{t} - \frac{1}{t+f(t)}\right) (r(\tau) + 1 + p + |s(\tau)|)\,\mathrm{d}\tau\\
&\le \frac{1}{t^2} \int_t^{t+f(t)} (r(\tau) + 1 + p + |s(\tau)|)\,\mathrm{d}\tau, \qquad t \in \mathbb{R}_\mathrm{e},
\end{aligned}$$

and

$$\limsup_{t\to\infty} \frac{1}{t} \int_t^{t+f(t)} (r(\tau) + 1 + p + |s(\tau)|)\, \mathrm{d}\tau$$

$$= \limsup_{t\to\infty} \frac{f(t)g(t)}{tf(t)g(t)} \int_t^{t+f(t)} (r(\tau) + 1 + p + |s(\tau)|)\, \mathrm{d}\tau$$

$$= \limsup_{t\to\infty} \frac{f(t)g(t)}{t} \left(\frac{\int_t^{t+f(t)} r(\tau)\, \mathrm{d}\tau}{f(t)g(t)} + \frac{1+p}{g(t)} + \frac{\int_t^{t+f(t)} |s(\tau)|\, \mathrm{d}\tau}{f(t)g(t)} \right) = 0.$$

Therefore,

$$\lim_{t\to\infty} \left| \frac{t}{f(t)} \int_t^{t+f(t)} \varphi'(\tau)\, \mathrm{d}\tau - \frac{1}{f(t)} \int_t^{t+f(t)} \Big(r(\tau)\cos^2\varphi(\tau) \right.$$
$$\left. - \frac{\log\tau - p}{\log\tau} \cos\varphi(\tau)\sin\varphi(\tau) + s(\tau)\sin^2\varphi(\tau) \Big) \mathrm{d}\tau \right| = 0. \quad (3.11)$$

Altogether, (3.10) and (3.11) give

$$\lim_{t\to\infty} \left| t\,\varphi_f'(t) - \frac{1}{f(t)} \int_t^{t+f(t)} \Big(r(\tau)\cos^2\varphi(\tau) \right.$$
$$\left. - \frac{\log\tau - p}{\log\tau} \cos\varphi(\tau)\sin\varphi(\tau) + s(\tau)\sin^2\varphi(\tau) \Big) \mathrm{d}\tau \right| = 0. \quad (3.12)$$

We use the Lipschitz continuity of the functions $y = \cos^2 x$, $y = \sin^2 x$, and $y = \cos x \sin x$ with the Lipschitz constant $L = 1$. Considering (3.1), (3.3), and (3.6) in Lemma 3.1, we have

$$\limsup_{t\to\infty} \left| \cos^2\varphi_f(t) \left(\frac{1}{f(t)} \int_t^{t+f(t)} r(\tau)\, \mathrm{d}\tau \right) - \frac{1}{f(t)} \int_t^{t+f(t)} r(\tau)\cos^2\varphi(\tau)\, \mathrm{d}\tau \right|$$

$$\le \limsup_{t\to\infty} \frac{1}{f(t)} \int_t^{t+f(t)} r(\tau) \left| \cos^2\varphi_f(t) - \cos^2\varphi(\tau) \right| \mathrm{d}\tau$$

$$\le \limsup_{t\to\infty} \frac{1}{f(t)} \int_t^{t+f(t)} r(\tau) \left| \varphi_f(t) - \varphi(\tau) \right| \mathrm{d}\tau$$

$$= \limsup_{t\to\infty} \frac{f(t)g^2(t)}{t} \left(\frac{1}{f(t)g(t)} \int_t^{t+f(t)} r(\tau) \frac{t\left|\varphi_f(t) - \varphi(\tau)\right|}{f(t)g(t)} \,\mathrm{d}\tau \right) = 0 \tag{3.13}$$

