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# Viterbo's conjecture as a worm problem

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**Abstract**
In this paper, we relate Viterbo's conjecture from symplectic geometry to Minkowski versions of worm problems which are inspired by the well-known Moser worm problem from geometry. For the special case of Lagrangian products this relation provides a connection to systolic Minkowski billiard inequalities and Mahler's conjecture from convex geometry. Moreover, we use the above relation in order to transfer Viterbo's conjecture to a conjecture for the longstanding open Wetzel problem which also can be expressed as a systolic Euclidean billiard inequality and for which we discuss an algorithmic approach in order to find a new lower bound. Finally, we point out that the above mentioned relation between Viterbo's conjecture and Minkowski worm problems has a structural similarity to the known relationship between Bellmann's lost-in-a-forest problem and the original Moser worm problem.

**Keywords** Viterbo's conjecture · EHZ-capacity · Shortest periodic orbit · Minkowski billiards · Worm problems

**Mathematics Subject Classification** 37C83

## 1 Introduction and main results

Worm problems have a long history. The earliest known problem of this type was posed by Moser in [44] (see also [45]) more than 50 years ago:

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*Moser's worm problem: Find a/the (convex) set of least area that contains a congruent copy of each arc in the plane of lenth one.*

Here, the unit arcs are sometimes called worms, while the problem has been phrased in many different ways in the literature: the architect's version (find the smallest comfortable living quarters for a unit worm), the humanitarian version (find the shape of the most efficient worm blanket), the sadistic version (find the shape of the best mallet head), and so on (see [53]). So far, despite a lot of research, only partial results are known, including the existence of such a minimum cover in the convex case (probably the first time proven in [39]), but its shape and area remain unknown. The best bounds presently known for its area $\mu$ are:[1]

$$0.23224 \leqslant \mu \leqslant 0.27091$$

(see [34] for the lower and [50] for the upper bound).

Worm problems can be formulated in considerable generality (see [53]):

*Given a collection $\mathcal{F}$ of $n$-dimensional figures $F$ and a transitive group $\mathcal{M}$ of motions $m$ on $\mathbb{R}^n$, find minimal convex target sets $K \subset \mathbb{R}^n$–minimal in the sense of having least volume, surface volume, or whatever–so that for each $F \in \mathcal{F}$ there is a motion $m \in \mathcal{M}$ with*

$$m(F) \subseteq K.$$

The existence of solutions to this problem can be guaranteed under certain natural hypotheses by fundamental compactness results like the Blaschke selection theorem (see [10, Sect. 18] for Blaschke's selection theorem and [33, 39] for its application; see also Theorem 3.8 and its application in Propositions 3.9, 3.13, 3.19, and 3.20).

When the problem does not permit an arc to be replaced by its mirror image, then it is appropriate to consider the subgroup of orientation preserving motions. For other problems, e.g., Moser's original worm problem, orientation reversing motions are permitted. Many problems whose motion group is the group of translations have been studied in the literature (see [8, 13, 52]).

In order to formulate the specific worm problem which is of main interest for our study, we introduce the following definition: Let $T \subset \mathbb{R}^n$ be a convex body, i.e., a compact convex set in $\mathbb{R}^n$ with nonempty interior, and $T^\circ$ its polar. Using the Minkowski functional

$$\mu_{T^\circ}(x) = \min\{t \geqslant 0 : x \in tT^\circ\}$$

with respect to $T$'s polar $T^\circ$, we define the $\ell_T$*-length* of a closed $H^1([0, \widetilde{T}], \mathbb{R}^n)$-curve[2] $\dot{q}$ (from now on, for the sake of simplicity, every closed curve is assumed to fulfill this Sobolev property), $\widetilde{T} \geqslant 0$, by

$$\ell_T(q) := \int_0^{\widetilde{T}} \mu_{T^\circ}(\dot{q}(t))\,\mathrm{d}t.$$

[1] We round all decimal numbers up to the fifth decimal place.

[2] This implies that $q$ is differentiable almost everywhere with $\dot{q} \in L^2([0, \widetilde{T}], \mathbb{R}^n)$.

The worm problem which is of main interest for our study we call the *Minkowski worm problem*. Referring to the above general worm problem formulation, for this for convex body $T \subset \mathbb{R}^n$, we consider $\mathcal{F} = \mathcal{F}(T, \alpha)$ as the set of closed curves of $\ell_T$-length $\alpha > 0$, $\mathcal{M}$ as the group of translations and the minimization in the sense of having minimal volume:

*Minkowski worm problem: Let $T \subset \mathbb{R}^n$ be a convex body. Find the volume-minimizing convex bodies $K \subset \mathbb{R}^n$ that contain a translate of every closed curve of $\ell_T$-length $\alpha$.*

So, in contrast to Moser's worm problem, we consider general dimension (instead of just dimension two), length-measuring with Minkowski functionals with respect to arbitrary convex bodies (instead of with respect to the Euclidean unit ball), closed curves (instead of not necessarily closed arcs), and translations (instead of congruence transformations). In other words and introducing a notation which will be useful throughout this paper: Let $cc(\mathbb{R}^n)$ be the set of closed curves in $\mathbb{R}^n$. Find the minimizers[3] of

$$\min_{K \in A(T,\alpha)} \operatorname{vol}(K),$$

where for convex body $T \subset \mathbb{R}^n$ and $\alpha > 0$, we define

$$A(T, \alpha) := \left\{K \subset \mathbb{R}^n \text{ convex body} : L_T(\alpha) \subseteq C(K)\right\}$$

with

$$L_T(\alpha) := \left\{q \in cc(\mathbb{R}^n) : \ell_T(q) = \alpha\right\}$$

and

$$C(K) := \left\{q \in cc(\mathbb{R}^n) : \exists k \in \mathbb{R}^n \text{ s.t. } q \subseteq k + K\right\},$$

where, for the sake of simplicity, we, in general, identify $q$ with its image.

The only Minkowski worm problem that has been investigated so far is the case when the dimension is 2, $T$ is the Euclidean unit ball in $\mathbb{R}^2$, and, without loss of generality, $\alpha = 1$ (one could say: the two-dimensional Euclidean worm problem). It is known as:

*Wetzel's problem: Find the area-minimizing convex bodies $K \subset \mathbb{R}^2$ that contain a translate of every closed curve of Euclidean length 1.*

So far, the minimal area for this problem is not known, but the best bounds presently known for the minimum are 0.15544 as lower (see [52], where an argument from [47] is used) and 0.16526 as upper bound (see [8]; note that in [52] it was claimed incorrectly an upper bound of 0.159). In comparison to that: The areas of the obvious covers of constant width, the ball of radius $1/4$ and the Reuleaux triangle of width $1/2$, are 0.19635 and 0.17619, respectively. Since, by the Blaschke-Lebesgue theorem, the Reuleaux triangle is the area-minimizing set of constant width (see [9, 40]; see [27] for a direct proof by analyzing the underlying variational problem), we can conclude that a minimizer for Wetzel's problem is not of constant width. We refer to Fig. 1 for

[3] In Proposition 3.9, we will prove that in fact there exists at least one minimizer.

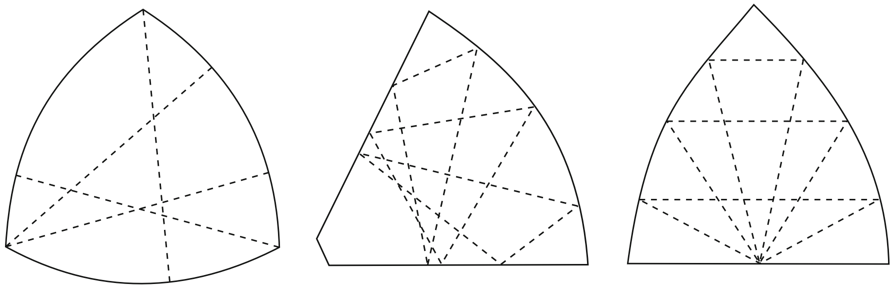

**Fig. 1** On the left side is the Reuleaux triangle with width $\frac{1}{2}$ and area 0.17619, in the middle is a convex body with area 0.17141 which was found by Wetzel in [52], and on the right is a convex body, looking a bit like a church window, with base length and height equal to $\frac{1}{2}$ and area $\frac{1}{6} \approx 0.16667$ (for both the middle and right convex body we refer to [8]). Some worms are drawn in in each case

three examples whose areas are approaching (not achieving) the minimum (clearly, the middle and right convex bodies are not of constant width).

Although we derive some results, the primary goal of our study will not be to solve these Minkowski worm problems, rather to relate them to Viterbo's conjecture from symplectic geometry (see [49]) which for convex bodies $C \subset \mathbb{R}^{2n}$ reads

$$\mathrm{vol}(C) \geqslant \frac{c_{EHZ}(C)^n}{n!}.$$

For that, we recall that the EHZ-capacity of a convex body $C \subset \mathbb{R}^{2n}$ can be defined[4] by

$$c_{EHZ}(C) = \min\{\mathbb{A}(x) : x \text{ closed characteristic on } \partial C\},$$

where a closed characteristic on $\partial C$ is an absolutely continuous loop in $\mathbb{R}^{2n}$ satisfying

$$\begin{cases} \dot{x}(t) \in J\partial H_C(x(t)) & \text{a.e.} \\ H_C(x(t)) = \frac{1}{2} \ \forall t \in \mathbb{T} \end{cases}$$

where

$$H_C(x) = \frac{1}{2}\mu_C(x)^2, \quad J = \begin{pmatrix} 0 & \mathbb{1} \\ -\mathbb{1} & 0 \end{pmatrix}, \quad \mathbb{T} = \mathbb{R}/\widetilde{T}\mathbb{Z}, \ \widetilde{T} > 0.$$

$\widetilde{T}$ is the period of the loop and by $\mathbb{A}$ we denote its action defined by

$$\mathbb{A}(x) = -\frac{1}{2}\int_0^{\widetilde{T}} \langle J\dot{x}(t), x(t)\rangle \, \mathrm{d}t.$$

[4] This definition is in fact the outcome of a historically grown study of symplectic capacities. Traced back–recalling that $c_{EHZ}$ in its present form is the generalization of a symplectic capacity by Künzle in [38] after applying the dual action functional introduced by Clarke in [12], the EHZ-capacity denotes the coincidence of the Ekeland-Hofer- and Hofer-Zehnder-capacities, originally constructed in [15] and [28], respectively.

The first main result of this paper addresses the special case of Lagrangian products

$$C = K \times T \subset \mathbb{R}^n_q \times \mathbb{R}^n_p \cong \mathbb{R}^{2n},$$

where $K$ and $T$ are convex bodies in $\mathbb{R}^n$.[5] We denote by $\mathcal{C}(\mathbb{R}^n)$ the set of convex bodies in $\mathbb{R}^n$.

**Theorem 1.1** *Viterbo's conjecture for convex Lagrangian products $K \times T \subset \mathbb{R}^n \times \mathbb{R}^n$*

$$\mathrm{vol}(K \times T) \geqslant \frac{c_{EHZ}(K \times T)^n}{n!}, \quad K, T \in \mathcal{C}(\mathbb{R}^n),$$

*is equivalent to the Minkowski worm problem*

$$\min_{K \in A(T,1)} \mathrm{vol}(K) \geqslant \frac{1}{n!\,\mathrm{vol}(T)}, \quad K, T \in \mathcal{C}(\mathbb{R}^n). \tag{1}$$

*Additionally, equality cases $K^* \times T^*$ of Viterbo's conjecture satisfying*

$$\mathrm{vol}(K^*) = \mathrm{vol}(T^*) = 1$$

*are composed of equality cases $(K^*, T^*)$ of (1). Conversely, equality cases $(K^*, T^*)$ of (1) form equality cases $K^* \times T^*$ of Viterbo's conjecture.*

This yields the following corollary, which seems to be more suitable in order to approach Viterbo's conjecture as an optimization problem (see Sect. 9).

**Corollary 1.2** *Viterbo's conjecture for convex Lagrangian products $K \times T \subset \mathbb{R}^n \times \mathbb{R}^n$*

$$\mathrm{vol}(K \times T) \geqslant \frac{c_{EHZ}(K \times T)^n}{n!}, \quad K, T \in \mathcal{C}(\mathbb{R}^n),$$

*is equivalent to*[6]

$$\min_{a_q \in \mathbb{R}^n} \mathrm{vol}\left(\mathrm{conv}\left\{\bigcup_{q \in L_T(1)} (q + a_q)\right\}\right) \geqslant \frac{1}{n!\,\mathrm{vol}(T)}, \quad T \in \mathcal{C}(\mathbb{R}^n), \tag{2}$$

*where the minimization runs for every $q \in L_T(1)$ over all possible translations in $\mathbb{R}^n$. Additionally, equality cases $K^* \times T^*$ of Viterbo's conjecture satisfying*

$$\mathrm{vol}(K^*) = \mathrm{vol}(T^*) = 1$$

[5] In this paper, whenever we write products of the form $K \times T$ for two convex bodies $K, T \subset \mathbb{R}^n$, we presume the natural symplectic structure of $\mathbb{R}^{2n} = \mathbb{R}^n_q \times \mathbb{R}^n_p$. So, every such product is a Lagrangian product.

[6] Here, we note that $K$ has been dissolved by replacing it by an expression that extremizes over all possible $K$s. The extremizing $K$ is of the form (3).

*are composed of equality cases $T^*$ of* (2) *with*

$$K^* = \operatorname{conv}\left\{ \bigcup_{q \in L_{T^*}(1)} (q + a_q^*) \right\}, \tag{3}$$

*where $a_q^*$ are the minimizers in* (2)*. Conversely, equality cases $T^*$ of* (2) *with $K^*$ as in* (3) *form equality cases $K^* \times T^*$ of Viterbo's conjecture.*

In analogy to Theorem 1.1, also Mahler's conjecture from convex geometry (see [43]), i.e.,

$$\operatorname{vol}(T)\operatorname{vol}(T^\circ) \geqslant \frac{4^n}{n!}, \quad T \in \mathcal{C}^{cs}(\mathbb{R}^n), \tag{4}$$

where by $\mathcal{C}^{cs}(\mathbb{R}^n)$ we denote the set of all centrally symmetric convex bodies in $\mathbb{R}^n$, can be expressed as a worm problem. As shown in [3], this is due to the fact that Mahler's conjecture is a special case of Viterbo's conjecture.

**Theorem 1.3** *Mahler's conjecture for centrally symmetric convex bodies*

$$\operatorname{vol}(T)\operatorname{vol}(T^\circ) \geqslant \frac{4^n}{n!}, \quad T \in \mathcal{C}^{cs}(\mathbb{R}^n), \tag{5}$$

*is equivalent to the Minkowski worm problem*

$$\min_{T \in A(T^\circ,1)} \operatorname{vol}(T) \geqslant \frac{1}{n!\operatorname{vol}(T^\circ)}, \quad T \in \mathcal{C}^{cs}(\mathbb{R}^n). \tag{6}$$

*Additionally, equality cases $T^*$ of Mahler's conjecture* (5) *satisfying*

$$\operatorname{vol}(T^*) = 1$$

*are equality cases of* (6)*. And conversely, equality cases $T^*$ of* (6) *are equality cases of Mahler's conjecture* (5)*.*

Furthermore, also *systolic Minkowski billiard inequalities* within the field of billiard dynamics can be related to worm problems.

In order to state this, let us recall some relevant notions from the theory of Minkowski billiards (see [36]): For convex bodies $K, T \subset \mathbb{R}^n$, we say that a closed polygonal curve[7] with vertices $q_1, ..., q_m$, $m \geqslant 2$, on the boundary of $K$ is a *closed weak $(K, T)$-Minkowski billiard trajectory* if for every $j \in \{1, ..., m\}$, there is a $K$-supporting hyperplane $H_j$ through $q_j$ such that $q_j$ minimizes

$$\mu_{T^\circ}(\overline{q}_j - q_{j-1}) + \mu_{T^\circ}(q_{j+1} - \overline{q}_j)$$

[7] For the sake of simplicity, whenever we talk of the vertices $q_1, ..., q_m$ of a closed polygonal curve, we assume that they satisfy $q_j \neq q_{j+1}$ and $q_j$ is not contained in the line segment connecting $q_{j-1}$ and $q_{j+1}$ for all $j \in \{1, ..., m\}$. Furthermore, whenever we settle indices $1, ..., m$, then the indices in $\mathbb{Z}$ will be considered as indices modulo $m$.

over all $\overline{q}_j \in H_j$. We encode this closed $(K,T)$-Minkowski billiard trajectory by $(q_1,...,q_m)$. Furthermore, we say that a closed polygonal curve with vertices $q_1,...,q_m$, $m \geqslant 2$, on the boundary of $K$ is a *closed (strong)* $(K,T)$*-Minkowski billiard trajectory* if there are points $p_1,...,p_m$ on $\partial T$ such that

$$\begin{cases} q_{j+1}-q_j \in N_T(p_j), \\ p_{j+1}-p_j = -N_K(q_{j+1}) \end{cases}$$

is fulfilled for all $j \in \{1,...,m\}$. We denote by $M_{n+1}(K,T)$ the set of closed $(K,T)$-Minkowski billiard trajectories with at most $n+1$ bouncing points.

Then, for convex body $K \subset \mathbb{R}^n$, introducing $F^{cp}(K)$ as the set of all closed polygonal curves in $\mathbb{R}^n$ that cannot be translated into $K$'s interior $\mathring{K}$, we have the following relations:

**Theorem 1.4** *Let* $T \subset \mathbb{R}^n$ *be a convex body and* $\alpha, c > 0$. *Then, the following statements are equivalent:*

(1)

$$\max_{\mathrm{vol}(K)=c} \min_{q \in F^{cp}(K)} \ell_T(q) \leqslant \alpha, \quad K \in \mathcal{C}(\mathbb{R}^n),$$

(2)

$$\max_{\mathrm{vol}(K)=c} c_{EHZ}(K \times T) \leqslant \alpha, \quad K \in \mathcal{C}(\mathbb{R}^n),$$

(3)

$$\max_{\mathrm{vol}(K)=c} \min_{q \in M_{n+1}(K,T)} \ell_T(q) \leqslant \alpha, \quad K \in \mathcal{C}(\mathbb{R}^n),$$

(4)

$$\min_{K \in A(T,\alpha)} \mathrm{vol}(K) \geqslant c, \quad K \in \mathcal{C}(\mathbb{R}^n),$$

(5)

$$\min_{a_q \in \mathbb{R}^n} \mathrm{vol}\left(\mathrm{conv}\left\{\bigcup_{q \in L_T(1)} (q+a_q)\right\}\right) \geqslant c, \quad K \in \mathcal{C}(\mathbb{R}^n).$$

*If* $T$ *is additionally assumed to be strictly convex, then the following systolic weak Minkowski billiard inequality can be added to the above list of equivalent expressions:*

(6)

$$\max_{\mathrm{vol}(K)=c} \min_{q \text{ cl. weak } (K,T)\text{-Mink. bill. traj.}} \ell_T(q) \leqslant \alpha, \quad K \in \mathcal{C}(\mathbb{R}^n).$$

*Moreover, every equality case $(K^*, T^*)$ of any of the above inequalities is also an equality case of all the others.*

Now, we turn our attention to the general Viterbo conjecture for convex bodies in $\mathbb{R}^{2n}$. For that, we first introduce the following definitions: We denote by $\mathcal{C}^p\left(\mathbb{R}^{2n}\right)$ the set of convex polytopes in $\mathbb{R}^{2n}$. For $P \in \mathcal{C}^p\left(\mathbb{R}^{2n}\right)$, we denote by

$$F_*^{cp}(P) \subset F^{cp}(P)$$

the set of all closed polygonal curves $q = (q_1, ..., q_m)$ in $F^{cp}(P)$ for which $q_j$ and $q_{j+1}$ are on neighbouring facets $F_j$ and $F_{j+1}$ of $P$ such that there are $\lambda_j, \mu_{j+1} \geqslant 0$ with

$$q_{j+1} = q_j + \lambda_j J\nabla H_P(x_j) + \mu_{j+1} J\nabla H_P(x_{j+1}),$$

where $x_j$ and $x_{j+1}$ are arbitrarily chosen interior points of $F_j$ and $F_{j+1}$, respectively. Later, we will see that the existence of such closed polygonal curves is guaranteed.

**Theorem 1.5** *Viterbo's conjecture for convex polytopes in* $\mathbb{R}^{2n}$

$$\mathrm{vol}(P) \geqslant \frac{c_{EHZ}(P)^n}{n!}, \quad P \in \mathcal{C}^p\left(\mathbb{R}^{2n}\right), \tag{7}$$

*is equivalent to the Minkowski worm problem*

$$\min_{P\in A(JP,1)} \mathrm{vol}(P) \geqslant \frac{(R_P)^n}{2^n n!}, \quad P \in \mathcal{C}^p\left(\mathbb{R}^{2n}\right), \tag{8}$$

*where we define*

$$R_P := \frac{\min_{q\in F_*^{cp}(P)} \ell_{\frac{JP}{2}}(q)}{\min_{q\in F^{cp}(P)} \ell_{\frac{JP}{2}}(q)} \geqslant 1.$$

*Additionally, $P^*$ is an equality case of Viterbo's conjecture for convex polytopes* (7) *satisfying*

$$\mathrm{vol}(P^*) = 1$$

*if and only if $P^*$ is an equality case of* (8).

When we look at the operator norm of the complex structure/symplectic matrix $J$ with respect to a convex body $C \subset \mathbb{R}^{2n}$ as map from

$$\left(\mathbb{R}^{2n}, ||\cdot||_{C^\circ}\right) \;\text{ to }\; \left(\mathbb{R}^{2n}, ||\cdot||_{C}\right)$$

as it has been done in [2] and [23], i.e.,

$$||J||_{C^\circ\to C} = \sup_{||v||_{C^\circ}\leqslant 1} ||Jv||_C,$$

then we derive the following theorem:

**Theorem 1.6** *Viterbo's conjecture for convex bodies in* $\mathbb{R}^{2n}$

$$\mathrm{vol}(C) \geqslant \frac{c_{EHZ}(C)^n}{n!}, \quad C \in \mathcal{C}\left(\mathbb{R}^{2n}\right), \tag{9}$$

*is equivalent to*

$$\min_{C \in A(C^\circ,1)} \mathrm{vol}(C) \geqslant \frac{(\widetilde{R}_C)^n}{n!}, \quad C \in \mathcal{C}\left(\mathbb{R}^{2n}\right), \tag{10}$$

*where*[8]

$$\widetilde{R}_C := \frac{c_{EHZ}(C)}{c_{EHZ}(C \times C^\circ)} \geqslant \frac{1}{2||J||_{C^\circ \to C}}.$$

*Additionally,* $C^*$ *is an equality case of Viterbo's conjecture for convex bodies in* $\mathbb{R}^{2n}$ (9) *satisfying*

$$\mathrm{vol}(C^*) = 1$$

*if and only if* $C^*$ *is an equality case of* (10).

