Monatshefte für Mathematik (2023) 200:163–178
https://doi.org/10.1007/s00605-022-01776-0

# Non-hyperbolic ergodic measures with the full support and positive entropy

Martha Łącka[1]

Received: 14 December 2020 / Accepted: 16 September 2022 / Published online: 27 September 2022


**Abstract**
The aim of this note is to give an alternative proof for the following result originally proved by Bonatti, Díaz and Kwietniak. For every $n \geq 3$ there exists a compact manifold without boundary $\mathbf{M}$ of dimension $n$ and a non-empty open set $U \subset \mathrm{Diff}(\mathbf{M})$ such that for every $f \in U$ there exists a non-hyperbolic measure $\mu$ invariant for $f$ with positive entropy and full support. We also investigate the connection between the Feldman-Katok convergence of measures and the Kuratowski convergence of their supports.

**Keywords** Entropy · Lyapunov exponents · Hyperbolic measures · F-bar pseudometric · Feldman-Katok pseudometric

**Mathematics Subject Classification** 37C05 · 37C20

## 1 Introduction

Thanks to the work of Abraham and Smale it is well known that hyperbolic diffeomorphisms are not dense in the space of all diffeomorphisms of a given manifold [1]. Even more is true: diffeomorphisms that do not admit any non-hyperbolic measure are not dense in this space as well. Let us recall that a measure is non-hyperbolic if it has at least one zero Lyapunov exponent (we recall the definition of the Lyapunov exponents in the preliminaries). This notion was introduced by Pesin [24] in the 70's. It is worth mentioning that every hyperbolic diffeomorphism admits only hyperbolic measures

Communicated by H. Bruin.

✉ Martha Łącka
martha.ubik@uj.edu.pl

1 Faculty of Mathematics and Computer Science, Jagiellonian University in Kraków, ul. Łojasiewicza 6, 30-348 Kraków, Poland

but there are non-hyperbolic diffeomorphisms such that all invariant measures with respect to them are hyperbolic [2, 9].

In 2005 Gorodetski, Ilyashenko, Kleptsyn and Nalsky introduced *royal measures*[1] and used them to construct non-hyperbolic diffeomorphisms on the skew product of Bernoulli shift with a circle as a fibre. They also conjectured that there exists a non-empty open subset $U$ of the set of all diffeomorphisms of the 3-dimensional torus such that every $f \in U$ admits at least one non-hyperbolic measure. Note that Lapunov exponents are sensitive under perturbation so one could suspect that the opposite is true. The conjecture was however proved by Kleptsyn and Nalsky in [19] and the idea of their work was somehow inspired by [15].

Royal measures were also under consideration of other authors, see for instance [6, 8, 10, 11, 19, 21, 28]. It was shown in [21] that the entropy of every royal measure vanishes. The main tool used for that was the *Feldman-Katok convergence* of measures. It allowed the authors to reprove also some already known properties of royal measures originally proved in [15] and in [6] in the more abstract setting. They did not mention however the connection between the Feldman-Katok convergence and the Kuratowski convergence of measures' supports. In this note we prove that under some additional assumptions the Feldman-Katok convergence implies the Kuratowski convergence of supports. This result can be applied to royal measures. We also give an example demonstrating that the sole Feldman-Katok convergence is not enough to conclude that the support of the limit measure is also a limit in the sense of Kuratowski.

Alternative to the construction of non-hyperbolic measures from [15] was the one introduced by Bochi, Bonatti and Díaz. In the series of papers [3–5] they strengthened the results of Kleptsyn and Nalsky showing the following:

- [5, Theorem 1] For every $n \geq 3$ there exists a compact manifold without boundary $\mathbf{M}$ of dimension $n$ and a non-empty open set $U \subset \mathrm{Diff}(\mathbf{M})$ such that for every $f \in U$ there exists a non-hyperbolic measure $\mu$ invariant for $f$ with positive entropy.
- [4, Theorem 9] For every $n \geq 3$ there exists a compact manifold without boundary $\mathbf{M}$ of dimension $n$ and a non-empty open set $U \subset \mathrm{Diff}(\mathbf{M})$ such that for every $f \in U$ there exists a non-hyperbolic measure $\mu$ invariant for $f$ with full support.

The above result ( [4, Theorem 9]) applies in fact to a wide class of manifolds, including any manifold carrying a transitive Anosov flow, in particular to the $n$-dimensional torus for every $n \geq 3$.

In [4, p. 3] authors conjectured that one may strengthen their results so that the resulting non-hyperbolic measure would have both: positive entropy and full support. That was shown in [7]. In this note we give an alternative, shorter proof of this fact. We demonstrate that actually the construction from [4] already leads to a measure with positive entropy.

[1] They did not give them any name, we use the name from [21] which is inspired by the fact that the first letters of the surnames of the authors of [15] form the word 'KING'.

## 2 Basic terminology and notation

### 2.1 The Kuratowski limit

In this section we outline Kuratowski limit, based on [20, Volume I]. For a sequence of non-empty, compact sets $A_1, A_2, \ldots \subset X$ we define the *lower Kuratowski limit* $\operatorname{Li\,top}_{n\to\infty} A_n$ and the *upper Kuratowski limit* $\operatorname{Ls\,top}_{n\to\infty} A_n$ of such a sequence by the conditions:

- $x \in \operatorname{Li\,top}_{n\to\infty} A_n$ if for every neighborhood $U$ of $x$ we have $A_m \cap U \neq \emptyset$ for all sufficiently large $m$.
- $x \in \operatorname{Ls\,top}_{n\to\infty} A_n$ if for every neighborhood $U$ of $x$ we have $A_m \cap U \neq \emptyset$ for infinitely many $m$.

Naturally, for every sequence of non-empty, compact sets $A_1, A_2, \ldots \subset X$ we have:

$$\operatorname{Li\,top}_{n\to\infty} A_n \subset \operatorname{Ls\,top}_{n\to\infty} A_n.$$

We say that $A_1, A_2, \ldots$ *converges to* $A \subset X$, if $A = \operatorname{Li\,top}_{n\to\infty} A_n = \operatorname{Ls\,top}_{n\to\infty}$. Such a limit is then denoted by

$$\operatorname{Lim\,top}_{n\to\infty} A_n = A.$$

The Kuratowski convergence is equivalent to the convergence with respect to the Hausdorff metric (defined on the family of closed and non-empty subsets of $X$) if $X$ is a compact metric space.

