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Expansion of eigenvalues of the perturbed discrete bilaplacian

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Abstract
We consider the family

$$\widehat{H}_\mu := \widehat{\Delta}\widehat{\Delta} - \mu\widehat{V}, \qquad \mu \in \mathbb{R},$$

of discrete Schrödinger-type operators in d-dimensional lattice $\mathbb{Z}^d$, where $\widehat{\Delta}$ is the discrete Laplacian and $\widehat{V}$ is of rank-one. We prove that there exist coupling constant thresholds $\mu_o, \mu^o \geq 0$ such that for any $\mu \in [-\mu^o, \mu_o]$ the discrete spectrum of $\widehat{H_\mu}$ is empty and for any $\mu \in \mathbb{R} \setminus [-\mu^o, \mu_o]$ the discrete spectrum of $\widehat{H_\mu}$ is a singleton $\{e(\mu)\}$, and $e(\mu) < 0$ for $\mu > \mu_o$ and $e(\mu) > 4d^2$ for $\mu < -\mu^o$. Moreover, we study the asymptotics of $e(\mu)$ as $\mu \searrow \mu_o$ and $\mu \nearrow -\mu^o$ as well as $\mu \to \pm\infty$. The asymptotics highly depends on d and $\widehat{V}$.

Keywords Discrete bilaplacian · Essential spectrum · Discrete spectrum · Eigenvalues · Asymptotics · Expansion

Mathematics Subject Classification 47A10 · 47A55 · 47A75 · 41A60

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1 Introduction

In this paper we investigate the spectral properties of the perturbed discrete biharmonic operator

$$\widehat{H}_\mu := \widehat{\Delta}\widehat{\Delta} - \mu\widehat{V}, \qquad \mu \in \mathbb{R}, \tag{1.1}$$

in the d-dimensional cubical lattice $\mathbb{Z}^d$, where $\widehat{\Delta}$ is the discrete Laplacian and $\widehat{V}$ is a is rank-one potential with a generating potential $\widehat{v}$. This model is associated to a one-particle system in $\mathbb{Z}^d$ with a potential field $\widehat{v}$, in which the particle freely "jumps" from a node X of the lattice not only to one of its nearest neighbors Y (similar to the discrete Laplacian case), but also to the nearest neighbors of the node Y. From the mathematical point of view, the discrete bilaplacian represents a discrete Schrödinger operator with a degenerate bottom, i.e., $\widehat{\Delta}\widehat{\Delta}$ is unitarily equivalent to a multiplication operator by a function $\mathfrak{e}$ which behaves as $o(|p-p_0|^2)$ close to its minimum point p_0.

The spectral properties of discrete Schrödinger operators with non-degenerate bottom (i.e., $\mathfrak{e}$ behaves as $O(|p-p_0|^2)$ close to its minimum point p_0), in particular with discrete Laplacian, have been extensively studied in recent years (see e.g. [1, 2, 7, 8, 10, 11, 20, 21, 23, 26, 28] and references therein) because of their applications in the theory of ultracold atoms in optical lattices [16, 24, 35, 36]. In particular, it is well-known that the existence of the discrete spectrum is strongly connected to the threshold phenomenon [18, 20–22], which plays an role in the existence the Efimov effect in three-body systems [31, 32, 34]: *if any two-body subsystem in a three-body system has no bound state below its essential spectrum and at least two two-body subsystem has a zero-energy resonance, then the corresponding three-body system has infinitely many bound states whose energies accumulate at the lower edge of the three-body essential spectrum.*

Recall that the Efimov effect may appear only for certain attractive systems of particles [29]. However, recent experimental results in the theory of ultracold atoms in an optical lattice have shown that two-particle systems can have repulsive bound states and resonances (see e.g. [36]), thus, one expects the Efimov effect to hold also for some repulsive three-particle systems in $\mathbb{Z}^3$.

The strict mathematical justification of the Effect effect including the asymptotics for the number of negative eigenvalues of the three-body Hamiltonian has been successfully established in 3-space dimensions (for both $\mathbb{R}^3$ and $\mathbb{Z}^3$) see e.g., [1, 4, 13, 19, 29, 31, 32, 34] and the references therein. In particular, the non-degeneracy of the bottom of the (reduced) one-particle Schrödinger operator played an important role in the study of resonance states of the associated two-body system [1, 31]. Another keypoint in the proof of the Efimov effect in $\mathbb{Z}^3$ was the asymptotics of the (unique) smallest eigenvalue of the (reduced) one-particle discrete Schrödinger operator which creates a singularity in the kernel of a Birman-Schwinger-type operator which used to obtain an asymptotics to the number of three-body bound states.

To the best of our knowledge, there are no published results related to the Efimov effect in lattice three-body systems in which associated (reduced) one-body Schrödinger operator has degenerate bottom.

We also recall that fourth order elliptic operators in $\mathbb{R}^d$ in particular, the biharmonic operator, play also a central role in a wide class of physical models such as linear elasticity theory, rigidity problems (for instance, construction of suspension bridges) and in streamfunction formulation of Stokes flows (see e.g. [9, 25, 27] and references therein). Moreover, recent investigations have shown that the Laplace and biharmonic operators have high potential in image compression with the optimized and sufficiently sparse stored data [15]. The need for corresponding numerical simulations has led to a vast literature devoted to a variety of discrete approximations to the solutions of fourth order equations [5, 12, 33]. The question of stability of such models is basically related to their spectral properties and therefore, numerous studies have been dedicated to the numerical evaluation of the eigenvalues [3, 6, 30].

The aim of the present paper is the study of the existence and asymptotics of eigenvalues as well as threshold resonance and bound states of $\widehat{H_\mu}$ defined in (1.1), which corresponds to the one-body Schrödinger operator with degenerate bottom. Namely, we study the discrete spectrum of $\widehat{H_\mu}$ depending on μ and on $\widehat{v}$. For simplicity we assume the generator $\widehat{v}$ of $\widehat{V}$ to decay exponentially at infinity, however, we urge that our methods can also be adjusted to less regular cases (see Remark 2.6). Since the spectrum of $\widehat{\Delta}$ consists of $[0, 2d]$ (see e.g., [1]), by the compactness of $\widehat{V}$ and Weyl's Theorem, the essential spectrum of $\widehat{H_\mu}$ fills the segment $[0, 4d^2]$ independently of μ. Moreover, the essential spectrum does not give birth to a new eigenvalue while μ runs in some real interval $[-\mu^o, \mu_o]$, and it turns out as soon as μ leaves this interval through μ_o resp. through $-\mu^o$, a unique negative resp. a unique positive eigenvalue $e(\mu)$ releases from the essential spectrum (Theorem 2.2).

Now we are interested in the absorption rate of $e(\mu)$ as $\mu \to \mu_o$ and $\mu \to -\mu^o$. The associated asymptotics are highly dependent not only on the dimension d of the lattice (as in the discrete Laplacian case [20, 21]), but also values on the multiplicity $2n_o$ and $2n^o$ of $0 \in \{v = 0\}$ (if $v(0) = 0$) and $\vec{\pi} \in \{v = 0\}$ (if $v(\vec{\pi}) = 0$), respectively. More precisely, depending on d and n_o, $e(\mu)$ has a convergent expansion

- in $(\mu - \mu_o)^{1/3}$ for $2n_o + d = 1, 7$;
- in $\mu - \mu_o$ for $2n_o + d = 3, 5$;
- in $(\mu - \mu_o)^{1/4}$ for $2n_o + d \geq 9$ with d odd;
- in $\mu - \mu_o$ and $-(\mu - \mu_o)\ln(\mu - \mu_o)$ for $2n_o + d = 2, 6$;
- in $\mu - \mu_o$ and $e^{-1/(\mu-\mu_o)}$ for $2n_o + d = 4$;
- in $(\mu - \mu_o)^{1/2}$, $-(\mu - \mu_o)\ln(\mu - \mu_o)$, $(-\frac{1}{\ln(\mu-\mu_o)})^{1/2}$ and $-\frac{\ln\ln(\mu-\mu_o)^{-1}}{\ln(\mu-\mu_o)}$ for $2n_o + d = 8$;
- in $(\mu - \mu_o)^{1/2}$ and $-(\mu - \mu_o)^{1/2}\ln(\mu - \mu_o)$ for $2n^o + d \geq 10$ with d even

(see Theorem 2.4). Moreover, resonance states of 0-energy, i.e. non-zero solutions f of $\widehat{H}_{\mu_o} f = 0$ not belonging to $\ell^2(\mathbb{Z}^d)$ appear if and only if $2n_o + d \in \{5, 6, 7, 8\}$. Recall that the emergence of 0-energy resonances in more lattice dimensions could allow the Efimov effect to be observed in other dimensions than $d = 3$.

Furthermore, observing that the top $\mathfrak{e}(\vec{\pi}) = 4d^2$ of the essential spectrum is non-degenerate, one expects the asymptotics of $e(\mu)$ as $\mu \to -\mu^o$ to be similar as in the discrete Laplacian case [20, 21]; more precisely, depending on d and n^o, $e(\mu)$ has a convergent expansion

- in $\mu + \mu^o$ for $2n^o + d = 1, 3$;

- in $(\mu+\mu^o)^{1/2}$ for $2n^o+d \geq 5$ with d odd;
- in $\mu+\mu^o$ and $e^{-1/(\mu+\mu^o)}$ for $2n_o+d=2$;
- in $\mu+\mu^o$, $-\frac{1}{\ln(\mu+\mu^o)}$ and $-\frac{\ln\ln(\mu+\mu^o)^{-1}}{\ln(\mu+\mu^o)}$ for $2n_o+d=4$;
- in $\mu+\mu^o$ and $-(\mu+\mu^o)\ln(\mu+\mu^o)$ for $2n^o+d \geq 6$ with d even

(see Theorem 2.5). Moreover, the resonance states of energy $4d^2$, i.e. non-zero solutions f of $\widehat{H}_{-\mu^o} f = 4d^2 f$ not belonging to $\ell^2(\mathbb{Z}^d)$ appear if and only if $2n_o+d=3,4$.