Finally, we turn to Wetzel's problem. For that, we keep the current state of things in mind:

**Theorem 1.7** (Wetzel in [52], '73; Bezdek and Connelly in [8], '89) *In dimension* $n = 2$*, we have*

$$\min_{K \in A\left(B_1^2,1\right)} \mathrm{vol}(K) \in (0.15544, 0.16526), \quad K \in \mathcal{C}(\mathbb{R}^2),$$

*where we denote by* $B_1^2$ *the Euclidean unit ball in* $\mathbb{R}^2$.

Then, as application of Theorem 1.1, we transfer Viterbo's conjecture onto Wetzel's problem. This results in the following conjecture:

**Conjecture 1.8** *We have*

$$\min_{K \in A\left(B_1^2,1\right)} \mathrm{vol}(K) \geqslant \frac{1}{2\pi} \approx 0.15915, \quad K \in \mathcal{C}(\mathbb{R}^2).$$

Applying [36, Theorem 3.12] and Theorem 1.4, we note that this conjecture can be equivalently expressed as *systolic Euclidean billiard inequality*:

**Conjecture 1.9** *We have*

$$\min_{q \text{ cl. } (K,B_1^2)\text{-Mink. bill. traj.}} \ell_{B_1^2}^2(q) \leqslant 2\pi \, \mathrm{vol}(K)$$

*for* $K \in \mathcal{C}(\mathbb{R}^2)$.

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[8] Here, by $C \times C^\circ$ we denote the Lagrangian product of $C$ and $C^\circ$, where we presume the natural symplectic structure on $\mathbb{R}^{2(2n)} \cong \mathbb{R}^{2n}_{q'} \times \mathbb{R}^{2n}_{p'}$ with $\mathbb{R}^{2n}_{q'} = \mathbb{R}^n_q \times \mathbb{R}^n_p$ and denote by $q'$ and $p'$ the local and momentum coordinates on $\mathbb{R}^{2n} \supset C$, respectively.

We remark that, for the configuration $(K, B_1^2)$, due to the strict convexity of $B_1^2$, the notions of weak and strong $(K, B_1^2)$-Minkowski billiards coincide and are equal to the one of billiards in the Euclidean sense.

Although much work has been done around Wetzel's problem and the systolic Euclidean billiard inequality, this shows that Viterbo's conjecture is even unsolved for the "trivial" configuration

$$K \times B_1^2 \subset \mathbb{R}^2 \times \mathbb{R}^2.$$

On the other hand, looking at these two problems from the symplectic point of view, can help us to conceptualize them from a very different point of view.

The Minkowski worm problems in Theorems 1.3, 1.5 and 1.6 seem to be very hard to solve (as it is expected from the perspective of Mahler's/Viterbo's conjecture). On the one hand, this is a consequence of the inner dependencies within

$$T \in A\left(T^\circ, 1\right), \;\; P \in A\left(JP, 1\right), \;\; \text{and} \;\; C \in A\left(C^\circ, 1\right),$$

on the other hand, the right hand sides in (6), (8), and (10)

$$\frac{1}{n!\,\mathrm{vol}(T^\circ)}, \;\; \frac{(R_P)^n}{2^n n!}, \;\; \text{and} \;\; \frac{(\widetilde{R}_C)^n}{n!}$$

also contain dependencies and, beyond specific configurations, do not seem to be so accessible. Nevertheless, perhaps it turns out to be fruitful to investigate worm problems of the following structure a little bit more in detail: Find

$$\min_{C \in A(C^\circ,1)} \mathrm{vol}(C) \;\; \text{and} \;\; \min_{C \in A(JC,1)} \mathrm{vol}(C).$$

Interestingly enough, from this perspective, Viterbo's and Mahler's conjecture are very similar in structure.

Motivated by a relationship between Moser's worm problem and a version of *Bellman's lost-in-a-forest problem* shown by Finch and Wetzel in [17], we further investigate whether it is possible also to relate Minkowski worm problems to versions of Bellman's lost-in-a-forest problem. And indeed, it will turn out that the relationship established in [17] is somewhat similar to the relationship between Minkowski worm problems and Viterbo's conjecture for convex Lagrangian products. However, before we will elaborate on this, we will give a short introduction to Bellman's lost-in-a-forest problem and general escape problems of this type.

In 1955, Bellman stated in [5] the following research problem (see also [6] and [7]):

> *We are given a region $R$ and a random point $P$ within the region. Determine the paths which (a) minimize the expected time to reach the boundary, or (b) minimize the maximum time required to reach the boundary.*

This problem can be phrased as:

*A hiker is lost in a forest whose shape and dimensions are precisely known to him. What is the best path for him to follow to escape from the forest?*

In other words: To solve the lost-in-a-forest problem one has to find the *best* escape path–the best in terms of minimizing the maximum or expected time required to escape the forest. A third interpretation of *best* has been given in [13]: Find the best escape path in terms of maximizing the probability of escape within a specified time period.

Bellman asked about two configurations in particular: on the one hand, the configuration in which the region is the infinite strip between two parallel lines a known distance apart, on the other hand, the configuration in which the region is a half-plane and the hiker's distance from the boundary is known. For the case when *best* is understood in terms of the maximum time to escape, both of these two configurations have been studied: for the first configuration, the best path was found in [55] ('61), for the second, in [31] ('57) (where a complete and detailed proof was not published until it was done in [32] ('80); see [18] for an english translation). In each of these two cases, the shortest escape path is unique up to congruence. Apart from that, not much is known for other interpretations of *best*. We refer to [51] for a detailed survey on the different types, results, and some related material.

Finch and Wetzel studied in [17] the case in which the *best* escape path is the shortest. As already mentioned above, in this case, they could show a fundamental relation to Moser's worm problem.

Before we further elaborate on this, it is worth mentioning to note that Williams in [54] has included lost-in-a-forest problems in his recent list "Million Buck Problems" of unsolved problems of high potential impact on mathematics. He justified the selection of these problems by mentioning that the techniques involved in their resolution will be worth at least one million dollars to mathematics.

Now, let's consider the case studied by Finch and Wetzel and take it a little more rigorously. For that, let $\gamma$ be a *path* in $\mathbb{R}^2$, i.e., a continuous and rectifiable mapping of $[0, 1]$ into $\mathbb{R}^2$. Let $\ell_{B_1^2}(\gamma)$ be its Euclidean length and $\{\gamma\}$ its trace $\gamma([0, 1])$. We call a *forest* a closed, convex region in the plane with nonempty interior. A path $\gamma$ is an *escape path* for a forest $K$ if a congruent copy of it meets the boundary $\partial K$ no matter how it is placed with its initial point in $K$, i.e., for each point $P \in K$ and each Euclidean motion (translation, rotation, reflection and combinations of them) $\mu$ for which $P = \mu(\gamma(0))$ the intersection $\mu(\{\gamma\}) \cap \partial K$ is nonempty. Then, among all the escape paths for a forest $K$, there is at least one whose length is the shortest. The *escape length* $\alpha$ of a forest $K$ is the length of one of these shortest escape paths for $K$. Based on these notions, Finch and Wetzel proved the following:

**Theorem 1.10** (Theorem 3 in [17]) *Let $K \subset \mathbb{R}^2$ be a convex body. The escape length $\alpha^*$ of $K$ is the largest $\alpha$ for which for every path $\gamma$ with length $\leqslant \alpha$, there is a Euclidean motion $\mu$ such that $K$ covers $\mu(\{\gamma\})$.*

For Finch and Wetzel, this theorem established the connection to Moser's worm problem. For that, we recall that in Moser's worm problem one tries to find a/the convex set of least area that contains a congruent copy of each arc in the plane of a certain length. Clearly, the condition of having a certain length can be replaced by the condition of having a length which is bounded from above by that certain length.

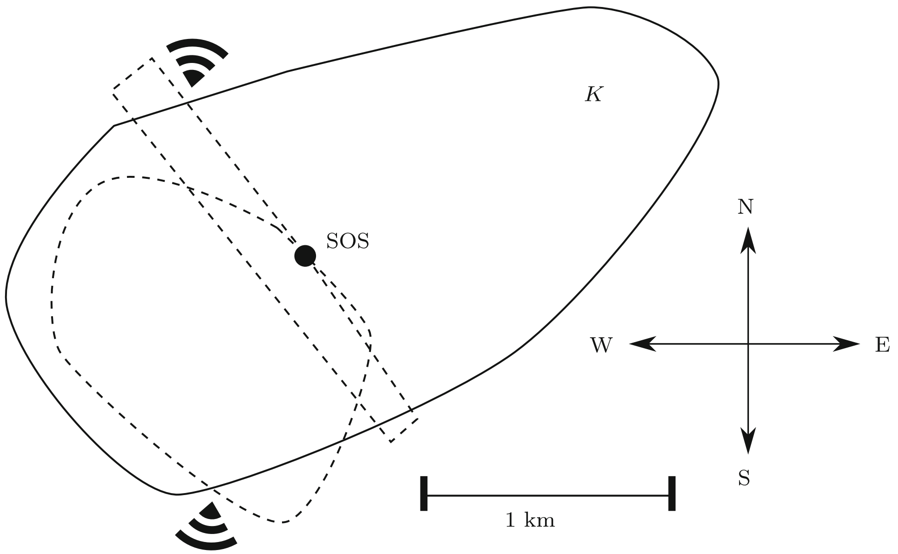


**Fig. 2** Visualization of the Minkowski escape problem for the special case of two dimensions with Euclidean measurement. This presents two possible Minkowski escape paths which, however, are not the length-minimizing one. For this $K$, the shortest Minkowski escape path is most likely a closed polygonal cruve with two vertices

Now, translated into our setting, we can derive a similar result. For that, we first have to define a version of a lost-in-a-forest problem which is compatible with the Minkowski worm problems discussed in the previous sections.

In order to indicate the connection to Minkowski worm problems in our setting, we will call the problem the *Minkowski escape problem*. We start by generalizing the problem to any dimension. So, we are considering higher dimensional "forests" which one aims to escape. We let $K \subset \mathbb{R}^n$ be a convex body, measure lengths by $\ell_T$, where $T \subset \mathbb{R}^n$ is a convex body, and we call $\gamma$ a *closed Minkowski escape path* for $K$ if $\gamma$ is a closed curve and for each point $P \in K$ and each translation $\mu$ for which $P = \mu(\gamma(0))$ the intersection $\mu(\{\gamma\}) \cap \partial K$ is nonempty. So, in contrast to considering not necessarily closed paths, allowing the motions to be Euclidean motions and measuring the lengths in the standard Euclidean sense in the escape problem of Finch and Wetzel, we only consider closed paths, translations and measure the lengths by the metric induced by the Minkowski functional with respect to the polar of $T$. Translating this problem into "our (mesocosmic) reality"—therefore, requiring $n = 2$ and Euclidean measurements, we get a slightly different problem (of course there are no limits to creativity) (see Fig. 2):

> *Two hikers walk in a forest. One of them gets injured and is in need of medical attention. The unharmed hiker would like to make the emergency call. Although he has his cell phone with him, there is only reception outside the forest. He has a map of the forest, i.e., the shape of the forest and its dimensions are known to him, and a compass to orient himself in terms of direction. Furthermore, he is able to measure the distance he has walked. However, he does not know exactly*

*where in the forest he is. What's the best way to get out of the forest, put off the emergency call, and then get back to the injured hiker?*

The fact that in our story the unharmed hiker knows the shape of the forest and has a compass to orient himself in terms of direction is due to the fact that in our Minkowski escape problem, translations are the only allowed motions. The condition of coming back to the injured hiker is a consequence of our demand to consider only closed curves.

We can prove the analogue to Theorem 1.10:

**Theorem 1.11** *Let $K, T \subset \mathbb{R}^n$ be convex bodies. Then, an/the $\ell_T$-minimizing closed Minkowski escape path for $K$ has $\ell_T$-length $\alpha^*$ if and only if $\alpha^*$ is the largest $\alpha$ for which*

$$K \in A(T, \alpha),$$

*i.e., for which for every closed path $\gamma$ of $\ell_T$-length $\leqslant \alpha$, there is a translation $\mu$ such that $K$ covers $\mu(\{\gamma\})$.*

Having in mind that Minkowski escape paths for a convex body $K \subset \mathbb{R}^n$ can be understood as closed curves which cannot be translated into the interior of $K$, we can use the Minkowski billiard characterization of shortest closed polygonal curves that cannot be translated into the interior of $K$, in order to directly conclude the following corollary. Note for this line of argumentation that shortest closed curves that cannot be translated into the interior of $K$ are in fact closed polygonal curves.

**Corollary 1.12** *Let $K, T \subset \mathbb{R}^n$ be convex bodies, where $T$ is additionally assumed to be strictly convex. An/The $\ell_T$-minimizing closed $(K, T)$-Minkowski billiard trajectory has $\ell_T$-length $\alpha^*$ if and only if $\alpha^*$ is the largest $\alpha$ for which*

$$K \in A(T, \alpha).$$

So, the unharmed hiker in our story can conceptualize his problem by searching for length-minimizing closed Euclidean billiard trajectories.

In general, the problem of minimizing over Minkowski escape problems in the sense of varying the forest while maintaining their volume in order to find the forest with minimal escape length becomes the problem of solving systolic Minkowski billiard inequalities, or equivalently, the problem of proving/investigating Viterbo's conjecture for Lagrangian products in $\mathbb{R}^n \times \mathbb{R}^n$.

This means: If the hikers want to play it safe from the outset by choosing, among forests of equal area, the one where the time needed to help an injured hiker is minimized, then it is useful for them to be familiar with symplectic geometry or billiard dynamics. Of course, they could have paid attention from the beginning to where they entered the forest from and how they designed their path. Then they do not have to solve too difficult problems.

This paper is organized as follows: In Sect. 2, we start with some relevant preliminaries before, in Sect. 3, we derive properties of Minkowski worm problems and the fundamental results in order to prove Theorems 1.1, 1.3, 1.4, 1.5, and 1.6 and Corollary 1.2 in Sects. 4, 5, and 6. In Sect. 7, we prove that it is justified to transfer a special case of Viterbo's conjecture into one for Wetzel's problem which becomes Conjecture 1.8. In Sect. 8, we prove Theorem 1.10 as analogue to the relationship between Moser's worm problem and Bellman's lost-in-a-forest problem. Finally, in Sect. 9, we discuss a computational approach for improving lower bounds in Minkowski worm problems, especially lower bounds for Wetzel's problem.

## 2 Preliminaries

We begin by collecting some results concerning the *Fenchel–Legendre transform* of a convex and continuous function $H:\mathbb{R}^n\to\mathbb{R}$, which for $x^*\in\mathbb{R}^n$ is defined by

$$H^*(x^*)=\sup_{x\in\mathbb{R}^n}(\langle x,x^*\rangle-H(x)).$$

**Proposition 2.1** (Proposition II.1.8 in [14]) *If $H^*$ is the Fenchel–Legendre transform of a convex and continuous function $H:\mathbb{R}^n\to\mathbb{R}$, then for $x\in\mathbb{R}^n$ we have*

$$H(x)=\sup_{x^*\in\mathbb{R}^n}(\langle x^*,x\rangle-H^*(x^*)).$$

The *subdifferential* of $H$ in $x\in\mathbb{R}^n$ is given by

$$\partial H(x)=\{x^*\in\mathbb{R}^n|H^*(x^*)=\langle x^*,x\rangle-H(x)\}$$

Then, we get the *Legendre recipocity formula*:

**Proposition 2.2** (Proposition II.1.15 in [14]) *For a convex and continuous function $H:\mathbb{R}^n\to\mathbb{R}$ the Legendre reciprocity formula is given by*

$$x^*\in\partial H(x)\;\Leftrightarrow\;H^*(x^*)+H(x)=\langle x^*,x\rangle\;\Leftrightarrow\;x\in\partial H^*(x^*),$$

*where $x,x^*\in\mathbb{R}^n$.*

We state the *generalized Euler identity*:

**Proposition 2.3** *Let $H:\mathbb{R}^n\to\mathbb{R}$ be a $p$-positively homogeneous, convex and continuous function of $\mathbb{R}^n$. Then, for each $x\in\mathbb{R}^n$ the following identity holds:*

$$\langle x^*,x\rangle=pH(x)\quad\forall x^*\in\partial H(x).$$

***Proof*** For each $x\in\mathbb{R}^n$, since $H$ is convex and continuous, we have

$$\partial H(x)\neq\emptyset.$$

For each

$$x^*\in\partial H(x)$$

Proposition 2.2 provides

$$H^*(x^*) + H(x) = \langle x^*, x\rangle \tag{11}$$

and from Proposition 2.1, i.e.,

$$H(x) = \sup_{x^* \in \mathbb{R}^n} (\langle x^*, x\rangle - H^*(x^*)),$$

we get

$$H(y) \geqslant \langle x^*, y\rangle - H^*(x^*) \quad \forall y \in \mathbb{R}^n. \tag{12}$$

Combining (11) and (12) we get

$$H(y) \geqslant \langle x^*, y - x\rangle + H(x) \quad \forall y \in \mathbb{R}^n. \tag{13}$$

Now, we set

$$y = \lambda x \quad (\lambda > 0)$$

and recognize to have equality in (13) for $\lambda \to 1$. Furthermore, we obtain by the $p$-homogeneity of $H$ for $\lambda \to 1$:

$$\lim_{\lambda \to 1} \frac{g(\lambda) - g(1)}{\lambda - 1} H(x) = \langle x^*, x\rangle,$$

where we introduced the function

$$g(x) := x^p.$$

Because of

$$g'(1) = p$$

we get

$$pH(x) = \langle x^*, x\rangle.$$

□

Noting that for convex body $C \subset \mathbb{R}^n$

$$H_C = \frac{1}{2}\mu_C^2$$

is a 2-positively homogeneous, convex and continuous function, we derive the following properties:

**Proposition 2.4** *For convex body $C \subset \mathbb{R}^n$ we have*

$$H_C^* = H_{C^\circ}.$$

***Proof*** For $\xi \in \mathbb{R}^n$ we have

$$\begin{aligned}\mu_{C^\circ}(\xi) &= \min\{t \geqslant 0 : \xi \in tC^\circ\} \\ &= \min\left\{t \geqslant 0 : \xi \in t\{\widehat{\xi} \in \mathbb{R}^n : \langle \widehat{\xi}, x\rangle \leqslant 1\, \forall x \in C\}\right\} \\ &= \min\left\{t \geqslant 0 : \xi \in \{\widehat{\xi} \in \mathbb{R}^n : \langle \widehat{\xi}, x\rangle \leqslant t\, \forall x \in C\}\right\} \\ &= \min\{t \geqslant 0 : \langle \xi, x\rangle \leqslant t\, \forall x \in C\} \\ &= \max_{x\in C} \langle \xi, x\rangle \\ &= \max_{\mu_C(x)=1} \langle \xi, x\rangle,\end{aligned}$$

and therefore

$$\begin{aligned}H_C^*(\xi) &= \sup_{x\in\mathbb{R}^n} \left(\langle \xi, x\rangle - H_C(x)\right) \\ &= \sup_{r\geqslant 0}\sup_{\mu_C(x)=1} \left(\langle \xi, rx\rangle - \frac{1}{2}\mu_C(rx)^2\right) \\ &= \sup_{r\geqslant 0}\left(r\left(\sup_{\mu_C(x)=1}\langle \xi, x\rangle\right) - \frac{r^2}{2}\right) \\ &= \max_{r\geqslant 0}\left(r\left(\max_{\mu_C(x)=1}\langle \xi, x\rangle\right) - \frac{r^2}{2}\right) \\ &= \max_{r\geqslant 0}\left(r\mu_{C^\circ}(\xi) - \frac{r^2}{2}\right) \\ &= \frac{\mu_{C^\circ}(\xi)^2}{2} \\ &= H_{C^\circ}(\xi).\end{aligned}$$

□

**Proposition 2.5** *Let $C \subset \mathbb{R}^n$ be a convex body. If $x^* \in \partial H_C(x)$ for $x \in \mathbb{R}^n$, then*

$$H_{C^\circ}(x^*) = H_C(x).$$

***Proof*** With Proposition 2.3 and the 2-homogeneity of $H_{C^\circ}$ we can write

$$2H_{C^\circ}(x^*) = \langle x', x^*\rangle,$$

where

$$x' \in \partial H_{C^\circ}(x^*),$$

which together with Propositions 2.2 and 2.4 and

$$(C^\circ)^\circ = C$$

is equivalent to
$$x^* \in \partial H^*_{C^\circ}(x') = \partial H_C(x').$$
Therefore, again using Proposition 2.3, we can conclude

$$2H_{C^\circ}(x^*) = \langle x', x^* \rangle = 2H_C(x').$$

In the following we show that

$$H_C(x') = H_C(x).$$

This would prove the claim.