### 2.2 Invariant measures

Let $\mathcal{M}_{\mathrm{T}}(X)$ be a set of all Borel probabilistic measures on $X$ that are invariant under the map $T$ acting on the space $X$ and $\mathcal{M}_T^e(X) \subset \mathcal{M}_T(X)$ be the set of measures that are ergodic as well. We consider the weak* topology on $\mathcal{M}_T(X)$, making it a compact and a metrizable space. A sequence $(\mu_n)_{n=1}^{\infty}$ converges to $\mu$ with respect to this topology if and only if for all continuous functions $\varphi\colon X \to \mathbb{R}$ a sequence $\int \varphi\, d\mu_n$ converges to $\int \varphi\, d\mu$ in $\mathbb{R}$. It is known that if $X$ is a compact metric space, then this topology is given by the Prokhorov metric $D_P$:

$$D_P(\mu, \nu) = \inf\left\{\varepsilon > 0 \,:\, \mu(B) \leq \nu(B^{\varepsilon}) + \varepsilon \text{ for every Borel set } B\right\},$$

where $B^{\varepsilon} = \{y \in X \,:\, \operatorname{dist}(y, B) < \varepsilon\}$ is an $\varepsilon$*-hull* of $B$. Moreover, the following portmanteau theorem holds (the portmanteau theorem says in fact more, but we cite only the part we will use in this paper).

**Theorem 1** *(*The portmanteau theorem) The following conditions are equivalent:

(i) A sequence $(\mu_n)_{n\in\mathbb{N}} \subset \mathcal{M}_{\mathrm{T}}(X)$ weakly* converges to $\mu \in \mathcal{M}_{\mathrm{T}}(X)$.

(ii) For each measurable set $B \subset Y$ for which $\mu(\partial B) = 0$ it holds that:

$$\lim_{n \to \infty} \mu_n(B) = \mu(B).$$

### 2.3 Empirical measures

Given $n \in \mathbb{N}$ and a sequence $\underline{x} = (x_i)^{i \in \mathbb{N}} \subset X$ we define the *empirical measure* $\mathfrak{m}(\underline{x}, n)$ as follows:

$$\mathfrak{m}(\underline{x}, n) = \frac{1}{n} \sum_{i=0}^{n} \delta_{x_i},$$

where $\delta_z$ denotes the Dirac measure supported at $z$. If we fix a map $T: X \to X$, then for $x \in X$ we put

$$\mathfrak{m}_T(x, n) = \mathfrak{m}((T^i(x))_{i \in \mathbb{N}}, n)$$

and we sometimes omit the subscript $T$ if it can be derived from the context. A point $x \in X$ is said to be *generic* for $\mu \in \mathcal{M}_{\mathrm{T}}(X)$ if $(\mathfrak{m}(x, n))_{n \in \mathbb{N}}$ converges to $\mu$ with respect to the weak* topology. We denote by $\hat{\omega}(x)$ the set of all weak* accumulation points of the sequence $(\mathfrak{m}(x, n))_{n \in \mathbb{N}}$ (if $x$ is generic for $\mu$ then $\hat{\omega}(x) = \{\mu\}$, but in general $\hat{\omega}(x)$ can contain more than one element).

### 2.4 Hyperbolic measures

Let $\mathbf{M}$ be a smooth $m$-dimensional manifold. For a diffeomorphism $T: \mathbf{M} \to \mathbf{M}$ and $T$-invariant ergodic measure $\mu$, there exists $\Lambda \subset \mathbf{M}$ as well as $\chi_\mu^1 \leq \ldots \leq \chi_\mu^m \in \mathbb{R}$, such that $\mu(\Lambda) = 1$ and for all $x \in \Lambda$ and $0 \neq v \in T_x M$ the following is true:

$$\lim_{n \to \infty} \frac{1}{n} \log \|DT_x^n(v)\| = \chi_\mu^i \quad \text{for some } i \in \{1, \ldots, m\}.$$

Numbers $\chi_\mu^i$ are the *Lyapunov exponents* and if they are all non-zero we call $\mu$ *hyperbolic*. If $\chi_\mu^i = 0$ for some $i \in \{1, \ldots, m\}$ then the measure $\mu$ is *non-hyperbolic*.

### 2.5 Royal measures

A $T$-periodic orbit $\Gamma$ approximates $(\gamma, \kappa)$-well a $T$-periodic orbit $\Lambda$ if there exist $\Delta \subset \Gamma$ with $|\Delta|/|\Gamma| \geq \kappa$ and a constant-to-one surjection $\psi: \Delta \to \Lambda$ such that for each $y \in \Delta$ and $0 \leq j < |\Lambda|$ we have

$$\rho(T^j(y), T^j(\psi(y))) < \gamma.$$

Such $\psi$ is called $(\gamma, \kappa)$-*projection*.

The following theorem was given in [6, Lemma 2.5]. The proof that $\mu$ is ergodic [6] uses Lemma 2 from [15].

**Theorem 2** *We are given a sequence of $T$-periodic orbits $(\Gamma_n)_{n\in\mathbb{N}}$ such that $|\Gamma_n|$ increases with n, and the ergodic measure $\mu_n$ supported on $\Gamma_n$. If there exist sequences of real positive numbers $(\gamma_n)_{n=1}^{\infty}$ and $(\kappa_n)_{n=1}^{\infty}$ that satisfy*

*(1) for each n the orbit $\Gamma_{n+1}$ is a $(\gamma_n, \kappa_n)$-good approximation of $\Gamma_n$,*

*(2) $\sum_{n=1}^{\infty} \gamma_n < \infty$,*

*(3) $\prod_{n=1}^{\infty} \kappa_n > 0$,*

*then $(\mu_n)_{n\in\mathbb{N}}$ weak* converges to an ergodic measure $\mu$ supported on the Kuratowski limit of $(\Gamma_n)_{n\in\mathbb{N}}$.*

The sequence of $T$-periodic orbits fulfilling the assumptions of Theorem 2 is called a *GIKN sequence*. *Royal measures* are (by the definition) weak* limits of GIKN-sequences.