and

$$\begin{aligned}
&\limsup_{t\to\infty} \left| \sin^2 \varphi_f(t) \left(\frac{1}{f(t)} \int_t^{t+f(t)} s(\tau)\,\mathrm{d}\tau \right) - \frac{1}{f(t)} \int_t^{t+f(t)} s(\tau)\sin^2\varphi(\tau)\,\mathrm{d}\tau \right| \\
&\le \limsup_{t\to\infty} \frac{1}{f(t)} \int_t^{t+f(t)} |s(\tau)| \left| \sin^2\varphi_f(t) - \sin^2\varphi(\tau) \right| \mathrm{d}\tau \\
&\le \limsup_{t\to\infty} \frac{1}{f(t)} \int_t^{t+f(t)} |s(\tau)| \left|\varphi_f(t) - \varphi(\tau)\right| \mathrm{d}\tau \\
&= \limsup_{t\to\infty} \frac{f(t)g^2(t)}{t} \left(\frac{1}{f(t)g(t)} \int_t^{t+f(t)} |s(\tau)| \frac{t\left|\varphi_f(t) - \varphi(\tau)\right|}{f(t)g(t)} \,\mathrm{d}\tau \right) = 0
\end{aligned} \tag{3.14}$$

together with

$$\begin{aligned}
&\limsup_{t\to\infty} \left| \frac{\log t - p}{\log t} \cos\varphi_f(t)\sin\varphi_f(t) - \frac{1}{f(t)} \int_t^{t+f(t)} \frac{\log\tau - p}{\log\tau} \cos\varphi(\tau)\sin\varphi(\tau)\,\mathrm{d}\tau \right| \\
&\le \limsup_{t\to\infty} \left(\left| \frac{\log t - p}{\log t} \cos\varphi_f(t)\sin\varphi_f(t) \right.\right. \\
&\quad \left. - \frac{1}{f(t)} \int_t^{t+f(t)} \frac{\log t - p}{\log t} \cos\varphi(\tau)\sin\varphi(\tau)\,\mathrm{d}\tau \right| \\
&\quad + \left| \frac{1}{f(t)} \int_t^{t+f(t)} \frac{\log t - p}{\log t} \cos\varphi(\tau)\sin\varphi(\tau)\,\mathrm{d}\tau \right. \\
&\quad \left.\left. - \frac{1}{f(t)} \int_t^{t+f(t)} \frac{\log\tau - p}{\log\tau} \cos\varphi(\tau)\sin\varphi(\tau)\,\mathrm{d}\tau \right| \right)
\end{aligned}$$

$$\leq (1+p) \limsup_{t\to\infty} \frac{1}{f(t)} \int_t^{t+f(t)} \left| \cos\varphi_f(t) \sin\varphi_f(t) - \cos\varphi(\tau)\sin\varphi(\tau) \right| \mathrm{d}\tau$$

$$+ \limsup_{t\to\infty} \left(\frac{1}{f(t)} \int_t^{t+f(t)} \frac{p}{\log t} - \frac{p}{\log \tau}\, \mathrm{d}\tau \right)$$

$$\leq (1+p) \limsup_{t\to\infty} \frac{1}{f(t)} \int_t^{t+f(t)} \left| \varphi_f(t) - \varphi(\tau) \right| \mathrm{d}\tau + p \limsup_{t\to\infty} \frac{\log(t+f(t)) - \log t}{\log^2 t}$$

$$= (1+p) \limsup_{t\to\infty} \frac{f(t)g(t)}{t} \left(\frac{1}{f(t)} \int_t^{t+f(t)} \frac{t \left| \varphi_f(t) - \varphi(\tau) \right|}{f(t)g(t)}\, \mathrm{d}\tau \right)$$

$$+ p \limsup_{t\to\infty} \frac{\log\left(1 + \frac{f(t)}{t}\right)}{\log^2 t} = 0.$$

Thus, using (3.12), (3.13), and (3.14), we obtain (3.7) for a continuous function $\psi : \mathbb{R}_e \to \mathbb{R}$ satisfying

$$\lim_{t\to\infty} \psi(t) = 0.$$

□

4 Oscillation criterion

To prove the main result (Theorem 4.1 below), we recall the following two known lemmas, which are proved in [14].