The threshold analysis for more general class of nonlocal discrete Schrödinger operators with δ-potential of type

$$\widehat{H}_\mu = \Psi(-\widehat{\Delta}) + \mu\delta_{x0},$$

can be found in [14], where Ψ is some strictly increasing C^1-function and δ_{x0} is the Dirac's delta-function supported at 0. Besides the existence of eigenvalues, authors of [14] classify (embedded) threshold resonances and threshold eigenvalues depending on the behaviour of Ψ at the edges of the essential spectrum of $-\widehat{\Delta}$ and on the lattice dimension d. The eigenvalue expansions for the discrete bilaplacian with δ-perturbation have been established in [17] for $d=1$ using the complex analytic methods.

The paper is organized as follows. In Sect. 2 after introducing some preliminaries we state the main results of the paper. In Theorem 2.2 we establish necessary and sufficient conditions for non-emptiness of the discrete spectrum of $\widehat{H_\mu}$, and in case of existence, we study the location and the uniqueness, analiticity, monotonicity and convexity properties of eigenvalues $e(\mu)$ as a function of μ. In particular, we study the asymptotics of $e(\mu)$ as $\mu \to \mu_o$ and $\mu \to -\mu^o$ as well as $\mu \to \pm\infty$. As discussed above in Theorems 2.4 and 2.5 we obtain expansions of $e(\mu)$ for small and positive $\mu-\mu_o$ and $\mu+\mu^o$. In Sect. 3 we prove the main results. The main idea of the proof is to obtain a nonlinear equation $\Delta(\mu; z)=0$ with respect to the eigenvalue $z=e(\mu)$ of $\widehat{H_\mu}$ and then study properties of $\Delta(\mu; z)$. Finally, in appendix Section A we obtain the asymptotics of certain integrals related to $\Delta(\mu; z)$ which will be used in the proofs of main results.

Data availability statement

We confirm that the current manuscript has no associated data.

2 Preliminary and main results

Let $\mathbb{Z}^d$ be the d-dimensional lattice and $\ell^2(\mathbb{Z}^d)$ be the Hilbert space of square-summable functions on $\mathbb{Z}^d$. Consider the family

$$\widehat{H_\mu} := \widehat{H}_0 - \mu\widehat{V}, \qquad \mu \geq 0,$$

of self-adjoint bounded discrete Schrödinger operators in $\ell^2(\mathbb{Z}^d)$. Here $\widehat{H}_0 := \widehat{\Delta}\widehat{\Delta}$ is discrete bilaplacian, where

$$\widehat{\Delta} f(x) = \frac{1}{2} \sum_{|s|=1} [f(x) - f(x+s)], \qquad f \in \ell^2(\mathbb{Z}^d),$$

is the discrete Laplacian, and $\widehat{V}$ is a rank-one operator

$$\widehat{V}\widehat{f}(x) = \widehat{v}(x) \sum_{y\in\mathbb{Z}^d} \widehat{v}(y)\widehat{f}(y),$$

where $\widehat{v} \in \ell^2(\mathbb{Z}^d) \setminus \{0\}$ is a given real-valued function.

Let $\mathbb{T}^d$ be the d-dimensional torus equipped with the Haar measure and $L^2(\mathbb{T}^d)$ be the Hilbert space of square-integrable functions on $\mathbb{T}^d$. By $\mathcal{F}$ we denote the the standard Fourier transform

$$\mathcal{F} : \ell^2(\mathbb{Z}^d) \to L^2(\mathbb{T}^d), \qquad \mathcal{F}\widehat{f}(p) = \frac{1}{(2\pi)^{d/2}} \sum_{x\in\mathbb{Z}^d} \widehat{f}(x)e^{ixp}.$$

Further we always assume that $\widehat{v}$ and its Fourier image

$$v(p) := \mathcal{F}\widehat{v}(p) = \frac{1}{(2\pi)^{d/2}} \sum_{x\in\mathbb{Z}^d} \widehat{v}(x)e^{ix\cdot p}$$

satisfy the following assumptions:

There exist reals $C, a > 0$ *and nonnegative integers* $n_o, n^o \geq 0$ *such that*

$$|\widehat{v}(x)| \leq Ce^{-a|x|} \quad \text{for all } x \in \mathbb{Z}^d, \tag{H1}$$

$$|v(0)|^2 = D^2|v(0)|^2 = \ldots = D^{2n_o-2}|v(0)|^2 = 0, \qquad D^{2n_o}|v(0)|^2 \neq 0, \tag{H2}$$

$$|v(\vec{\pi})|^2 = D^2|v(\vec{\pi})|^2 = \ldots = D^{2n^o-2}|v(\vec{\pi})|^2 = 0, \qquad D^{2n^o}|v(\vec{\pi})|^2 \neq 0, \tag{H3}$$

here $D^j f(p)$ is the j-th order differential of f at p, i.e. the j-th order symmetric tensor

$$D^j f(p)[\underbrace{w, \ldots, w}_{j-times}] = \sum_{i_1+\ldots+i_d=j, i_k\geq 0} \frac{\partial^j f(p)}{\partial^{i_1} p_1 \ldots \partial^{i_d} p_d} w_1^{i_1} \ldots w_d^{i_d},$$
$$w = (w_1, \ldots, w_d) \in \mathbb{R}^d,$$

and $\vec{\pi} = (\pi, \ldots, \pi) \in \mathbb{T}^d$. Notice that under assumption (H1), v is analytic on $\mathbb{T}^d$.

Recall that $\sigma(\widehat{\Delta}) = \sigma_{\mathrm{ess}}(\widehat{\Delta}) = [0, 2d]$ (see e.g. [1]). Hence, $\sigma(\widehat{H}_0) = \sigma_{\mathrm{ess}}(\widehat{H}_0) = [0, 4d^2]$, and by the compactness of $\widehat{V}$ and Weyl's Theorem,

$$\sigma_{\mathrm{ess}}(\widehat{H_\mu}) = \sigma_{\mathrm{ess}}(\widehat{H}_0) = [0, 4d^2]$$

for any $\mu \in \mathbb{R}$.

Before stating the main results let us introduce the constants

$$\mu_o := \Big(\int_{\mathbb{T}^d} \frac{|v(q)|^2 dq}{\mathfrak{e}(q)}\Big)^{-1}, \qquad \mu^o := \Big(\int_{\mathbb{T}^d} \frac{|v(q)|^2 dq}{4d^2 - \mathfrak{e}(q)}\Big)^{-1}, \tag{2.2}$$

$$\widehat{c_v} := \int_{\mathbb{T}^d} \frac{|v(q)|^2 dq}{\mathfrak{e}(q)^2}, \qquad \widehat{C_v} := \int_{\mathbb{T}^d} \frac{|v(q)|^2 dq}{(4d^2 - \mathfrak{e}(q))^2}, \tag{2.3}$$

and

$$c_v := \frac{2^{2n_o+d}}{(2n_o)!} \int_{\mathbb{S}^{d-1}} D^{2n_o} |v(0)|^2 [w, \dots, w] \, d\mathcal{H}^{d-1}(w), \tag{2.4}$$

$$C_v := \frac{2^{2n^o+d-1}}{(8d)^{n^o+d/2}\,(2n^o)!} \int_{\mathbb{S}^{d-1}} D^{2n^o} |v(\vec{\pi})|^2 [w, \dots, w] \, d\mathcal{H}^{d-1}(w), \tag{2.5}$$

where $\mathbb{S}^{d-1}$ is the unit sphere in $\mathbb{R}^d$ and

$$\mathfrak{e}(q) := \Big(\sum_{i=1}^d (1 - \cos q_i)\Big)^2.$$

Remark 2.1 Under assumptions (H1)–(H3), $\mu_o, \mu^o \geq 0$, $c_v, C_v > 0$, and $\widehat{c_v}, \widehat{C_v} \in (0, +\infty]$. Moreover, by Propositions A.1 and A.2:

- $\mu_o = 0$ (resp. $\mu^o = 0$) if and only if $2n_o + d \leq 4$ (resp. $2n^o + d \leq 2$);
- $\widehat{c_v} < \infty$ (resp. $\widehat{C_v} < \infty$) if $2n_o + d \geq 9$ (resp. $2n^o + d \geq 5$).

2.1 Main results

First we concern with the existence of the discrete spectrum of $\widehat{H_\mu}$.

Theorem 2.2 *Let $\mu_o, \mu^o \geq 0$ be given by (2.2). Then $\sigma_{\mathrm{disc}}(\widehat{H_\mu}) = \emptyset$ for any $\mu \in [-\mu^o, \mu_o]$ and $\sigma_{\mathrm{disc}}(\widehat{H_\mu})$ is a singleton $\{e(\mu)\}$ for any $\mu \in \mathbb{R} \setminus [-\mu^o, \mu_o]$. Moreover, the associated eigenfunction $\widehat{f_\mu}$ to $e(\mu)$ is given by $\widehat{f_\mu} := \mathcal{F}^* f_\mu$, where*

$$f_\mu(p) = \frac{v(p)}{\mathfrak{e}(p) - e(\mu)}.$$

Furthermore, if $\mu < -\mu^o$ (resp. $\mu > \mu_o$), then $e(\mu) > 4d^2$ (resp. $e(\mu) < 0$). Moreover, the function $\mu \in \mathbb{R} \setminus [-\mu^o, \mu_o] \mapsto e(\mu)$ is real-analytic strictly decreasing, convex in $(-\infty, -\mu^o)$ and concave in $(\mu_o, +\infty)$, and satisfies

$$\lim_{\mu \searrow \mu_o} e(\mu) = 0 \quad \text{and} \quad \lim_{\mu \nearrow -\mu^o} e(\mu) = 4d^2 \tag{2.6}$$

and

$$\lim_{\mu\to\pm\infty}\frac{e(\mu)}{\mu} = -\int_{\mathbb{T}^d}|v(q)|^2dq. \tag{2.7}$$

Next we study the threshold resonances of $\widehat{H_\mu}$.

Theorem 2.3 *Let $n_o, n^o \geq 0$ be given by (H2)–(H3).*

(a) Let $2n_o + d \geq 5$. Then $\widehat{f} := \mathcal{F}^* f \in c_0(\mathbb{Z}^d)$, i.e., $\widehat{f}(x) \to 0$ as $|x| \to +\infty$, where

$$f(p) = \frac{v(p)}{\mathfrak{e}(p)} \in L^1(\mathbb{T}^d).$$

Moreover, $\widehat{f} \in c_0(\mathbb{Z}^d)\setminus \ell^2(\mathbb{Z}^d)$ for $2n_o + d \in \{5,6,7,8\}$, $\widehat{f} \in \ell^2(\mathbb{Z}^d)$ for $2n_o + d \geq 9$, and $\widehat{f}$ solves the equation $\widehat{H}_{\mu_o} f = 0$.