### 2.6 The symbolic dynamics

Throughout this paper $\mathscr{A}$ is a finite set with the discrete topology. We define a *shift space over an alphabet* $\mathcal{A}$ as $\mathcal{A}^{\infty}$ equipped with the product topology together with the shift map $\sigma$ given by the formula $\sigma((x_n)_{n=0}^{\infty}) = (x_n)_{n=1}^{\infty}$. We denote by $\mathcal{A}^n$ the set of the words of *length-n over* $\mathcal{A}$. Let $\mathcal{A}^* = \bigcup_{n\geq 1} \mathcal{A}^n$ and let $|u|$ be the word-length of $u \in \mathcal{A}^*$. Each word $u \in \mathcal{A}^*$ defines its *cylinder set* $[u] \subset \mathcal{A}^{\infty}$ consisting of all the sequences in $\mathcal{A}^{\infty}$ which are prefixed by $u$. The cylindric sets form a open-closed basis of the topology for the shift space $\mathcal{A}^{\infty}$. For $A = a_1 \cdots a_n \in \mathscr{A}^*$ and $k \in \mathbb{N}$ let

$$A^k := \underbrace{a_1 \cdots a_n \quad a_1 \cdots a_n \quad \cdots \quad a_1 \cdots a_n}_{k \text{ times}},$$

$$A^{\infty} := a_1 \cdots a_n \quad a_1 \cdots a_n \quad a_1 \cdots a_n \quad a_1 \cdots a_n \quad \cdots$$

Let also $\mathfrak{p}(A) \in \mathcal{M}_{\sigma}^{e}(\mathscr{A}^{\infty})$ be a periodic measure generated by a periodic sequence $A^{\infty}$.

### 2.7 Entropy

Let $\mathcal{P}$ be a finite and a measurable division of $X$ and $\mu \in \mathcal{M}_T(X)$. The *entropy* $\mathcal{P}$ *with respect to* $\mu$ *and* $T$ is denoted by $h(X, T, \mu, \mathcal{P}) = h(\mu, \mathcal{P})$ and the *entropy* $\mu$ *with respect to* $T$ - by $h(X, T, \mu) = h(\mu)$, that is $h(\mu) = \sup_{\mathcal{P}} h(\mu, \mathcal{P})$. In [12] the reader will find more information about the entropy.

### 2.8 The $\bar{f}$ pseudometric

In this section we introduce the $\bar{f}$ pseudometric that was defined by Feldman [14] and independently by Katok [17].

Given $\underline{u} = u_0 u_1 \cdots u_{n-1}$, $\underline{w} = w_0 w_1 \cdots w_{n-1} \in \mathscr{A}^n$ let

$$\bar{f}_n(u, w) = 1 - \frac{k}{n},$$

where $k$ is such that for some

$$0 \leq i_1 < i_2 < \cdots < i_k < n \text{ and } 0 \leq j_1 < j_2 < \cdots < j_k < n$$

we have $u_{i_s} = w_{j_s}$ for all $s = 1, \cdots, k$ and there is no larger $k$ with this property. The $\bar{f}$ pseudometric is defined for $\underline{u}, \underline{w} \in \mathscr{A}^\infty$ as

$$\bar{f}(\underline{u}, \underline{w}) = \limsup_{n \to \infty} \bar{f}_n(u_0 \cdots u_{n-1}, w_0 \cdots w_{n-1}).$$

If $A = a_1 \cdots a_n$, $A' = a'_1 \cdots a'_{n'}$ and $m \leq |A|, |A'|$, then we define

$$\bar{f}_m(A, A') = \bar{f}_m(a_1 \cdots a_m, \quad a'_1 \cdots a'_m).$$

The following theorem comes from [13].

**Theorem 3** *Given a finite set $\mathscr{A}$ and $\varepsilon > 0$ one can find $\delta > 0$ such that if $\underline{x}, \underline{z} \in \mathscr{A}^{\mathbb{N}}$ satisfy $\bar{f}(\underline{x}, \underline{z}) < \delta$ and $\hat{\omega}(\underline{x}) = \{\mu\}$, $\hat{\omega}(\underline{z}) = \{\nu\}$ for some (not necessarily ergodic) measures $\mu$ and $\nu$, then $|h(\mu) - h(\nu)| < \varepsilon$.*

For the further properties of $\bar{f}$ we refer to [14, 16, 17, 23, 26].

### 2.9 The Feldman-Katok pseudometric $\bar{\Phi}$

Let $(X, \rho)$ be a compact metric space. Fix $\underline{x} = (x_j)_{j=0}^\infty$, $\underline{z} = (z_j)_{j=0}^\infty \in X^\infty$, $\delta > 0$, and $n \in \mathbb{N}$. An $(n, \delta)$-*match* of $\underline{x}$ and $\underline{z}$ is an increasing bijection $\pi : \mathcal{D}(\pi) \to \mathcal{R}(\pi)$ such that $\mathcal{D}(\pi), \mathcal{R}(\pi) \subset \{0, 1, \cdots, n-1\}$ and for every $i \in \mathcal{D}(\pi)$ we have $\rho(x_i, z_{\pi(i)}) < \delta$. An $(n, \delta)$-match $\pi : \mathcal{D}(\pi) \to \mathcal{R}(\pi)$ is *maximal* if $|\mathcal{D}(\pi)|$ the largest possible. The $(n, \delta)$-*gap* between $\underline{x}$ and $\underline{z}$ is defined by

$$\bar{f}_{n,\delta}(\underline{x}, \underline{z}) = 1 - \frac{\max\{|\mathcal{D}(\pi)| : \pi \text{ is an } (n, \delta) - \text{match of } \underline{x} \text{ with } \underline{z}\}}{n}.$$

The $\bar{f}_\delta$-*pseudometric* between $\underline{x}$ and $\underline{z}$ is given by

$$\bar{f}_\delta(\underline{x}, \underline{z}) = \limsup_{n \to \infty} \bar{f}_{n,\delta}(\underline{x}, \underline{z}).$$

Finally, we define the *Feldman-Katok pseudometric on* $X^\infty$ as follows

$$\bar{\Phi}(\underline{x}, \underline{z}) = \inf\{\delta > 0 : \bar{f}_\delta(\underline{x}, \underline{z}) < \delta\}.$$

The above formula induces a pseudometric on $X$ (which we also call the *Feldman-Katok pseudometric* and denote by $\bar{\Phi}$) in the following way:

$$\bar{\Phi}(x,z) := \bar{\Phi}\big((T^i(x))_{i\in\mathbb{N}},\ (T^i(x))_{i\in\mathbb{N}}\big).$$

We define $\bar{f}_{n,\delta}(x,z)$ and $\bar{f}_\delta(x,z)$ in the obvious way.