Lemma 4.1 *If there exists a solution $\varphi : \mathbb{R}_e \to \mathbb{R}$ of Eq. (3.4) satisfying*

$$\lim_{t\to\infty} \varphi(t) = \infty, \tag{4.1}$$

then Eq. (1.2) is oscillatory.

Lemma 4.2 *Let $A, B > 0$. Let $\eta : \mathbb{R}_e \to \mathbb{R}$ be a solution of the equation*

$$\eta'(t) = \frac{1}{t} \left(A\cos^2\eta(t) - \frac{\log t - p}{\log t} \cos\eta(t)\sin\eta(t) + B\sin^2\eta(t) \right).$$

If $4AB > 1$, then

$$\lim_{t\to\infty} \eta(t) = \infty.$$

Theorem 4.1 *Let a continuously differentiable function* $f : \mathbb{R}_e \to (0, \infty)$ *and a continuous function* $g : \mathbb{R}_e \to [1, \infty)$ *satisfy* (3.1)*. Let us consider Eq.* (1.2)*, where continuous functions* $r : \mathbb{R}_e \to (0, \infty)$ *and* $s : \mathbb{R}_e \to \mathbb{R}$ *satisfy* (3.3)*. Let*

$$r_f := \liminf_{t\to\infty} \frac{1}{f(t)} \int_t^{t+f(t)} r(\tau)\,d\tau \in \mathbb{R}, \qquad s_f := \liminf_{t\to\infty} \frac{1}{f(t)} \int_t^{t+f(t)} s(\tau)\,d\tau \in \mathbb{R}. \tag{4.2}$$

If $4r_f s_f > 1$*, then Eq.* (1.2) *is oscillatory.*

Proof Considering Lemma 4.1, it suffices to show that a solution $\varphi : \mathbb{R}_e \to \mathbb{R}$ of Eq. (3.4) satisfies (4.1). Thus, we consider an arbitrary solution $\varphi : \mathbb{R}_e \to \mathbb{R}$ of Eq. (3.4) and the corresponding function φ_f introduced in (3.5). From Remark 1, we know that it suffices to prove $\lim_{t\to\infty} \varphi_f(t) = \infty$ which gives (4.1).

Using (3.7) in Lemma 3.2 and (4.2), for all $t > e$, we have

$$\begin{aligned}\varphi_f'(t) &= \frac{1}{t}\left(\cos^2\varphi_f(t)\left(\frac{1}{f(t)}\int_t^{t+f(t)} r(\tau)\,d\tau\right)\right.\\ &\qquad \left. - \frac{\log t - p}{\log t}\cos\varphi_f(t)\sin\varphi_f(t) + \sin^2\varphi_f(t)\left(\frac{1}{f(t)}\int_t^{t+f(t)} s(\tau)\,d\tau\right) + \psi(t)\right)\\ &\ge \frac{1}{t}\left(r_f\cos^2\varphi_f(t) - \frac{\log t - p}{\log t}\cos\varphi_f(t)\sin\varphi_f(t) + s_f\sin^2\varphi_f(t) + \Psi(t)\right),\end{aligned}$$

where $\Psi : \mathbb{R}_e \to \mathbb{R}$ is a continuous function with the property that $\lim_{t\to\infty}\Psi(t) = 0$. There exists a sufficiently small $\varepsilon > 0$ that satisfies $4(r_f - \varepsilon)(s_f - \varepsilon) > 1$ from $4r_f s_f > 1$. Therefore, we obtain the inequality

$$\begin{aligned}\varphi_f'(t) > \frac{1}{t}\Big(&\big(r_f - \varepsilon\big)\cos^2\varphi_f(t) - \frac{\log t - p}{\log t}\cos\varphi_f(t)\sin\varphi_f(t)\\ &+ \big(s_f - \varepsilon\big)\sin^2\varphi_f(t)\Big)\end{aligned}$$

for t greater than or equal to a sufficiently large T. From the standard comparison theorem, we have