(b) Let $2n^o + d \geq 3$. Then $\widehat{g} := \mathcal{F}^* g \in \ell^0(\mathbb{Z}^d)$, where

$$g(p) = \frac{v(p)}{4d^2 - \mathfrak{e}(p)}.$$

Moreover, $\widehat{g} \in \ell^0(\mathbb{Z}^d)\setminus \ell^2(\mathbb{Z}^d)$ for $2n^o + d \in \{3,4\}$, $\widehat{g} \in \ell^2(\mathbb{Z}^d)$ for $2n^o + d \geq 5$, and $\widehat{g}$ solves the equation $\widehat{H}_{-\mu^o} f = 4d^2 f$.

We recall that in the literature the non-zero solutions of equations $\widehat{H_\mu}\widehat{f} = 0$ and $\widehat{H_\mu}\widehat{g} = 4d^2\widehat{g}$ not belonging to $\ell^2(\mathbb{Z}^d)$ are called the *resonance states* [1, 2].

Now we study the rate of the convergences in (2.6).

Theorem 2.4 (Expansions of $e(\mu)$ at $\mu = \mu_o$) *For $\mu > \mu_o$ let $e(\mu) < 0$ be the eigenvalue of $\widehat{H_\mu}$.*

(a) Suppose that d is odd:

(a1) if $2n_o + d = 1, 3$, then $\mu_o = 0$ and for sufficiently small and positive μ,

$$(-e(\mu))^{1/4} = \begin{cases} \left(\frac{\pi c_v}{4}\right)^{1/3}\mu^{1/3} + \sum\limits_{n\geq1} c_{1,n}\mu^{\frac{n+1}{3}}, & 2n_o + d = 1, \\ \frac{\pi c_v}{8}\,\mu + \sum\limits_{n\geq1} c_{3,n}\mu^{n+1}, & 2n_o + d = 3, \end{cases}$$

where $\{c_{1,n}\}$ and $\{c_{3,n}\}$ are some real coefficients;

(a2) if $2n_o + d = 5, 7$, then $\mu_o > 0$ and for sufficiently small and positive $\mu - \mu_o$,

$$(-e(\mu))^{1/4} = \begin{cases} \frac{8}{\pi c_v\mu_o^2}(\mu-\mu_o) + \sum\limits_{n\geq1} c_{5,n}(\mu-\mu_o)^{n+1}, & 2n_o + d = 5, \\ \left(\frac{8}{\pi c_v\mu_o^2}\right)^{1/3}(\mu-\mu_o)^{1/3} + \sum\limits_{n\geq1} c_{7,n}(\mu-\mu_o)^{\frac{n+1}{3}}, & 2n_o + d = 7, \end{cases}$$

where $\{c_{5,n}\}$ and $\{c_{7,n}\}$ are some real coefficients;

(a3) if $2n_o + d \geq 9$*, then* $\mu_o > 0$ *and for sufficiently small and positive* $\mu - \mu_o$*,*

$$(-e(\mu))^{1/4} = (\mu_o^2 \widehat{c}_v)^{-1/4} (\mu - \mu_o)^{1/4} + \sum_{n \geq 1} c_{9,n} (\mu - \mu_o)^{n/4},$$

where $\{c_{9,n}\}$ *are some real coefficients.*

(b) Suppose that d is even:

(b1) if $2n_o + d = 2, 4$*, then* $\mu_o = 0$ *and for sufficiently small and positive* μ*,*

$$(-e(\mu))^{1/2} = \begin{cases} \frac{\pi c_v}{8}\,\mu + \sum\limits_{n+m\geq 1, n,m\geq 0} c_{2,nm}\mu^{n+1}(-\mu \ln \mu)^m, & 2n_o + d = 2, \\ ce^{-\frac{8}{c_v \mu}} + \sum\limits_{n+m\geq 1, n,m\geq 0} c_{4,nm}\mu^{n+1}\left(\frac{1}{\mu}\, e^{-\frac{8}{c_v\mu}}\right)^{m+1}, & 2n_o + d = 4, \end{cases}$$

where $\{c_{2,nm}\}$ *and* $\{c_{4,nm}\}$ *are some real coefficients and* $c > 0$*;*

(b2) if $2n_o + d = 6, 8$*, then* $\mu_o > 0$ *and for sufficiently small and positive* $\mu - \mu_o$*,*

$$(-e(\mu))^{1/2} = \begin{cases} \frac{8}{\pi c_v \mu_o^2}\,\tau^2 + \sum\limits_{n+m\geq 1, n,m\geq 0} c_{6,nm}\tau^{2n+2}\theta^m, & 2n_o + d = 6, \\ \left(\frac{8}{c_v \mu_o^2}\right)^{1/2} \tau\sigma + \sum\limits_{n+m+k\geq 1, n,m,k\geq 0} c_{8,nmk}\tau^{n+1}\sigma^{m+1}\eta^k, & 2n_o + d = 8, \end{cases}$$

where $\{c_{4,nm}\}$ *and* $\{c_{8,nmk}\}$ *are some real coefficients and*

$$\tau := (\mu - \mu_o)^{1/2}, \quad \theta := -\tau^2 \ln \tau, \quad \sigma := \left(-\frac{1}{\ln \tau}\right)^{1/2}, \quad \eta := -\frac{\ln \ln \tau^{-1}}{\ln \tau}, \tag{2.8}$$

(b3) if $2n_o + d \geq 10$*, then* $\mu_o > 0$ *and for sufficiently small and positive* $\mu - \mu_o$*,*

$$(-e(\mu))^{1/2} = (\mu_o^2 \widehat{c}_v)^{-1/2}\,\tau + \sum_{n+m\geq 1, n,m\geq 0} c_{10,nm}\tau^{n+1}\theta^m,$$

where $\{c_{10,nm}\}$ *are some real coefficients.*

Here $c_v > 0$ *and* $\widehat{c}_v > 0$ *are given by (2.4) and (2.3), respectively.*

Theorem 2.5 *(**Expansions of*** $e(\mu)$ **at** $\mu = -\mu^o$*) For let* $\mu < -\mu^o$ *let* $e(\mu) > 4d^2$ *be the eigenvalue of* $\widehat{H_\mu}$*.*

(a) Suppose that d is odd:

(a1) if $2n^o + d = 1$, *then* $\mu^o = 0$ *and for sufficiently small and negative* μ,

$$(e(\mu) - 4d^2)^{1/2} = -\pi C_v \, \mu + \sum_{n \geq 1} C_{1,n} \mu^{n+1},$$

where $\{C_{1,n}\}$ *are some real coefficients;*

(a2) if $2n^o + d = 3$, *then* $\mu^o > 0$ *and for sufficiently small and positive* $\mu + \mu^o$,

$$(e(\mu) - 4d^2)^{1/2} = (\pi C_v \mu^{o2})^{-1} \, (\mu + \mu^o) + \sum_{n \geq 1} C_{3,n} (\mu + \mu^o)^{n+1},$$

where $\{C_{3,n}\}$ *and* $\{C_{7,n}\}$ *are some real coefficients;*

(a3) if $2n^o + d \geq 5$, *then* $\mu^o > 0$ *and for sufficiently small and positive* $\mu + \mu^o$,

$$(e(\mu) - 4d^2)^{1/2} = (\widehat{C}_v \mu^{o2})^{-1/2} \, (\mu + \mu^o)^{1/2} + \sum_{n \geq 1} C_{5,n} (\mu + \mu^o)^{(n+1)/2},$$

where $\{C_{5,n}\}$ *are some real coefficients.*

(b) Suppose that d is even:

(b1) if $2n_o + d = 2$, *then* $\mu_o = 0$ *and for sufficiently small and negative* μ,

$$e(\mu) - 4d^2 = C \, e^{\frac{1}{C_v \mu}} + \sum_{n+m \geq 1, n,m \geq 0} C_{2,nm} \mu^{n+1} \left(-\frac{1}{\mu} \, e^{\frac{1}{C_v \mu}} \right)^{m+1},$$

where $\{C_{2,nm}\}$ *are some real coefficients and* $C > 0$;

(b2) if $2n_o + d = 4$, *then* $\mu^o > 0$ *and for sufficiently small and positive* $\mu + \mu^o$,

$$e(\mu) - 4d^2 = (C_v \mu^{o2})^{-1} \, \mu\sigma + \sum_{n+m+k \geq 1, n,m,k \geq 0} C_{4,nmk} \tau^{n+1} \sigma^{m+1} \eta^k,$$

where $\{C_{4,nm}\}$ *are some real coefficients and*

$$\tau := \mu + \mu^o, \qquad \sigma := -\frac{1}{\ln \tau}, \qquad \eta := -\frac{\ln \ln \tau^{-1}}{\ln \tau};$$

(b3) if $2n_o + d \geq 6$, *then* $\mu^o > 0$ *and for sufficiently small and positive* $\mu + \mu^o$,

$$\begin{aligned} e(\mu) - 4d^2 &= (\widehat{C}_v \mu^{o2})^{-1} \, (\mu + \mu^o) \\ &\quad + \sum_{n+m \geq 1, n,m \geq 0} C_{6,nm} (\mu + \mu^o)^{n+1} [-(\mu + \mu^o) \ln(\mu + \mu^o)]^m, \end{aligned}$$

where $\{C_{6,nm}\}$ *are some real coefficients.*

Here C_v *and* $\widehat{C}_v$ *are given by (2.5) and (2.3), respectively.*

Remark 2.6 Few comments on the main results are in order.

1. The assertions of Theorem 2.2 hold in fact for any $\widehat{v} \in \ell^2(\mathbb{Z}^d)$ (see Remark 3.2);
2. Similar expansions of $e(\mu)$ in Theorems 2.4 and 2.5 at $\mu = \mu_o$ and $\mu = -\mu^o$, respectively, still hold for any exponentially decaying $\widehat{v} : \mathbb{Z}^d \to \mathbb{C}$ (see Remark 3.3);
3. If $\widehat{v}$ decays at most polynomially at infinity, i.e. $\widehat{v}(x) = O(|x|^{-\alpha})$ for some $\alpha > 0$, then instead of the expansions in Theorem 2.4 and 2.5 we obtain only asymptotics of $e(\mu)$ (see Remark 3.4).