It is not known if the pseudometric $\bar{\Phi}$ induces somehow a (pseudo)metric on the simplex of invariant measures $\mathcal{M}_{\mathrm{T}}(X)$. It gives us however the notion of convergence as explained below. We say that $\underline{z} = (z_n)_{n=0}^\infty \in X^\infty$ is a *quasi-orbit* for $T$ if $\bar{d}(\{n \geq 0 : z_{n+1} \neq T(z_n)\}) = 0$, where $\bar{d}$ denotes the upper asymptotic density. A sequence of measures $(\mu_n)_{n=1}^\infty \subset \mathcal{M}_{\mathrm{T}}(X)$ *converges in* $\bar{\Phi}$ or $\bar{\Phi}$*-converges* to $\mu \in \mathcal{M}_{\mathrm{T}}(X)$ if there exists a sequence of quasi-orbits $(\underline{x}^{(n)})_{n=1}^\infty \subset X^\infty$ with $\hat{\omega}(\underline{x}^{(n)}) = \{\mu_n\}$ such that for some $\mu$-generic quasi-orbit $\underline{z} \in X^\infty$ we have $\bar{\Phi}(\underline{z}, \underline{x}^{(n)}) \to 0$ as $n \to \infty$.

We cite the below lemmas for the future reference. The details and other properties of the Feldman-Katok topologies can be found in [21].

**Lemma 4** *If* $\bar{f}_{n,\delta}(\underline{x},\underline{z}) < \varepsilon$*, then* $D_P(\mathfrak{m}(\underline{x},n), \mathfrak{m}(\underline{z},n)) < \max\{\delta,\varepsilon\}$*. Consequently, the Feldman-Katok convergence implies the weak* convergence.*

**Lemma 5** *If* $i \geq 0$*, then* $\bar{f}_\delta(\underline{x},\underline{z}) = \bar{f}_\delta(\underline{x},\sigma^i(\underline{z}))$*. In particular,* $\bar{f}_\delta(\underline{x},\sigma^i(\underline{x})) = 0$.

## 3 Non-hyperbolic measures with the full support and positive entropy

To prove Theorem 7 (which together with the theorem of Bochi, Bonatti and Díaz implies our main result — Theorem 9 concerning the existence of non-hyperbolic measures with the full support and positive entropy) we need the below technical lemma. Informally, it says the following. Assume that two weak* convergent sequences of periodic measures on the shift space are given. If the periodic orbits generating these sequences are Feldman-Katok close to each other, then using them one can build generic points for the limit measures for which the Feldman-Katok pseudometric is small.

**Lemma 6** *Let* $(\underline{s}^n)_{n\in\mathbb{N}}, (\underline{w}^{(n)})_{n\in\mathbb{N}} \subset \mathscr{A}^*$*,* $\mu,\nu \in \mathcal{M}_\sigma(\mathscr{A}^\infty)$ *and* $\delta > 0$*. Assume also that:*

*(1)* $\mathfrak{p}(\underline{s}^{(n)}) \to \mu$ *and* $\mathfrak{p}(\underline{w}^{(n)}) \to \nu$ *as* $n \to \infty$*,*
*(2) for every* $n \in \mathbb{N}$ *one has* $|\underline{s}^{(n)}| < |\underline{s}^{(n+1)}|$ *and* $|\underline{w}^{(n)}| < |\underline{w}^{(n+1)}|$*,*
*(3) there exists* $M \in \mathbb{N}$ *such that for all sufficiently large* $n \in \mathbb{N}$ *and* $m \geq M$ *and for all* $k$ *satisfying* $k \cdot \min\{|\underline{s}^{(n)}|, |\underline{w}^{(n)}|\} > m$ *the following inequality holds* $\bar{f}_m((\underline{s}^{(n)})^k, (\underline{w}^{(n)})^k) \leq \delta$.

*Then there exist* $\underline{s}', \underline{w}' \in \mathscr{A}^\infty$ *such that* $\underline{s}'$ *is generic for* $\mu$*,* $\underline{w}'$ *is generic for* $\nu$ *and* $\bar{f}(\underline{s}', \underline{w}') \leq \delta$.

***Proof*** For every $n \in \mathbb{N}$ pick $M_n \in \mathbb{N}$ such that for all $m \geq M_n$ one has $\min\{|\underline{s}^{(m)}|, |\underline{w}^{(m)}|\} > 2^n$. Since the cylinder sets are clopen it follows from the portmanteau theorem that for every $n \in \mathbb{N}$ there is $k_n \in \mathbb{N}$ such that for all $k \geq k_n$ and

$A \in \mathscr{A}^n$ the following inequalities hold:

$$|\mathfrak{p}(s^{(k)})([A]) - \mu([A])| < 1/2^n \text{ and } |\mathfrak{p}(w^{(k)})([A]) - \nu([A])| < 1/2^n. \tag{1}$$

Let $(l_n)_{n\in\mathbb{N}} \subset \mathbb{N}$ satisfy for every $n \in \mathbb{N}$ the following condition:

$$l_n \geq 2^n \cdot \max\left\{|\underline{s}^{(k_{n+1})}|,\ \sum_{i=0}^{n-1} l_i|\underline{s}^{(k_i)}|,\ |\underline{w}^{(k_{n+1})}|,\ \sum_{i=0}^{n-1} l_i|\underline{w}^{(k_i)}|\right\}. \tag{2}$$

We will show that the sequences

$$\underline{s}' = \underbrace{\underline{s}^{(k_1)}\underline{s}^{(k_1)}\cdots\underline{s}^{(k_1)}}_{l_1 \text{ times}}\underbrace{\underline{s}^{(k_2)}\underline{s}^{(k_2)}\cdots\underline{s}^{(k_2)}}_{l_2 \text{ times}}\underbrace{\underline{s}^{(k_3)}\underline{s}^{(k_3)}\cdots\underline{s}^{(k_3)}}_{l_3 \text{ times}}\cdots,$$

$$\underline{w}' = \underbrace{\underline{w}^{(k_1)}\underline{w}^{(k_1)}\cdots\underline{w}^{(k_1)}}_{l_1 \text{ times}}\underbrace{\underline{w}^{(k_2)}\underline{w}^{(k_2)}\cdots\underline{w}^{(k_2)}}_{l_2 \text{ times}}\underbrace{\underline{w}^{(k_3)}\underline{w}^{(k_3)}\cdots\underline{w}^{(k_3)}}_{l_3 \text{ times}}\cdots$$

are generic for $\mu$ and $\nu$, respectively. Since cylinder sets form a basis for the product topology on $\mathscr{A}^\infty$ and are clopen it is enough to prove that for every $A \in \mathscr{A}^*$ one has

$$\lim_{p\to\infty} |\mathfrak{m}(\underline{s}', p)([A]) - \mu([A])| = 0 \text{ and } \lim_{p\to\infty} |\mathfrak{m}(\underline{w}', p)([A]) - \nu([A])| = 0.$$