$$\varphi_f(t) \ge \eta(t), \qquad t \ge T,$$

where η is the solution of the equation

$$\eta'(t) = \frac{1}{t}\left(\big(r_f - \varepsilon\big)\cos^2\eta(t) - \frac{\log t - p}{\log t}\cos\eta(t)\sin\eta(t) + \big(s_f - \varepsilon\big)\sin^2\eta(t)\right)$$

satisfying $\eta(T) = \varphi_f(T)$. Lemma 4.2 guarantees that $\lim_{t\to\infty} \eta(t) = \infty$. Consequently, we have

$$\lim_{t\to\infty} \varphi_f(t) = \infty.$$

This completes the proof. □

Remark 2 The used processes differ from the previous ones applied in the oscillation research of Eq. (1.2). Thus, Theorem 4.1 is not a simple generalization of any previous result. Especially, Theorem 1.2 does not follow from Theorem 4.1. Indeed, in the statement of Theorem 1.2, the auxiliary functions are $f(t) = 1$, $g(t) = \sqrt{t}$, $t \in \mathbb{R}_e$. For these functions, we have

$$\lim_{t\to\infty} \frac{f(t)g^2(t)}{t} = 1,$$

i.e., (3.1) is not satisfied. At the same time, for this choice, (1.3) from Theorem 1.2 is replaced by the constraint

$$\lim_{t\to\infty} \frac{\int_t^{t+1} r(\tau)\,\mathrm{d}\tau}{t^\beta} = \lim_{t\to\infty} \frac{\int_t^{t+1} |s(\tau)|\,\mathrm{d}\tau}{t^\beta} = 0$$

for some $\beta < 1/2$.

Remark 3 We briefly discuss the role of the second auxiliary function g which is linked to the first auxiliary function f. Roughly speaking, the smaller f is, the larger g may become, and vice versa. This fact is documented in Corollary 4.3 and Example 3 below.

Remark 4 We conjecture that it is not possible to decide the oscillation behavior of Eq. (1.2) for general coefficients $r : \mathbb{R}_e \to (0, \infty)$, $s : \mathbb{R}_e \to \mathbb{R}$ satisfying

$$\lim_{t\to\infty} \left(\frac{1}{f(t)} \int\limits_t^{t+f(t)} r(\tau)\,\mathrm{d}\tau \right) \left(\frac{1}{f(t)} \int\limits_t^{t+f(t)} s(\tau)\,\mathrm{d}\tau \right) = \frac{1}{4}$$

for treated auxiliary functions f. This conjecture is based on oscillation results about perturbed differential equations in [4, 8].

To explain the novelty of our main result, we mention the corollaries and examples below. These oscillation criteria and examples are not covered by any previously known result.

Corollary 4.1 *Let $\alpha, \beta \in (0, 1)$ be such that $\alpha + 2\beta < 1$. Let us consider Eq. (1.2), where continuous functions $r : \mathbb{R}_e \to (0, \infty)$ and $s : \mathbb{R}_e \to \mathbb{R}$ satisfy*

$$\limsup_{t\to\infty} \frac{\int_t^{t+t^\alpha} r(\tau)\,\mathrm{d}\tau}{t^{\alpha+\beta}} < \infty, \qquad \limsup_{t\to\infty} \frac{\int_t^{t+t^\alpha} |s(\tau)|\,\mathrm{d}\tau}{t^{\alpha+\beta}} < \infty. \tag{4.3}$$

Let

$$r_{t^\alpha} := \liminf_{t\to\infty} \frac{1}{t^\alpha} \int\limits_t^{t+t^\alpha} r(\tau)\,\mathrm{d}\tau \in \mathbb{R}, \qquad s_{t^\alpha} := \liminf_{t\to\infty} \frac{1}{t^\alpha} \int\limits_t^{t+t^\alpha} s(\tau)\,\mathrm{d}\tau \in \mathbb{R}.$$

If $4r_{t^\alpha}s_{t^\alpha} > 1$, *then Eq.* (1.2) *is oscillatory.*

Proof The corollary follows from Theorem 4.1 for $f(t) = t^\alpha$, $g(t) = t^\beta$, $t \in \mathbb{R}_e$. For these auxiliary functions f, g, (3.1) is true and (3.3) reduces to (4.3). □

Remark 5 In the statement of Corollary 4.1, the finiteness of r_{t^α} and s_{t^α} is considered. In fact, for $r_{t^\alpha} > 0$ and $s_{t^\alpha} = \infty$ or $r_{t^\alpha} = \infty$ and $s_{t^\alpha} > 0$, Eq. (1.2) is oscillatory as well. It follows from the famous Sturm comparison theorem (see, e.g., [34]).