3 Proof of main results

In this section we prove the main results. By the Birman-Schwinger principle and the Fredholm Theorem we have

Lemma 3.1 *A complex number $z \in \mathbb{C} \setminus [0, 4d^2]$ is an eigenvalue of $\widehat{H_\mu}$ if and only if*

$$\Delta(\mu; z) := 1 - \mu \int_{\mathbb{T}^d} \frac{|v(q)|^2 dq}{\mathfrak{e}(q) - z} = 0.$$

Proof of Theorem 2.2 By the definition of μ_o, for any $\mu < -\mu^o$:

$$\lim_{z \nearrow -\mu^o} \Delta(\mu; z) = 1 + \frac{\mu}{\mu^o} < 0, \qquad \lim_{z \to +\infty} \Delta(\mu; z) = 1.$$

Since $\Delta(\mu; z) > 1$ for $z < 0$ and $\mu > -\mu^o$, in view of the strict monotonicity $\Delta(\mu; \cdot)$ in $(4d^2, \infty)$, for any $\mu < -\mu^o$ there exists a unique $e(\mu) \in (4d^2, +\infty)$ such that $\Delta(\mu; e(\mu)) = 0$. Analogously, for any $\mu > \mu_o$ there exists a unique $e(\mu) \in (-\infty, 0)$ such that $\Delta(\mu; e(\mu)) = 0$. By the Implicit Function Theorem the function $\mu \in \mathbb{R} \setminus [-\mu^o, \mu_o] \mapsto e(\mu)$ is real-analytic. Moreover, computing the derivatives of the implicit function $e(\mu)$ we find:

$$e'(\mu) = -\frac{1}{\mu} \int_{\mathbb{T}^d} \frac{|v(q)|^2 dq}{\mathfrak{e}(q) - e(\mu)} \Big(\int_{\mathbb{T}^d} \frac{|v(q)|^2 dq}{(\mathfrak{e}(q) - e(\mu))^2} \Big)^{-1}, \qquad \mu \neq 0, \tag{3.1}$$

thus, using $\mu(\mathfrak{e}(q) - e(\mu)) > 0$ we get $e'(\mu) < 0$, i.e. $e(\cdot)$ is strictly decreasing in $\mathbb{R} \setminus \{0\}$. Differentiating (3.1) one more time we get

$$e''(\mu) = \frac{2e'(\mu)}{\mu} \left(1 - \mu e'(\mu) \int_{\mathbb{T}^d} \frac{|v(q)|^2 dq}{(\mathfrak{e}(q) - e(\mu))^3} \left(\int_{\mathbb{T}^d} \frac{|v(q)|^2 dq}{(\mathfrak{e}(q) - e(\mu))^2} \right)^{-1} \right).$$

Therefore, $e''(\mu) > 0$ (i.e. $e(\cdot)$ is strictly convex) for $\mu < 0$ and $e''(\mu) < 0$ (i.e. $e(\cdot)$ is strictly concave) for $\mu > 0$.

To prove (2.7), first we let $\mu \to \pm\infty$ in

$$1 = \mu \int_{\mathbb{T}^d} \frac{|v(q)|^2 dq}{\mathfrak{e}(q) - e(\mu)} \tag{3.2}$$

and find $\lim_{\mu\to\pm\infty} e(\mu) = \mp\infty$. In particular, if $|\mu|$ is sufficiently large, $|\frac{\mathfrak{e}(q)}{e(\mu)}| < \frac{1}{2}$ and hence, by (3.2) and the Dominated Convergence Theorem,

$$\lim_{\mu\to\pm\infty} \frac{e(\mu)}{\mu} = -\lim_{\mu\to\pm\infty} \int_{\mathbb{T}^d} \frac{|v(q)|^2 dq}{1 - \frac{\mathfrak{e}(q)}{e(\mu)}} = -\int_{\mathbb{T}^d} |v(q)|^2 dq.$$

To prove that $\widehat{f}_\mu$ solves $\widehat{H}_\mu \widehat{f}_\mu = e(\mu)\widehat{f}_\mu$ we consider the equivalent equality $\mathcal{F}\widehat{H}_\mu\mathcal{F}^* f_\mu = e(\mu) f_\mu$, which is easily reduced to the equality $\Delta(\mu; e(\mu)) = 0$. □

Remark 3.2 In view of Lemma 3.1 and the proof of Theorem 2.2, their assertions still hold for any $v \in \ell^2(\mathbb{Z}^d)$.

Proof of Theorem 2.3 We prove only (a), the proof of (b) being similar. Repeating the proof of the continuity (resp. differentiability) of $\mathfrak{l}_f$ at $z = 0$ in Proposition A.1 one can show that $f \in L^1(\mathbb{T}^d) \setminus L^2(\mathbb{T}^d)$ for $2n_o + d \in \{5, 6, 7, 8\}$ and $f \in L^2(\mathbb{T}^d)$ for $2n_o + d \geq 9$. Thus, by the Riemann-Lebesgue Lemma, $\widehat{f} \in \ell^0(\mathbb{Z}^d)$. To show that $\widehat{H}_{\mu_o}\widehat{f} = 0$ it suffices to observe that $\mathcal{F}\widehat{H}_{\mu_0}\mathcal{F}^* f = 0$. □

Proof of Theorem 2.4 Since

$$|v(p)|^2 = (2\pi)^{-d}\left(\sum_{x\in\mathbb{Z}^d} \widehat{v}(x)\cos p\cdot x\right)^2 + (2\pi)^{-d}\left(\sum_{x\in\mathbb{Z}^d} \widehat{v}(x)\sin p\cdot x\right)^2, \tag{3.3}$$

the function $p \in \mathbb{T}^d \mapsto |v(p)|^2$ is nonnegative even real-analytic function. Notice also that if $n_o \geq 1$, then by the nonnegativity of $|v|^2$, $p = 0$ is a global minimum for $|v|^2$. Therefore, the tensor $D^{2n_o}|v(0)|^2$ is positively definite and

$$c_v := \frac{2^{2n_o+d}}{(2n_o)!} \int_{\mathbb{S}^{d-1}} D^{2n_o}|v(0)|^2[w, \ldots, w]d\mathcal{H}^{d-1} > 0.$$

Note that

$$\widehat{c}_v = \mathfrak{l}'_{|v|^2}(0) = \int_{\mathbb{T}^d} \frac{|v(q)|^2 dq}{\mathfrak{e}(q)^2},$$

where $\mathfrak{l}_f$ is defined in (A.1). By Proposition A.1, $f(p) = \frac{v(p)}{\mathfrak{e}(p)} \in L^2(\mathbb{T}^d)$ if and only if $2n_o + d \geq 9$. Moreover, by definition, $\mu_o > 0$ and $\Delta(\mu_o; 0) = 0$ for $2n_o + d \geq 5$, and hence, as in the proof of Lemma 3.1 for such d one can show that $H_{\mu_o} f = 0$.

In view of the strict monotonicity and (2.6) there exists a unique $\mu_1 > 0$ such that $e(\mu) \in (-\frac{1}{128}, 0)$ for any $\mu \in (0, \mu_1)$. Since

$$\mu = (\mathfrak{l}_{|v|^2}(e(\mu)))^{-1}, \tag{3.4}$$

we can use Proposition A.1 with $f = |v|^2$ and $e := e(\mu)$, to find the expansions of the inverse function $\mu := \mu(e)$. Then applying the appropriate versions of the Implicit Function Theorem in analytical case we get the expansions of $e = e(\mu)$. Notice that from (A.3) and (A.4) as well as (3.5) it follows that $\mu_o = 0$ for $2n_o + d \le 4$ and $\mu_o = \left(\int_{\mathbb{T}^d} \frac{|v(q)|^2 dq}{\mathfrak{e}(q)}\right)^{-1} > 0$ for $2n_o + d \ge 5$.

(a) Suppose that d is odd. In view of the expansions (A.3) of $\mathfrak{l}_f$, in this case, (3.4) is reduced to the inverting the equation

$$\mu = g(\alpha), \tag{3.5}$$

where $\alpha := (-e)^{1/4}$ and g is an analytic function around $\alpha = 0$.

Case $2n_o + d = 1$. In this case by (A.3),

$$g(\alpha) := \frac{\alpha^3}{c_1^3 + \sum_{n\ge1} a_n \alpha^n},$$

where $\{a_n\} \subset \mathbb{R}$ and $c_1 := (\pi c_v/4)^{1/3}$ and (3.5) is equivalently represented as

$$\alpha = \mu \left(c_1^3 + \sum_{n\ge1} a_n \alpha^n\right)^{1/3}, \tag{3.6}$$

where $\mu = \mu^{1/3}$. Now setting

$$\alpha = \mu(c_1 + u), \tag{3.7}$$

and using the Taylor series of $(c_1^3 + x)^{1/3}$, for μ and u sufficiently small we rewrite (3.6) as

$$F(u, \mu) := u - \sum_{n\ge1} \tilde{a}_n \mu^n (c_1 + u)^n = 0, \tag{3.8}$$

where $F(\cdot,\cdot)$ is analytic at $(u, \mu) = (0, 0)$, $F(0, 0) = 0$ and $F_u(0, 0) = 1$. Hence, by the Implicit Function Theorem, there exists $\gamma_1 > 0$ such that for $|\mu| < \gamma_1$, (3.8) has a unique real-analytic solution $u = u(\mu)$ which can be represented as an absolutely convergent series $u = \sum_{n\ge1} b_n \mu^n$. Putting this in (3.7) and recalling the definitions of α and μ we get the expansion of $(-e(\mu))^{1/4}$ for $\mu > 0$ small.

Case $2n_o + d = 3$. By (A.3),

$$g(\alpha) = \alpha \left(c_3 + \sum_{n\ge1} a_n \alpha^n\right)^{-1}, \tag{3.9}$$

where $\{a_n\} \subset \mathbb{R}$ and $c_3 := \pi c_v/8$, and hence, (3.5) is represented as

$$\alpha = \mu \left(c_3 + \sum_{n \geq 1} a_n \alpha^n \right).$$

Then setting $\alpha = \mu(c_3 + u)$ we rewrite (3.9) in the form (3.8), and as in the case of $2n_o + d = 1$, we get the expansion of $(-e(\mu))^{1/4}$.