Fix $A \in \mathscr{A}^*$ and $\varepsilon > 0$. Let $N \in \mathbb{N}$ be such that $\varepsilon > (N+2)/2^N$ and $|A| \leq N$. Pick $n \geq N$. Let $c(A, n)$ denote the frequency with ihich $A$ occurs as a subword of $\underline{s}^{(k_n)}$. Then

$$|\underline{s}^{k_n}| \cdot \mathfrak{p}(\underline{s}^{k_n})([A]) \leq |\underline{s}^{k_n}| \cdot c(A, n) \leq |\underline{s}^{k_n}| \cdot \mathfrak{p}(\underline{s}^{k_n})([A]) + (|A| - 1).$$

Since $|\underline{s}^{(k_n)}| > 2^N$ and $|A| \leq N$ it follows from (1) that

$$\mu([A]) - N/2^N \leq c(A, n) \leq \mu([A]) + 1/2^N. \tag{3}$$

Define

$$P_0 = \sum_{i=1}^{N} l_i |s^{(k_i)}|.$$

Fix $p \geq P_0$ and put

$$R = \max\left\{P \in \mathbb{N} : \sum_{i=1}^{P} l_i |s^{(k_i)}| \leq p\right\}.$$

In other words prefix of $\underline{s}'$ of the length $p$ is of the form

$$\underbrace{\underline{s}^{(k_1)}\underline{s}^{(k_1)}\dots\underline{s}^{(k_1)}}_{l_1 \text{ times}}\cdots\underbrace{\underline{s}^{(k_R)}\underline{s}^{(k_R)}\dots\underline{s}^{(k_R)}}_{l_R \text{ times}}\underbrace{\underline{s}^{(k_{R+1})}\underline{s}^{(k_{R+1})}\dots\underline{s}^{(k_{R+1})}}_{0\le l<l_{R+1} \text{ times}}\underline{t},$$

where $\underline{t}$ is (possibly, empty) prefix of $\underline{s}^{(k_{R+1})}$ (different from $\underline{s}^{(k_{R+1})}$). Since $p \ge P_0$, one has $R \ge N$. Consequently, it follows from (3) that

$$\mu([A]) - N/2^N \le c(A, R) \le \mu([A]) + 1/2^N$$
$$\text{and } \mu([A]) - N/2^N \le c(A, R+1) \le \mu([A]) + 1/2^N.$$

The above inequalities imply that the frequency with which $A$ occurs in the block

$$\underbrace{\underline{s}^{(k_R)}\underline{s}^{(k_R)}\dots\underline{s}^{(k_R)}}_{l_R \text{ times}}\underbrace{\underline{s}^{(k_{R+1})}\underline{s}^{(k_{R+1})}\dots\underline{s}^{(k_{R+1})}}_{l \text{ times}}$$

is not smaller than $\mu([A]) - N/2^N$ and not larger than $\mu([A]) + 1/2^N$. What is more, the condition (2) says that the length of this block is at least $2^N/(2^N+2)$ times more than the length of the prefix of $\underline{s}'$ of the length $p$. Therefore

$$\mu([A]) - \varepsilon < \mu([A]) - (N+2)/2^N < \mathfrak{m}(\underline{s}', p)([A])$$
$$\le \mu([A]) + (N+2)/2^N < \mu([A]) + \varepsilon.$$

Since $\varepsilon$ is arbitrary we have that $|\,\mathfrak{m}(\underline{s}', p)([A]) - \mu([A])| \to 0$ as $p \to \infty$. Analogously one can show that $|\,\mathfrak{m}(\underline{w}', p)([A]) - \nu([A])| \to 0$ as $p \to \infty$.

To finish the proof it is enough to notice that $\bar{f}(\underline{s}', \underline{w}') \le \delta$. This is however obvious since for sufficiently large $n$ the match $\pi$ of $n$-prefixes of $\underline{s}'$ and $\underline{w}'$ satisfying $|\mathcal{D}(\pi)| \ge n(1-\delta)$ can be obtained by taking a concatenation of the optimal matches of words $\underline{s}^{(k_i)}$ and $\underline{w}^{(k_i)}$ from these prefixes. □

**Theorem 7** *Let $(X, T)$ be a dynamical system, $\alpha$ — a positive constant, $\psi : X \to \mathbb{R}$ — a continuous function, $(M_n)_{n\in\mathbb{N}}$ — an increasing sequence of positive integers, $(\underline{s}^{(n)})_{n\in\mathbb{N}}$ — a sequence of finite words over the alphabet $\{+,-\}$, $(x_n)_{n\in\mathbb{N}}$ — a sequence of points from the space $X$, and let $K(+), K(-) \subset X$ be compact sets. Fix a measure $\mu \in \mathcal{M}_\sigma(\{+,-\}^\infty)$ with the positive entropy and $\varepsilon \in (0, h(\mu))$ and pick $\delta > 0$ for this $\varepsilon$ as in Theorem 3. Assume also that:*

*(1) $\psi|_{K(+)} > \alpha$ and $\psi|_{K(-)} < -\alpha$,*
*(2) $M_1$ divides $M_n$ for every $n \in \mathbb{N}$,*
*(3) for every $n \in \mathbb{N}$ there exists a set $I_n \subset \{0, 1, \cdots M_n/M_1\}$ such that $|\underline{s}^{(n)}| = |I_n|$, the inequality $|I_n| > (1-\delta)M_n/M_1$ is satisfied and for every $j \in I_n$ one has*

$$T^{jM_1}(x_n) \in K(s^{(n)}_{\Lambda_n(j)}),$$

*where the function* $\Lambda_n\colon \{0, 1, \cdots, |I_n| - 1\} \to \mathbb{N}$ *is given by*

$$\Lambda_n(j) = \left|\{0 \leq i < j \,:\, i \in I_n\}\right|,$$

*(4)* $\mathfrak{p}(\underline{s}^{(n)}) \to \mu$ *as* $n \to \infty$.

