The atomic Hardy space for a general Bessel operator

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Abstract

We study Hardy spaces associated with a general multidimensional Bessel operator $\mathbb{B}_\nu$. This operator depends on a multiparameter of type ν that is usually restricted to a product of half-lines. Here we deal with the Bessel operator in the general context, with no restrictions on the type parameter. We define the Hardy space H^1 for $\mathbb{B}_\nu$ in terms of the maximal operator of the semigroup of operators $\exp(-t\mathbb{B}_\nu)$. Then we prove that, in general, H^1 admits an atomic decomposition of local type.

Keywords Bessel operator · Hardy space · Local atomic decomposition

Mathematics Subject Classification 42B30 (primary) · 42B25 · 47D03 (secondary)

1 Introduction

Let $\nu = (\nu_1, \dots, \nu_d) \in \mathbb{R}^d$, $d \geq 1$, and consider the multidimensional Bessel differential operator

$$\mathbb{B}_\nu f(x) = \sum_{j=1}^d -\partial_j^2 f(x) - \frac{2\nu_j + 1}{x_j} \partial_j f(x), \quad (1.1)$$

acting on functions on $\mathbb{R}_+^d = (0, \infty)^d$. Operator $\mathbb{B}_\nu$ is formally symmetric in $L^2(\mathbb{R}_+^d, d\mu_\nu)$, where

$$d\mu_\nu(x) = x^{2\nu+1} dx := x_1^{2\nu_1+1} \dots x_d^{2\nu_d+1} dx_1 \dots dx_d.$$

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1.1 Hardy spaces associated with the classical Bessel operator

When $\nu \in (-1, \infty)^d$ there exists a classical self-adjoint extension of $\mathbb{B}_\nu$ (acting initially on $C_c^2(\mathbb{R}_+^d)$), from now on denoted by $\mathbb{B}_\nu^{\text{cls}}$ [16, 19]. The operator $\mathbb{B}_\nu^{\text{cls}}$ is the infinitesimal generator of the classical Bessel semigroup of operators $\mathbb{W}_{t,\nu}^{\text{cls}} = \exp(-t\mathbb{B}_\nu^{\text{cls}})$, which has the integral representation $\mathbb{W}_{t,\nu}^{\text{cls}} f(x) = \int_{\mathbb{R}_+^d} \mathbb{W}_{t,\nu}^{\text{cls}}(x, y) f(y) d\mu_\nu(y)$, $t > 0$. It is well known that

$$\mathbb{W}_{t,\nu}^{\text{cls}}(x, y) = \prod_{j=1}^d \frac{1}{2t} (x_j y_j)^{-\nu_j} I_{\nu_j} \left(\frac{x_j y_j}{2t} \right) \exp \left(-\frac{x_j^2 + y_j^2}{4t} \right) =: \prod_{j=1}^d W_{t,\nu_j}^{\text{cls}}(x_j, y_j),$$

for $x, y \in \mathbb{R}_+^d$ and $t > 0$, where I_τ denotes the modified Bessel function of the first kind and order $\tau > -1$, cf. [26]. The classical Bessel kernel satisfies the lower and upper Gaussian bounds, i.e.

$$\frac{C_1}{\mu_\nu(B(x, \sqrt{t}))} \exp \left(-\frac{|x-y|^2}{c_1 t} \right) \leq \mathbb{W}_{t,\nu}^{\text{cls}}(x, y) \leq \frac{C_2}{\mu_\nu(B(x, \sqrt{t}))} \exp \left(-\frac{|x-y|^2}{c_2 t} \right), \quad (1.2)$$

with some constants $c_1, c_2, C_1, C_2 > 0$, where $B(x, \sqrt{t}) = \{y \in \mathbb{R}_+^d : |x-y| < \sqrt{t}\}$.

Recently, harmonic analysis related to the classical Bessel operator has been extensively developed, see e.g. [1–6, 8, 12, 14, 19, 20] and references therein. In particular, the Hardy space

$$H^1(\mathbb{B}_\nu^{\text{cls}}) = \left\{ f \in L^1(\mathbb{R}_+^d, d\mu_\nu) : \|f\|_{H^1(\mathbb{B}_\nu^{\text{cls}})} := \left\| \sup_{t>0} |\mathbb{W}_{t,\nu}^{\text{cls}} f| \right\|_{L^1(\mathbb{R}_+^d, d\mu_\nu)} < \infty \right\}, \quad (1.3)$$

associated with $\mathbb{B}_\nu^{\text{cls}}$ and its characterizations have been studied. An especially useful result is the characterization of $H^1(\mathbb{B}_\nu^{\text{cls}})$ by atomic decomposition, which was proved in [5] in the one-dimensional case and then extended to higher dimensions in [12]. This atomic characterization also follows from a more general result from [11]. This result states, in particular, that every function $f \in H^1(\mathbb{B}_\nu^{\text{cls}})$ can be represented as $f = \sum_k \lambda_k a_k$, where $\sum_k |\lambda_k| \simeq \|f\|_{H^1(\mathbb{B}_\nu^{\text{cls}})}$ and a_k are the classical atoms on the space of homogeneous type $(\mathbb{R}_+^d, |\cdot|, d\mu_\nu)$, that is there exist balls B_k such that

$$\text{supp } a_k \subseteq B_k, \quad \|a_k\|_\infty \leq \mu_\nu(B_k)^{-1}, \quad \int a_k d\mu_\nu = 0. \quad (1.4)$$

We say that the atoms a_k satisfy localization, size and cancellation conditions [9].