Again, using Propositions 2.2 and 2.4, the fact

$$x^* \in \partial H_C(x)$$

is equivalent to
$$x \in \partial H^*_C(x^*) = \partial H_{C^\circ}(x^*).$$
All previous informations now can be summarized by the following two equations:

$$H_C(x) + H_{C^\circ}(x^*) = \langle x, x^* \rangle, \quad H_{C^\circ}(x^*) + H_C(x') = \langle x', x^* \rangle.$$

The difference yields
$$H_C(x) - H_C(x') = \langle x - x', x^* \rangle,$$
which implies

$$H_C(x') = H_C(x) - \langle x - x', x^* \rangle = H_C(x) - \langle x, x^* \rangle + \langle x', x^* \rangle.$$

The conditions
$$x \in \partial H_{C^\circ}(x^*) \;\text{ and }\; x' \in \partial H_{C^\circ}(x^*)$$
imply, applying Proposition 2.3,

$$-\langle x, x^* \rangle + \langle x', x^* \rangle = -2H_{C^\circ}(x^*) + 2H_{C^\circ}(x^*) = 0,$$

therefore
$$H_C(x') = H_C(x).$$

□

The following proposition is the generalization of [36, Proposition 3.11] from closed polygonal curves to closed curves:

**Proposition 2.6** *Let $T \subset \mathbb{R}^n$ be a convex body, $q \in cc(\mathbb{R}^n)$ and $\lambda > 0$. Then, we have*

$$\ell_T(\lambda q) = \ell_{\lambda T}(q) = \lambda \ell_T(q).$$

***Proof*** From
$$\mu_{T^\circ}(\lambda x)=\mu_{(\lambda T)^\circ}(x)=\lambda\mu_{T^\circ}(x),\quad x\in\mathbb{R}^n,$$
(see [36, Proposition 2.3(iii)]) we conclude

$$\ell_T(\lambda q)=\int_0^{\widetilde{T}}\mu_{(T)^\circ}((\dot{\lambda q})(t))\,\mathrm{d}t=\int_0^{\widetilde{T}}\mu_{(\lambda T)^\circ}(\dot{q}(t))\,\mathrm{d}t=\ell_{\lambda T}(q)$$

and
$$\ell_T(\lambda q)=\int_0^{\widetilde{T}}\mu_{(T)^\circ}((\dot{\lambda q})(t))\,\mathrm{d}t=\lambda\int_0^{\widetilde{T}}\mu_{(T)^\circ}((\dot{q})(t))\,\mathrm{d}t=\lambda\ell_T(q).$$

□

We continue by recalling [48, Theorem 1.1] which will be useful throughout this paper:

**Theorem 2.7** *Let $K,T\subset\mathbb{R}^n$ be convex bodies. Then, we have*

$$c_{EHZ}(K\times T)=\min_{q\in F^{cp}(K)}\ell_T(q)=\min_{p\in F^{cp}(T)}\ell_K(p)=\min_{q\in M_{n+1}(K,T)}\ell_T(q).$$

We note that in [48, Theorem 1.1] actually appear $F_{n+1}^{cp}(K)$ and $F_{n+1}^{cp}(T)$ instead of $F^{cp}(K)$ and $F^{cp}(T)$, respectively. However, for the purposes within this paper, we only need this more general formulation which is valid since there are no $\ell_T/\ell_K$-minimizing closed polygonal curves in $F^{cp}(K)/F^{cp}(T)$ with more than $n+1$ vertices and shorter $\ell_T/\ell_K$-length than the $\ell_T/\ell_K$-minimizing closed polygonal curves in $F_{n+1}^{cp}(K)/F_{n+1}^{cp}(T)$ (see the proof of point (1) in the proof of [48, Theorem 2.2]).

We collect some invariance properties of Viterbo's as well as of Mahler's conjecture:

**Proposition 2.8** *Viterbo's conjecture is invariant under translations.*

***Proof*** Translations
$$t_a:\mathbb{R}^n\to\mathbb{R}^n,\quad \xi\mapsto\xi+a,\ a\in\mathbb{R}^n,$$
are symplectomorphism because of

$$\mathrm{d}t_a(\xi)=\mathbb{1}$$

and therefore
$$\mathrm{d}t_a(\xi)^T J\,\mathrm{d}t_a(\xi)=J.$$

Finally, we recall that Viterbo's conjecture is invariant under symplectomorphisms, since symplectomorphisms in the above convex setting preserve the volume as well as the action and therefore the EHZ-capacity. □

**Proposition 2.9** *Let $C\subset\mathbb{R}^{2n}$ and $K,T\subset\mathbb{R}^n$ be convex bodies. Then*

$$\operatorname{vol}(C)\geqslant\frac{c_{EHZ}(C)^n}{n!}\ \Leftrightarrow\ \operatorname{vol}(\lambda C)\geqslant\frac{c_{EHZ}(\lambda C)^n}{n!}$$

*for* $\lambda > 0$, *and*

$$\operatorname{vol}(K \times T) \geqslant \frac{c_{EHZ}(K \times T)^n}{n!} \Leftrightarrow \operatorname{vol}(\lambda K \times \mu T) \geqslant \frac{c_{EHZ}(\lambda K \times \mu T)^n}{n!}$$

*for* $\lambda, \mu > 0$. *If*

$$\Phi : \mathbb{R}^n \to \mathbb{R}^n$$

*is an invertible linear transformation, then*

$$\begin{aligned} &\operatorname{vol}(K \times T) \geqslant \frac{c_{EHZ}(K \times T)^n}{n!} \\ \Leftrightarrow\ &\operatorname{vol}\left(\Phi(K) \times \left(\Phi^T\right)^{-1}(T)\right) \geqslant \frac{c_{EHZ}\left(\Phi(K) \times \left(\Phi^T\right)^{-1}(T)\right)^n}{n!}. \end{aligned}$$

***Proof*** We have

$$\operatorname{vol}(\lambda C) = \lambda^{2n} \operatorname{vol}(C)$$

and

$$c_{EHZ}(\lambda C) = \lambda^2 c_{EHZ}(C)$$

due to the 2-homogeneity of the action. Further,

$$\operatorname{vol}(\lambda K \times \mu T) = \operatorname{vol}(\lambda K) \operatorname{vol}(\mu K) = \lambda^n \mu^n \operatorname{vol}(K) \operatorname{vol}(T) = \lambda^n \mu^n \operatorname{vol}(K \times T)$$

and

$$c_{EHZ}(\lambda K \times \mu T) = \min_{q \in F^{cp}(\lambda K)} \ell_{\mu T}(q) = \lambda\mu \min_{q \in F^{cp}(K)} \ell_T(q)$$

due to Theorem 2.7 and [36, Proposition 3.11(ii) and (iv)] (see also Lemma 3.12).

Furthermore,

$$\Phi \times \left(\Phi^T\right)^{-1}$$

is a symplectomorphism, i.e.,

$$\left(\Phi \times \left(\Phi^T\right)^{-1}\right)^T J \left(\Phi \times \left(\Phi^T\right)^{-1}\right) = J.$$

Indeed, for $a, b \in \mathbb{R}^n$, we calculate

$$\begin{aligned} \left(\Phi \times \left(\Phi^T\right)^{-1}\right)^T J \left(\Phi \times \left(\Phi^T\right)^{-1}\right)(a, b) &= \left(\Phi \times \left(\Phi^T\right)^{-1}\right)^T J \left(\Phi(a), \left(\Phi^T\right)^{-1}(b)\right) \\ &= \left(\Phi \times \left(\Phi^T\right)^{-1}\right)^T \left(\left(\Phi^T\right)^{-1}(b), -\Phi(a)\right) \\ &= \left(\Phi^T \times \Phi^{-1}\right)\left(\left(\Phi^T\right)^{-1}(b), -\Phi(a)\right) \\ &= \left(\Phi^T\left(\left(\Phi^T\right)^{-1}(b)\right), \Phi^{-1}(-\Phi(a))\right) \end{aligned}$$

$$
\begin{aligned}
&= (b, -a) \\
&= J(a, b),
\end{aligned}
$$

where we used the facts

$$\left(\Phi^T\right)^T = \Phi \text{ and } \left(\Phi^T\right)^{-1} = \left(\Phi^{-1}\right)^T.$$

Finally, we recall that, in the above convex setting, every symplectomorphism preserves the volume as well as the action and therefore the EHZ-capacity.

□

**Proposition 2.10** *If $T \subset \mathbb{R}^n$ is a centrally symmetric convex body and*

$$\Phi : \mathbb{R}^n \to \mathbb{R}^n$$

*an invertible linear transformation, then*

$$\operatorname{vol}(T) \operatorname{vol}(T^\circ) \geqslant \frac{4^n}{n!} \;\Leftrightarrow\; \operatorname{vol}(\Phi(T)) \operatorname{vol}(\Phi(T)^\circ) \geqslant \frac{4^n}{n!}.$$

***Proof*** Because of

$$\Phi(T)^\circ = \left(\Phi^T\right)^{-1}\left(T^\circ\right)$$

and the volume preservation of

$$\Phi \times \left(\Phi^T\right)^{-1},$$

we have

$$
\begin{aligned}
\operatorname{vol}(\Phi(T)) \operatorname{vol}\left((\Phi(T))^\circ\right) &= \operatorname{vol}\left(\Phi(T) \times \Phi(T)^\circ\right) \\
&= \operatorname{vol}\left(\Phi(T) \times \left(\Phi^T\right)^{-1}(T^\circ)\right) \\
&= \operatorname{vol}\left(\left(\Phi \times \left(\Phi^T\right)^{-1}\right)(T \times T^\circ)\right) \\
&= \operatorname{vol}(T \times T^\circ) \\
&= \operatorname{vol}(T) \operatorname{vol}(T^\circ).
\end{aligned}
$$

□

## 3 Properties of Minkowski worm problems

We begin by concluding some basic properties of the set

$$A(T, \alpha), \quad T \in \mathcal{C}(\mathbb{R}^n),\ \alpha > 0.$$

We note that all of the following properties can be easily extended to the case $\alpha \geqslant 0$. Nevertheless, for the sake of simplicity and in order to avoid trivial case distinctions when it is not possible to divide by $\alpha$, for the following we just treat the case $\alpha > 0$.

**Proposition 3.1** *Let $T \subset \mathbb{R}^n$ be a convex body and $\alpha, \lambda, \mu > 0$. Then we have*

$$A(\lambda T, \mu\alpha) = \frac{\mu}{\lambda} A(T, \alpha).$$

***Proof*** We have

$$A(\lambda T, \mu\alpha) = \{K \in \mathcal{C}(\mathbb{R}^n) : L_{\lambda T}(\mu\alpha) \subseteq C(K)\}.$$

Because of

$$\ell_{\lambda T}(q) = \ell_T(\lambda q)$$

(see Proposition 2.6) we conclude

$$q \in L_{\lambda T}(\mu\alpha) \;\Leftrightarrow\; \lambda q \in L_T(\mu\alpha)$$

which together with

$$q \in C(K) \;\Leftrightarrow\; \lambda q \in C(\lambda K)$$

implies

$$\begin{aligned} A(\lambda T, \mu\alpha) &= \{K \in \mathcal{C}(\mathbb{R}^n) : q \in L_{\lambda T}(\mu\alpha) \Rightarrow q \in C(K)\} \\ &= \{K \in \mathcal{C}(\mathbb{R}^n) : \lambda q \in L_T(\mu\alpha) \Rightarrow \lambda q \in C(\lambda K)\} \\ &\underset{(q^*=\lambda q)}{\overset{(K^*=\lambda K)}{=}} \left\{\frac{1}{\lambda} K^* \in \mathcal{C}(\mathbb{R}^n) : q^* \in L_T(\mu\alpha) \Rightarrow q^* \in C(K^*)\right\} \\ &= \frac{1}{\lambda} A(T, \mu\alpha). \end{aligned}$$

Again referring to Proposition 2.6 we conclude

$$\ell_T(q) = \mu\alpha \;\Leftrightarrow\; \ell_T\left(\frac{q}{\mu}\right) = \alpha,$$

and therefore

$$q \in L_T(\mu\alpha) \;\Leftrightarrow\; \frac{q}{\mu} \in L_T(\alpha).$$

This implies

$$\begin{aligned} A(T, \mu\alpha) &= \{K \in \mathcal{C}(\mathbb{R}^n) : q \in L_T(\mu\alpha) \Rightarrow q \in C(K)\} \\ &= \left\{K \in \mathcal{C}(\mathbb{R}^n) : \frac{q}{\mu} \in L_T(\alpha) \Rightarrow \frac{q}{\mu} \in C\left(\frac{K}{\mu}\right)\right\} \\ &\underset{(q^*=\frac{q}{\mu})}{\overset{(K^*=\frac{K}{\mu})}{=}} \{\mu K^* \in \mathcal{C}(\mathbb{R}^n) : q^* \in L_T(\alpha) \Rightarrow q^* \in C(K^*)\} \end{aligned}$$

$$=\mu A(T, \alpha).$$

$\square$

**Proposition 3.2** *Let $T \subset \mathbb{R}^n$ be a convex body and $\alpha_1, \alpha_2 > 0$. Then, we have*

$$\alpha_1 \begin{Bmatrix} \leqslant \\ < \\ = \end{Bmatrix} \alpha_2 \;\Rightarrow\; A(T, \alpha_1) \begin{Bmatrix} \supseteq \\ \supsetneq \\ = \end{Bmatrix} A(T, \alpha_2).$$

***Proof*** We find $\mu > 0$ such that

$$\mu\alpha_1 = \alpha_2.$$

Then, using Proposition 3.1 we have

$$A(T, \alpha_2) = A(T, \mu\alpha_1) = \mu A(T, \alpha_1). \tag{14}$$

This implies

$$\alpha_1 \begin{Bmatrix} \leqslant \\ < \end{Bmatrix} \alpha_2 \;\Leftrightarrow\; \mu \begin{Bmatrix} \geqslant \\ > \end{Bmatrix} 1 \;\Leftrightarrow\; A(T, \alpha_1) \begin{Bmatrix} \supseteq \\ \supsetneq \\ = \end{Bmatrix} A(T, \alpha_2),$$

where the last equivalence follows from the following considerations: If we have (14) with $\mu \geqslant 1$, then

$$K \in A(T, \alpha_2) = \mu A(T, \alpha_1)$$

means that

$$\frac{1}{\mu} K \in A(T, \alpha_1),$$

i.e.,

$$L_T(\alpha_1) \subseteq C\left(\frac{1}{\mu} K\right) \subseteq C(K).$$

This implies

$$K \in A(T, \alpha_1)$$

and therefore

$$A(T, \alpha_2) \subseteq A(T, \alpha_1).$$

The case $\mu > 1$ follows by considering that in this case there can be find a convex body $K^*$ with

$$K^* \in A(T, \alpha_1) \setminus A(T, \alpha_2).$$

Indeed, for

$$K \in A(T, \alpha_2)$$

we define

$$\widehat{K} := \widehat{\lambda} K, \quad \widehat{\lambda} := \min\{0 < \lambda \leqslant 1 : \lambda K \in A(T, \alpha_2)\}.$$

Then, one has

$$\widehat{K} \in A(T, \alpha_2)$$

and with (14)

$$\frac{1}{\mu}\widehat{K} \in A(T, \alpha_1).$$

with

$$K^* := \frac{1}{\mu}\widehat{K}$$

it follows

$$K^* \in A(T, \alpha_1) \setminus A(T, \alpha_2)$$

by the definition of $\widehat{K}$. □

For convex body $T \subset \mathbb{R}^n$ and $\alpha > 0$ we define the set

$$A^{\leqslant}(T, \alpha) = \left\{K \in \mathcal{C}(\mathbb{R}^n) : L_T^{\leqslant}(\alpha) \subseteq C(K)\right\},$$

where

$$L_T^{\leqslant}(\alpha) = \left\{q \in cc(\mathbb{R}^n) : 0 < \ell_T(q) \leqslant \alpha\right\} = \bigcup_{0<\widetilde{\alpha}\leqslant\alpha} L_T\left(\widetilde{\alpha}\right).$$

Then, we have the following identity:

**Proposition 3.3** *Let $T \subset \mathbb{R}^n$ be a convex body and $\alpha > 0$. Then, we have*

$$A(T, \alpha) = A^{\leqslant}(T, \alpha).$$

***Proof*** By definition it is clear that

$$A^{\leqslant}(T, \alpha) \subseteq A(T, \alpha).$$

Indeed, if

$$K \in A^{\leqslant}(T, \alpha),$$

then this means

$$L_T(\widetilde{\alpha}) \subseteq C(K), \quad \text{for all } 0 < \widetilde{\alpha} \leqslant \alpha.$$

For $\widetilde{\alpha} = \alpha$ it follows

$$L_T(\alpha) \subseteq C(K)$$

and therefore

$$K \in A(T, \alpha).$$

Let $0 < \widetilde{\alpha} \leqslant \alpha$. Then, it follows from Proposition 3.2 that

$$A(T, \alpha) \subseteq A\left(T, \widetilde{\alpha}\right), \quad \text{for all } 0 < \widetilde{\alpha} \leqslant \alpha.$$

This implies

$$A(T,\alpha) \subseteq \bigcap_{0<\widetilde{\alpha}\leqslant\alpha} A\left(T,\widetilde{\alpha}\right) = A^{\leqslant}(T,\alpha).$$

□

**Proposition 3.4** *Let $\alpha > 0$ and $T_1, T_2 \subset \mathbb{R}^n$ be two convex bodies. Then, we have*

$$T_1 \subseteq T_2 \;\Rightarrow\; A(T_1,\alpha) \subseteq A(T_2,\alpha).$$

***Proof*** Let

$$T_1 \subseteq T_2.$$

If

$$K \in A(T_1,\alpha),$$

then it follows from Proposition 3.3 that

$$L^{\leqslant}_{T_1}(\alpha) \subseteq C(K).$$

Because of

$$\ell_{T_1}(q) \leqslant \ell_{T_2}(q) \;\text{ for all }\; q \in cc(\mathbb{R}^n),$$

as consequence of

$$\mu_{T_1^\circ}(x) \leqslant \mu_{T_2^\circ}(x) \quad \forall x \in \mathbb{R}^n,$$

it follows that

$$L^{\leqslant}_{T_2}(\alpha) \subseteq L^{\leqslant}_{T_1}(\alpha)$$

and therefore

$$L^{\leqslant}_{T_2}(\alpha) \subseteq C(K).$$

With Proposition 3.3 this implies

$$K \in A\left(T_2,\alpha\right).$$

Consequently, it follows

$$A(T_1,\alpha) \subseteq A(T_2,\alpha).$$

□

**Lemma 3.5** *Let $T \subset \mathbb{R}^n$ be a convex body and $\alpha > 0$. Further, let $K_1, K_2 \subset \mathbb{R}^n$ be two convex bodies with*

$$K_1 \subseteq K_2.$$

*Then it holds:*

$$K_1 \in A(T,\alpha) \;\Rightarrow\; K_2 \in A(T,\alpha).$$

***Proof*** Let

$$K_1 \in A(T,\alpha),$$

i.e.,

$$L_T(\alpha) \subseteq C(K_1).$$

It obviously holds

$$K_1 \subseteq K_2 \;\Rightarrow\; C(K_1) \subseteq C(K_2).$$

Therefore

$$L_T(\alpha) \subseteq C(K_2),$$

i.e.,

$$K_2 \in A(T, \alpha).$$

□

For the next lemma we recall the following: If $(M, d)$ is a metric space and $P(M)$ the set of all nonempty compact subsets of $M$, then $(P(M), d_H)$ is a metric space, where by $d_H$ we denote the *Hausdorff metric* $d_H$ which for nonempty compact subsets $X, Y$ of $(M, d)$ is defined by

$$d_H(X, Y) = \max\left\{\sup_{x\in X}\inf_{y\in Y} d(x, y), \sup_{y\in Y}\inf_{x\in X} d(x, y)\right\}.$$

Then, $(cc(\mathbb{R}^n), d_H)$ is a metric subspace of the complete metric space $(P(\mathbb{R}^n), d_H)$ which is induced by the Euclidean space $(\mathbb{R}^n, |\cdot|)$. For convex body $K \subset \mathbb{R}^n$ we consider

$$(F^{cc}(K), d_H) \;\text{ and }\; (C(K), d_H)$$

as metric subspaces of $(cc(\mathbb{R}^n), d_H)$. We have that

$$F^{cc}(K) \setminus C(K) \;\text{ and }\; C(K)$$

are complements of each other in $cc(\mathbb{R}^n)$.

**Lemma 3.6** *Let $K \subset \mathbb{R}^n$ be a convex body. Then, $(C(K), d_H)$ is a closed metric subspace of $(cc(\mathbb{R}^n), d_H)$.*

***Proof*** Since $C(K)$ is a subset of $cc(\mathbb{R}^n)$, $(C(K), d_H)$ is a metric subspace of the metric space $(cc(\mathbb{R}^n), d_H)$. It remains to show that $C(K)$ is a closed subset of $cc(\mathbb{R}^n)$. For that let $(q_j)_{j\in\mathbb{N}}$ be a sequence of closed curves in $C(K)$ $d_H$-converging to the closed curve $q^*$. If

$$q^* \notin C(K),$$

then $q^*$ cannot be translated into $K$. Using the closedness of $K$ in $\mathbb{R}^n$ this means

$$\min_{k\in\mathbb{R}^n} d_H\left(\partial\text{conv}\{K + k, q^*\}, \partial(K + k)\right) =: m > 0.$$

Then, we can find a $j_0 \in \mathbb{N}$ such that

$$d_H(p_j, q^*) < m$$

for all $j \geqslant j_0$. But this implies

$$\min_{k\in\mathbb{R}^n} d_H\left(\partial\text{conv}\{K+k, q_j\}, \partial(K+k)\right) > 0$$

for all $j \geqslant j_0$, i.e., $p_j$ cannot be translated into $K$ for all $j \geqslant j_0$, a contradiction to

$$q_j \in C(K) \quad \forall j \in \mathbb{N}.$$

Therefore, it follows

$$q^* \in C(K),$$

and consequently, $(C(K), d_H)$ is a closed metric subspace of $(cc(\mathbb{R}^n), d_H)$. □

**Lemma 3.7** *Let $T \subset \mathbb{R}^n$ be a convex body and $(\alpha_k)_{k\in\mathbb{N}}$ an increasing sequence of positive real numbers converging to $\alpha > 0$ for $k \to \infty$. If*

$$K \in A(T, \alpha_k) \quad \forall k \in \mathbb{N},$$

*then also*

$$K \in A(T, \alpha).$$

***Proof*** Let

$$K \in A(T, \alpha_k) \quad \forall k \in \mathbb{N},$$

i.e.,

$$L_T(\alpha_k) \subseteq C(K) \quad \forall k \in \mathbb{N}. \tag{15}$$

This means for all $k \in \mathbb{N}$ that for all

$$q \in cc(\mathbb{R}^n) \ \text{ with } \ \ell_T(q) = \alpha_k$$

holds

$$q \in C(K).$$

Let us assume that

$$K \notin A(T, \alpha),$$

i.e.,

$$L_T(\alpha) \nsubseteq C(K).$$

This means that there is a $q^* \in cc(\mathbb{R}^n)$ with

$$\ell_T(q^*) = \alpha \ \text{ and } \ q^* \in F^{cc}(K) \setminus C(K).$$

Due to Lemma 3.6 $(C(K), d_H)$ is a closed metric subspace of $(cc(\mathbb{R}^n), d_H)$. Since $F^{cc}(K) \setminus C(K)$ is the complement of $C(K)$ in $cc(\mathbb{R}^n)$ it follows that $F^{cc}(K) \setminus C(K)$ is an open subset of the metric space $(cc(\mathbb{R}^n), d_H)$. Consequently there is a $k_0 \in \mathbb{N}$ sufficiently big such that

$$q^* \frac{\alpha_k}{\alpha} \in F^{cc}(K) \setminus C(K) \quad \forall k \geqslant k_0. \tag{16}$$

But with [36, Proposition 3.11(iv)] it is

$$\ell_T\left(q^*\frac{\alpha_k}{\alpha}\right) = \ell_T\left(q^*\right)\frac{\alpha_k}{\alpha} = \alpha_k \quad \forall k \in \mathbb{N},$$

i.e.,

$$q^*\frac{\alpha_k}{\alpha} \in L_T(\alpha_k) \quad \forall k \in \mathbb{N},$$

which produces a contradiction between (15) and (16).