*Let* $\omega$ *be an accumulation point with respect to the weak*$^*$ *topology of the sequence of empirical measures*

$$\left(\frac{1}{M_n}\sum_{j=0}^{M_n-1}\delta_{T^j(x_n)}\right)_{n\in\mathbb{N}}.$$

*Then* $h(\omega) > (h(\mu) - \varepsilon)/M_1 > 0$.

***Proof*** Note that $\omega$ is a $T$-invariant measure concentrated on the orbit's closure of the point $x$. For every $n \in \mathbb{N}$ and $0 \leq j < M_n/M_1$ put $l_j^{(n)} = jM_1$. Since $\omega$ is a finite measure and the interval $(-\alpha, \alpha)$ is uncountable, there is a real number $\beta$ such that $-\alpha < \beta < \alpha$ and

$$\omega\left(\partial\psi^{-1}\big((-\infty, \beta)\big)\right) = \omega\left(\partial\psi^{-1}\big([\beta, \infty)\big)\right) = 0.$$

Denote $P(-) = \psi^{-1}(-\infty, \beta))$ and $P(+) = \psi^{-1}\big([\beta, \infty)\big)$. Notice that $K(+) \subset P(+)$, $K(-) \subset P(-)$ and $\mathcal{P} = \{P(-), P(+)\}$ is a measurable finite partition of $X$. For $n \in \mathbb{N}$ define the words $\underline{t}^{(n)} \in \{+,-\}^{M_n}$ and $\underline{w}^{(n)} \in \{+,-\}^{M_n/M_1}$ by the formulas:

$$t_i^{(n)} = *, \;\text{ if } T^i(x_n) \in P_*, \quad \text{ where } 0 \leq i < M_n \;\text{ and } * \in \{+,-\},$$
$$w_j^{(n)} = *, \;\text{ if } T^{jM_1}(x_n) \in P_*, \quad \text{ where } 0 \leq j < M_n/M_1 \;\text{ and } * \in \{+,-\}.$$

Passing to a subsequence if necessary we can assume that $\omega$ is a limit (not only an accumulation point) of the sequence

$$\left(\frac{1}{M_n}\sum_{j=0}^{M_n-1}\delta_{T^j(x_n)}\right)_{n\in\mathbb{N}},$$

$(\mathfrak{p}(w^{(n)}))_{n\in\mathbb{N}} \to \tilde{\omega}$ and $(\mathfrak{m}(\underline{w}^{(n)}))_{n\in\mathbb{N}} \to \tilde{\nu}$ for some $\tilde{\nu}, \tilde{\omega} \in \mathcal{M}_\sigma(\{+,-\}^\infty)$. We will show that $h(\omega, \mathcal{P}) > 0$. To this end note that the sequences $(\underline{s}^{(n)})_{n\in\mathbb{N}}$ and $(\underline{w}^{(n)})_{n\in\mathbb{N}}$ satisfy the assumptions of Lemma 6 for $\delta$. Therefore we can use Theorem 3 to conclude that $h(\tilde{\nu}) > h(\mu) - \varepsilon$ and consequently $h(\omega, \mathcal{P}) = h(\tilde{\omega}) \geq h(\tilde{\nu})/M_1 > (h(\mu) - \varepsilon)/M_1 > 0$. □

***Remark 8*** Bochi, Bonatti and Díaz proved in [5] that for every $n \geq 3$ there exists a compact manifold without boundary $\mathbf{M}$ of dimension $n$ such that for every $\delta > 0$ one can find a non-empty open set $U \subset \mathrm{Diff}(\mathbf{M})$ such that for every map $T \in U$ one of the Lapunov exponents with respect to $T$ is given by the integral of some continuous function $\psi_T\colon \mathbf{M} \to \mathbb{R}$ and there are non-empty compact sets $K(+), K(-) \subset \mathbf{M}$, a number $\alpha > 0$ and an increasing sequence $(M_n)_{n\in\mathbb{N}} \subset \mathbb{N}$ satisfying the following properties:

(1) $\psi_T|_{K(+)} > \alpha,\ \psi_T|_{K(-)} < -\alpha$,
(2) $M_1$ divides $M_n$ for every $n \in \mathbb{N}$,
(3) there are sets $I_n \subset \{0, 1, \cdots M_n/M_1\}$ for $n \in \mathbb{N}$ such that:

   (i) $|I_n| > (1-\delta)M_n/M_1$ for every $n \in \mathbb{N}$,
   (ii) if we pick a sequence $\underline{s} \in \{+,-\}^\infty$ in such a way that $|\underline{s}^{(n)}| = |I_n|$ for every $n \in \mathbb{N}$ and $s^{(m)}$ is a prefix of $s^{(n)}$ for all $m < n$, then we can find a point $x \in X$ such that for every $j \in I_n$ we have

$$T^{jM_1}(x) \in K\left(s_{\Lambda_n(j)}\right),$$

   where the function $\Lambda_n\colon \{0, 1, \cdots, |I_n|-1\} \to \mathbb{N}$ is given by the formula

$$\Lambda_n(j) = \big|\{0 \le i < j\ :\ i \in I_n\}|,$$

(4) In addition, for the above point $x$ the following holds. For every measure $\omega \in \hat{\omega}(x)$ and for $\omega$-almost every point $z \in \mathbf{M}$ the conditions below are satisfied:

   i. $\displaystyle \frac{1}{N}\sum_{j=0}^{N-1} \psi_T^j(z) \to 0 \quad \text{as } N \to \infty,$
   ii. the orbit of $z$ is dense in $\mathbf{M}$..

We will now prove the main theorem of this section.