Example 1 For arbitrarily given numbers $a > 1/4$ and $b > 0$ and for all $t \in \mathbb{R}_e$, we define

$$r(t) := \begin{cases} 1 - \frac{1}{2n}\left(t - 2^n\right), & t \in \left[2^n, 2^n + n\right),\ n \in \mathbb{N}; \\ 1 - \frac{1}{2n}\left(2^n + 2n - t\right), & t \in \left[2^n + n, 2^n + 2n\right],\ n \in \mathbb{N}; \\ 1, & t \in \left(2^n + 2n, 2^{n+1}\right),\ n \in \mathbb{N}, \end{cases}$$

and

$$s(t) := \begin{cases} a - b\left(t - 2^n\right), & t \in \left[2^n, 2^n + n\right),\ n \in \mathbb{N}; \\ a - b\left(2^n + 2n - t\right), & t \in \left[2^n + n, 2^n + 2n\right],\ n \in \mathbb{N}; \\ a, & t \in \left(2^n + 2n, 2^{n+1}\right),\ n \in \mathbb{N}. \end{cases}$$

For these functions, let us consider Eq. (1.2) and apply Corollary 4.1 for arbitrary $\alpha, \beta \in (0, 1)$ satisfying $\alpha + 2\beta < 1$. One can easily compute

$$\limsup_{t\to\infty} \frac{\int_t^{t+t^\alpha} r(\tau)\,\mathrm{d}\tau}{t^{\alpha+\beta}} \le \lim_{t\to\infty} \frac{1}{t^\beta} = 0$$

and

$$\begin{aligned} \limsup_{t\to\infty} \frac{\int_t^{t+t^\alpha} |s(\tau)|\,\mathrm{d}\tau}{t^{\alpha+\beta}} &\le \limsup_{t\to\infty} \frac{\int_t^{t+t^\alpha} a\,\mathrm{d}\tau}{t^{\alpha+\beta}} + \limsup_{n\to\infty} \frac{\int_{2^n}^{2^n+2^{n\alpha}} |s(\tau) - a|\,\mathrm{d}\tau}{2^{n(\alpha+\beta)}} \\ &\le \lim_{t\to\infty} \frac{a}{t^\beta} + \limsup_{n\to\infty} \frac{\int_{2^n}^{2^n+2n} bn\,\mathrm{d}\tau}{2^{n(\alpha+\beta)}} \\ &= 0 + \limsup_{n\to\infty} \frac{2bn^2}{2^{n(\alpha+\beta)}} = 0, \end{aligned}$$

i.e., (4.3) is valid. In addition,

$$\begin{aligned} 1 \geq r_{t^\alpha} &= \liminf_{t\to\infty} \frac{1}{t^\alpha} \int_t^{t+t^\alpha} r(\tau)\,\mathrm{d}\tau \\ &\geq 1 - \limsup_{n\to\infty} \frac{1}{2^{n\alpha}} \int_{2^n}^{2^n+2n} \frac{1}{2}\,\mathrm{d}\tau \\ &= 1 - \limsup_{n\to\infty} \frac{n}{2^{n\alpha}} = 1 \end{aligned}$$

and

$$\begin{aligned} a \geq s_{t^\alpha} &= \liminf_{t\to\infty} \frac{1}{t^\alpha} \int_t^{t+t^\alpha} s(\tau)\,\mathrm{d}\tau \\ &\geq a - \limsup_{n\to\infty} \frac{1}{2^{n\alpha}} \int_{2^n}^{2^n+2n} bn\,\mathrm{d}\tau \\ &= a - \limsup_{n\to\infty} \frac{2bn^2}{2^{n\alpha}} = a, \end{aligned}$$

i.e., $4r_{t^\alpha}s_{t^\alpha} = 4a > 1$. Hence, the treated equation is oscillatory. Note that its oscillation does not follow from any previously known result for any $a > 1/4$.

