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Potency in soluble groups

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Abstract
We prove in particular that if G is a soluble group with no non-trivial locally finite normal subgroups, then G is p-potent for every prime p for which G has no Prüfer p-sections. (A group G is p-potent if for every power n of p and for any element x of G of infinite order or of finite order divisible by n there is a normal subgroup N of G of finite index such that the order of x modulo N is n. A Prüfer p-group is an infinite locally cyclic p-group.) This extends to soluble groups in general, and gives a more direct proof of, recent results of Azarov on polycyclic groups and soluble minimax groups.

Keywords Soluble groups · Residual properties

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1 Introductions

Define the spectrum σ(G) of any group G to be the set of all primes p such that G has a Prüfer p-section. As usual σ(G)´ denotes the set of all primes not in σ(G). If G is also soluble, it is easy to see that for each p in σ(G) there is a factor in the derived series of G with a Prüfer p-image. Thus this definition of spectrum extends to all groups the standard notion of the spectrum of a soluble minimax group (e.g. see [4] pages 86 & 87).

Following [1] say that a group G is π-potent for some set π of primes, if for all x in G and all positive π-numbers n, with n dividing the order |x| of x if |x| should be finite, there is a homomorphism ϕ of G into a finite group with $|x\phi| = n$. If π is the set of all primes, just say G is potent. For a history of this and related concepts, going

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back to 1971, see the introduction to [1]. Further τ (G) denotes the unique maximal, locally finite, normal subgroup of G. The following is the main result of this paper.

Theorem 1 *Let G be a soluble group with τ(G) finite. Then there exists a normal σ(K)´-potent subgroup K of G of finite index.*

Suppose H is a subgroup of the group G. If G is π-potent, clearly H is also π-potent. If H has finite index in G, then σ(H) = σ(G) and so if G is σ(G)´-potent, then H is σ(H)´-potent.

Corollary 1 *Any group with G/τ(G) soluble-by-finite is (locally finite)-by-(σ(G/τ(G))´-potent)-by-finite.*

For clearly in the Theorem σ(G) = σ(K). The following corollary is due to Azarov, see Theorem 2 and Corollary 2.1 of [1]; it was this paper that first led the author to consider potency. Note that the finite residual of a soluble-by-finite minimax group is divisible abelian and τ(G) is Chernikov, see [4] 5.3.1. Further any polycyclic group G has τ(G) finite with σ(G) empty. Thus the following is immediate from Corollary 1.

Corollary 2 (Azarov [1]). *If G is a residually finite, soluble-by-finite minimax group, then G/τ(G) is (σ(G/τ(G))´-potent)-by-finite. If G is polycyclic-by-finite, then G is potent-by-finite.*

Corollary 3 *If G is a <L, P´>(**AF**)-group of finite rank, then G is (locally finite)-by-(π-potent)-by-finite for π= σ(G/τ(G))´.*

Here <L, P´>(**AF**) denotes the very wide class of groups that is defined as containing all the abelian and all the finite groups, being closed under the local and ascending series operators and being minimal with these properties. (Some authors call such groups elementary amenable.) By Theorem 1 of [7] for G as in Corollary 3, G/τ(G) is soluble-by-finite; hence Corollary 1 above applies and Corollary 3 follows.

A group G has finite Hirsch number the non-negative integer h if G has an ascending series running from <1> to G with exactly h of the factors infinite cyclic, all the remaining factors being locally finite. (This is equivalent in the terminology of [2] to G having 0-rank h with all the periodic sections of G being locally finite.) Clearly, any group G with finite Hirsch number h lies in the class <L, P´>(**AF**). Also G/τ(G) has finite rank (at most [7h/2]+1 by Theorem 3 of [7]). Thus the following is immediate from Corollary 3.

Corollary 4 *Any group G with finite Hirsch number is (locally finite)-by-(π-potent)-by-finite for π = σ(G/τ(G))´.*

Corollary 5 *Any linear group G is (locally finite)-by-(π-potent)-by-finite for π = σ(G/τ(G))´.*

The only ø-number is 1, so trivially every group is ø-potent. Thus to prove Corollary 5 we may assume π ≠ ø If X is a non-cyclic free group, then X´ is a free group of infinite rank and hence σ(X)´ = ø If G is a linear group of characteristic q ≥ 0, then from Tits's Theorem (e.g. see [5] 10.17) G has a soluble normal subgroup S with G/S finite if $q = 0$ and only locally finite if $q > 0$. In particular if $q = 0$, Corollary 5 follows at once from Corollary 1.

Suppose $q > 0$, $G \leq GL(n, F)$ and F is algebraically closed. If G is completely reducible, then the connected component S° of S is triangularizable ([5] 5.8), completely reducible ([5] 1.8) and so diagonalizable. Also G/S° is locally finite. Set $C = C_G(S^\circ)$. Then $(G : C) \leq n!$ by [5]1.12 and C is centre-by-(locally finite). By Schur's Theorem (actually here a very special case of [5] 10.20) C' is locally finite and G is (locally finite)-by-abelian-by-finite. In general if U denotes the unipotent radical of G, then U is a nilpotent normal q-subgroup of G with G/U isomorphic to a completely reducible subgroup of GL(n, F). By the above G/U is (locally finite)-by-abelian-by-finite. Hence G is (locally finite)-by-abelian-by-finite and again Corollary 1 applies.