Case $2n_o + d = 5$. In this case by (A.3)

$$g(\alpha) = \left(\frac{1}{\mu_o} - \frac{\pi c_v \alpha}{8} \left(1 + \sum_{n \geq 1} a_n \alpha^n \right) \right)^{-1},$$

where $\{a_n\} \subset \mathbb{R}$, and hence, by (3.5),

$$\frac{\mu - \mu_o}{\mu \mu_o} = \frac{\pi c_v \alpha}{8} \left(1 + \sum_{n \geq 1} a_n \alpha^n \right). \tag{3.10}$$

Note that if $|\mu - \mu_o| < \mu_o$, then

$$\frac{\mu - \mu_o}{\mu \mu_o} = \frac{\mu - \mu_o}{{\mu_o}^2 + \mu_o(\mu - \mu_o)} = \frac{\mu - \mu_o}{{\mu_o}^2} \sum_{n \geq 0} \left(\frac{\mu - \mu_o}{\mu_o} \right)^n, \tag{3.11}$$

thus from (3.10) we get

$$\alpha = (\mu - \mu_o) \left(c_5 + c_5 \sum_{n \geq 1} \mu_o^{-n} (\mu - \mu_o)^n \right) \left(1 + \sum_{n \geq 1} a_n \alpha^n \right)^{-1}.$$

and $c_5 := 8/(\pi c_v \mu_o^2)$. Now setting $\alpha = (\mu - \mu_o)\,(c_5 + u)$ for sufficiently small and positive $\mu - \mu_o$ we get

$$u = \sum_{n,m \geq 1} \tilde{c}_{n,m} (\mu - \mu_o)^n (c_5 + u)^m,$$

where $\tilde{c}_{n,m} \subset \mathbb{R}$. By the Implicit Function Theorem, for sufficiently small $\mu - \mu_o$ there exists a unique real-analytic function $u = u(\mu)$ given by the absolutely convergent series $u(\mu) = \sum_{n \geq 1} b_n (\mu - \mu_o)^n$. By the definition of α, this implies the expansion of $(-e(\mu))^{1/4}$.

Case $2n_o + d = 7$. As the previous case, by (A.3) and (3.11), the equation (3.5) is represented as

$$(\mu - \mu_o)\left(c_7^3 + c_7^3 \sum_{n\geq 1} \mu_o^{-n}(\mu - \mu_o)^n\right) = \alpha^3 \left(1 + \sum_{n\geq 1} a_n\alpha^n\right), \qquad (3.12)$$

where $\{a_n\} \subset \mathbb{R}$ and $c_7 := [8/(\pi c_v \mu_o^2)]^{1/3}$. When $\mu - \mu_o > 0$ is small enough, by the Taylor series of $(1 + x)^{\pm 1/3}$ at $x = 0$, (3.12) is equivalently rewritten as

$$\alpha = (\mu - \mu_o)^{1/3}\left(c_7 + \sum_{n\geq 1} \tilde{c}_n(\mu - \mu_o)^n\right)\left(1 + \sum_{n\geq 1} \tilde{a}_n\alpha^n\right), \qquad (3.13)$$

Thus, for $\rho = (\mu - \mu_o)^{1/3}$, setting $\alpha = \rho\,(c_7 + u)$ in (3.13), for sufficiently small and positive ρ we get

$$u = \sum_{n,m\geq 1} \tilde{c}_{n,m}\rho^n(c_7 + u)^m.$$

By the Implicit Function Theorem, this equation has a unique real-analytic solution $u = u(\rho)$ given by the absolutely convergent series $u = \sum\limits_{n\geq 1} b_n\rho^n$. This, definitions of α and ρ imply the expansion of $(-e(\mu))^{1/4}$.

Case $2n_o + d = 9$. In this case by (A.3) and (3.11)

$$(\mu - \mu_o)\left(c_9^4 + c_9^4 \sum_{n\geq 1} \mu_o^{-n}(\mu - \mu_o)^n\right) = \alpha^4 \left(1 + \sum_{n\geq 1} a_n\alpha^n\right), \qquad (3.14)$$

where $\{a_n\} \subset \mathbb{R}$ and $c_9 := (\mu_o^2\widehat{c}_v)^{-1/4}$. Thus, for sufficiently small and positive $\mu - \mu_o$ using the Taylor series of $(1 + x)^{\pm 1/4}$ at $x = 0$, this equation can also be represented as

$$\alpha = \rho\left(c_9 + \sum_{n\geq 1} \tilde{b}_n\rho^{4n}\right)\left(1 + \sum_{n\geq 1} \tilde{a}_n\alpha^n\right),$$

where $\rho := (\mu - \mu_o)^{1/4}$. Now setting $\alpha = \rho(c_9 + u)$ in (3.14) we get

$$u = \sum_{n,m\geq 1} \tilde{c}_{n,m}\rho^n(c_9 + u)^m,$$

and the expansion of $(-e(\mu))^{1/4}$ follows as in the case of $2n_o + d = 7$.

(b) Suppose that d is even. In view of the expansion (A.3) of $\mathfrak{l}_f$, in this case, (3.4) is reduced to the inverting the equation

$$\mu = \frac{\alpha^l}{g(\alpha) + h(\alpha)\ln\alpha}, \tag{3.15}$$

where $\alpha := (-e)^{1/2}$, $l \in \mathbb{N}_0$, and g and h are analytic around $\alpha = 0$. Presence of $\ln\alpha$ implies that unlike the case of odd dimensions, α is not necessarily analytic with respect to μ^s. Therefore, we need to introduce new variables dependent on $\ln\mu$ to reduce the problem to the Implicit Function Theorem.

Case $2n_o + d = 2$. By (A.4), in this case for $c_2 := \pi c_v/8$

$$l = 1, \qquad g(\alpha) = c_2 + \sum_{n\ge1} a_n\alpha^n, \qquad h(\alpha) = \sum_{n\ge1} b_n\alpha^{2n}.$$

Hence, setting

$$\alpha = \mu(c_2 + u) \tag{3.16}$$

and $\tau = -\mu\ln\mu$ we represent (3.15) as

$$\begin{aligned} F(u,\mu,\tau) := u - \sum_{n\ge1} a^n\mu^n(c_2+u)^n \qquad &+ \ln(c_2+u)\sum_{n\ge1} b^n\mu^n(c_2+u)^n \\ - \tau\sum_{n\ge1} b^n\mu^{n-1}(c_2+u)^n = 0, \end{aligned}$$

where F is analytic around $(0,0,0)$, $F(0,0,0) = 0$, $F_u(0,0,0) = 1$. Hence, by the Implicit Function Theorem, there exists a unique real-analytic function $u = u(\mu,\tau)$ given by the convergent series $u(\mu,\tau) = \sum_{n+m\ge1,n,m\ge0} \tilde{c}_{n,m}\mu^n\tau^m$ for sufficiently small $|\mu|$ and $|\tau|$, which satisfies $F(u(\mu,\tau),\mu,\tau) \equiv 0$. Inserting u in (3.16) we get the expansion of $\alpha = (-e)^{1/2}$.

Case $2n_o + d = 4$. In this case, by (A.4) for $c_4 := 8/c_v$

$$l = 0, \qquad g(\alpha) = \sum_{n\ge0} a_n\alpha_n, \qquad h(\alpha) = -c_4 + \sum_{n\ge1} b_n\alpha^{2n}.$$

Letting $\alpha = e^{-\frac{1}{c_4\mu}}(c+u)$, where $c = e^{a_0/c_4} > 0$, we represent (3.15) as

$$\begin{aligned} \ln(c+u) - b_0 =& \frac{1}{\mu}e^{-\frac{1}{c_4\mu}}\sum_{n\ge1} a^n e^{-\frac{n-1}{c_4\mu}}(c+u)^n \\ &+ \ln(c+u)\sum_{n\ge1} b^n e^{-\frac{n}{c_4\mu}}(c+u)^n - \sum_{n\ge1} a^n e^{-\frac{n}{c_4\mu}}(c+u)^n = 0. \end{aligned} \tag{3.17}$$

Writing $\tau := \frac{1}{\mu} e^{-\frac{1}{c_4 \mu}}$ so that $e^{-\frac{1}{c_4 \mu}} = \mu\tau$, (3.17) is represented as

$$F(u, \mu, \tau) := \ln(c+u) - b_0 - \mu \sum_{n\geq 1} a^n \mu^{n-1} \tau^{n-1} (c+u)^n \\ - \ln(c+u) \sum_{n\geq 1} b^n \mu^n \tau^n (c+u)^n + \sum_{n\geq 1} a^n \mu^n \tau^n (c+u)^n = 0,$$

where F is analytic around $(0,0,0)$, $F(0,0,0) = 0$, and $F_u(0,0,0) = \frac{1}{c} > 0$. Thus, by the Implicit Function Theorem, for $|\mu|$, $|\tau|$ and $|u|$ small there exists a unique real analytic function $u = u(\mu, \tau)$ given by the convergent series $u = \sum_{n+m\geq 1, n,m\geq 0} \tilde{c}_{n,m} \mu^n \tau^m$ such that $F(u(\mu,\tau), \mu, \tau) \equiv 0$. Since $\tau = \frac{1}{\mu} e^{-\frac{1}{c_4\mu}}$, this implies

$$\alpha = e^{-\frac{1}{c_4 \mu}} (c+u) = c e^{-\frac{1}{c_4 \mu}} + \sum_{n+m\geq 1, n,m\geq 0} \tilde{c}_{n,m} \mu^{n+1} \left(\frac{1}{\mu} e^{-\frac{1}{c_4 \mu}} \right)^{m+1}.$$

Case $2n_o + d = 6$. In this case, by (A.4), for $c_6 := 8/(\pi c_v \mu_o^2)$

$$l = 0, \qquad g(\alpha) = \frac{1}{\mu_o} - \frac{1}{c_6 \mu_o^2} \left(\alpha + \sum_{n\geq 2} a_n \alpha^n \right), \qquad h(\alpha) = \frac{1}{c_6 \mu_o^2} \sum_{n\geq 1} b_n \alpha^{2n},$$

and hence, (3.15) is represented as

$$\frac{1}{\mu} - \frac{1}{\mu_o} = \frac{1}{c_6 \mu_o^2} \left(\alpha + \sum_{n\geq 2} a_n \alpha^n + \ln\alpha \sum_{n\geq 1} b_n \alpha^{2n} \right),$$

or equivalently, by (3.11),

$$\alpha = c_6 (\mu - \mu_o) \sum_{n\geq 0} \left(\frac{\mu - \mu_o}{\mu_o} \right)^n - \sum_{n\geq 2} a_n \alpha^n - \ln\alpha \sum_{n\geq 1} b_n \alpha^{2n}. \tag{3.18}$$