Therefore it follows

$$K \in A(T, \alpha).$$

□

The next proposition justifies to write "min" instead of "inf" within the Minkowski worm problem. The main ingredient of its proof will be Blaschke's selection theorem (see [10, Sect. 18]).

**Theorem 3.8** (Blaschke selection theorem) *Let $(C_k)_{k\in\mathbb{N}}$ be a sequence of convex bodies in $\mathbb{R}^n$ satisfying*

$$C_k \subset B_R^n, \quad R > 0,$$

*for all $k \in \mathbb{N}$. Then there is a subsequence $(C_{k_l})_{l\in\mathbb{N}}$ and a convex body $C$ in $\mathbb{R}^n$ such that $C_{k_l}$ $d_H$-converges to $C$ for $l \to \infty$.*

**Proposition 3.9** *Let $T$ be a convex body and $\alpha > 0$. Then we have*

$$\inf_{K\in A(T,\alpha)} \mathrm{vol}(K) = \min_{K\in A(T,\alpha)} \mathrm{vol}(K).$$

***Proof*** Let $(K_k)_{k\in\mathbb{N}}$ be a minimizing sequence of

$$\inf_{K\in A(T,\alpha)} \mathrm{vol}(K). \tag{17}$$

Then, there is a $k_0 \in \mathbb{N}$ and a sufficiently big $R > 0$ such that

$$K_k \subset B_R^n \quad \forall k \geqslant k_0.$$

Indeed, if this is not the case, then there is a subsequence $\left(K_{k_j}\right)_{j\in\mathbb{N}}$ such that

$$R_j := \max\{R > 0 : K_{k_j} \in F(B_R^n)\} \to \infty \quad (j \to \infty). \tag{18}$$

Guaranteeing

$$K_{k_j} \in A(T, \alpha) = \{K \in \mathcal{C}(\mathbb{R}^n) : L_T(\alpha) \subseteq C(K)\} \quad \forall j \in \mathbb{N}$$

means that

$$V_j := \mathrm{vol}\left(K_{k_j}\right) \to \infty \quad (j \to \infty). \tag{19}$$

The latter follows together with (18) and the convexity of $K_{k_j}$ for all $j \in \mathbb{N}$ from the fact that due to

$$L_T(\alpha) \subseteq C\left(K_{k_j}\right) \quad \forall j \in \mathbb{N}$$

there is no direction in which $K_{k_j}$ can be shrunk. But (19) is not possible since $\left(K_{k_j}\right)_{j \in \mathbb{N}}$ is a minimizing sequence of (17).

Applying Theorem 3.8, there is a subsequence $(K_{k_l})_{l \in \mathbb{N}}$ and a convex body $K \subset \mathbb{R}^n$ such that $K_{k_l}$ $d_H$-converges to $K$ for $l \to \infty$. It remains to show that

$$K \in A(T, \alpha).$$

The fact that $K_{k_l}$ $d_H$-converges to $K$ for $l \to \infty$ implies that for every $\varepsilon > 0$ there is an $l_0 \in \mathbb{N}$ such that

$$(1 - \varepsilon)K \subseteq K_{k_l} \subseteq (1 + \varepsilon)K \quad \forall l \geqslant l_0. \tag{20}$$

Then, with

$$K_{k_l} \in A(T, \alpha) \quad \forall l \in \mathbb{N}$$

it follows from the second inclusion in (20) together with Lemma 3.5 that

$$(1 + \varepsilon)K \in A(T, \alpha).$$

Applying Proposition 3.1 this means

$$K \in A\left(T, \frac{\alpha}{1 + \varepsilon}\right).$$

We define the sequence

$$\alpha_k := \frac{\alpha}{1 + \frac{1}{k}} \quad \forall k \in \mathbb{N}.$$

Then, $(\alpha_k)_{k \in \mathbb{N}}$ is an increasing sequence of positive numbers converging to $\alpha$ for $k \to \infty$ and together with the aboved mentioned ($\varepsilon > 0$ can be chosen arbitrarily) we have

$$K \in A\left(T, \alpha_k\right) \quad \forall k \in \mathbb{N}.$$

Applying Lemma 3.7 it follows

$$K \in A(T, \alpha).$$

$\square$

**Proposition 3.10** *Let $T \subset \mathbb{R}^n$ be a convex body and $\alpha, \lambda, \mu > 0$. Then we have*

$$\min_{K \in A(\lambda T, \mu\alpha)} \operatorname{vol}(K) = \frac{\mu^n}{\lambda^n} \min_{K \in A(T, \alpha)} \operatorname{vol}(K).$$

***Proof*** From

$$A(\lambda T, \mu\alpha) = \frac{1}{\lambda} A(T, \mu\alpha)$$

(see Proposition 3.1) it follows

$$K \in A(\lambda T, \mu\alpha) \;\Leftrightarrow\; \lambda K \in A(T, \mu\alpha)$$

and therefore

$$\min_{K \in A(\lambda T, \mu\alpha)} \mathrm{vol}(K) \overset{(K^* = \lambda K)}{=} \min_{K^* \in A(T, \mu\alpha)} \mathrm{vol}\left(\frac{K^*}{\lambda}\right) = \frac{1}{\lambda^n} \min_{K^* \in A(T, \mu\alpha)} \mathrm{vol}(K^*).$$

From

$$A(T, \mu\alpha) = \mu A(T, \alpha)$$

(see Proposition 3.1) it follows

$$K \in A(T, \mu\alpha) \;\Leftrightarrow\; \frac{K}{\mu} \in A(T, \alpha)$$

and therefore

$$\min_{K \in A(T, \mu\alpha)} \mathrm{vol}(K) \overset{(K^* = \frac{K}{\mu})}{=} \min_{K^* \in A(T, \alpha)} \mathrm{vol}(\mu K^*) = \mu^n \min_{K^* \in A(T, \alpha)} \mathrm{vol}(K^*).$$

□

**Proposition 3.11** *Let $T \subset \mathbb{R}^n$ be a convex body and $\alpha_1, \alpha_2 > 0$. Then, we have*

$$\alpha_1 \begin{Bmatrix} \leqslant \\ < \\ = \end{Bmatrix} \alpha_2 \;\Leftrightarrow\; \min_{K \in A(T, \alpha_1)} \mathrm{vol}(K) \begin{Bmatrix} \leqslant \\ < \\ = \end{Bmatrix} \min_{K \in A(T, \alpha_2)} \mathrm{vol}(K)$$

***Proof*** We find $\mu > 0$ such that

$$\mu\alpha_1 = \alpha_2.$$

Then, we apply Proposition 3.10. □

Now, for convex bodies $K, T \subset \mathbb{R}^n$ we will turn our attention to the minimization problem

$$\min_{q \in F^{cp}(K)} \ell_T(q).$$

The existence of the minimum is guaranteed by Theorem 2.7.

**Lemma 3.12** *Let $K, T \subset \mathbb{R}^n$ be convex bodies and $\lambda > 0$. Then*

$$\min_{q \in F^{cp}(\lambda K)} \ell_T(q) = \min_{q \in F^{cp}(K)} \ell_T(\lambda q) = \lambda \min_{q \in F^{cp}(K)} \ell_T(q).$$

***Proof*** Similar to [36, Proposition 3.11(ii)] we have

$$q \in F^{cp}(\lambda K) \ \Leftrightarrow\ \frac{q}{\lambda} \in F^{cp}(K)$$

and using [36, Proposition 3.11(iv)] therefore

$$\min_{q \in F^{cp}(\lambda K)} \ell_T(q) = \min_{\frac{q}{\lambda} \in F^{cp}(K)} \ell_T(q) \overset{(q^*=\frac{q}{\lambda})}{=} \min_{q^* \in F^{cp}(K)} \ell_T(\lambda q^*) = \lambda \min_{q^* \in F^{cp}(K)} \ell_T(q^*).$$

□

In the following for convex body $T \subset \mathbb{R}^n$ and $c > 0$ we consider the minimax problem[9]

$$\max_{\mathrm{vol}(K)=c} \min_{q \in F^{cp}(K)} \ell_T(q).$$

The following proposition guarantees the existence of its maximum:

**Proposition 3.13** *Let $T \subset \mathbb{R}^n$ be a convex body and $c > 0$. Then, we have*

$$\sup_{\mathrm{vol}(K)=c} \min_{q \in F^{cp}(K)} \ell_T(q) = \max_{\mathrm{vol}(K)=c} \min_{q \in F^{cp}(K)} \ell_T(q).$$

***Proof*** Let $(K_k)_{k \in \mathbb{N}}$ be a maximizing sequence of

$$\sup_{\mathrm{vol}(K)=c} \min_{q \in F^{cp}(K)} \ell_T(q). \tag{21}$$

Then, there is a $k_0 \in \mathbb{N}$ and an $R > 0$ such that

$$K_k \subset B_R^n \quad \forall k \geqslant k_0. \tag{22}$$

Indeed, if this is not the case, then there is a subsequence $\left(K_{k_j}\right)_{j \in \mathbb{N}}$ such that

$$R_j := \max\{R > 0 : K_{k_j} \in F(B_R^n)\} \to \infty \quad (j \to \infty). \tag{23}$$

But this implies

$$L_j := \min\left\{\ell_T(q) : q \in F^{cp}(K_{k_j})\right\} \to 0 \quad (j \to \infty). \tag{24}$$

This follows from the fact that for every $j \in \mathbb{N}$ we can find a

$$q_j \in F^{cp}(K_{k_j})$$

[9] Whenever we write

$$\max_{\mathrm{vol}(K)=c} \min_{q \in F^{cp}(K)} \ell_T(q)$$

the maximum is understood to consider only convex bodies $K \subset \mathbb{R}^n$. This is implicitly indicated by the fact that we defined $F^{cp}(K)$ only for convex bodies $K \subset \mathbb{R}^n$.

with

$$\ell_T(q_j) \to 0 \quad (j \to \infty).$$

The latter is a consequence of (23) and the constraint

$$\mathrm{vol}\left(K_{k_j}\right) = c \quad \forall j \in \mathbb{N}, \tag{25}$$

i.e., due to the convexity of $K_{k_j}$ for all $j \in \mathbb{N}$ guaranteeing (25) there are directions from the origin in which $K_{k_j}$ has to shrink for $j \to \infty$ and which are suitable in order to construct convenient $q_j$. But (24) is not possible since $\left(K_{k_j}\right)_{j\in\mathbb{N}}$ is a maximizing sequence of (21).

Then, we apply Theorem 3.8 and find a subsequence $(K_{k_l})_{l\in\mathbb{N}}$ and a convex body $K \subset \mathbb{R}^n$ such that $K_{k_l}$ $d_H$-converges to $K$ for $l \to \infty$. It remains to prove that

$$\mathrm{vol}(K) = c,$$

but this is an immediate consequence of the $d_H$-continuity of the volume function. □

**Proposition 3.14** *Let $T \subset \mathbb{R}^n$ be a convex body. Then,*

$$\max_{\mathrm{vol}(K)=c} \min_{q\in F^{cp}(K)} \ell_T(q)$$

*increases/decreases strictly if and only if this is the case for $c > 0$.*

***Proof*** We make use of the implication

$$\max_{\mathrm{vol}(K)=c_1} \min_{q\in F^{cp}(K)} \ell_T(q) = \max_{\mathrm{vol}(K)=c_2} \min_{q\in F^{cp}(K)} \ell_T(q) \;\Rightarrow\; c_1 = c_2 \tag{26}$$

for all $c_1, c_2 > 0$.

Let us verify (26): We assume

$$\max_{\mathrm{vol}(K)=c_1} \min_{q\in F^{cp}(K)} \ell_T(q) = \max_{\mathrm{vol}(K)=c_2} \min_{q\in F^{cp}(K)} \ell_T(q) \tag{27}$$

and without loss of generality $c_1 < c_2$. Let the pair

$$(K_1^*, q_1^*) \;\text{ with }\; \mathrm{vol}(K_1^*) = c_1 \;\text{ and }\; q_1^* \in F^{cp}(K_1^*)$$

be a maximizer of the left side in (27), i.e.,

$$\max_{\mathrm{vol}(K)=c_1} \min_{q\in F^{cp}(K)} \ell_T(q) = \min_{q\in F^{cp}(K_1^*)} \ell_T(q) = \ell_T(q_1^*).$$

With

$$q_1^* \in F^{cp}(K_1^*)$$

similar to [36, Proposition 3.11(ii)] we have

$$\sqrt[n]{\frac{c_2}{c_1}} q_1^* \in F^{cp}\left(\widetilde{K}\right)$$

for

$$\widetilde{K} := \sqrt[n]{\frac{c_2}{c_1}} K_1^*.$$

From

$$\min_{q \in F^{cp}(K_1^*)} \ell_T(q) = \ell_T(q_1^*)$$

it follows together with Lemma 3.12 that

$$\min_{q \in F^{cp}(\widetilde{K})} \ell_T(q) = \min_{q \in F^{cp}\left(\sqrt[n]{\frac{c_2}{c_1}} K_1^*\right)} \ell_T(q) = \sqrt[n]{\frac{c_2}{c_1}} \min_{q \in F^{cp}(K_1^*)} \ell_T(q) = \sqrt[n]{\frac{c_2}{c_1}} \ell_T(q_1^*).$$

Since

$$\mathrm{vol}\left(\widetilde{K}\right) = \mathrm{vol}\left(\sqrt[n]{\frac{c_2}{c_1}} K_1^*\right) = \frac{c_2}{c_1} \mathrm{vol}(K_1^*) = c_2,$$

we conclude

$$\begin{aligned} \max_{\mathrm{vol}(K)=c_1} \min_{q \in F^{cp}(K)} \ell_T(q) &= \min_{q \in F^{cp}(K_1^*)} \ell_T(q) = \ell_T(q_1^*) \\ &< \sqrt[n]{\frac{c_2}{c_1}} \ell_T(q_1^*) \\ &= \min_{q \in F^{cp}(\widetilde{K})} \ell_T(q) \\ &\leqslant \max_{\mathrm{vol}(K)=c_2} \min_{q \in F^{cp}(K)} \ell_T(q), \end{aligned}$$

which is a contradiction to (27). Therefore, noting that the assumption $c_1 > c_2$ would have led analogously to the same contradiction, it follows

$$c_1 = c_2.$$

We now prove the equivalence

$$\max_{\mathrm{vol}(K)=c_1} \min_{q \in F^{cp}(K)} \ell_T(q) < \max_{\mathrm{vol}(K)=c_2} \min_{q \in F^{cp}(K)} \ell_T(q) \Leftrightarrow c_1 < c_2 \tag{28}$$

for $c_1, c_2 > 0$.

If

$$\max_{\mathrm{vol}(K)=c_1} \min_{q \in F^{cp}(K)} \ell_T(q) < \max_{\mathrm{vol}(K)=c_2} \min_{q \in F^{cp}(K)} \ell_T(q), \tag{29}$$

then from the first part of the proof it necessarily follows $c_1 \neq c_2$. Let us assume $c_1 > c_2$. We further assume that the pair

$$(K_2^*, q_2^*) \text{ with } \operatorname{vol}(K_2^*) = c_2 \text{ and } q_2^* \in F^{cp}(K_2^*)$$

is a maximizer of the right side in (29), i.e.,

$$\max_{\operatorname{vol}(K)=c_2} \min_{q \in F^{cp}(K)} \ell_T(q) = \min_{q \in K_2^*} \ell_T(q) = \ell_T(q_2^*).$$

We define

$$\widehat{K} := \sqrt[n]{\frac{c_1}{c_2}} K_2^*.$$

From

$$\min_{q \in K_2^*} \ell_T(q) = \ell_T(q_2^*)$$

it follows together with Lemma 3.12 that

$$\min_{q \in F^{cp}(\widehat{K})} \ell_T(q) = \min_{q \in F^{cp}\left(\sqrt[n]{\frac{c_1}{c_2}} K_2^*\right)} \ell_T(q) = \sqrt[n]{\frac{c_1}{c_2}} \min_{q \in F^{cp}(K_2^*)} \ell_T(q) = \sqrt[n]{\frac{c_1}{c_2}} \ell_T(q_2^*).$$

Since

$$\operatorname{vol}\left(\widehat{K}\right) = \operatorname{vol}\left(\sqrt[n]{\frac{c_1}{c_2}} K_2^*\right) = \frac{c_1}{c_2} \operatorname{vol}(K_2^*) = c_1,$$

we conclude

$$\begin{aligned} \max_{\operatorname{vol}(K)=c_2} \min_{q \in F^{cp}(K)} \ell_T(q) = \min_{q \in F^{cp}(K_2^*)} \ell_T(q) &= \ell_T(q_2^*) \\ &< \sqrt[n]{\frac{c_1}{c_2}} \ell_T(q_2^*) \\ &= \min_{q \in F^{cp}(\widehat{K})} \ell_T(q) \\ &\leqslant \max_{\operatorname{vol}(K)=c_1} \min_{q \in F^{cp}(K)} \ell_T(q), \end{aligned}$$

which is a contradiction to (29). Therefore, we conclude $c_1 < c_2$.

Conversely, let $c_1 < c_2$. From (26) we conclude

$$\max_{\operatorname{vol}(K)=c_1} \min_{q \in F^{cp}(K)} \ell_T(q) \neq \max_{\operatorname{vol}(K)=c_2} \min_{q \in F^{cp}(K)} \ell_T(q). \tag{30}$$

If the strict inequality ">" holds in (30), then we conclude from the above proven implication "⇒" in (28) that $c_1 > c_2$, a contradiction. Therefore, it follows

$$\max_{\operatorname{vol}(K)=c_1} \min_{q \in F^{cp}(K)} \ell_T(q) < \max_{\operatorname{vol}(K)=c_2} \min_{q \in F^{cp}(K)} \ell_T(q).$$

□

**Proposition 3.15** *Let $K, T \subset \mathbb{R}^n$ be convex bodies with $q^*$ as minimizer of*

$$\min_{q \in F^{cp}(K)} \ell_T(q).$$

*Then, it follows*

$$K \in A(T, \ell_T(q^*)) = A\left(T, \min_{q \in F^{cp}(K)} \ell_T(q)\right).$$

***Proof*** Let $q^*$ be a minimizer of

$$\min_{q \in F^{cp}(K)} \ell_T(q).$$

Then, it follows

$$L_T(\ell_T(q^*)) \subseteq C(K).$$

Indeed, otherwise, if there is

$$\widetilde{q} \in L_T(\ell_T(q^*)) \setminus C(K),$$

i.e.,

$$\ell_T(\widetilde{q}) = \ell_T(q^*) \text{ and } \widetilde{q} \in F^{cc}(K) \setminus C(K),$$

then, due to the openess of

$$F^{cc}(K) \setminus C(K) \text{ in } cc(\mathbb{R}^n)$$

with respect to $d_H$ (see Lemma 3.6), there is a $\lambda < 1$ such that

$$\lambda\widetilde{q} \in F^{cc}(K) \setminus C(K).$$

Then, using [36, Proposition 3.11(iv)], we conclude

$$\ell_T(\lambda\widetilde{q}) = \lambda\ell_T(\widetilde{q}) < \ell_T(\widetilde{q}) = \ell_T(q^*) = \min_{q \in F^{cp}(K)} \ell_T(q).$$

Because of the $d_H$-density of $F^{cp}(K)$ in $F^{cc}(K)$ and the $d_H$-continuity of $\ell_T$ on $F^{cc}(K)$ (see [36, Proposition 3.11(v)]–which is also valid for closed curves) then we can find a

$$\widehat{q} \in F^{cp}(K)$$

with

$$\ell_T(\widehat{q}) < \min_{q \in F^{cp}(K)} \ell_T(q),$$

a contradiction.