We highlight that Corollary 4.1 gives new results in many special cases (for example, for any $p > 0$). To illustrate this fact, we mention the following new result with a concrete choice of p, when the leading coefficient is constant and the second one is bounded and positive.

Corollary 4.2 *Let $a > 1$. Let us consider the equation*

$$\left(\log t\, x'(t)\right)' + \frac{\log t}{t^2} s(t)x(t) = 0, \tag{4.4}$$

where $s : \mathbb{R}_e \to (0, \infty)$ is a continuous and bounded function. If

$$\liminf_{t\to\infty} \frac{\log_a t}{t} \int_t^{t+\frac{t}{\log_a t}} s(\tau)\,\mathrm{d}\tau > \frac{1}{4}, \tag{4.5}$$

then Eq. (4.4) is oscillatory.

Proof It suffices to consider Eq. (1.2) and Theorem 4.1 for $p = 1$ and $r(t) = 1$, $f(t) = t/\log_a t$, $g(t) = 1$, $t \in \mathbb{R}_e$. Especially, (3.3) follows from the boundedness of r and s and

$$g(t) = 1, \qquad \frac{f(t)}{t} = \frac{1}{\log_a t}, \qquad t \in \mathbb{R}_e,$$
$$f'(t) = \frac{\log_a t - \frac{1}{\log a}}{\log_a^2 t}, \qquad t > e,$$

give (3.1). Because of $r_f = 1$, the inequality $4r_f s_f > 1$ reduces to (4.5). □

In the example below, we solve the oscillation of a simple equation which is not covered by any previous result (including Corollary 4.1).

Example 2 For arbitrarily given numbers $c > 1/4$ and $d \in (-c, -c + 1/4]$ and for all $t \in \mathbb{R}_e$, we define

$$s(t) := \begin{cases} c, & t \in [e, 128)\,; \\ c + d\,(t - 2^n)\,, & t \in \left[2^n, 2^n + 1\right)\,, \; n \ge 7, \; n \in \mathbb{N}; \\ c + d, & t \in \left[2^n + 1, 2^n + \frac{2^n}{n^2} - 1\right)\,, \; n \ge 7, \; n \in \mathbb{N}; \\ c + d\left(2^n + \frac{2^n}{n^2} - t\right), & t \in \left[2^n + \frac{2^n}{n^2} - 1, 2^n + \frac{2^n}{n^2}\right)\,, \; n \ge 7, \; n \in \mathbb{N}; \\ c, & t \in \left[2^n + \frac{2^n}{n^2}, 2^{n+1}\right)\,, \; n \ge 7, \; n \in \mathbb{N}. \end{cases}$$

For this function, let us consider Eq. (4.4) and apply Corollary 4.2 for $a = 2$. Using

$$\begin{aligned} c &\ge \limsup_{t\to\infty} \frac{\log_2 t}{t} \int_t^{t+\frac{t}{\log_2 t}} s(\tau)\,\mathrm{d}\tau \\ &\ge \liminf_{t\to\infty} \frac{\log_2 t}{t} \int_t^{t+\frac{t}{\log_2 t}} s(\tau)\,\mathrm{d}\tau \\ &= \liminf_{n\to\infty} \frac{\log_2 2^n}{2^n} \int_{2^n}^{2^n+\frac{2^n}{\log_2 2^n}} s(\tau)\,\mathrm{d}\tau \\ &= \liminf_{n\to\infty} \frac{n}{2^n} \int_{2^n}^{2^n\left(1+\frac{1}{n}\right)} s(\tau)\,\mathrm{d}\tau \end{aligned}$$