**Theorem 9** *For every $n \ge 3$ there exist a compact $n$-dimensional manifold without boundary $\mathbf{M}$ and a non-empty open set $U \subset \mathit{Diff}(\mathbf{M})$ such that for every $T \in U$ there exists an non-hyperbolic measure $\mu \in \mathcal{M}_T^e(\mathbf{M})$ with the full support and positive entropy.*

***Proof*** Pick $\underline{s} \in \{+.-\}^\infty$ which is generic for some measure $\mu \in \mathcal{M}_\sigma(\{+,-\}^\infty)$ with positive entropy. Fix $\varepsilon \in (0, h(\mu))$ and let $\delta > 0$ be chosen for this $\varepsilon$ as in Theorem 3. Let $\mathbf{M}$ and $\mathcal{U} \subset \mathrm{Diff}(\mathbf{M})$ be defined as in Remark 8. Fix $T \in U$. Let $K(+), K(-) \subset \mathbf{M}$, $\alpha > 0$, $\psi_T \in \mathcal{C}(\mathbf{M})$ and the sequences $(M_n)_{n\in\mathbb{N}}$ and $(I_n)_{n\in\mathbb{N}}$ be defined as in Remark 8 for $T$ and $\delta$.

Use Theorem 7 for the following set of data:

(1) sets $K(+), K(-) \subset \mathbf{M}$, $\alpha > 0$, $\psi_T \in \mathcal{C}(\mathbf{M})$ and the sequence $(M_n)_{n\in\mathbb{N}} \subset \mathbb{N}$,
(2) the sequence $(\underline{s}^{(n)})_{n\in\mathbb{N}}$ of the form $\underline{s}^{(n)} = s_0 s_1 \cdots s_{|I_n|-1}$,
(3) the measure $\mu$ and the numbers $\varepsilon > 0$ and $\delta > 0$,
(4) the sequence $(x_n)_{n\in\mathbb{N}} \subset \mathbf{M}$ constantly equal to $x$,
(5) the measure $\omega \in \mathcal{M}_T(\mathbf{M})$ which is an accumulation point of the sequence

$$\left(\frac{1}{M_n}\sum_{j=0}^{M_n-1} \delta_{T^j(x)}\right)_{n\in\mathbb{N}}.$$

We get $h(\omega) > (h(\mu) - \varepsilon)/M_1 > 0$. Note that $\omega \in \hat{\omega}(x)$ and hence it follows from Remark 8 that for $\omega$-almost every point $z \in \mathbf{M}$ the following conditions are satisfied:

(i) $\displaystyle\frac{1}{N}\sum_{j=0}^{N-1} \psi_T^j(z) \to 0, \quad \text{as } N \to \infty,$

(ii) the orbit of $z$ is dense in $\mathbf{M}$.

Let $\omega'$ be a measure from the ergodic decomposition of $\omega$ with a positive entropy (such a measure exists since $h(\omega) > 0$). Let $Z \subset \mathbf{M}$ be the set of points for which both conditions (i) and (ii) are satisfied. It follows from the ergodic decomposition theorem that $\omega'(Z) = 1$. The Birkhoff ergodic theorem together with the condition (i) imply that $\int_{\mathbf{M}} \psi_T \, \mathrm{d}\omega' = 0$, and so $\omega'$ is a non-hyperbolic measure. Moreover, from the condition (ii) we get that $\omega'$ is fully supported. □

## 4 Feldman-Katok convergence of measures vs kuratowski convergence of supports

The aim of this section is to describe the relationship between the Feldman–Katok convergence of measures and the Kuratowski convergence of their supports.

**Theorem 10** *Let $(x_n)_{n\in\mathbb{N}} \subset X$ be a Cauchy sequence of periodic points with respect to $\bar{\Phi}$. For $m, n \in \mathbb{N}$ denote by $p_n$ the period of $x_n$, by $\mu_n$ the measure generated by $x_n$ and by $\pi_{m,n}^{(p)} \colon \mathcal{D}(\pi_{m,n}^{(p)}) \to \mathcal{R}(\pi_{m,n}^{(p)})$ the maximal $(p, \bar{\Phi}(x_m, x_n))$-match of $x_m$ and $x_n$. Assume that*

$$\inf_{m\geq n} \liminf_{p\to\infty} \frac{|\mathcal{R}(\pi_{m,n}^{(p)}) \cap (p_n \cdot \mathbb{Z} + j)|}{p} > 0 \textit{ for every } 0 \leq j < p_n.$$

*Denote by $\mu$ the limit of the sequence $(\mu_n)_{n\in\mathbb{N}}$.*
*Then*

$$\operatorname{supp} \mu = \operatorname{Lim\,top}_{n\to\infty} \operatorname{supp} \mu_n.$$

***Proof*** We will prove that $\operatorname{Ls\,top}_{n\to\infty} \operatorname{supp} \mu_n \subset \operatorname{supp} \mu$ which is enough as the inclusion $\operatorname{supp} \mu \subset \operatorname{Li\,top}_{n\to\infty} \operatorname{supp} \mu_n$ follows from the weak* convergence of the sequence $(\mu_n)_{n\in\mathbb{N}}$ (see [22, Theorem 1.59]). Choose $z \in \operatorname{Ls\,top}_{n\to\infty} \operatorname{supp} \mu_n$.Let $k_n \nearrow \infty$ and $(x_n')_{n\in\mathbb{N}}$ be such that for every $n \in \mathbb{N}$ one has $x_n' \in \operatorname{supp} \mu_{k_n}$ and $\rho(z, x_n') \to 0$ as $n \to \infty$. Fix $\varepsilon > 0$. We will show that $\mu(B(z, \varepsilon)) > 0$. To this end choose $n$ such that

(i) $x_n' \in B(z, \varepsilon/3)$,

(ii) $\sup_{m\geq n} \bar{\Phi}(x_m', x_n') < \varepsilon/3$ (this condition is satisfied for $n$ large enough as it follows from Lemma 5 that $\bar{\Phi}(x_n', x_m') = \bar{\Phi}(x_{k_n}, x_{k_m})$).