Recalling the definitions of τ and θ in (2.8), setting $\alpha = \tau^2 (c_6 + u)$, we represent (3.18) as

$$F(u, \tau, \theta) := u - c_6 \sum_{n\geq 1} \frac{\tau^{2n}}{\mu_o^n} - \sum_{n\geq 2} a_n \tau^{2n-2} (c_6+u)^n \\ - \ln(c_6+u) \sum_{n\geq 1} b_n \tau^{4n} (c_6+u)^{2n} - \theta \sum_{n\geq 1} b_n \tau^{4n-4} (c_6+u)^{2n} = 0,$$

where F is real-analytic around $(0,0,0)$, $F(0,0,0)=0$ and $F_u(0,0,0)=1$, and F is even in τ. Thus, by the Implicit Function Theorem, for $|u|$, $|\tau|$ and $|\theta|$ small there exists a unique real analytic function $u=u(\tau,\theta)$, even in τ, given by the convergent series $u=\sum\limits_{n+m\geq 1,n,m\geq 0}\tilde{c}_{n,m}\tau^{2n}\theta^m$ such that $F(u(\tau,\theta),\tau,\theta)\equiv 0$. Thus,

$$\alpha=\tau^2(c_6+u)=c_6\sigma+\sum_{n+m\geq 1,n,m\geq 0}\tilde{c}_{n,m}\tau^{2n+2}\theta^m.$$

Case $2n_o+d=8$. By (A.4), for $c_8:=[8/c_v\mu_o^2]^{-1/2}$,

$$l=0,\qquad g(\alpha)=\frac{1}{\mu_o^2c_8^2}\sum_{n\geq 2}a_n\alpha^n,\qquad h(\alpha)=\frac{1}{\mu_o^2c_8^2}\left(\alpha^2+\sum_{n\geq 2}b_n\alpha^{2n}\right),$$

thus, as in the case of $2n_o+d=6$, (3.15) is represented as

$$c_8^2(\mu-\mu_o)\sum_{n\geq 0}\left(\frac{\mu-\mu_o}{\mu_o}\right)^n=\alpha^2\ln\alpha+\ln\alpha\sum_{n\geq 2}b_n\alpha^{2n}+\sum_{n\geq 2}a_n\alpha^n. \quad (3.19)$$

For τ, σ and η given in (2.8) set $\alpha=\tau\sigma(c_8+u)$ and represent (3.19) as

$$\begin{aligned}2c_8u+u^2=&c_8^2\sum_{n\geq 1}\frac{\tau^{2n}}{\mu_o^n}+\sum_{n\geq 2}a_n\tau^{n-1}\sigma^{n+1}(c_8+u)^{n+2}\\&-\sum_{n\geq 2}b_n(\tau\sigma)^{2n-2}(c_8+u)^{2n+2}\\&+\left(\sigma^2\ln(c_8+u)-\frac{\eta}{2}\right)\left((c_8+u)^2+\sum_{n\geq 2}b_n(\tau\sigma)^{2n-2}(c_8+u)^{2n+2}\right).\end{aligned}$$

This equation is represented as $F(u,\tau,\sigma,\eta)=0$, where F is real-analytic in a neighborhood of $(0,0,0,0)$, $F(0,0,0,0)=0$ and $F_u(0,0,0,0)=2c_8>0$. Hence, for $|u|$, $|\tau|$, $|\sigma|$ and $|\eta|$ small, by the Implicit Function Theorem, there exists a unique real-analytic function $u=u(\tau,\sigma,\eta)$ given by the convergent series $u=\sum\limits_{n+m+k\geq 1,n,m,k\geq 0}\tilde{c}_{n,m,k}\tau^n\sigma^m\mu^k$ such that $F(u(\tau,\sigma,\eta),\tau,\sigma,\eta)\equiv 0$. Thus,

$$\alpha=\tau\sigma(c_8+u)=c_8\tau\sigma+\sum_{n+m+k\geq 1,n,m,k\geq 0}\tilde{c}_{n,m,k}\tau^{n+1}\sigma^{m+1}\eta^k.$$

Case $2n_o+d\geq 10$. By (A.4) for $c_{10}:=(\mu_o^2\widehat{c}_v)^{-1/2}$,

$$l=0,\qquad g(\alpha)=\frac{1}{\mu_o}+\widehat{c}_v\alpha^2+\sum_{n\geq 2}a_n\alpha^{n+2},\qquad h(\alpha)=\sum_{n\geq 2}b_n\alpha^{2n},$$

and as in the case of $2n_o + d = 6$, (3.15) is represented as

$$\frac{\mu - \mu_o}{\mu_o^2} \sum_{n \geq 0} \left(\frac{\mu - \mu_o}{\mu_o} \right)^n = \widehat{c}_v \alpha^2 + \sum_{n \geq 2} a_n \alpha^{n+2} + \ln \alpha \sum_{n \geq 2} b_n \alpha^{2n}. \tag{3.20}$$

Recalling the definitions of τ and θ in (2.8), we set $\alpha = \tau(c_{10} + u)$. Then (3.20) is represented as

$$\begin{aligned} F(u, \tau, \theta) :=& 2c_{10}u + u^2 - c_{10}^2 \sum_{n \geq 1} \frac{\tau^{2n}}{\mu_o^n} + \sum_{n \geq 2} a_n \tau^n (c_{10} + u)^{n+2} \\ & - \theta \sum_{n \geq 2} b_n \tau^{2n-4} (c_8 + u)^{2n} + \ln(c_{10} + u) \sum_{n \geq 2} b_n \tau^{2n-2} (c_8 + u)^{2n} = 0, \end{aligned}$$

where F is analytic at $(0, 0, 0)$, $F(0, 0, 0) = 0$ and $F_u(0, 0, 0) = 2c_{10} > 0$. Thus, by the Implicit Function Theorem, for $|u|$, $|\tau|$ and $|\theta|$ small there exists a unique real-analytic function $u = u(\tau, \theta)$ given by the convergent series $u = \sum_{n+m \geq 1, n,m \geq 0} \tilde{c}_{n,m} \tau^n \theta^n$ such that $F(u(\tau, \theta), \tau, \theta) \equiv 0$. Then

$$\alpha = \mu(c_{10} + u) = c_{10}\mu + \sum_{n+m \geq 1, n,m \geq 0} \tilde{c}_{n,m} \mu^{n+1} \theta^n.$$

Theorem is proved. □

Proof of Theorem 2.5 From (3.3) it follows that the map $p \in \mathbb{T}^d \mapsto |v|^2(\vec{\pi} + p)$ is even. Now the expansions of $e(\mu)$ at $\mu = -\mu^o$ can be proven along the same lines of Theorem 2.4 using Proposition A.2 with $f = |v|^2$. □

Remark 3.3 Let $\widehat{v} : \mathbb{Z}^d \to \mathbb{C}$ satisfy (H1). Since $\mathfrak{e}(\cdot)$ is even,

$$\int_{\mathbb{T}^d} \frac{|v(p)|^2 dp}{\mathfrak{e}(p) - z} = \frac{1}{(2\pi)^d} \int_{\mathbb{T}^d} \frac{f(p) dp}{\mathfrak{e}(p) - z},$$

where

$$\begin{aligned} f(p) :=& \left(\sum_{x \in \mathbb{Z}^d} \widehat{v}_1(x) \cos p \cdot x \right)^2 + \left(\sum_{x \in \mathbb{Z}^d} \widehat{v}_2(x) \cos p \cdot x \right)^2 \\ & + \left(\sum_{x \in \mathbb{Z}^d} \widehat{v}_1(x) \sin p \cdot x \right)^2 + \left(\sum_{x \in \mathbb{Z}^d} \widehat{v}_2(x) \sin p \cdot x \right)^2 \end{aligned}$$

and $\widehat{v} = \widehat{v}_1 + i\widehat{v}_2$ for some $\widehat{v}_1, \widehat{v}_2 : \mathbb{Z}^d \to \mathbb{R}$. By Lemma 3.1, the unique eigenvalue $e(\mu)$ of H_μ solves

$$1 - \mu \int_{\mathbb{T}^d} \frac{f(p)dp}{\mathfrak{e}(p) - e(\mu)} = 0.$$

Since both $p \in \mathbb{T}^d \mapsto f(p)$ and $p \in \mathbb{T}^d \mapsto f(\vec{\pi} + p)$ are even analytic functions, we can still apply Propositions A.1 and A.2 to find the expansions of $z \mapsto \int_{\mathbb{T}^d} \frac{f(p)dp}{\mathfrak{e}(p)-z}$ and thus, repeating the same arguments of the proofs of Theorems 2.4 and 2.5 one can obtain the corresponding expansions of $e(\mu)$.

Remark 3.4 When

$$|\widehat{v}(x)| = O(|x|^{2n_0+d+1}) \qquad \text{as } |x| \to \infty$$

for some $n_0 \geq 1$, in view of Remark A.3, we need to solve equation (3.4) with respect to μ using only that left-hand side is an asymptotic sum (not a convergent series). This still can be done using appropriate modification of the Implicit Function Theorem for differentiable functions. As a result, we obtain only (Taylor-type) asymptotics of $e(\mu)$.

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Appendix A. Asymptotics of some integrals

In this section we study the behaviour of the integral

$$\mathfrak{l}_f(z) := \int_{\mathbb{T}^d} \frac{f(q)dq}{\mathfrak{e}(q) - z}, \qquad z \in \mathbb{C} \setminus [0, 4d^2], \tag{A.1}$$

as $z \to 0$ and $z \to 4d^2$, where $f : \mathbb{T}^d \to \mathbb{R}$ is a real-analytic even function on $\mathbb{T}^d$. Further we denote by $W_r(\xi) \subset \mathbb{C}$ the complex disc of radius $r > 0$ centered at $\xi \in \mathbb{C}$.