$$\geq \liminf_{n\to\infty} \frac{n}{2^n} \left(\int\limits_{2^n}^{2^n\left(1+\frac{1}{n^2}\right)} c + d \, \mathrm{d}\tau + \int\limits_{2^n\left(1+\frac{1}{n^2}\right)}^{2^n\left(1+\frac{1}{n}\right)} c \, \mathrm{d}\tau \right)$$

$$= \lim_{n\to\infty} \frac{n}{2^n} \cdot \frac{2^n}{n^2} (c + d) + \lim_{n\to\infty} \frac{n}{2^n} \cdot 2^n \left(\frac{1}{n} - \frac{1}{n^2}\right) c = c,$$

we obtain

$$\lim_{t\to\infty} \frac{\log_2 t}{t} \int\limits_{t}^{t+\frac{t}{\log_2 t}} s(\tau) \, \mathrm{d}\tau = c > \frac{1}{4}.$$

Thus, the considered equation is oscillatory. We add that we cannot use Theorem 1.2, because

$$\liminf_{t\to\infty} \frac{1}{\sigma} \int\limits_{t}^{t+\sigma} s(\tau) \, \mathrm{d}\tau = c + d \leq \frac{1}{4}$$

for any $\sigma > 0$.

To illustrate the role of the auxiliary function g, we mention the last corollary and example.

Corollary 4.3 *Let us consider the equation*

$$\left(\frac{\log^2 t}{r(t)} x'(t) \right)' + \frac{\log^2 t}{t^2} s(t) x(t) = 0, \tag{4.6}$$

where continuous functions $r : \mathbb{R}_\mathrm{e} \to (0, \infty)$ *and* $s : \mathbb{R}_\mathrm{e} \to \mathbb{R}$ *satisfy*

$$\limsup_{t\to\infty} \log t \int\limits_{t}^{t+\frac{1}{t}} r(\tau) \, \mathrm{d}\tau < \infty, \qquad \limsup_{t\to\infty} \log t \int\limits_{t}^{t+\frac{1}{t}} |s(\tau)| \, \mathrm{d}\tau < \infty. \tag{4.7}$$

Let

$$r_{t^{-1}} := \liminf_{t\to\infty} t \int\limits_{t}^{t+\frac{1}{t}} r(\tau) \, \mathrm{d}\tau \in \mathbb{R}, \qquad s_{t^{-1}} := \liminf_{t\to\infty} t \int\limits_{t}^{t+\frac{1}{t}} s(\tau) \, \mathrm{d}\tau \in \mathbb{R}.$$

If $4 r_{t^{-1}} s_{t^{-1}} > 1$, *then Eq. (4.6) is oscillatory.*

Proof The corollary is a consequence of Theorem 4.1 for $p = 2$ and $f(t) = 1/t$, $g(t) = t/\log t$, $t \in \mathbb{R}_e$. These auxiliary functions satisfy (3.1) and (3.3) reduces to (4.7). □

Example 3 We put

$$r(t) := 2 + \cos\left(3^t + 1\right), \qquad t \in \mathbb{R}_e,$$

and

$$s(t) := a + \sqrt{t}\sin 2^t, \qquad t \in \mathbb{R}_e,$$

where $a > 1/8$. For these functions, let us consider Eq. (4.6) and use Corollary 4.3. One can easily verify that (4.7) is valid and that

$$r_{t^{-1}} = \lim_{t\to\infty} t \int_t^{t+\frac{1}{t}} r(\tau)\,\mathrm{d}\tau = 2, \qquad s_{t^{-1}} = \lim_{t\to\infty} t \int_t^{t+\frac{1}{t}} s(\tau)\,\mathrm{d}\tau = a.$$

Considering $4r_{t^{-1}}s_{t^{-1}} = 8a > 1$, we obtain the oscillation of the equation, where its oscillation does not follow from any previously known result. In particular, we cannot apply Theorem 1.2, because

$$\lim_{t\to\infty} \frac{\int_t^{t+1} |s(\tau)|\,\mathrm{d}\tau}{\sqrt{t}} = \frac{2}{\pi} \neq 0,$$

i.e., (1.3) is not satisfied.

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