Let $x_n' = T^j(x_{k_n})$ for some $0 \leq j < p_{k_n}$. Denote

$$\inf_{m\geq n} \liminf_{p\to\infty} \frac{|\mathcal{R}(\pi_{k_m,k_n}^{(p)}) \cap (p_{k_n} \cdot \mathbb{Z} + j)|}{p} =: \alpha > 0. \tag{4}$$

Fix $m \geq n$. Note that for every $p \in \mathbb{N}$ we can consider the match $\pi^{(p)}_{k_m,k_n}$ as $(p, \bar{\Phi}(x'_m, x'_n))$-match $\pi^{(p)}$ of the points $x'_n$ and $x'_m$ that satisfies the following: $|\pi^{(p)}| > |\pi^{(p)}_{k_m,k_n}| - p_{k_m}$. It follows from (4) that if $p$ is large enough, then

$$\left\{ j \in \mathcal{D}(\pi^{(p)}) \;:\; T^{\pi(j)}(x_m) = x_n \right\} \geq p\alpha/2.$$

This together with (i) imply that

$$\left\{ i \in \mathcal{D}(\pi^{(p)}) \;:\; T^{\pi(i)}(x_m) \in B(z, \varepsilon/3) \right\} \geq p\alpha/2.$$

Therefore:

$$\begin{aligned} \mathfrak{m}_T(x'_m, p)(B(z, 2\varepsilon/3)) &= \frac{1}{p} \left| \left\{ 0 \leq i < p \;:\; T^i(x'_m) \in B(z, 2\varepsilon/3) \right\} \right| \\ &\geq \frac{1}{p} \left| \left\{ 0 \leq i < p \;:\; T^i(x'_n) \in B(z, \varepsilon/3) \text{ and } i \in \mathcal{R}(\pi^{(p)}_{k_m,k_n}) \right\} \right| \geq \frac{\alpha}{2}. \end{aligned}$$

Consequently,

$$\mu_{k_m}(\overline{B}(z, 2\varepsilon/3)) \geq \limsup_{p \to \infty} \mathfrak{m}_T(x'_m, p)(\overline{B}(z, 2\varepsilon/3)) \geq \alpha/2$$

and hence

$$\mu(B(z, \varepsilon)) \geq \mu(\overline{B}(z, 2\varepsilon/3)) \geq \limsup_{m \to \infty} \mu_{k_m}(\overline{B}(z, 2\varepsilon/3)) \geq \alpha/2 > 0.$$

Because $\varepsilon$ is arbitrary, we get that $z \in \operatorname{supp} \mu$. □

***Remark 11*** As a corollary of the above Theorem 10 we obtain a part of Theorem 2: the support of every royal measure equals the Kuratowski limit of supports of periodic measures used for the construction of that measure. The notion of the Feldman-Katok convergence allows to give an alternative proof for the whole Theorem 2, see [21] for the details.

***Example 12*** The Feldman-Katok convergence of measures does not imply the Kuratowski convergence of their supports. To see that consider a point $x$, whose trajectory is convergent to $z$ with respect to the natural topology (we assume that $x$ is not a fixed point). Let $(x_n)_{n\in\mathbb{N}}$ be a sequence of periodic points such that for every $n \in \mathbb{N}$ the following holds:

(i) the period of the point $x_n$ equals $2(n^2 + n)$,
(ii) for every $0 \leq j < 2n$ one has $\rho\big(T^j(x_n), T^j(x)\big) < 1/n$,
(iii) for every $2n \leq j \leq 2n^2 + 2n$ one has $\rho\big(T^j(x_n), z\big) < 1/n$.

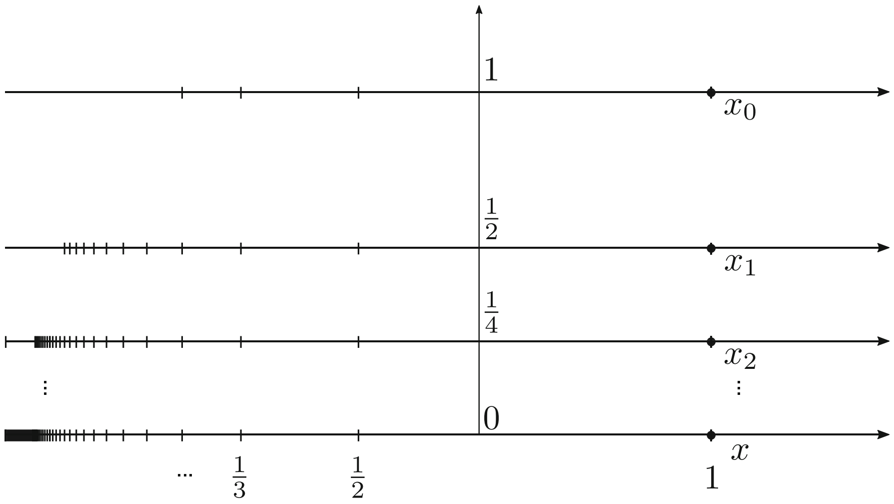


**Fig. 1** A dynamical system illustrating that the sole Feldman-Katok convergence of measures does not imply the Kuratowski convergence of its supports

Then $\bar{\Phi}(x_n, z) \leq 1/n$ and so the sequence $(x_n)_{n\in\mathbb{N}}$ tends to $z$ with respect to $\bar{\Phi}$. On the other hand, $z$ is a generic point for the atomic measure supported at $\{z\}$, while the Kuratowski limit of supports of measures generated by $x_n$ contains the whole orbit of $x$.

It is easy to see that a dynamical system that admits the above described situation can be constructed. For example (see Fig. 1), let

$$X = \big(\{0\} \cup \{1/k \,:\, k \geq 1\}\big) \times \{0\} \cup \bigcup_{n\geq 1} \{1/j \,:\, 1 \leq j \leq 2(n^2+n)\} \times \{1/2^n\}.$$

We equip $X$ with the maximum metric (induced from $\mathbb{R}^2$). Let $T\colon X \to X$ be given by the formula

$$T((m,n)) = \begin{cases} \big(0,\ 0\big), & \text{if } m = 0 \text{ and } n = 0, \\ \big(1/(1/m+1),\ 0\big), & \text{if } m \neq 0 \text{ and } n = 0, \\ \big(1/(1/m+1 \mod 2n^2+2n),\ n\big), & \text{if } m \neq 0 \text{ and } n \neq 0. \end{cases}$$

We define $x = (1, 0)$ and $x_n = (1, 1/2^n)$ for $n \in \mathbb{N}$. It obvious that such a system satisfies the requested conditions.

**Acknowledgements** Martha Łącka acknowledges support of the National Science Centre (NCN), Poland, the Doctoral Scholarship No. 2017/24/T/ST1/00372. The results in this paper are part of the PhD thesis defended at the Jagiellonian University under the supervision of Dominik Kwietniak and were created during the internship at the University of Burgundy in Dijon under the supervision of Christian Bonatti. The author would like to thank both professors for their patience, enthusiasm for mathematics, important remarks and help.

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