Finally, from

$$L_T(\ell_T(q^*)) \subseteq C(K)$$

it follows
$$K \in A(T, \ell_T(q^*)) = A\left(T, \min_{q \in F^{cp}(K)} \ell_T(q)\right).$$

□

**Lemma 3.16** *Let $K \subset \mathbb{R}^n$ be a convex body and $\lambda > 1$. If*
$$q \in F^{cc}(K) \cap C(K), \tag{31}$$
*then it follows that*
$$\lambda q \in F^{cc}(K) \setminus C(K). \tag{32}$$

***Proof*** If we assume (31) but (32) does not hold. Then it follows
$$q, \lambda q \in C(K)$$
and due to $\lambda > 1$ therefore
$$q \in C\left(\mathring{K}\right).$$
But this is a contradiction to
$$q \in F^{cc}(K).$$
Therefore, it follows (32). □

**Proposition 3.17** *Let $T \subset \mathbb{R}^n$ be a convex body and $\alpha > 0$. If $K^*$ is a minimizer of*
$$\min_{K \in A(T,\alpha)} \mathrm{vol}(K), \tag{33}$$
*then*
$$\min_{q \in F^{cp}(K^*)} \ell_T(q) = \alpha.$$

***Proof*** If $q^*$ is a minimizer of
$$\min_{q \in F^{cp}(K^*)} \ell_T(q),$$
then it follows from Proposition 3.15 that
$$K^* \in A(T, \ell_T(q^*)).$$
This means
$$\min_{K \in A(T,\ell_T(q^*))} \mathrm{vol}(K) \leqslant \mathrm{vol}(K^*) = \min_{K \in A(T,\alpha)} \mathrm{vol}(K).$$
Proposition 3.11 implies
$$\ell_T(q^*) \leqslant \alpha.$$
If
$$\ell_T(q^*) < \alpha,$$

then with Proposition [36, Proposition 3.11(iv)] there is $\lambda > 1$ such that

$$\ell_T(\lambda q^*) = \alpha.$$

Together with

$$F^{cp}(K^*) \subseteq F^{cc}(K^*)$$

and Lemma 3.16 the fact

$$q^* \in F^{cp}(K^*)$$

implies

$$\lambda q^* \in F^{cc}(K^*) \setminus C(K^*),$$

therefore, there is no translate of $K^*$ that covers $\lambda q^*$. Consequently,

$$K^* \notin A(T, \ell_T(\lambda q^*)) = A(T, \alpha),$$

a contradiction to the fact that $K^*$ is a minimizer of (33). Therefore, it follows that

$$\min_{q \in F^{cp}(K^*)} \ell_T(q) = \ell_T(q^*) = \alpha.$$

$\square$

The idea which underlies the following theorem leads to the heart of this paper.

**Theorem 3.18** *Let $T \subset \mathbb{R}^n$ be a convex body. If $K^*$ is a minimizer of*

$$\min_{K \in A(T,\alpha)} \operatorname{vol}(K) \tag{34}$$

*for $\alpha > 0$, then $K^*$ is a maximizer of*

$$\max_{\operatorname{vol}(K)=c} \min_{q \in F^{cp}(K)} \ell_T(q) \tag{35}$$

*for*

$$c := \operatorname{vol}(K^*)$$

*with*

$$\min_{q \in F^{cp}(K^*)} \ell_T(q) = \alpha.$$

*Conversely, if $K^*$ is a maximizer of (35) for $c > 0$, then $K^*$ is a minimizer of (34) for*

$$\alpha := \min_{q \in F^{cp}(K^*)} \ell_T(q)$$

*and with*

$$\operatorname{vol}(K^*) = c.$$

*Consequently, for* $\alpha, c > 0$ *we have the equivalence*

$$\min_{K \in A(T,\alpha)} \operatorname{vol}(K) = c \;\Leftrightarrow\; \max_{\operatorname{vol}(K)=c} \; \min_{q \in F^{cp}(K)} \ell_T(q) = \alpha$$

*and moreover*

$$\min_{K \in A(T,\alpha)} \operatorname{vol}(K) \geqslant c \;\Leftrightarrow\; \max_{\operatorname{vol}(K)=c} \; \min_{q \in F^{cp}(K)} \ell_T(q) \leqslant \alpha. \tag{36}$$

***Proof*** Let $K^*$ be a minimizer of (34) for $\alpha > 0$. If $K^*$ is not a maximizer of (35) for

$$c = \operatorname{vol}(K^*),$$

then there is a convex body

$$K^{**} \subset \mathbb{R}^n \;\text{ with }\; \operatorname{vol}(K^{**}) = c$$

and a

$$q^{**} \in F^{cp}(K^{**})$$

such that

$$\ell_T(q^{**}) = \min_{q \in F^{cp}(K^{**})} \ell_T(q) > \min_{q \in F^{cp}(K^*)} \ell_T(q) = \ell_T(q^*), \tag{37}$$

where by $q^*$ we denote a minimizer of

$$\min_{q \in F^{cp}(K^*)} \ell_T(q).$$

From Proposition 3.15 it follows

$$K^{**} \in A(T, \ell_T(q^{**})), \tag{38}$$

and further from Proposostion 3.17 that

$$\min_{q \in F^{cp}(K^*)} \ell_T(q) = \ell_T(q^*) = \alpha. \tag{39}$$

From (37), (38) and (39) together with Proposition 3.11 we conclude

$$\begin{aligned} c = \operatorname{vol}(K^{**}) \overset{(38)}{\geqslant} \min_{K \in A(T,\ell_T(q^{**}))} \operatorname{vol}(K) &\overset{(37)}{>} \min_{K \in A(T,\ell_T(q^*))} \operatorname{vol}(K) \\ &\overset{(39)}{=} \min_{K \in A(T,\alpha)} \operatorname{vol}(K) \\ &= \operatorname{vol}(K^*) \\ &= c, \end{aligned}$$

a contradiction. Therefore, $K^*$ is a maximizer of (35) for

$$c = \mathrm{vol}(K^*).$$

Conversely, let $K^*$ be a maximizer of (35) for $c > 0$ with

$$q^* \in F^{cp}(K^*)$$

such that

$$\max_{\mathrm{vol}(K)=c} \min_{q \in F^{cp}(K)} \ell_T(q) = \min_{q \in F^{cp}(K^*)} \ell_T(q) = \ell_T(q^*) =: \alpha.$$

Then, from Proposition 3.15 it follows that

$$K^* \in A(T, \alpha),$$

and consequently

$$c = \mathrm{vol}(K^*) \geqslant \min_{K \in A(T,\alpha)} \mathrm{vol}(K).$$

If $K^*$ is not a minimizer of (34) for

$$\alpha = \ell_T(q^*),$$

then there is a

$$K^{**} \in A(T, \alpha)$$

with

$$c = \mathrm{vol}(K^*) > \min_{K \in A(T,\alpha)} \mathrm{vol}(K) = \mathrm{vol}(K^{**}). \tag{40}$$

Then, from Proposition 3.17 it follows that

$$\min_{q \in F^{cp}(K^{**})} \ell_T(q) = \alpha.$$

This implies

$$\begin{aligned}
\max_{\mathrm{vol}(K)=c} \min_{q \in F^{cp}(K)} \ell_T(q) = \min_{q \in F^{cp}(K^*)} \ell_T(q) &= \ell_T(q^*) \\
&= \alpha \\
&= \min_{q \in F^{cp}(K^{**})} \ell_T(q) \\
&\leqslant \max_{\mathrm{vol}(K)=\mathrm{vol}(K^{**})} \min_{q \in F^{cp}(K)} \ell_T(q),
\end{aligned}$$

which because of (40) is a contradiction to Proposition 3.14. We conclude that $K^*$ is a minimizer of (34) for

$$\alpha = \ell_T(q^*).$$

From the before proven it clearly follows the equivalence

$$\min_{K\in A(T,\alpha)} \operatorname{vol}(K)=c \;\Leftrightarrow\; \max_{\operatorname{vol}(K)=c}\; \min_{q\in F^{cp}(K)} \ell_T(q)=\alpha$$

for $\alpha, c>0$. In order to prove (36) it remains to show

$$\min_{K\in A(T,\alpha)} \operatorname{vol}(K)>c \;\Leftrightarrow\; \max_{\operatorname{vol}(K)=c}\; \min_{q\in F^{cp}(K)} \ell_T(q)<\alpha.$$

Let $K^*$ be a minimizer of

$$\min_{K\in A(T,\alpha)} \operatorname{vol}(K),$$

where $c>0$ is chosen such that

$$\operatorname{vol}(K^*)>c. \tag{41}$$

Then we know from the above reasoning that $K^*$ is a maximizer of

$$\max_{\operatorname{vol}(K)=\operatorname{vol}(K^*)}\; \min_{q\in F^{cp}(K)} \ell_T(q)$$

with

$$\min_{q\in F^{cp}(K^*)} \ell_T(q)=\alpha.$$

From (41) and Proposition 3.14 it follows

$$\max_{\operatorname{vol}(K)=c}\; \min_{q\in F^{cp}(K)} \ell_T(q) < \max_{\operatorname{vol}(K)=\operatorname{vol}(K^*)}\; \min_{q\in F^{cp}(K)} \ell_T(q)=\alpha.$$

Conversely, let $K^*$ be a maximizer of

$$\max_{\operatorname{vol}(K)=c}\; \min_{q\in F^{cp}(K)} \ell_T(q),$$

where $\alpha>0$ is chosen such that

$$\min_{q\in F^{cp}(K^*)} \ell_T(q)=:\widetilde{\alpha}<\alpha.$$

Then we know from the above reasoning that $K^*$ is a minimizer of

$$\min_{T\in A(T,\widetilde{\alpha})} \operatorname{vol}(K),$$

and from Proposition 3.11 it follows

$$\min_{K\in A(T,\alpha)} \operatorname{vol}(K) > \min_{K\in A(T,\widetilde{\alpha})} \operatorname{vol}(K)=\operatorname{vol}(K^*)=c.$$

$\square$

Hereinafter we will deal with the following two minimax problems[10]: For $\alpha, d > 0$ we will consider

$$\min_{\mathrm{vol}(T)=d} \min_{K \in A(T,\alpha)} \mathrm{vol}(K),$$

and for $c, d > 0$ we will consider

$$\max_{\mathrm{vol}(T)=d} \max_{\mathrm{vol}(K)=c} \min_{q \in F^{cp}(K)} \ell_T(q).$$

It is indeed justified to write "min" and "max" respectively:

**Proposition 3.19** *Let $\alpha, d > 0$. Then we have*

$$\inf_{\mathrm{vol}(T)=d} \min_{K \in A(T,\alpha)} \mathrm{vol}(K) = \min_{\mathrm{vol}(T)=d} \min_{K \in A(T,\alpha)} \mathrm{vol}(K).$$

***Proof*** Let $(T_k)_{k\in\mathbb{N}}$ be a minimizing sequence of

$$\inf_{\mathrm{vol}(T)=d} \min_{K \in A(T,\alpha)} \mathrm{vol}(K). \tag{42}$$

Then there is an $R > 0$ and a $k_0 \in \mathbb{N}$ such that

$$T_k \subset B_R^n \quad \forall k \geqslant k_0.$$

Indeed, if this is not the case, then there is a subsequence $\left(T_{k_j}\right)_{j\in\mathbb{N}}$ such that

$$R_j := \max\left\{R > 0 : T_{k_j} \in F\left(B_R^n\right)\right\} \to \infty \quad (j \to \infty). \tag{43}$$

This implies

$$\begin{aligned} V_j :&= \min\left\{\mathrm{vol}(K) : K \in A\left(T_{k_j}, \alpha\right)\right\} \\ &= \min\left\{\mathrm{vol}(K) : K \in \mathcal{C}(\mathbb{R}^n),\ L_{T_{k_j}}(\alpha) \subseteq C(K)\right\} \\ &\to \infty \quad (j \to \infty). \end{aligned}$$

The latter follows from the fact that—(43) together with the convexity of $T_{k_j}$ for all $j \in \mathbb{N}$ and the constraint

$$\mathrm{vol}\left(T_{k_j}\right) = d \quad \forall j \in \mathbb{N}$$

[10] Whenever we write

$$\min_{\mathrm{vol}(T)=d} \min_{K \in A(T,\alpha)} \mathrm{vol}(K)$$

or

$$\max_{\mathrm{vol}(T)=d} \max_{\mathrm{vol}(K)=c} \min_{q \in F^{cp}(K)} \ell_T(q)$$

the minimum/maximum is understood to consider only convex bodies $T \subset \mathbb{R}^n$. This is implicitly indicated by the fact that we defined $A(\cdot, \alpha)$ and $\ell.(q)$ only for convex bodies $T \subset \mathbb{R}^n$.

means that there are directions from the origin in which $T_{k_j}$ has to shrink for $j \to \infty$– for every $j \in \mathbb{N}$ we can find

$$q_j \in L_{T_{k_j}}(\alpha)$$

($q_j$ can be constructed by using the aforementioned directions) for which

$$\ell_{T_{k_j}}(q_j) = \alpha$$

means

$$\max_{t\in[0,\widetilde{T}_j]} |q_j(t)| \to \infty \quad (j \to \infty),$$

where by $\widetilde{T}_j$ we denote the period of the closed curve $q_j$, and for every convex body $K_j \subset \mathbb{R}^n$ minimizing

$$\min \left\{ \mathrm{vol}(K) : K \in \mathcal{C}(\mathbb{R}^n),\ L_{T_{k_j}}(\alpha) \subseteq C(K) \right\}$$

means

$$V_j = \mathrm{vol}(K_j) \to \infty \quad (j \to \infty).$$

But this is not possible since $(T_k)_{k\in\mathbb{N}}$ is a minimizing sequence of (42).

Then, we can apply Theorem 3.8: There is a subsequence $\left(T_{k_l}\right)_{l\in\mathbb{N}}$ and a convex body $T \subset \mathbb{R}^n$ such that $T_{k_l}$ $d_H$-converges to $T$ for $l \to \infty$. We clearly have

$$\mathrm{vol}(T) = \mathrm{vol}\left(\lim_{l\to\infty} T_{k_l}\right) = \lim_{l\to\infty} \mathrm{vol}\left(T_{k_l}\right) = d.$$

Therefore, $T$ is a minimizer of (42). □

**Proposition 3.20** *Let $c, d > 0$. Then we have*

$$\sup_{\mathrm{vol}(T)=d} \max_{\mathrm{vol}(K)=c} \min_{q\in F^{cp}(K)} \ell_T(q) = \max_{\mathrm{vol}(T)=d} \max_{\mathrm{vol}(K)=c} \min_{q\in F^{cp}(K)} \ell_T(q). \tag{44}$$

***Proof*** Let $\alpha > 0$ and let us consider the minimax problem

$$\min_{\mathrm{vol}(T)=d} \min_{K\in A(T,\alpha)} \mathrm{vol}(K). \tag{45}$$

Let the pair

$$(K^*, T^*) \ \text{ with } \ \mathrm{vol}(T^*) = d \ \text{ and } \ K^* \in A(T^*, \alpha)$$

be a minimizer of (45), i.e., it is

$$\min_{\mathrm{vol}(T)=d} \min_{K\in A(T,\alpha)} \mathrm{vol}(K) = \min_{K\in A(T^*,\alpha)} \mathrm{vol}(K) = \mathrm{vol}(K^*) =: \widetilde{c}.$$

By Theorem 3.18 $K^*$ is a maximizer of

$$\max_{\mathrm{vol}(K)=\widetilde{c}} \min_{q\in F^{cp}(K)} \ell_{T^*}(q)$$

with

$$\min_{q \in F^{cp}(K^*)} \ell_{T^*}(q) = \alpha.$$

Then, due to

$$\mathrm{vol}(T^*) = d$$

we clearly have

$$\alpha = \max_{\mathrm{vol}(K)=\widetilde{c}} \min_{q \in F^{cp}(K)} \ell_{T^*}(q) \leqslant \sup_{\mathrm{vol}(T)=d} \max_{\mathrm{vol}(K)=\widetilde{c}} \min_{q \in F^{cp}(K)} \ell_T(q). \tag{46}$$

If this is a strict inequality, then there is a pair of convex bodies

$$(K^{**}, T^{**}) \text{ with } \mathrm{vol}(T^{**}) = d \text{ and } \mathrm{vol}(K^{**}) = \widetilde{c}$$

such that

$$\alpha < \max_{\mathrm{vol}(K)=\widetilde{c}} \min_{q \in F^{cp}(K)} \ell_{T^{**}}(q) = \min_{q \in F^{cp}(K^{**})} \ell_{T^{**}}(q) =: \widetilde{\alpha}.$$

Then, by Theorem 3.18 $K^{**}$ is a minimizer of

$$\min_{K \in A(T^{**}, \widetilde{\alpha})} \mathrm{vol}(K)$$

with

$$\min_{K \in A(T^{**}, \widetilde{\alpha})} \mathrm{vol}(K) = \mathrm{vol}(K^{**}) = \widetilde{c}.$$

Now, $\widetilde{\alpha} > \alpha$ together with Proposition 3.11 implies

$$\begin{aligned} \widetilde{c} = \mathrm{vol}(K^{**}) = \min_{K \in A(T^{**}, \widetilde{\alpha})} \mathrm{vol}(K) &\geqslant \min_{\mathrm{vol}(T)=d} \min_{K \in A(T, \widetilde{\alpha})} \mathrm{vol}(K) \\ &> \min_{\mathrm{vol}(T)=d} \min_{K \in A(T, \alpha)} \mathrm{vol}(K) \\ &= \min_{K \in A(T^*, \alpha)} \mathrm{vol}(K) \\ &= \mathrm{vol}(K^*) \\ &= \widetilde{c}, \end{aligned}$$

a contradiction. Therefore, it follows that the inequality in (46) is in fact an equality, i.e.,

$$\sup_{\mathrm{vol}(T)=d} \max_{\mathrm{vol}(K)=\widetilde{c}} \min_{q \in F^{cp}(K)} \ell_T(q) = \alpha = \min_{q \in F^{cp}(K^*)} \ell_{T^*}(q).$$

This means that the pair $(K^*, T^*)$ is a maximizer of

$$\sup_{\mathrm{vol}(T)=d} \max_{\mathrm{vol}(K)=\widetilde{c}} \min_{q \in F^{cp}(K)} \ell_T(q).$$

Since it is sufficient to prove the claim (44) for one $c > 0$, we are done. □

**Theorem 3.21** *If the pair* $(K^*, T^*)$ *is a minimizer of*

$$\min_{\mathrm{vol}(T)=d} \min_{K \in A(T,\alpha)} \mathrm{vol}(K) \tag{47}$$

*for* $\alpha, d > 0$*, then* $(K^*, T^*)$ *is a maximizer of*

$$\max_{\mathrm{vol}(T)=d} \max_{\mathrm{vol}(K)=c} \min_{q \in F^{cp}(K)} \ell_T(q) \tag{48}$$

*for*

$$c := \mathrm{vol}(K^*)$$

*with*

$$\min_{q \in F^{cp}(K^*)} \ell_{T^*}(q) = \alpha.$$

*Conversely, if the pair* $(K^*, T^*)$ *is a maximizer of (48) for* $c, d > 0$*, then* $(K^*, T^*)$ *is a minimizer of (47) for*

$$\alpha := \min_{q \in F^{cp}(K^*)} \ell_{T^*}(q)$$

*with*

$$\mathrm{vol}(K^*) = c.$$

*Consequently, for* $\alpha, c, d > 0$ *we have the equivalence*

$$\min_{\mathrm{vol}(T)=d} \min_{K \in A(T,\alpha)} \mathrm{vol}(K) = c \;\Leftrightarrow\; \max_{\mathrm{vol}(T)=d} \max_{\mathrm{vol}(K)=c} \min_{q \in F^{cp}(K)} \ell_T(q) = \alpha$$

*and moreover*

$$\min_{\mathrm{vol}(T)=d} \min_{K \in A(T,\alpha)} \mathrm{vol}(K) \geqslant c \;\Leftrightarrow\; \max_{\mathrm{vol}(T)=d} \max_{\mathrm{vol}(K)=c} \min_{q \in F^{cp}(K)} \ell_T(q) \leqslant \alpha. \tag{49}$$

***Proof*** Let the pair $(K^*, T^*)$ be a minimizer of (47) for $\alpha, d > 0$, i.e., it is

$$\mathrm{vol}(T^*) = d \;\text{ and }\; K^* \in A(T^*, \alpha)$$

such that

$$\min_{\mathrm{vol}(T)=d} \min_{K \in A(T,\alpha)} \mathrm{vol}(K) = \min_{K \in A(T^*,\alpha)} \mathrm{vol}(K) = \mathrm{vol}(K^*).$$

Then, in the proof of Proposition 3.20 we have seen that $(K^*, T^*)$ is a maximizer of (48) for

$$c := \mathrm{vol}(K^*) \;\text{ with }\; \min_{q \in F^{cp}(K^*)} \ell_{T^*}(q) = \alpha.$$

Conversely, let the pair $(K^*, T^*)$ be a maximizer of (48) for $c, d > 0$, i.e., $K^*, T^* \subset \mathbb{R}^n$ are convex bodies of volume $c$ and $d$, respectively, such that

$$\max_{\mathrm{vol}(T)=d} \max_{\mathrm{vol}(K)=c} \min_{q \in F^{cp}(K)} \ell_T(q) = \max_{\mathrm{vol}(K)=c} \min_{q \in F^{cp}(K)} \ell_{T^*}(q)$$

$$
\begin{aligned}
&= \min_{q \in F^{cp}(K^*)} \ell_{T^*}(q) \\
&=: \alpha.
\end{aligned}
$$

By Theorem 3.18 $K^*$ minimizes

$$\min_{K \in A(T^*,\alpha)} \mathrm{vol}(K)$$

with

$$\mathrm{vol}(K^*) = c.$$

Then, we clearly have

$$c = \mathrm{vol}(K^*) = \min_{K \in A(T^*,\alpha)} \mathrm{vol}(K) \geqslant \min_{\mathrm{vol}(T)=d} \min_{K \in A(T,\alpha)} \mathrm{vol}(K).$$

If this is a strict inequality, then there is a pair $(K^{**}, T^{**})$ with

$$c > \min_{\mathrm{vol}(T)=d} \min_{K \in A(T,\alpha)} \mathrm{vol}(K) = \min_{K \in A(T^{**},\alpha)} \mathrm{vol}(K) = \mathrm{vol}(K^{**}) =: \widetilde{c},$$

where

$$K^{**} \in A(T, \alpha)$$

and $T^{**} \subset \mathbb{R}^n$ is a convex body of volume $d$. Then, by Theorem 3.18 $K^{**}$ is a maximizer of

$$\max_{\mathrm{vol}(K)=\widetilde{c}} \min_{q \in F^{cp}(K)} \ell_{T^{**}}(q)$$

with

$$\max_{\mathrm{vol}(K)=\widetilde{c}} \min_{q \in F^{cp}(K)} \ell_{T^{**}}(q) = \min_{q \in F^{cp}(K^{**})} \ell_{T^{**}}(q) = \alpha.$$

Now, $\widetilde{c} < c$ together with Proposition 3.14 implies

$$
\begin{aligned}
\alpha = \min_{q \in F^{cp}(K^{**})} \ell_{T^{**}}(q) &= \max_{\mathrm{vol}(K)=\widetilde{c}} \min_{q \in F^{cp}(K)} \ell_{T^{**}}(q) \\
&\leqslant \max_{\mathrm{vol}(T)=d} \max_{\mathrm{vol}(K)=\widetilde{c}} \min_{q \in F^{cp}(K)} \ell_T(q) \\
&< \max_{\mathrm{vol}(T)=d} \max_{\mathrm{vol}(K)=c} \min_{q \in F^{cp}(K)} \ell_T(q) \\
&= \max_{\mathrm{vol}(K)=c} \min_{q \in F^{cp}(K)} \ell_{T^*}(q) \\
&= \min_{q \in F^{cp}(K^*)} \ell_{T^*}(q) \\
&= \alpha,
\end{aligned}
$$

a contradiction. Therefore,

$$\min_{\mathrm{vol}(T)=d} \min_{K \in A(T,\alpha)} \mathrm{vol}(K) = c = \mathrm{vol}(K^*) = \min_{K \in A(T^*,\alpha)} \mathrm{vol}(K),$$

i.e., the pair $(K^*, T^*)$ is a minimizer of (47).