Let A be a torsion-free abelian group. If $\sigma(A) = \emptyset$, then a very special case of the Theorem is that A has a potent subgroup of finite index. The converse does not hold. For example, suppose A is a free abelian group of infinite rank. Then A is potent (easy to see directly, but it also can be derived from Lemma 2 below). However, $\sigma(A)$ is not empty; actually, it is the set of all primes.

I am indebted to the referee for introducing the author to [3] as a paper related to our work here. It discusses potency (so π the set of all primes), under the name 'cyclic separation', in soluble torsion-free groups G. Clearly $\tau(G) = <1>$ for such G and hence our theorem above is applicable. In particular [3] determines those soluble torsion-free groups G having a normal subgroup H of finite index such that H and every torsion-free quotient of H is potent.

We start our proof of the Theorem with the following easy remark.

Lemma 1 *For any set π of primes, a group G is π-potent if and only if G is p-potent for every p in π.*

Proof Clearly π-potent groups are p-potent for all p in π. Let $x \in G$; note we are not assuming x has infinite order. Suppose $n = \prod_\kappa p^{e(p)}$ for some finite subset κ of π. If N_p is a normal subgroup of G of finite index with $|xN_p| = p^{e(p)}$ for each p in κ, set $N = \cap_\kappa N_p$. Then N is also a normal subgroup of G of finite index and clearly $|xN| = \prod_\kappa |xN_p| = \prod_\kappa p^{e(p)} = n$. The lemma follows.

Notice that if each N_p is characteristic in G, then N is characteristic in G and if each G/N_p is a π-group, then G/N is a π-group.

Lemma 2 *Let A be an (additive) torsion-free abelian group of finite rank. Then A is $\sigma(A)'$-potent. Moreover, if $a \in A\backslash<0>$ and if n is a $\sigma(A)'$-number, then there exists a characteristic subgroup B of A with A/B a finite $\sigma(A)'$-group such that $|aB| = n$.*

Proof Set $\kappa = \sigma(A)'$, which we may assume is not empty. Suppose $p \in \kappa$ and $a \in \cap_i p^i A \setminus \{0\}$. Then $a = p^i a_i$ for some a_i in A and all $i \geq 0$. Clearly $p^i(a_i - pa_{i+1}) = a - a = 0$ and A is torsion-free, so $a_i = pa_{i+1}$ for each i and therefore $<a_i : i \geq 0>/<a>$ is a Prüfer p-group. But then $p \notin \kappa$, a contradiction yielding $\cap_i p^i A = \{0\}$ for all p in κ.

Now let $a \in A\backslash<0>$ and consider $n = p^e$. By the previous paragraph there exists a positive integer j such that $<a> \cap p^j A = p^e <a>$. Then a has order n modulo $p^j A$. Set $B = p^j A$. Then B is characteristic in A and A/B is a finite p-group, the latter since

$|A/p^jA|$ divides $p^{j(\text{rank } A)}$. In particular A is p-potent and hence A is $\sigma(A)'$-potent by Lemma 1. The final claim of Lemma 2 follows from the remark at the end of the proof of Lemma 1.

Lemma 3 *Let* $<1> = G_0 \leq G_1 \leq \ldots \leq G_r = G$ *be a normal series of the group G with each* G_i/G_{i-1} *torsion-free abelian of finite rank. Then G is* $\sigma(G)'$*-potent.*

Proof Let $a \in G_1 \backslash <1>$ and let n be a positive $\sigma(G)'$-number. By Lemma 2 there is a characteristic subgroup B of G_1 such that G_1/B is finite and a has order n modulo B. Now B is normal in G. By [4] 5.3.1 the intersection D/B of the normal subgroups of G/B of finite index is divisible and nilpotent. Since G_1/B is finite and G/G_1 is torsion-free, so $G_1 \cap D = B$ and $DG_1/G_1 \cong D/B$. If $D \neq B$ then D/B contains a copy of the additive group of the rationals, which implies $\sigma(G)'$ is empty and $n = 1$. If $D = B$, then there exists N, a normal subgroup of G of finite index such that $G_1 \cap N = B$. Consequently the order of a modulo N is equal to n, the order of a modulo B.

Now assume that $a \in G \backslash G_1$. Since G/G_1 is torsion-free $<a> \cap G_1 = <1>$. By induction on r (the case $r = 1$ being Lemma 2) there exists a normal subgroup $N \geq G_1$ of G of finite index such that the order of the image of $aG_1 \in G/G_1$ in G/N is exactly n. Therefore the order of a modulo N is also n. The lemma is proved.

1.1 Proof of the Theorem

Here G is soluble. Assume $\sigma(G)'$ is not empty. Since a free abelian group of infinite rank has a Prüfer q-image for every prime q, so each torsion-free abelian section of G has finite rank and G has finite Hirsch number. By hypothesis $\tau(G)$ is finite. Hence G has a normal series $<1> = G_0 \leq G_1 \leq \ldots \leq G_r \leq G$ with G/G_r finite and each G_i/G_{i-1} torsion-free abelian of finite rank (e.g. use 5.2.4 and 5.2.5 of [4] or Lemmas 5 and 6 of [6]). Set $K = G_r$. Then K is normal in G of finite index and K is $\sigma(K)'$-potent by Lemma 3.

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