Proposition A.1 *Let $f : \mathbb{T}^d \to \mathbb{R}$ be a real-analytic even function such that*

$$f(0) = D^2 f(0) = \ldots = D^{2n_o-2} f(0) = 0, \qquad D^{2n_o}(0) \neq 0 \tag{A.2}$$

for some $n_o \geq 0$. *Then:*

- $\mathfrak{l}_f$ *is continuous at* 0 *if and only if* $2n_o + d \geq 5$;
- $\mathfrak{l}_f$ *is continuously differentiable at* 0 *if and only if* $2n + d \geq 9$, *in this case,*

$$\mathfrak{l}'_f(0) := \int_{\mathbb{T}^d} \frac{f(q)dq}{(\mathfrak{e}(q))^2} = \lim_{z \searrow 0} \int_{\mathbb{T}^d} \frac{f(q)dq}{(\mathfrak{e}(q) - z)^2}.$$

Moreover, for any $z \in (-\frac{1}{64}, 0)$:

(a) if d is odd, then

$$\mathfrak{l}_f(z) = \begin{cases} \frac{\pi}{4(-z)^{3/4}} \left(c_f + \sum\limits_{n\geq 1} a_n^d \, (-z)^{n/4} \right), & 2n_o + d = 1, \\ \frac{\pi}{8(-z)^{1/4}} \left(c_f + \sum\limits_{n\geq 1} a_n^d \, (-z)^{n/4} \right), & 2n_o + d = 3, \\ \mathfrak{l}_f(0) - \frac{\pi(-z)^{1/4}}{8} \left(c_f + \sum\limits_{n\geq 1} a_n^d \, (-z)^{n/4} \right), & 2n_o + d = 5, \\ \mathfrak{l}_f(0) - \frac{\pi(-z)^{3/4}}{8} \left(c_f + \sum\limits_{n\geq 1} a_n^d \, (-z)^{n/4} \right), & 2n_o + d = 7, \\ \mathfrak{l}_f(0) + z \left(\mathfrak{l}'_f(0) + \sum\limits_{n\geq 1} a_n^d \, (-z)^{n/4} \right), & 2n_o + d \geq 9, \end{cases} \tag{A.3}$$

(b) if d is even, then

$$\mathfrak{l}_f(z) = \begin{cases} \frac{\pi}{8(-z)^{1/2}} \left(c_f + \sum\limits_{n\geq 1} b_n^d \, (-z)^{n/2} \right) - \frac{1}{16} \ln(-z) \sum\limits_{n\geq 0} c_n^d z^n, & 2n_o + d = 2, \\ -\frac{1}{16} \ln(-z) \left(c_f + \sum\limits_{n\geq 1} c_n^d z^n \right) + \sum\limits_{n\geq 0} b_n^d \, (-z)^{n/2}, & 2n_o + d = 4, \\ \mathfrak{l}_f(0) - \frac{\pi(-z)^{1/2}}{8} \left(c_f + \sum\limits_{n\geq 1} b_n^d \, (-z)^{n/2} \right) + z \ln(-z) \sum\limits_{n\geq 0} c_n^d z^n, & 2n_o + d = 6, \\ \mathfrak{l}_f(0) - \frac{z}{16} \ln(-z) \left(c_f + \sum\limits_{n\geq 1} c_n^d z^n \right) + \sum\limits_{n\geq 2} b_n^d \, (-z)^{n/2}, & 2n_o + d = 8, \\ \mathfrak{l}_f(0) + z \left(\mathfrak{l}'_f(0) + \sum\limits_{n\geq 1} b_n^d \, (-z)^{n/2} \right) + z^2 \ln(-z) \sum\limits_{n\geq 0} c_n^d \, z^n, & 2n_o + d \geq 10, \end{cases} \tag{A.4}$$

where $\{a_n^d\}$, $\{b_n^d\}$ *and* $\{c_n^d\}$ *are some real coefficients,*

$$c_f := \frac{2^{2n_o+d}}{(2n_o)!} \int_{\mathbb{S}^{d-1}} D^{2n_o} f(0)[w, \ldots, w] d\mathcal{H}^{d-1}; \tag{A.5}$$

and all series in (A.3) and (A.4) converge absolutely for $z \in W_{1/64}(0) \subset \mathbb{C}$.

Proof Given $\gamma \in (0, \frac{1}{\sqrt{2}}]$, let $\varphi : B_\gamma(0) \subset \mathbb{R}^d \to \varphi(B_\gamma(0)) \subset \mathbb{R}^d$ be the smooth diffeomorphism

$$\varphi_i(y) = 2\arcsin y_i, \qquad i = 1, \ldots, d.$$

Note that

$$\mathfrak{e}(\varphi(y)) = \left(\sum_{i=1}^d (1 - \cos(2\arcsin(y_i)))\right)^2 = 4\left(\sum_{i=1}^d y_i^2\right)^2 = 4y^4, \tag{A.6}$$

therefore,

$$\mathfrak{e}(q) \geq 4\gamma^4 \qquad \text{for any } q \in \mathbb{T}^d \setminus \varphi(B_\gamma). \tag{A.7}$$

We rewrite $\mathfrak{l}_f(z)$ as

$$\mathfrak{l}_f(z) := \int_{\varphi(B_\gamma(0))} \frac{f(q)dq}{\mathfrak{e}(q) - z} + \int_{\mathbb{T}^d \setminus \varphi(B_\gamma(0))} \frac{f(q)dq}{\mathfrak{e}(q) - z} := \mathfrak{l}^*(z) + \mathfrak{l}^{**}(z).$$

By virtue of (A.7),

$$\mathfrak{l}^{**}(z) = \int_{\mathbb{T}^d \setminus \varphi(B_\gamma(0))} \frac{f(q)}{\mathfrak{e}(q)} \left(1 - \frac{z}{\mathfrak{e}(q)}\right)^{-1} dq = \sum_{n \geq 0} z^n \int_{\mathbb{T}^d \setminus \varphi(B_\gamma(0))} \frac{f(q)dq}{(\mathfrak{e}(q))^{n+1}}, \tag{A.8}$$

i.e. $\mathfrak{l}^{**}(\cdot)$ is analytic in $W_{2\gamma^4}(0)$. In $\mathfrak{l}^*$ making the change of variables $q = \varphi(y)$ and using (A.6) we get

$$\mathfrak{l}^*(z) = \int_{B_\gamma(0)} \frac{f(\varphi(y))\, J(\varphi(y))\, dy}{4y^4 - z}, \tag{A.9}$$

where $y^4 := (y^2)^2$ with $y^2 := \sum_{i=1}^d y_i^2$, and

$$J(\varphi(y)) = \prod_{i=1}^d \frac{2}{\sqrt{1 - y_i^2}} \tag{A.10}$$

is the Jacobian of φ. Since f is an even analytic function satisfying (A.2), even each coordinate, from the Taylor series for f it follows that

$$f(p) = \sum_{n \geq n_o} \frac{1}{(2n)!} D^{2n} f(0)[\underbrace{p, \ldots, p}_{2n\text{-times}}], \tag{A.11}$$

and by the analyticity of f in $B_\pi(0) \subset \mathbb{R}^d$, the series converges absolutely in $p \in B_\pi(0)$. By the definition of φ, $\varphi(rw) \subset B_\pi(0)$ for any $r \in (0, \gamma)$ and $w = (w_1, \ldots, w_d) \in \mathbb{S}^{d-1}$, where $\mathbb{S}^{d-1}$ is the unit sphere in $\mathbb{R}^d$. Then letting $p = \varphi(rw)$ and using the Taylor series

$$\varphi_i(rw) = 2rw_i + \frac{r^3 w_i^3}{3} + \sum_{n\ge 3} \tilde{c}_n r^{2n-1} w_i^{2n-1}$$

of $2\arcsin(\cdot)$, which is absolutely convergent for $|rw_i| < 1$, from (A.11) we obtain

$$f(\varphi(rw)) = \sum_{n\ge n_o} \tilde{C}_n(w)\, r^{2n}, \tag{A.12}$$

where $\tilde{C}_n : \mathbb{S}^{d-1} \to \mathbb{R}$ is a homogeneous polynomial of $w \in \mathbb{S}^{d-1}$ of degree $2n$, and

$$\tilde{C}_{n_o}(w) = \frac{2^{2n_o}}{(2n_o)!} D^{2n_o} f(0) \Bigg[\underbrace{w, \ldots, w}_{2\mathrm{n_o}\text{ - times}} \Bigg]$$

Next consider $J(\varphi(y))$. Inserting the Taylor series of $(1-t)^{-1/2}$ into (A.10) we obtain

$$J(\varphi(rw)) = 2^d \left(1 + \sum_{n\ge 1} \widehat{C}_n(w) r^{2n} \right), \tag{A.13}$$

where $\widehat{C}_n : \mathbb{S}^{d-1} \to \mathbb{R}$ is a homogeneous symmetric polynomial of $w \in \mathbb{S}^{d-1}$ of degree $2n$, and the series converges absolutely.

Now passing to polar coordinates by $y = rw$ in (A.9) and using (A.12) and (A.13) as well as the absolute convergence of the series we get

$$\mathfrak{l}^*(z) = 2^d \int_0^\gamma \frac{r^{d-1}}{4r^4 - z} \left(\sum_{n\ge n_o} \int_{\mathbb{S}^{d-1}} C_n(w)\, r^{2n} \right) d\mathcal{H}^{d-1} dr = \sum_{n\ge n_o} \widehat{c}_n \int_0^\gamma \frac{r^{2n+d-1} dr}{4r^4 - z}, \tag{A.14}$$

where $C_n : \mathbb{S}^{d-1} \to \mathbb{R}$ is a homogeneous polynomial of $w \in \mathbb{S}^{d-1}$ of degree $2n$ and

$$\widehat{c}_n := 2^d \int_{\mathbb{S}^{d-1}} C_n(w) d\mathcal{H}^{d-1}.$$

Note that $\widehat{c}_{n_o} = c_f$, where c_f is given by (A.5) and the last series in (A.14) uniformly converges in any compact subset of $\mathbb{C} \setminus [0, 4]$ since $\mathfrak{l}^*$ and

$$z \in \mathbb{C} \setminus [0, 4] \mapsto \mathfrak{j}_{2n+d-1}(z) := \int_0^\gamma \frac{r^{2n+d-1} dr}{4r^4 - z}$$

are analytic functions in $\mathbb{C} \setminus [0,4]$ and all series in (A.14) converge pointwise[1]. Note that for any $m \geq 0$, there exist $c_m \in \mathbb{R}$ and an analytic function f_m in the ball $W_{\gamma^4}(0) \subset \mathbb{C}$ such that for any $z \in (-\gamma^4, 0)$,

$$\mathrm{j}_m(z) = z^n\, \mathrm{j}_l^o(z) + c_m + z^\nu f_m((-z)^{1/2}), \tag{A.15}$$

where $n := [\frac{m}{4}]$, $l := m - 4n \in \{0,1,2,3\}$, $\nu = \frac{1}{2}$ for $m = 0,2$ and $\nu = 1$ for $m = 1,3$ or $m \geq 4$, and

$$\mathrm{j}_l^o(z) := \begin{cases} \frac{\pi}{4}\,(-z)^{-3/4} & \text{if} \quad l = 0, \\ \frac{\pi}{8}\,(-z)^{-1/2} & \text{if} \quad l = 1, \\ \frac{\pi}{8}\,(-z)^{-1/4} & \text{if} \quad l = 2, \\ -\frac{1}{16}\,\ln(-z) & \text{if} \quad l = 3. \end{cases}$$