From the before proven it clearly follows the equivalence

$$\min_{\mathrm{vol}(T)=d} \min_{K\in A(T,\alpha)} \mathrm{vol}(K) = c \;\Leftrightarrow\; \max_{\mathrm{vol}(T)=d} \max_{\mathrm{vol}(K)=c} \min_{q\in F^{cp}(K)} \ell_T(q) = \alpha.$$

for $\alpha, c, d > 0$.

In order to prove (49) it is sufficient to show

$$\min_{\mathrm{vol}(T)=d} \min_{K\in A(T,\alpha)} \mathrm{vol}(K) > c \;\Leftrightarrow\; \max_{\mathrm{vol}(T)=d} \max_{\mathrm{vol}(K)=c} \min_{q\in F^{cp}(K)} \ell_T(q) < \alpha.$$

Let the pair $(K^*, T^*)$ be a minimizer of

$$\min_{\mathrm{vol}(T)=d} \min_{K\in A(T,\alpha)} \mathrm{vol}(K),$$

where $c > 0$ is chosen such that

$$c < \min_{\mathrm{vol}(T)=d} \min_{K\in A(T,\alpha)} \mathrm{vol}(K) = \min_{K\in A(T^*,\alpha)} \mathrm{vol}(K) = \mathrm{vol}(K^*) =: \widetilde{c}.$$

From above reasoning we know that $(K^*, T^*)$ maximizes (48) (for $c$ replaced by $\widetilde{c}$), i.e., $K^*, T^* \subset \mathbb{R}^n$ are convex bodies of volume $\widetilde{c}$ and $d$, respectively, such that

$$\begin{aligned}
\max_{\mathrm{vol}(T)=d} \max_{\mathrm{vol}(K)=\widetilde{c}} \min_{q\in F^{cp}(K)} \ell_T(q) &= \max_{\mathrm{vol}(K)=\widetilde{c}} \min_{q\in F^{cp}(K)} \ell_{T^*}(q) \\
&= \min_{q\in F^{cp}(K^*)} \ell_{T^*}(q) \\
&= \alpha.
\end{aligned}$$

Now, $c < \widetilde{c}$ together with Proposition 3.14 implies

$$\max_{\mathrm{vol}(T)=d} \max_{\mathrm{vol}(K)=c} \min_{q\in F^{cp}(K)} \ell_T(q) < \max_{\mathrm{vol}(T)=d} \max_{\mathrm{vol}(K)=\widetilde{c}} \min_{q\in F^{cp}(K)} \ell_T(q) = \alpha.$$

Conversely, let $(K^*, T^*)$ be a maximizer of

$$\max_{\mathrm{vol}(T)=d} \max_{\mathrm{vol}(K)=c} \min_{q\in F^{cp}(K)} \ell_T(q),$$

i.e., $K^*, T^* \subset \mathbb{R}^n$ are convex bodies of volume $c$ and $d$, respectively, where $\alpha > 0$ is chosen such that

$$\begin{aligned}
\alpha > \max_{\mathrm{vol}(T)=d} \max_{\mathrm{vol}(K)=c} \min_{q\in F^{cp}(K)} \ell_T(q) &= \max_{\mathrm{vol}(K)=c} \min_{q\in F^{cp}(K)} \ell_{T^*}(q) \\
&= \min_{q\in F^{cp}(K^*)} \ell_{T^*}(q) \\
&=: \widetilde{\alpha}.
\end{aligned}$$

Then we know from above reasoning that $(K^*, T^*)$ minimizes (47) (for $\alpha$ replaced by $\widetilde{\alpha}$), i.e.,

$$\min_{\mathrm{vol}(T)=d}\ \min_{K\in A(T,\widetilde{\alpha})} \mathrm{vol}(K) = \min_{K\in A(T^*,\widetilde{\alpha})} \mathrm{vol}(K) = \mathrm{vol}(K^*) = c.$$

Now, $\alpha > \widetilde{\alpha}$ together with Proposition 3.11 implies

$$\min_{\mathrm{vol}(T)=d}\ \min_{K\in A(T,\alpha)} \mathrm{vol}(K) > \min_{\mathrm{vol}(T)=d}\ \min_{K\in A(T,\widetilde{\alpha})} \mathrm{vol}(T) = c.$$

□

## 4 Proofs of Theorems 1.1, 1.3, 1.4 and Corollary 1.2

In the following, we mainly make use of Theorems 3.18 and 3.21. However, we begin by rewriting Viterbo's conjecture for convex Lagrangian products:

**Proposition 4.1** *Viterbo's conjecture for convex Lagrangian products* $K \times T \subset \mathbb{R}^n \times \mathbb{R}^n$

$$\mathrm{vol}(K \times T) \geqslant \frac{c_{EHZ}(K \times T)^n}{n!}, \quad K, T \in \mathcal{C}(\mathbb{R}^n),$$

*is equivalent to*

$$\max_{\mathrm{vol}(K)=1}\ \max_{\mathrm{vol}(T)=1}\ \min_{q\in F^{cp}(K)} \ell_T(q) \leqslant \sqrt[n]{n!}, \quad K, T \in \mathcal{C}(\mathbb{R}^n).$$

***Proof*** Using Proposition 2.9, Viterbo's conjecture for convex Lagrangian products is equivalent to

$$\max_{\mathrm{vol}(K)=1}\ \max_{\mathrm{vol}(T)=1} c_{EHZ}(K \times T) \leqslant \sqrt[n]{n!}, \quad K, T \in \mathcal{C}(\mathbb{R}^n).$$

By Theorem 2.7, this is equivalent to

$$\max_{\mathrm{vol}(K)=1}\ \max_{\mathrm{vol}(T)=1}\ \min_{q\in F^{cp}(K)} \ell_T(q) \leqslant \sqrt[n]{n!}, \quad K, T \in \mathcal{C}(\mathbb{R}^n).$$

□

Now, we can prove Theorems 1.1, 1.3, 1.4 and Corollary 1.2 which we will recall for the sake of overview, respectively.

**Theorem** (Theorem 1.1) *Viterbo's conjecture for convex Lagrangian products* $K \times T \subset \mathbb{R}^n \times \mathbb{R}^n$

$$\mathrm{vol}(K \times T) \geqslant \frac{c_{EHZ}(K \times T)^n}{n!}, \quad K, T \in \mathcal{C}(\mathbb{R}^n),$$

*is equivalent to the Minkowski worm problem*

$$\min_{K\in A(T,1)} \mathrm{vol}(K) \geqslant \frac{1}{n!\,\mathrm{vol}(T)}, \quad K,T\in \mathcal{C}(\mathbb{R}^n). \tag{50}$$

*Additionally, equality cases $K^*\times T^*$ of Viterbo's conjecture satisfying*

$$\mathrm{vol}(K^*)=\mathrm{vol}(T^*)=1$$

*are composed of equality cases $(K^*, T^*)$ of (50). Conversely, equality cases $(K^*, T^*)$ of (50) form equality cases $K^*\times T^*$ of Viterbo's conjecture.*

***Proof*** Using Proposition 4.1, Viterbo's conjecture for convex Lagrangian products is equivalent to

$$\max_{\mathrm{vol}(K)=1}\ \max_{\mathrm{vol}(T)=1}\ \min_{q\in F^{cp}(K)} \ell_T(q) \leqslant \sqrt[n]{n!}, \quad K,T\in \mathcal{C}(\mathbb{R}^n).$$

After applying Theorem 3.21, it is further equivalent to

$$\min_{\mathrm{vol}(T)=1}\ \min_{K\in A\left(T,\sqrt[n]{n!}\right)} \mathrm{vol}(K)\geqslant 1, \quad K,T\in \mathcal{C}(\mathbb{R}^n).$$

Using Proposition 3.10, this can be written as

$$\min_{K\in A(T,1)} \mathrm{vol}(K) \geqslant \frac{1}{n!\,\mathrm{vol}(T)}, \quad K,T\in \mathcal{C}(\mathbb{R}^n).$$

By similar reasoning, Theorem 3.21 also guarantees the equivalence of the equality case of Viterbo's conjecture for convex Lagrangian products $K\times T\subset \mathbb{R}^n\times\mathbb{R}^n$

$$\mathrm{vol}(K\times T)=\frac{c_{EHZ}(K\times T)^n}{n!}, \quad K,T\in \mathcal{C}(\mathbb{R}^n), \tag{51}$$

i.e.,

$$\max_{\mathrm{vol}(K)=1}\ \max_{\mathrm{vol}(T)=1}\ \min_{q\in F^{cp}(K)} \ell_T(q) = \sqrt[n]{n!}, \quad K,T\in \mathcal{C}(\mathbb{R}^n),$$

and

$$\min_{K\in A(T,1)} \mathrm{vol}(K) = \frac{1}{n!\,\mathrm{vol}(T)}, \quad K,T\in \mathcal{C}(\mathbb{R}^n). \tag{52}$$

Moreover, Theorem 3.21 guarantees the following: If $K^*\times T^*$ is a solution of (51) satisfying

$$\mathrm{vol}(K^*)=\mathrm{vol}(T^*)=1 \tag{53}$$

(note that, applying Proposition 2.9, the property of being a solution of (51) is invariant under scaling), then the pair $(K^*, T^*)$ is a solution of (52). And conversely, if the pair $(K^*, T^*)$ is a solution of (52), then $K^*\times T^*$ is a solution of (51). □

**Corollary** (Corollary 1.2) *Viterbo's conjecture for convex Lagrangian products $K \times T \subset \mathbb{R}^n \times \mathbb{R}^n$*

$$\mathrm{vol}(K \times T) \geqslant \frac{c_{EHZ}(K \times T)^n}{n!}, \quad K, T \in \mathcal{C}(\mathbb{R}^n),$$

*is equivalent to*[11]

$$\min_{a_q \in \mathbb{R}^n} \mathrm{vol}\left(\mathrm{conv}\left\{\bigcup_{q \in L_T(1)} (q + a_q)\right\}\right) \geqslant \frac{1}{n!\,\mathrm{vol}(T)}, \quad T \in \mathcal{C}(\mathbb{R}^n), \tag{54}$$

*where the minimization runs for every $q \in L_T(1)$ over all possible translations in $\mathbb{R}^n$. Additionally, equality cases $K^* \times T^*$ of Viterbo's conjecture satisfying*

$$\mathrm{vol}(K^*) = \mathrm{vol}(T^*) = 1$$

*are composed of equality cases $T^*$ of (54) with*

$$K^* = \mathrm{conv}\left\{\bigcup_{q \in L_{T^*}(1)} (q + a_q^*)\right\}, \tag{55}$$

*where $a_q^*$ are the minimizers in (54). Conversely, equality cases $T^*$ of (54) with $K^*$ as in (55) form equality cases $K^* \times T^*$ of Viterbo's conjecture.*

***Proof*** In view of the proof of Theorem 1.1, for convex bodies $K, T \subset \mathbb{R}^n$, it is sufficient to prove the following equality:

$$\min_{\mathrm{vol}(T)=1} \min_{K \in A(T,1)} \mathrm{vol}(K) = \min_{\mathrm{vol}(T)=1} \min_{a_q \in \mathbb{R}^n} \mathrm{vol}\left(\mathrm{conv}\left\{\bigcup_{q \in L_T(1)} (q + a_q)\right\}\right). \tag{56}$$

But this follows from the following gradually observation: First, we notice that the volume-minimizing convex cover for a set of closed curves is, equivalently, the volume-minimizing convex hull of these closed curves. So, if we ask for lower bounds of

$$\min_{K \in A(T,1)} \mathrm{vol}(K),$$

we note that for $q_1, ..., q_k \in L_T(1)$, we have

$$\min_{(a_1,...,a_k) \in (\mathbb{R}^n)^k} \mathrm{vol}\left(\mathrm{conv}\{q_1 + a_1, ..., q_k + a_k\}\right) \leqslant \min_{K \in A(T,1)} \mathrm{vol}(K).$$

This estimate can be further improved by

$$\max_{q_1,...,q_k \in L_T(1)} \min_{(a_1,...,a_k) \in (\mathbb{R}^n)^k} \mathrm{vol}(\mathrm{conv}\{q_1 + a_1, ..., q_k + a_k\}) \leqslant \min_{K \in A(T,1)} \mathrm{vol}(K),$$

[11] Here, we note that $K$ has been dissolved by replacing it by an expression that extremizes over all possible $K$s. The extremizing $K$ is of the form (3).

so that eventually we get

$$\min_{a_q \in \mathbb{R}^n} \operatorname{vol}\left(\operatorname{conv}\left\{\bigcup_{q \in L_T(1)} (q + a_q)\right\}\right) = \min_{K \in A(T,1)} \operatorname{vol}(K),$$

where the minimum on the left runs for every $q \in L_T(1)$ over all possible translations in $\mathbb{R}^n$. Minimizing this equation over all convex bodies $T \subset \mathbb{R}^n$ of volume 1, we get (56). □

**Theorem** (Theorem 1.3) *Mahler's conjecture for centrally symmetric convex bodies*

$$\operatorname{vol}(T)\operatorname{vol}(T^\circ) \geqslant \frac{4^n}{n!}, \quad T \in \mathcal{C}^{cs}(\mathbb{R}^n), \tag{57}$$

*is equivalent to the Minkowski worm problem*

$$\min_{T \in A(T^\circ,1)} \operatorname{vol}(T) \geqslant \frac{1}{n!\operatorname{vol}(T^\circ)}, \quad T \in \mathcal{C}^{cs}(\mathbb{R}^n). \tag{58}$$

*Additionally, equality cases $T^*$ of Mahler's conjecture (57) satisfying*

$$\operatorname{vol}(T^*) = 1$$

*are equality cases of (58). And conversely, equality cases $T^*$ in (58) are equality cases of Mahler's conjecture (57).*

***Proof*** Because of

$$c_{EHZ}(T \times T^\circ) = 4$$

for all centrally symmetric convex bodies $T \subset \mathbb{R}^n$ (see [3]), Mahler's conjecture for centrally symmetric convex bodies is equivalent to

$$\operatorname{vol}(T \times T^\circ) \geqslant \frac{c_{EHZ}(T \times T^\circ)^n}{n!}, \quad T \in \mathcal{C}^{cs}(\mathbb{R}^n). \tag{59}$$

Fixing

$$\operatorname{vol}(T) = 1,$$

which is without loss of generality due to Proposition 2.10, and using Theorem 2.7, (59) is equivalent to

$$\sqrt[n]{n!\operatorname{vol}(T^\circ)} \geqslant c_{EHZ}(T \times T^\circ) = \min_{q \in F^{cp}(T)} \ell_{T^\circ}(q), \quad T \in \mathcal{C}^{cs}(\mathbb{R}^n).$$

This can be written as

$$\max_{\operatorname{vol}(T)=1} \min_{q \in F^{cp}(T)} \ell_{T^\circ}(q) \leqslant \sqrt[n]{n!\operatorname{vol}(T^\circ)}, \quad T \in \mathcal{C}^{cs}(\mathbb{R}^n),$$

which, by Theorem 3.18, is equivalent to

$$\min_{T \in A\left(T^\circ,\, \sqrt[n]{n!\,\mathrm{vol}(T^\circ)}\right)} \mathrm{vol}(T) \geqslant 1, \quad T \in \mathcal{C}^{cs}(\mathbb{R}^n).$$

Applying Proposition 3.10, we finally conclude that Mahler's conjecture for centrally symmetric convex bodies is equivalent to

$$\min_{T \in A(T^\circ,1)} \mathrm{vol}(T) \geqslant \frac{1}{n!\,\mathrm{vol}(T^\circ)}, \quad T \in \mathcal{C}^{cs}(\mathbb{R}^n).$$

By similar reasoning, Theorem 3.18 also guarantees the equivalence of the equality case of Mahler's conjecture for centrally symmetric convex bodies $T \subset \mathbb{R}^n$

$$\mathrm{vol}(T)\,\mathrm{vol}(T^\circ) = \frac{4^n}{n!}, \tag{60}$$

i.e.,

$$\max_{\mathrm{vol}(T)=1} \min_{q \in F^{cp}(T)} \ell_{T^\circ}(q) = \sqrt[n]{n!\,\mathrm{vol}(T^\circ)}, \quad T \in \mathcal{C}^{cs}(\mathbb{R}^n),$$

and

$$\min_{T \in A(T^\circ,1)} \mathrm{vol}(T) = \frac{1}{n!\,\mathrm{vol}(T^\circ)}, \quad T \in \mathcal{C}^{cs}(\mathbb{R}^n). \tag{61}$$

Moreover, Theorem 3.18 guarantees the following: If $T^*$ is a solution of (60) satisfying

$$\mathrm{vol}(T^*) = 1$$

(note that, applying Proposition 2.10, the property of being a solution of (60) is invariant under scaling), then it is a solution of (61). And conversely, if $T^*$ is a solution of (61), then it is also a solution of (60). □

**Theorem** (Theorem 1.4) *Let $T \subset \mathbb{R}^n$ be a convex body and $\alpha, c > 0$. Then, the following statements are equivalent:*

(1)

$$\max_{\mathrm{vol}(K)=c} \min_{q \in F^{cp}(K)} \ell_T(q) \leqslant \alpha, \quad K \in \mathcal{C}(\mathbb{R}^n),$$

(2)

$$\max_{\mathrm{vol}(K)=c} c_{EHZ}(K \times T) \leqslant \alpha, \quad K \in \mathcal{C}(\mathbb{R}^n),$$

(3)

$$\max_{\mathrm{vol}(K)=c} \min_{q \in M_{n+1}(K,T)} \ell_T(q) \leqslant \alpha, \quad K \in \mathcal{C}(\mathbb{R}^n),$$

(4)

$$\min_{K \in A(T,\alpha)} \mathrm{vol}(K) \geqslant c, \quad K \in \mathcal{C}(\mathbb{R}^n),$$

(5)

$$\min_{a_q\in\mathbb{R}^n}\mathrm{vol}\left(\mathrm{conv}\left\{\bigcup_{q\in L_T(1)}(q+a_q)\right\}\right)\geqslant c,\quad K\in\mathcal{C}(\mathbb{R}^n).$$

*If $T$ is additionally assumed to be strictly convex, then the following systolic weak Minkowski billiard inequality can be added to the above list of equivalent expressions:*

(6)

$$\max_{\mathrm{vol}(K)=c}\quad\min_{q\ \textit{cl. weak}\ (K,T)\textit{-Mink. bill. traj.}}\ell_T(q)\leqslant\alpha,\quad K\in\mathcal{C}(\mathbb{R}^n).$$

*Moreover, every equality case $(K^*,T^*)$ of any of the above inequalities is also an equality case of all the others.*

***Proof*** The equivalence of (1), (2), and (3) follows from Theorem 2.7. The equivalence of (1) and (4) follows from Theorem 3.18. The equivalence of (4) and (5) can be concluded as within the proof of Corollary 1.2. For the case of strictly convex $T\subset\mathbb{R}^n$, the equivalence of (1) and (6) follows from [36, Theorem 1.3].

The addition that every equality case $(K^*,T^*)$ of any of the inequalities is also an equality case of all the others is guaranteed by Theorem 3.18. □

## 5 Proof of Theorem 1.5

We start by recalling Theorem 1.5:

**Theorem** (Theorem 1.5) *Viterbo's conjecture for convex polytopes in $\mathbb{R}^{2n}$*

$$\mathrm{vol}(P)\geqslant\frac{c_{EHZ}(P)^n}{n!},\quad P\in\mathcal{C}^p\left(\mathbb{R}^{2n}\right),\tag{62}$$

*is equivalent to the Minkowski worm problem*

$$\min_{P\in A(JP,1)}\mathrm{vol}(P)\geqslant\frac{(R_P)^n}{2^n n!},\quad P\in\mathcal{C}^p\left(\mathbb{R}^{2n}\right),\tag{63}$$

*where we define*

$$R_P:=\frac{\min_{q\in F_*^{cp}(P)}\ell_{\frac{JP}{2}}(q)}{\min_{q\in F^{cp}(P)}\ell_{\frac{JP}{2}}(q)}\geqslant 1.$$

*Additionally, $P^*$ is an equality case of Viterbo's conjecture for convex polytopes (62) satisfying*

$$\mathrm{vol}(P^*)=1$$

*if and only if $P^*$ is an equality case of (63).*

Now, we recall a sligthly rephrased version of the main result of Haim-Kislev in [26]:

**Theorem 5.1** *Let $P \subset \mathbb{R}^{2n}$ be a convex polytope. Then, there is an action-minimizing closed characteristic $x$ on $\partial P$ which is a closed polygonal curve consisting of finitely many segments*

$$[x(t_j), x(t_{j+1})]$$

*given by*

$$x(t_{j+1}) = x(t_j) + \lambda_j J\nabla H_P(x_j), \quad \lambda_j > 0,$$

*while $x_j \in \mathring{F}_j$, $F_j$ is a facet of $P$ and $x$ visits every facet $F_j$ at most once.*

For the proof of Theorem 1.5, we need the following theorem:

**Theorem 5.2** *If $P \subset \mathbb{R}^{2n}$ is a convex polytope, then we have*

$$c_{EHZ}(P) = \min_{q \in F_*^{cp}(P)} \ell_{\frac{JP}{2}}(q) = R_P \min_{q \in F^{cp}(P)} \ell_{\frac{JP}{2}}(q)$$

*with*

$$R_P = \frac{\min_{q \in F_*^{cp}(P)} \ell_{\frac{JP}{2}}(q)}{\min_{q \in F^{cp}(P)} \ell_{\frac{JP}{2}}(q)} \geqslant 1.$$

If we consider $P \times \frac{1}{2} JP$ as a Lagrangian product (in the light of Footnote 8 within Theorem 1.6), then the combination of Theorem 2.7 and Theorem 5.2 implies the following relationship between the EHZ-capacity of $P$ and the EHZ-capacity of the Lagrangian product $P \times \frac{1}{2} JP$:

$$c_{EHZ}(P) = R_P c_{EHZ}\left(P \times \frac{1}{2} JP\right).$$

For the proof of Theorem 5.2, we need the following proposition. We remark that in the proof of Theorem 5.2, we need it only in the case of action-minimizing closed characteristics on the boundary of a polytope. However, we will state it in full generality which has relevance beyond its use in the proof of Theorem 5.2 (which we will briefly address below).

**Proposition 5.3** *Let $C \subset \mathbb{R}^{2n}$ be a convex body. Let $x$ be any closed characteristic on $\partial C$. Then, the action of $x$ equals its $\ell_{\frac{JC}{2}}$-length:*

$$\mathbb{A}(x) = \ell_{\frac{JC}{2}}(x).$$

Proposition 5.3 implies a noteworthy connection between closed characteristics and closed Finsler geodesics: Every closed characteristic on $\partial C$ can be interpreted as a closed Finsler geodesic with respect to the Finsler metric determined by $\mu_{2JC^\circ}$ and which is parametrized by arc length. This raises a number of questions; for example, which closed Finsler geodesics are closed characteristics (we note that there are more closed geodesics than those which, by the least action principle and Proposition 5.3, can be associated to closed characteristics) and the length-minimizing closed Finsler geodesics of which class are of this kind. Following this line of thought, would lead to the question whether it is possible to deduce Viterbo's conjecture from systolic inequalities for certain closed Finsler geodesics. However, we leave these questions for further research.

***Proof of Proposition 5.3*** By

$$\dot{x}(t) \in J\partial H_C(x(t)) \quad \text{a.e.},$$

we conclude

$$\begin{aligned}\frac{1}{2}\left(\mu_{2JC^\circ}(\dot{x}(t))\right)^2 &= H_{2JC^\circ}(\dot{x}(t)) \in H_{2JC^\circ}(J\partial H_C(x(t))) \\ &= \frac{1}{4} H_{C^\circ}(\partial H_C(x(t))) \quad \text{a.e.},\end{aligned}$$

where we used the facts

$$J^{-1} = -J, \quad H_C(Jx) = H_{J^{-1}C}(x)$$

and

$$H_{\lambda C}(x) = H_C\left(\frac{1}{\lambda}x\right) = \frac{1}{\lambda^2} H_C(x), \quad \lambda \neq 0,$$

(see [36, Proposition 2.3(iii)]). From Proposition 2.5, we therefore conclude

$$\frac{1}{2}\left(\mu_{2JC^\circ}(\dot{x}(t))\right)^2 = \frac{1}{4} H_C(x(t)) = \frac{1}{8} \quad \text{a.e.}$$

and consequently

$$\mu_{2JC^\circ}(\dot{x}(t)) = \frac{1}{2} \quad \text{a.e.}$$

Considering

$$(2JC^\circ)^\circ = \frac{1}{2}JC$$

(see [36, Proposition 2.1]), we obtain

$$\ell_{\frac{JC}{2}}(x) = \int_0^T \mu_{\left(\frac{JC}{2}\right)^\circ}(\dot{x}(t))\,\mathrm{d}t = \int_0^T \mu_{2JC^\circ}(\dot{x}(t))\,\mathrm{d}t = \int_0^T \frac{1}{2}\,\mathrm{d}t = \frac{T}{2} = \mathbb{A}(x),$$

where the last equality follows from

$$\mathbb{A}(x) = -\frac{1}{2}\int_0^T \langle J\dot{x}(t), x(t)\rangle\,\mathrm{d}t \in \frac{1}{2}\int_0^T \langle \partial H_C(x(t)), x(t)\rangle\,\mathrm{d}t$$

which by Proposition 2.3 and the 2-homogeneity of $H_C$ implies

$$\mathbb{A}(x) = \int_0^T H_C(x(t))\,\mathrm{d}t = \frac{T}{2}.$$

□

Then, we come to the proof of Theorem 5.2:

***Proof of Theorem 5.2*** The idea behind the proof is to associate action-minimizing closed characteristics on $\partial P$ in the sense of Theorem 5.1 with $\ell_{\frac{1}{2}JP}$-minimizing closed $(P, \frac{JP}{2})$-Minkowski billiard trajectories.

Let $x$ be an action-minimizing closed characteristic on $\partial P$ in the sense of Theorem 5.1. Let us assume $x$ is moving on the facets of $P$ according to the order

$$F_1 \to F_2 \to \cdots \to F_m \to F_1,$$

while the linear flow on every facet is given by the $J$-rotated normal vector at the interior of this facet. Out of every trajectory segment

$$\mathrm{orb}(x) \cap \mathring{F}_j,$$

we choose one point $q_j$ arbitrarily (on the whole requiring $q_i \neq q_j$ for $i \neq j$) and connect these points by straight lines (by maintaining the order of the corresponding facets) constructing a closed polygonal curve

$$q := (q_1, ..., q_m)$$

within $P$ which has its vertices on $\partial P$. From Lemma 5.4 (which we provide subsequently), we derive

$$\ell_{\frac{JP}{2}}(q) = \ell_{\frac{JP}{2}}(x)$$

since the trajectory segment of $x$ between the two consecutive points $q_j$ and $q_{j+1}$–let us call it $\mathrm{orb}(x)_{q_j \to q_{j+1}}$–together with the line from $q_j$ to $q_{j+1}$ (as trajectory segment of $q$)–let us call it $[q_j, q_{j+1}]$–builds a triangle with the property that

$$\mu_{2JP^\circ}\left(\mathrm{orb}(x)_{q_j \to q_{j+1}}\right) = \mu_{2JP^\circ}([q_j, q_{j+1}]).$$

We therefore conclude from Proposition 5.3 that

$$\ell_{\frac{JP}{2}}(q) = \mathbb{A}(x).$$

Because of the arbitrariness of the choice of $q_j$ within $\mathrm{orb}(x) \cap \mathring{F}_j$, we can assign infinitely many different closed polygonal curves of the above kind to one action-minimizing closed characteristic fulfilling the demanded conditions.

Each of these closed polygonal curves $q$ is a closed $(P \times \frac{1}{2}JP)$-Minkowski billiard trajectory: This follows from the fact that $q$ fulfills

$$\begin{cases} q_{j+1} - q_j \in N_{\frac{1}{2}JP}(p_j), \\ p_{j+1} - p_j \in -N_P(q_{j+1}), \end{cases}$$

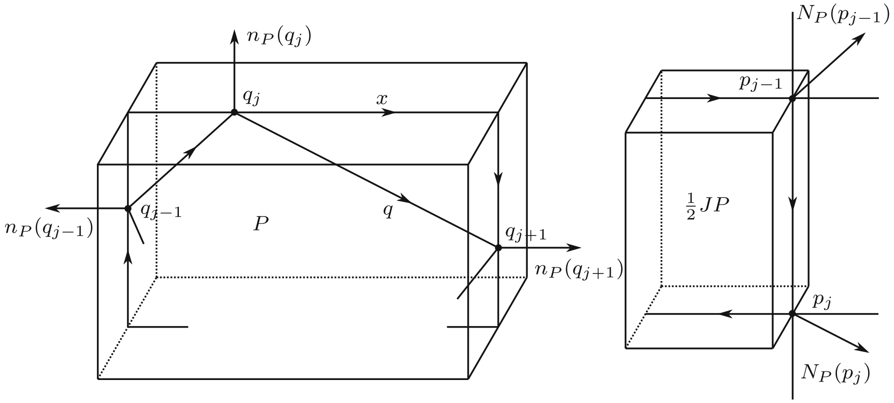


**Fig. 3** $q = (q_1, ..., q_m)$ is a closed $(P, \frac{1}{2}JP)$-Minkowski billiard trajectory with $p = (p_1, ..., p_m)$ as its dual billiard trajectory in $\frac{1}{2}JP$

for the closed polygonal curve $p = (p_1, ..., p_m)$ in $\frac{1}{2}JP$ with

$$p_{j-1} \in \partial\left(\frac{1}{2}JP\right)$$

given as the intersection point

$$\frac{1}{2}J\left(\{q_{j-1} + tJ\nabla H_P(q_{j-1}) : t \in \mathbb{R}\} \cap \{q_j + tJ\nabla H_P(q_j) : t \in \mathbb{R}\}\right) \subset \frac{JF_{j-1}}{2} \cap \frac{JF_j}{2}$$

for all $j \in \{2, ..., m+1\}$.

Indeed, from the definition of $p$, it follows

$$p_{j+1} - p_j \in -N_P(q_{j+1}) \quad \forall j \in \{1, ..., m\} \tag{64}$$

since by construction, $p_{j+1} - p_j$ is a multiple of the outer normal vector at $P$ in $q_j$ rotated by twofold multiplication with $J$ ($J^2 = -\mathbb{1}$ produces the minus sign in (64)). Since, by construction,

$$J^{-1}(q_j - q_{j-1})$$

is in the normal cone at $P$ in the intersection point

$$\{q_{j-1} + tJ\nabla H_P(q_{j-1}) : t \in \mathbb{R}\} \cap \{q_j + tJ\nabla H_P(q_j) : t \in \mathbb{R}\} \subset F_{j-1} \cap F_j,$$

roation by $\frac{1}{2}J$ then implies that $q_j - q_{j-1}$ is in the normal cone at $\frac{1}{2}JP$ in $p_{j-1}$. This implies

$$q_j - q_{j-1} \in N_P(p_{j-1}) \quad \forall j \in \{1, ..., m\}.$$

From [36, Proposition 3.9], it follows that $q$ cannot be translated into $\mathring{P}$, i.e.,

$$q \in F^{cp}(P).$$

From the construction of $q$, we moreover know

$$q \in F_*^{cp}(P), \tag{65}$$

where we recall that $F_*^{cp}(P)$ as subset of $F^{cp}(P)$ was defined as the set of all closed polygonal curves $q = (q_1, ..., q_m)$ in $F^{cp}(P)$ for which $q_j$ and $q_{j+1}$ are on neighbouring facets $F_j$ and $F_{j+1}$ of $P$ such that there are $\lambda_j, \mu_{j+1} \geqslant 0$ with

$$q_{j+1} = q_j + \lambda_j J\nabla H_P(x_j) + \mu_{j+1} J\nabla H_P(x_{j+1}),$$

where $x_j$ and $x_{j+1}$ are arbitrarily chosen interior points of $F_j$ and $F_{j+1}$, respectively.

Because of (65), we have

$$\ell_{\frac{JP}{2}}(q) \geqslant \min_{\widetilde{q} \in F_*^{cp}(P)} \ell_{\frac{JP}{2}}(\widetilde{q}).$$

Since, by definition and the above considerations, every closed polygonal curve in $F_*^{cp}(P)$ is associated with a closed characteristic on $\partial P$, where the $\ell_{\frac{JP}{2}}$-length of the former coincides with the action of the latter, and $x$ (to which $q$ is associated) was chosen to be action-minimizing, we actually have

$$\ell_{\frac{JP}{2}}(q) = \min_{\widetilde{q} \in F_*^{cp}(P)} \ell_{\frac{JP}{2}}(\widetilde{q}).$$

Altogether, this implies

$$c_{EHZ}(P) = \mathbb{A}(x) = \ell_{\frac{JP}{2}}(x) = \ell_{\frac{JP}{2}}(q) = \min_{\widetilde{q} \in F_*^{cp}(P)} \ell_{\frac{JP}{2}}(\widetilde{q}) = R_P \min_{\widetilde{q} \in F^{cp}(P)} \ell_{\frac{JP}{2}}(\widetilde{q})$$

for

$$R_P = \frac{\min_{q \in F_*^{cp}(P)} \ell_{\frac{JP}{2}}(q)}{\min_{q \in F^{cp}(P)} \ell_{\frac{JP}{2}}(q)} \geqslant 1.$$

□

**Lemma 5.4** *Let $P \subset \mathbb{R}^{2n}$ be a convex polytope. If*

$$y = \lambda_i J\nabla H_P(x_i) + \lambda_j J\nabla H_P(x_j),\ \ \lambda_i, \lambda_j \geqslant 0,$$

*where $F_i$ and $F_j$ are neighbouring facets of $P$ with $x_i \in \mathring{F}_i$ and $x_j \in \mathring{F}_j$, then*

$$\mu_{2JP^\circ}(y) = \lambda_i \mu_{2JP^\circ}(J\nabla H_P(x_i)) + \lambda_j \mu_{2JP^\circ}(J\nabla H_P(x_j)) = \frac{1}{2}(\lambda_i + \lambda_j).$$

***Proof*** We first notice that

$$\nabla H_P(x_i) \ \text{ and } \ \nabla H_P(x_j)$$

are neighbouring vertices of $P^\circ$, i.e.,

$$t\nabla H_P(x_i)+(1-t)\nabla H_P(x_j)\in\partial P^\circ \quad \forall t\in[0,1].$$

Indeed, from the fact that $\nabla H_P(x_i)$ and $\nabla H_P(x_j)$ are elements of the one dimensional normal cone at $\mathring{F}_i$ and $\mathring{F}_j$, we conclude by the properties of the polar of convex polytopes (see [21, Chapter 3.3]) that they point into the direction of two neigbouring vertices of $P^\circ$. Using Proposition 2.5, we calculate

$$H_{P^\circ}(\nabla H_P(x_i))=H_P(x_i)=\frac{1}{2}$$

and

$$H_{P^\circ}(\nabla H_P(x_j))=H_P(x_j)=\frac{1}{2}$$

and conclude that $\nabla H_P(x_i)$ and $\nabla H_P(x_j)$ actually are these two neighbouring vertices of $P^\circ$.

Using for convex body $C\subset\mathbb{R}^{2n}$ and $\lambda>0$ the properties

$$\mu_{\lambda C}(x)=\frac{1}{\lambda}\mu_C(x)\;\text{ and }\;\mu_{JC}(Jx)=\mu_C(x),\quad x\in\mathbb{R}^{2n},$$

(see [36, Proposition 2.3(iii)]), we derive

$$\begin{aligned}\mu_{2JP^\circ}(y)&=\mu_{2JP^\circ}(\lambda_i J\nabla H_P(x_i)+\lambda_j J\nabla H_P(x_j))\\&=\mu_{2P^\circ}(\lambda_i\nabla H_P(x_i)+\lambda_j\nabla H_P(x_j))\\&=\frac{1}{2}\left(\mu_{P^\circ}(\lambda_i\nabla H_P(x_i)+\lambda_j\nabla H_P(x_j))\right)\\&\overset{(\star)}{=}\frac{1}{2}\left(\mu_{P^\circ}(\lambda_i\nabla H_P(x_i))+\mu_{P^\circ}(\lambda_j\nabla H_P(x_j))\right)\\&=\frac{1}{2}\left(\lambda_i\mu_{P^\circ}(\nabla H_P(x_i))+\lambda_j\mu_{P^\circ}(\nabla H_P(x_j))\right)\\&=\frac{1}{2}(\lambda_i+\lambda_j),\end{aligned}$$

where in $(\star)$ we used that, by the choice of $x_i$ and $x_j$ and the properties of polar bodies, $\nabla H_P(x_i)$ and $\nabla H_P(x_j)$ are neighbouring vertices of $P^\circ$ and, therefore, in $(\star)$, the initial term can be splitted linearly. □

***Proof of Theorem 1.5*** Viterbo's conjecture for convex polytopes in $\mathbb{R}^{2n}$ can be written as

$$\mathrm{vol}(P)\geqslant\frac{c_{EHZ}(P)^n}{n!},\quad P\in\mathcal{C}^p\left(\mathbb{R}^{2n}\right),$$

which by Theorem 5.2, is equivalent to

$$\mathrm{vol}(P)\geqslant\frac{R_P^n}{2^n n!}c_{EHZ}(P\times JP)^n,\quad P\in\mathcal{C}^p\left(\mathbb{R}^{2n}\right).$$

By referring to Proposition 2.9, we can assume

$$\mathrm{vol}(P) = 1$$

without loss of generality and get

$$c_{EHZ}(P \times JP) \leqslant \frac{2\sqrt[n]{n!}}{R_P}, \quad P \in \mathcal{C}^p\left(\mathbb{R}^{2n}\right),$$

which by Theorem 2.7, is equivalent to

$$\max_{\mathrm{vol}(P)=1} \min_{q \in F^{cp}(P)} \ell_{JP}(q) \leqslant \frac{2\sqrt[n]{n!}}{R_P}, \quad P \in \mathcal{C}^p\left(\mathbb{R}^{2n}\right).$$

By Theorem 3.18, this is equivalent to

$$\min_{P \in A\left(JP, \frac{2\sqrt[n]{n!}}{R_P}\right)} \mathrm{vol}(P) \geqslant 1, \quad P \in \mathcal{C}^p\left(\mathbb{R}^{2n}\right),$$

and after applying Proposition 3.10, to

$$\min_{P \in A(JP,1)} \mathrm{vol}(P) \geqslant \frac{(R_P)^n}{2^n n!}, \quad P \in \mathcal{C}^p\left(\mathbb{R}^{2n}\right).$$

By similar reasoning, Theorem 3.18 also guarantees the equivalence of

$$\max_{\mathrm{vol}(P)=1} \min_{q \in F^{cp}(P)} \ell_{JP}(q) = \frac{2\sqrt[n]{n!}}{R_P}, \quad P \in \mathcal{C}^p\left(\mathbb{R}^{2n}\right), \tag{66}$$

and

$$\min_{P \in A(JP,1)} \mathrm{vol}(P) = \frac{(R_P)^n}{2^n n!}, \quad P \in \mathcal{C}^p\left(\mathbb{R}^{2n}\right). \tag{67}$$

Moreover, Theorem 3.18 guarantees the following: $P^*$ is a solution of (66) if and only if $P^*$ is a solution of (67). □

## 6 Proof of Theorem 1.6

We start by recalling Theorem 1.6:

**Theorem** (Theorem 1.6) *Viterbo's conjecture for convex bodies in* $\mathbb{R}^{2n}$

$$\mathrm{vol}(C) \geqslant \frac{c_{EHZ}(C)^n}{n!}, \quad C \in \mathcal{C}\left(\mathbb{R}^{2n}\right), \tag{68}$$

*is equivalent to the Minkowski worm problem*

$$\min_{C\in A(C^\circ,1)} \mathrm{vol}(C) \geqslant \frac{(\widetilde{R}_C)^n}{n!}, \quad C \in \mathcal{C}\left(\mathbb{R}^{2n}\right), \tag{69}$$

*where*

$$\widetilde{R}_C := \frac{c_{EHZ}(C)}{c_{EHZ}(C\times C^\circ)} \geqslant \frac{1}{2||J||_{C^\circ\to C}}.$$

*Additionally, $C^*$ is an equality case of Viterbo's conjecture for convex bodies in $\mathbb{R}^{2n}$* (68) *satisfying*

$$\mathrm{vol}(C^*) = 1$$

*if and only if $C^*$ is an equality case of* (69).

In order to prove Theorem 1.6, we need the following propositon:

**Proposition 6.1** *Let $C \subset \mathbb{R}^{2n}$ be a convex body and $x$ a closed characteristic on $\partial C$. Then, $x$ cannot be translated into $\mathring{C}$.*

***Proof*** Let us assume that $x$ can be translated into $\mathring{C}$. Let $\widetilde{T} > 0$ be the period of $x$. Because of

$$\dot{x}(t) \in J\partial H_C(x(t)) \quad \text{a.e. on } [0,\widetilde{T}],$$

there is a vector-valued function $n_C$ on $\partial C$ such that

$$\dot{x}(t) = Jn_C(x(t)) \quad \text{a.e. on } [0,\widetilde{T}]$$

with

$$n_C(x(t)) \in \partial H_C(x(t))$$

for all $t \in [0,\widetilde{T}]$ for which $\dot{x}(t)$ exists and

$$n_C(x(t)) = 0$$

for all $t \in [0,\widetilde{T}]$ for which $\dot{x}(t)$ does not exist.