Inserting (A.15) into (A.14) we obtain

$$\mathfrak{l}^*(z) = \sum_{n \geq n_o} \widehat{c}_n \Big(z^{[\frac{2n+d-1}{4}]}\, \mathrm{j}^o_{2n+d-1-4[\frac{2n+d-1}{4}]}(z) + c_{2n+d-1} + \widehat{c}_n (-z)^{\nu_n} f_{2n+d-1}((-z)^{1/2}) \Big),$$

where $\{c_{2n+d-1}\} \subset \mathbb{R}$ and $\{f_{2n+d-1}\}$ is a sequence of analytic functions in $W_{\gamma^4}(0)$ and

$$\nu_n := \begin{cases} \frac{1}{2}, & 2n+d = 1,3, \\ 1, & \text{otherwise}. \end{cases}$$

Since (A.14) converges locally uniformly in $\mathbb{C} \setminus [0,4]$, $C := \sum\limits_{n \geq n_o} \widehat{c}_n c_{2n+d-1}$ is finite and

$$\sum_{n \geq n_o} \widehat{c}_n (-z)^{\nu_n} f_{2n+d-1}((-z)^{1/2}) = (-z)^\nu g((-z)^{1/2}),$$

where g is analytic in $W_{\gamma^2}(0)$ and $\nu = \frac{1}{2}$ for $2n_o + d = 1,3$ and $\nu = 1$ otherwise. Hence,

$$\mathfrak{l}^*(z) = C + (-z)^\nu g((-z)^{1/2}) + \sum_{n \geq n_o} \widehat{c}_n\, z^{[\frac{2n+d-1}{4}]}\, \mathrm{j}^o_{2n+d-1-4[\frac{2n+d-1}{4}]}(z), \tag{A.16}$$

[1] If $\{h_n\}$ is an equi-bounded sequence of analytic functions in a connected open set $\Omega \subset \mathbb{C}$ converging pointwise to a function $h : \Omega \to \mathbb{C}$, then h is analytic and h_n converges uniformly to h in compact subsets of Ω.

If $0 \le 2n_o + d - 1 \le 3$, then by (A.16),

$$\mathfrak{l}^*(z) = C + (-z)^\nu g((-z)^{1/2}) + \widehat{c}_{n_o}\mathfrak{j}^o_{2n_o+d-1}(z) + \sum_{n\ge n_o+1} \widehat{c}_n z^{[\frac{2n+d-1}{4}]}\mathfrak{j}^o_{2n+d-1-4[\frac{2n+d-1}{4}]}(z). \tag{A.17}$$

In view of (A.8) and the definition of $\mathfrak{j}^o_l$, from (A.17) we obtain the expansions (A.3) and (A.4) of $\mathfrak{l}_f$ for $2n_o + d \le 4$. In particular, since $[\frac{2n+d-1}{4}] \ge 1$ for $n \ge n_o + 1$, letting $z \to 0$ in (A.17) we get

$$\lim_{z\to 0} \mathfrak{l}^*(z) = +\infty. \tag{A.18}$$

If $2n_o + d - 1 \ge 4$, then $[\frac{2n+d-1}{4}] \ge 1$ for any $n \ge n_o$. Therefore, by (A.16), $\mathfrak{l}^*(0) := \lim_{z\to 0} \mathfrak{l}^*(z)$ exists and equals to C. In particular, for $2n_o + d - 1 \le 7$, one has

$$\mathfrak{l}^*(z) = \mathfrak{l}^*(0) - zg((-z)^{1/2}) + \widehat{c}_{n_o} z\mathfrak{j}^o_{2n_o+d-1}(z) + \sum_{n\ge n_o+1} \widehat{c}_n z^{[\frac{2n+d-1}{4}]}\mathfrak{j}^o_{2n+d-1-4[\frac{2n+d-1}{4}]}(z), \tag{A.19}$$

from which and (A.8) we deduce the expansions (A.3) and (A.4) of $\mathfrak{l}_f$ for $5 \le 2n_o+d \le 8$. In particular, by virtue of (A.18) and analyticity of $\mathfrak{l}^{**}$ at $z = 0$, $\mathfrak{l}_f$ is continous at 0 if and only if $2n_o + d \ge 5$. Notice also by (A.19)

$$\lim_{z\to 0} \frac{\mathfrak{l}^*(z) - \mathfrak{l}^*(0)}{z} = +\infty, \tag{A.20}$$

i.e. $\mathfrak{l}^*$ (and hence $\mathfrak{l}_f$) is not differentiable at $z = 0$.

Finally, if $2n_o + d - 1 \ge 8$, then $[\frac{2n+d-1}{4}] \ge 2$ for any $n \ge n_o$. Therefore, by (A.16) there exists

$$\mathfrak{l}^{*\prime}(0) := \lim_{z\to 0} \frac{\mathfrak{l}^*(z) - \mathfrak{l}^*(0)}{z} = -g(0).$$

Now using the Taylor series of g at 0 we get

$$zg((-z)^{1/2}) = \mathfrak{l}^{*\prime}(0)z + z\sum_{n\ge 1} \frac{g^{(n)}(0)}{n!}(-z)^{n/2}.$$

Inserting this in (A.16), using the definition of $\mathfrak{j}^o_l$ and the analyticity of $\mathfrak{l}^{**}$ we get the expansions (A.3) and (A.4) of $\mathfrak{l}_f$ for $2n_o + d \ge 9$.

By (A.18) and (A.20), $\mathfrak{l}_f$ is continously differentiable at 0 if and only if $2n_o+d \ge 9$. Now the choice $\gamma = \frac{1}{\sqrt{2}}$ completes the proof. □

Proposition A.2 *Let $f : \mathbb{T}^d \to \mathbb{R}$ be a real-analytic function such that $q \in \mathbb{T}^d \mapsto f(\vec{\pi} + q)$ is even and*

$$f(\vec{\pi}) = D^2 f(\vec{\pi}) = \ldots = D^{2n_o-2} f(\vec{\pi}) = 0, \qquad D^{2n_o}(\vec{\pi}) \neq 0$$

for some $n_o \in \mathbb{N}_0$. Then:

- $\mathfrak{l}_f$ *is continuous at $z = 4d^2$ if and only if for $2n_o + d \geq 3$,*
- $\mathfrak{l}_f$ *is continuously differentiable at $z = 4d^2$ if and only if for $2n_o + d \geq 5$, in this case*

$$\mathfrak{l}'_f(4d^2) := \int_{\mathbb{T}^d} \frac{f(q)dq}{(\mathfrak{e}(q) - 4d^2)^2} = \lim_{z \searrow 4d^2} \int_{\mathbb{T}^d} \frac{f(q)dq}{(\mathfrak{e}(q) - z)^2}$$

exists.

Moreover, if $z - 4d^2 \in (0, \frac{1}{16})$, $\mathfrak{l}_f(z)$ is represented as:

(a) if d is odd, then

$$\mathfrak{l}_f(z) = \begin{cases} -\frac{\pi C_f}{\sqrt{z-4d^2}} + \sum\limits_{k\geq 0} a_k^d (z-4d^2)^{k/2}, & 2n_o + d = 1, \\ \mathfrak{l}_f(4d^2) + \pi C_f \sqrt{z-4d^2} + \sum\limits_{k\geq 2} a_k^d (z-4d^2)^{k/2}, & 2n_o + d = 3, \\ \mathfrak{l}_f(4d^2) + \mathfrak{l}'_f(4d^2)\,(z-4d^2) + \sum\limits_{k\geq 3} a_k^d (z-4d^2)^{k/2}, & 2n_o + d \geq 5; \end{cases} \tag{A.21}$$

(b) if d is even, then

$$\mathfrak{l}_f(z) = \begin{cases} C_f \ln\alpha + \ln\alpha \sum\limits_{k\geq 1} b_k^d \alpha^k + \sum\limits_{k\geq 0} c_k^d \alpha^k, & 2n_o + d = 2, \\ \mathfrak{l}_f(4d^2) - C_f \alpha \ln\alpha + \ln\alpha \sum\limits_{k\geq 2} b_k^d \alpha^k + \sum\limits_{k\geq 1} c_k^d \alpha^k, & 2n_o + d = 4, \\ \mathfrak{l}_f(4d^2) + \mathfrak{l}'_f(4d^2)\,\alpha + \ln\alpha \sum\limits_{k\geq 2} b_k^d \alpha^k + \sum\limits_{k\geq 2} c_k^d \alpha^k, & 2n_o + d \geq 6, \end{cases} \tag{A.22}$$

where $\alpha := z - 4d^2$, $\{a_k^d\}, \{b_k^d\}, \{c_k^d\} \subset \mathbb{R}$ and

$$C_f := \frac{2^{2n_o+d-1}}{(8d)^{n_o+d/2}\,(2n_o)!} \int_{\mathbb{S}^{d-1}} D^{2n_o} f(\vec{\pi})[w, \ldots, w] d\mathcal{H}^{d-1}.$$

Proof Since $4d^2 - \mathfrak{e}(\cdot)$ has a unique non-degenerate minimum at $\vec{\pi}$, the asymptotics of $\mathfrak{l}_f(z)$ as $z \searrow 4d^2$ can be done along the lines of, for instance, [22,Lemma 4.1], hence, we skip the proof. □

Remark A.3 When

$$|\widehat{v}(x)| = O(|x|^{2n_0+d+1}) \qquad \text{as } |x| \to \infty$$

for some $n_0 \geq 1$, one has $v \in C^{2n_0}(\mathbb{T}^d)$. In this case the Taylor series of f becomes only asymptotics of order $2n_0 - 1$ and thus, instead of expansions (A.3)-(A.4) and (A.21)-(A.22) of $\mathfrak{l}_f$ one has only asymptotics up to order $2n_0 - 1$.

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