Then, the convex cone $U$ spanned by

$$n_C(x(t)) \in N_C(x(t)),\ t \in \left[0,\widetilde{T}\right],$$

has the property

$$\forall u \in U\setminus\{0\}:\ -u \notin U,$$

since otherwise, one could find points on $x$ and $C$-supporting hyperplanes through these points with the property that the intersection of the $C$-containing half-spaces bounded by these hyperplanes is nearly bounded (what would imply that $x$ cannot be translated into $\mathring{C}$). By the convexity of $U$, this implies that

$$\int_0^{\widetilde{T}} n_C(x(t))\,\mathrm{d}t \neq 0,$$

and therefore

$$\int_0^{\widetilde{T}} Jn_C(x(t))\,\mathrm{d}t \neq 0.$$

Since $x$ is a closed characteristic on $\partial C$, $x$ fulfills $x(0) = x(\widetilde{T})$. This implies

$$0 = x(\widetilde{T}) - x(0) = \int_0^{\widetilde{T}} \dot{x}(t)\,\mathrm{d}t = \int_0^{\widetilde{T}} Jn_C(x(t))\,\mathrm{d}t \neq 0,$$

a contradiction. Therefore, it follows that $x$ cannot be translated into $\mathring{C}$. □

We now consider the operator norm of the complex structure/symplectic matrix $J$. It is given by:

$$||J||_{C^\circ \to C} = \sup_{||v||_{C^\circ} \leqslant 1} ||Jv||_C = \sup_{\mu_{C^\circ}(v) \leqslant 1} \mu_C(Jv).$$

We derive the following lemma:

**Lemma 6.2** *Let $C \subset \mathbb{R}^{2n}$ be a convex body and $x$ a closed characteristic on $\partial C$ which has period $\widetilde{T} > 0$. Then, we have*

$$\mu_C(\dot{x}(t)) \leqslant ||J||_{C^\circ \to C} \quad \textit{a.e. on } [0, \widetilde{T}].$$

***Proof*** Since $x$ is a closed characteristic on $\partial C$, we have

$$\dot{x}(t) \in J\partial H_C(x(t)) \quad \text{a.e. on } [0, \widetilde{T}].$$

This implies

$$H_{C^\circ}(-J\dot{x}(t)) \in H_{C^\circ}(\partial H_C(x(t))) \quad \text{a.e. on } [0, \widetilde{T}].$$

Using Proposition 2.5, we conclude

$$H_{C^\circ}(-J\dot{x}(t)) = H_C(x(t)) = \frac{1}{2} \quad \text{a.e. on } [0, \widetilde{T}],$$

i.e.,

$$\mu_{C^\circ}(-J\dot{x}(t)) = 1 \quad \text{a.e. on } [0, \widetilde{T}].$$

Therefore, for

$$v(t) := -J\dot{x}(t) \quad \text{a.e. on } [0, \widetilde{T}],$$

we have

$$\mu_{C^\circ}(v(t)) = 1 \;\text{ and }\; Jv(t) = \dot{x}(t) \quad \text{a.e. on } [0, \widetilde{T}]$$

and consequently

$$\mu_C(\dot{x}(t)) \leqslant \sup_{\mu_{C^\circ}(v) \leqslant 1} \mu_C(Jv) = ||J||_{C^\circ \to C} \quad \text{a.e. on } [0, \widetilde{T}].$$

□

***Proof of Theorem 1.6*** By Theorem 2.7, we have

$$c_{EHZ}(C \times C^{\circ}) = \min_{q \in F^{cp}(C)} \ell_{C^{\circ}}(q). \tag{70}$$

Let $x$ be an action-minimizing closed characteristic on $\partial C$, i.e., $x$ fulfills

$$\dot{x} \in J\partial H_C(x) \quad \text{a.e.}$$

and minimizes the action with

$$\mathbb{A}(x) = -\frac{1}{2}\int_0^{\widetilde{T}} \langle J\dot{x}(t), x(t)\rangle \, \mathrm{d}t = \int_0^{\widetilde{T}} H_C(x) \, \mathrm{d}t = \frac{\widetilde{T}}{2}, \tag{71}$$

where we used Euler's identity (see Proposition 2.3) to derive

$$\langle y, x(t)\rangle = H_C(x(t)) \quad \forall y \in \partial H_C(x(t)).$$

Then, since $x$ is in $\partial C$ and, by Proposition 6.1, cannot be translated into $\mathring{C}$, (70) together with

$$\min_{q \in F^{cp}(C)} \ell_{C^{\circ}}(q) = \min_{q \in F^{cc}(C)} \ell_{C^{\circ}}(q) \tag{72}$$

(see Proposition 8.2) implies that

$$c_{EHZ}(C \times C^{\circ}) \leqslant \ell_{C^{\circ}}(x) = \int_0^{\widetilde{T}} \mu_C(\dot{x}(t)) \, \mathrm{d}t.$$

Using Lemma 6.2 and (71), we conclude

$$\begin{aligned} c_{EHZ}(C \times C^{\circ}) \leqslant \int_0^{\widetilde{T}} \mu_C(\dot{x}(t)) \, \mathrm{d}t \leqslant \int_0^{\widetilde{T}} ||J||_{C^{\circ} \to C} \, \mathrm{d}t \\ = \widetilde{T}||J||_{C^{\circ} \to C} \\ = 2\mathbb{A}(x)||J||_{C^{\circ} \to C} \\ = 2c_{EHZ}(C)||J||_{C^{\circ} \to C}. \end{aligned}$$

This implies

$$\widetilde{R}_C = \frac{c_{EHZ}(C)}{c_{EHZ}(C \times C^{\circ})} \geqslant \frac{1}{2||J||_{C^{\circ} \to C}}.$$

Therefore, Viterbo's conjecture for convex bodies in $\mathbb{R}^{2n}$ is equivalent to

$$\mathrm{vol}(C) \geqslant \frac{c_{EHZ}(C)^n}{n!} = \frac{\widetilde{R}_C^n c_{EHZ}(C \times C^{\circ})^n}{n!}, \quad C \in \mathcal{C}\left(\mathbb{R}^{2n}\right).$$

By referring to Proposition 2.9, we can assume

$$\mathrm{vol}(C) = 1$$

without loss of generality and get

$$c_{EHZ}(C \times C^{\circ}) \leqslant \frac{\sqrt[n]{n!}}{\widetilde{R}_C}, \quad C \in \mathcal{C}\left(\mathbb{R}^{2n}\right),$$

which, by Theorem 2.7, is equivalent to

$$\max_{\mathrm{vol}(C)=1} \min_{q \in F^{cp}(C)} \ell_{C^{\circ}}(q) \leqslant \frac{\sqrt[n]{n!}}{\widetilde{R}_C}, \quad C \in \mathcal{C}\left(\mathbb{R}^{2n}\right).$$

By Theorem 3.18, this is equivalent to

$$\min_{C \in A\left(C^{\circ}, \frac{\sqrt[n]{n!}}{\widetilde{R}_C}\right)} \mathrm{vol}(T) \geqslant 1, \quad C \in \mathcal{C}\left(\mathbb{R}^{2n}\right),$$

and, after applying Proposition 3.10, to

$$\min_{C \in A(C^{\circ},1)} \mathrm{vol}(T) \geqslant \frac{(\widetilde{R}_C)^n}{n!}, \quad C \in \mathcal{C}\left(\mathbb{R}^{2n}\right).$$

By similar reasoning, Theorem 3.18 also guarantees the equivalence of

$$\max_{\mathrm{vol}(C)=1} \min_{q \in F^{cp}(C)} \ell_{C^{\circ}}(q) = \frac{\sqrt[n]{n!}}{\widetilde{R}_C}, \quad C \in \mathcal{C}\left(\mathbb{R}^{2n}\right), \tag{73}$$

and

$$\min_{C \in A(C^{\circ},1)} \mathrm{vol}(T) = \frac{(\widetilde{R}_C)^n}{n!}, \quad C \in \mathcal{C}\left(\mathbb{R}^{2n}\right). \tag{74}$$

Moreover, Theorem 3.18 guarantees the following: $C^*$ is a solution of (73) if and only if $C^*$ is a solution of (74). □

## 7 Justification of Conjectures 1.8 and 1.9

We start by recalling Conjectures 1.8 and 1.9:

**Conjecture** (Conjecture 1.8) *We have*

$$\min_{K \in A\left(B_1^2,1\right)} \mathrm{vol}(K) \geqslant \frac{1}{2\pi} \approx 0.15915, \quad K \in \mathcal{C}(\mathbb{R}^2).$$

**Conjecture** (Conjecture 1.9) *We have*

$$\min_{q \text{ cl. } (K,B_1^2)\text{-Mink. bill. traj.}} \ell^2_{B_1^2}(q) \leqslant 2\pi \,\mathrm{vol}(K)$$

*for* $K \in \mathcal{C}(\mathbb{R}^2)$.

We transfer Viterbo's conjecture onto Wetzel's problem. For that, we define

$$y := \min_{K \in A\left(B_1^2, 1\right)} \operatorname{vol}(K)$$

and let $K^* \subset \mathbb{R}^2$ be an arbitrarily chosen convex body of volume $y$. Then, applying Theorems 2.7 and 3.18, we have

$$\begin{aligned} \frac{c_{EHZ}\left(K^* \times B_1^2\right)^2}{2} &\leqslant \max_{\operatorname{vol}(K)=y} \frac{c_{EHZ}\left(K \times B_1^2\right)^2}{2} \\ &= \max_{\operatorname{vol}(K)=y} \min_{q \in F^{cp}(K)} \frac{\ell_{B_1^2}(q)^2}{2} \\ &= \frac{1}{2}. \end{aligned}$$

Further, we have

$$\operatorname{vol}\left(K^* \times B_1^2\right) = \pi y.$$

The truth of Viterbo's conjecture requires

$$\operatorname{vol}\left(K^* \times B_1^2\right) \geqslant \frac{c_{EHZ}\left(K^* \times B_1^2\right)^2}{2},$$

i.e., $\pi y \geqslant \frac{1}{2}$, which means

$$y \geqslant \frac{1}{2\pi} \approx 0.15915.$$

Theorem 3.18 also guarantees the sharpness of this estimate.

Together with Theorem 2.7, this justifies the formulation of Conjectures 1.8 and 1.9.

## 8 Proofs of Theorem 1.11 and Corollary 1.12

We start by recalling Theorem 1.11 and Corollary 1.12:

**Theorem** (Theorem 1.11) *Let* $K, T \subset \mathbb{R}^n$ *be convex bodies. Then, an/the* $\ell_T$*-minimizing closed Minkowski escape path for* $K$ *has* $\ell_T$*-length* $\alpha^*$ *if and only if* $\alpha^*$ *is the largest* $\alpha$ *for which*

$$K \in A(T, \alpha),$$

*i.e., for which for every closed path* $\gamma$ *of* $\ell_T$*-length* $\leqslant \alpha$*, there is a translation* $\mu$ *such that* $K$ *covers* $\mu(\{\gamma\})$.

**Corollary** (Corollary 1.12) *Let $K, T \subset \mathbb{R}^n$ be convex bodies, where $T$ is additionally assumed to be strictly convex. An/The $\ell_T$-minimizing closed $(K,T)$-Minkowski billiard trajectory has $\ell_T$-length $\alpha^*$ if and only if $\alpha^*$ is the largest $\alpha$ for which*

$$K \in A(T,\alpha).$$

In order to prove Theorem 1.11, we start with the two following obvious observations:

**Proposition 8.1** *Let $K \subset \mathbb{R}^n$ be a convex body. Then we have*

$$\{\textit{closed Minkowski escape paths for } K\} = F^{cc}(K).$$

***Proof*** The statement follows directly by recalling that a closed Minkowski escape path is a closed curve whose all translates intersect $\partial K$ and therefore, equivalently, cannot be translated into $\mathring{K}$. □

**Proposition 8.2** *Let $K, T \subset \mathbb{R}^n$ be convex bodies. Then we have*

$$\min_{q\in F^{cc}(K)} \ell_T(q) = \min_{q\in F^{cp}(K)} \ell_T(q).$$

***Proof*** Since

$$F^{cp}(K) \subset F^{cc}(K),$$

it suffices to find for every closed curve $q \in F^{cc}(K)$ a closed polygonal curve $\widetilde{q} \in F^{cp}(K)$ with

$$\ell_T(\widetilde{q}) \leqslant \ell_T(q). \tag{75}$$

If $q$ cannot be translated into $\mathring{K}$, then by the remark beyond [35, Lemma 2.1], there are $n+1$ points on $q$ that cannot be translated into $\mathring{K}$. By connecting these points, we obtain a closed polygonal curve in $F^{cp}(K)$ which we call $\widetilde{q}$. By the subadditivity of the Minkowski functional, it follows (75). □

Based on these propositions, we can prove the analogue to Theorem 1.10:

***Proof of Theorem 1.11*** We first use Proposition 8.1 in order to reduce the statement of Theorem 1.11 to: An/The $\ell_T$-minimizing closed curve in $F^{cc}(K)$ has $\ell_T$-length $\alpha^*$ if and only if $\alpha^*$ is the largest $\alpha$ for which

$$K \in A(T,\alpha). \tag{76}$$

First, let us asssume that $\alpha^*$ is the $\ell_T$-length of an/the $\ell_T$-minimizing closed curve in $F^{cc}(K)$. Then, from Proposition 8.2, we know that there is a closed polygonal curve

$$q^* \in F^{cp}(K) \;\text{ with }\; \ell_T(q^*) = \alpha^*,$$

i.e., $q^*$ is a minimizer of

$$\min_{q\in F^{cp}(K)} \ell_T(q).$$

Then it follows from Proposition 3.15 that

$$K \in A(T, \ell_T(q^*)) = A(T, \alpha^*).$$

Let $\alpha > \alpha^*$. If

$$K \in A(T, \alpha), \tag{77}$$

then

$$L_T(\alpha) \subseteq C(K),$$

i.e., every closed curve of $\ell_T$-length $\alpha$ can be covered by a translate of $K$. This implies that every closed curve of $\ell_T$-length $\lambda\alpha$, $\lambda < 1$, can be covered by a translate of $\mathring{K}$. From this we conclude

$$q^* \notin F^{cp}(K).$$

Therefore, there is no $\alpha > \alpha^*$ for which (77) is fulfilled, i.e., $\alpha^*$ is the largest $\alpha$ for which (76) holds.

Conversely, if $\alpha^*$ is the largest $\alpha$ for which (76) holds. Then, there is a closed curve $q^*$ with

$$q^* \in F^{cc}(K) \cap C(K) \text{ and } \ell_T(q^*) = \alpha^*. \tag{78}$$

Otherwise, if not, then one has

$$q \in C(K) \setminus F^{cc}(K)$$

for all closed curves $q$ of $\ell_T$-length $\alpha^*$. This implies

$$q \in C(\mathring{K})$$

for all closed curves $q$ of $\ell_T$-length $\alpha^*$. But then there is a $\lambda > 1$ such that

$$\lambda q \in C(\mathring{K})$$

for all closed curves of $\ell_T$-length $\alpha^*$. But this is a contradiction to the fact that $\alpha^*$ is the largest $\alpha$ for which (76) holds.

Now, if

$$\min_{q \in F^{cc}(K)} \ell_T(q) =: \widetilde{\alpha} < \alpha^*$$

and $\widetilde{q}$ is a minimizer of the left side, then it follows

$$\widetilde{q} \in C(K)$$

because, due to Proposition 3.2, with $\widetilde{\alpha} < \alpha^*$ one has

$$K \in A(T, \alpha^*) \subseteq A(T, \widetilde{\alpha}).$$

Then, with Lemma 3.16, there is a $\lambda > 1$ such that

$$\lambda\widetilde{q} \in F^{cc}(K) \setminus C(K)$$

with

$$\ell_T(\lambda\widetilde{q}) < \alpha^*.$$

But this is a contradiction to the fact that every closed curve of $\ell_T$-length $\leqslant \alpha^*$ can be covered by a translate of $K$. Therefore, it follows

$$\min_{q\in F^{cc}(K)} \ell_T(q) \geqslant \alpha^*,$$

and together with (78), we conclude that

$$\min_{q\in F^{cc}(K)} \ell_T(q) = \alpha^*.$$

□

The proof of Corollary 1.12 follows immediately:

***Proof of Corollary 1.12*** The proof follows directly by combining Proposition 8.2, [36, Theorem 3.12], and Theorem 1.11. □

## 9 Computational approach for improving the lower bound in Wetzel's problem

In this section, we aim to present a computational approach for improving the best lower bound in Wetzel's problem, which, as stated in Theorem 1.7, is due to Wetzel himself (see [52]). But not only that, our approach most likely also allows to find, more generally, lower bounds in Minkowski worm problems. By Theorem 3.18, these lower bounds eventually translate into upper bounds for systolic Minkowski billiard inequalities as well as for Viterbo's conjecture for convex Lagrangian products.

The main idea of this approach is inspired by a series of works related to the search for area-minimizing convex hulls of closed curves in the plane which are allowed to be translated and rotated. Since the area-minimizing convex cover for a set of closed curves is, equivalently, the area-minimizing convex hull of these closed curves (note that this observation has already used within the proof of Corollary 1.2), these works treat the question of lower bounds for the following version of Moser's worm problem in which *closed* arcs are considered:

*Find a/the convex set of least area that contains a congruent copy of each closed arc in the plane of length one.*

In [11] (applying results from [16]), the first lower bound for the area was found considering the convex hull of a circle and a line segment. In [20], this lower bound was improved by first considering a circle and a certain rectangle and later a circle and a curvilinear rectangle. The latest improvements are due to Grechuk and Som-am who in [24] considered the convex hull of a circle, an equilateral triangle and a certain rectangle, and in [25] the convex hull of a circle, a certain rectangle, and a line segment.

However, in order to adapt these approaches to our setting, in the details, we have to make some changes.

But let us first start with some underlying considerations (as in the proof of Corollary 1.2) in the most general case: For arbitrary convex body $T \subset \mathbb{R}^n$, we ask for lower bounds of

$$\min_{K \in A(T,1)} \operatorname{vol}(K). \tag{79}$$

By referring to the above mentioned main idea, we start by noting that for

$$q_1, ..., q_k \in L_T(1)$$

we have

$$\min_{(a_1,...,a_k) \in (\mathbb{R}^n)^k} \operatorname{vol}(\operatorname{conv}\{q_1 + a_1, ..., q_k + a_k\}) \leqslant \min_{K \in A(T,1)} \operatorname{vol}(K). \tag{80}$$

This estimate can be further improved by

$$\max_{q_1,...,q_k \in L_T(1)} \min_{(a_1,...,a_k) \in (\mathbb{R}^n)^k} \operatorname{vol}(\operatorname{conv}\{q_1 + a_1, ..., q_k + a_k\}) \leqslant \min_{K \in A(T,1)} \operatorname{vol}(K),$$

so that, eventually, we get

$$\min_{a_q \in \mathbb{R}^n} \operatorname{vol}\left(\operatorname{conv}\left\{\bigcup_{q \in L_T(1)} (q + a_q)\right\}\right) = \min_{K \in A(T,1)} \operatorname{vol}(K),$$

where the minimum on the left runs for every $q \in L_T(1)$ over all possible translations in $\mathbb{R}^n$.

Let us now exemplary show how (80) can be used to calculate lower bounds of (79) within the setting of Wetzel's problem, i.e., $n = 2$ and $T = B_1^2$.

Let $q_1$ be the boundary of $B^2_{\frac{1}{2\pi}}$,

$$q_2 = q_2(t_1, t_2, \theta)$$

the boundary of an equilateral triangle $T_{t_1,t_2,\frac{1}{3},\theta}$ with mass point $(t_1, t_2)$, side length $\frac{1}{3}$, and angle $\theta$ between one of the sides and the horizontal line, and let

$$q_3 = q_3(r_1, r_2, \widehat{q})$$

be the boundary of a rectangle $R_{r_1,r_2,1,\widehat{q}}$ with middle point $(r_1, r_2)$, perimeter 1, and quotient of the side lengths $\widehat{q}$.

Then, by definition, we have

$$q_1,\ q_2(t_1, t_2, \theta),\ q_3(r_1, r_2, \widehat{q}) \in L_{B_1^2}(1)$$

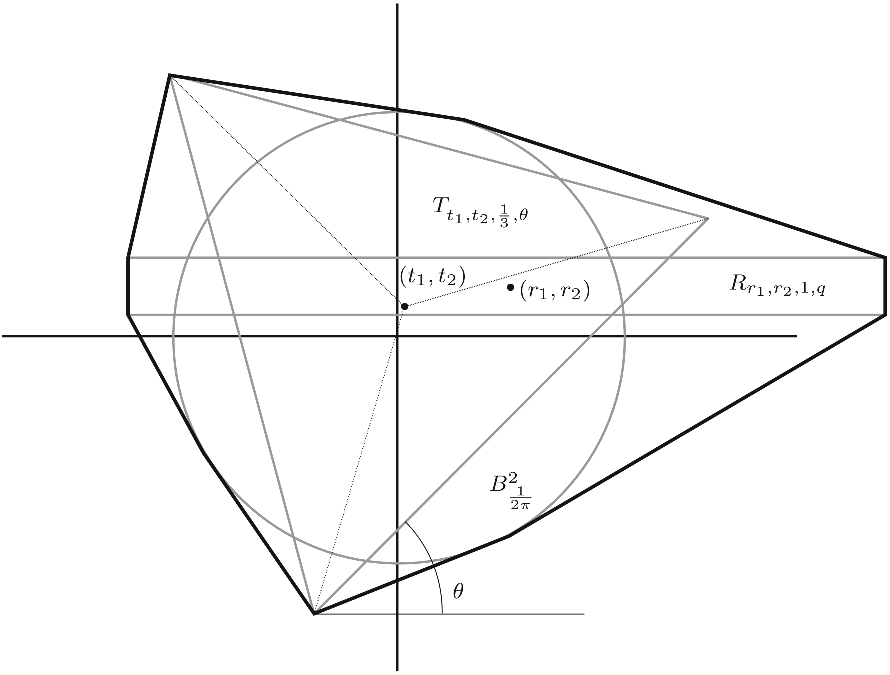


**Fig. 4** Illustration of the convex hull of $B^2_{\frac{1}{2\pi}}$, $R_{r_1,r_2,1,\widehat{q}}$ and $T_{t_1,t_2,\frac{1}{3},\theta}$

for all

$$t_1, t_2 \in \mathbb{R},\ \theta \in \left[0, \frac{3\pi}{4}\right],\ r_1, r_2 \geqslant 0,\ \widehat{q} > 0$$

and (80) (because of $\theta \in \left[0, \frac{3\pi}{4}\right]$ and $\widehat{q} > 0$, one has $k = \infty$) becomes

$$\max_{\theta\in[0,\frac{3\pi}{4}],\ \widehat{q}>0}\ \min_{t_1,t_2\in\mathbb{R},\ r_1,r_2\geqslant 0}\ \mathrm{vol}\left(\mathrm{conv}\left\{B^2_{\frac{1}{2\pi}}, T_{t_1,t_2,\frac{1}{3},\theta}, R_{r_1,r_2,1,\widehat{q}}\right\}\right) \leqslant \min_{K\in A(B^2_1,1)} \mathrm{vol}(K).$$

Then, one can define

$$f(t_1, t_2, r_1, r_2, \theta, \widehat{q}) := \mathrm{vol}\left(\mathrm{conv}\left\{B^2_{\frac{1}{2\pi}}, T_{t_1,t_2,\frac{1}{3},\theta}, R_{r_1,r_2,1,\widehat{q}}\right\}\right)$$

which is a convex function with respect to the first four coordinates $(t_1, t_2, r_1, r_2)$ (this can be shown similar to in [24]) and compute

$$\max_{\theta\in[0,\frac{3\pi}{4}],\ \widehat{q}>0}\ \min_{t_1,t_2\in\mathbb{R},\ r_1,r_2\geqslant 0}\ f(t_1, t_2, r_1, r_2, \theta, \widehat{q}).$$

We leave it at that, starting with (80), gives us the ability to tackle many different Minkowski worm problems–in any dimension, for many different $T$s and by using diverse closed curves

$$q_1, ..., q_k \in L_T(1).$$

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