The main goal of this paper is to prove an atomic decomposition theorem for the Hardy space associated with the multidimensional Bessel operator $\mathbb{B}_\nu$ in the general situation, admitting all $\nu \in \mathbb{R}^d$. For a precise definition of the operator $\mathbb{B}_\nu$ see in Sect. 1.3. According to the best of the author's knowledge the theory of Hardy spaces for the Bessel operator for the full range of the parameter has never been studied. The most general results are known for $\nu_j > -1$, and a vast majority of them are

restricted to $v_j \geq -1/2$, $j = 1, \dots, d$. Our atomic decomposition theorem is a starting point for developing this theory in a general Bessel context and the first step towards proving other characterizations of $H^1(\mathbb{B}_v)$. Note that one of the difficulties in the general situation is that the measure is not locally finite, in particular it does not satisfy the standard doubling condition. Theory of Hardy spaces or Calderón-Zygmund operators for spaces with non-doubling measure is far more difficult and less known. It has been developed only quite recently, see e.g. [17,18,23,24,27]. Usually an essential assumption in this theory is the polynomial growth condition $\mu(B(x, r)) \leq Cr^n$. But this condition fails for μ_v unless $v \in (-1, \infty)^d$, since otherwise there are balls of arbitrarily small radius and infinite measure. For the same reason, our measure does not satisfy the property known in the literature as a local doubling condition, considered for instance in [7]. Another obstacle is that, in general, operators considered in the literature are generators of semigroups of operators whose integral kernels satisfy the upper Gaussian bounds. This is not the case of $\mathbb{B}_v$ if $v \notin (-1, \infty)^d$. Nevertheless we overcome these difficulties and give an atomic decomposition of the Hardy space associated with the general multidimensional Bessel operator $\mathbb{B}_v$ for any $v \in \mathbb{R}^d$.

1.2 The exotic Bessel operator

Recall that $d\mu(x) = x^{2v+1}dx$. When we take $v \in ((-\infty, 1) \setminus \{0\})^d$ in (1.1), then there exists a self-adjoint extension of $\mathbb{B}_v$, denoted by $\mathbb{B}_v^{\text{exo}}$ and called the exotic Bessel operator [19]. It is quite remarkable that for the parameter $v \in ((-\infty, 1) \setminus \{0\})^d$ there exist both, the classical and the exotic, self-adjoint extensions of $\mathbb{B}_v$, which differ significantly. We denote by $\mathbb{W}_{t,v}^{\text{exo}} = \exp(-t\mathbb{B}_v^{\text{exo}})$ the semigroup of operators generated by $\mathbb{B}_v^{\text{exo}}$. It is known that the exotic semigroup kernel can be expressed in a simple way in terms of the classical one, namely

$$\mathbb{W}_{t,v}^{\text{exo}}(x, y) = \prod_{j=1}^d W_{t,v_j}^{\text{exo}}(x_j, y_j) = \prod_{j=1}^d (x_j y_j)^{-2v_j} W_{t,-v_j}^{\text{cls}}(x_j, y_j). \quad (1.5)$$

It turns out that if $0 < v_j < 1$ then the so-called pencil phenomenon occurs. In the one-dimensional situation this means that for any fixed $t > 0$ the operator W_{t,v_j}^{exo} is well defined on $L^p(\mathbb{R}_+, d\mu_{v_j})$ and maps this space into itself if and only if $v_j + 1 < p < (v_j + 1)/v_j$. Therefore there is no reason in studying the theory of Hardy spaces for the exotic Bessel operator in the case $v \in (0, 1)^d$. For more details see the discussion in [19, Sec. 4] or in [4]. Thus, from now on we shall consider the operator $\mathbb{B}_v^{\text{exo}}$ only for $v \in (-\infty, 0)^d$.

For the sake of convenience we shall write $\mathbb{B}_{-v}^{\text{exo}}$, where $v \in (0, \infty)^d$. Hence, we will use such notation in the rest of the paper.

1.3 Main results

We consider a general multidimensional Bessel operator $\mathbb{B}_v = L_1 + \dots + L_d$, $d \geq 1$, where each L_i is either one-dimensional classical Bessel operator $B_{v_i}^{\text{cls}}$ for $v_i \in$

$(-1, \infty)$ or one-dimensional exotic Bessel operator $B_{-v_i}^{\text{exo}}$ for $v_i \in (0, \infty)$ (acting on the i th coordinate variable, see Sect. 4.1 for details). Since $B_{v_i}^{\text{cls}}$ and $B_{-v_i}^{\text{exo}}$ are self-adjoint operators, $\mathbb{B}_v$ is well defined and essentially self-adjoint [21, Thm. 7.23]. To simplify the notation, by changing the coordinates, we shall consider $\mathbb{B}_v = \mathbb{B}_{v_c}^{\text{cls}} + \mathbb{B}_{-v_e}^{\text{exo}}$, where $v = (v_c, -v_e) \in (-1, \infty)^{d_1} \times (-\infty, 0)^{d_2}$ and $d_1 + d_2 = d$. Then

$$\mathbb{B}_v f(\mathbf{x}) = \mathbb{B}_{v_c}^{\text{cls}} f(\cdot, \mathbf{x}_2) + \mathbb{B}_{-v_e}^{\text{exo}} f(\mathbf{x}_1, \cdot) \quad \text{for } \mathbf{x} = (\mathbf{x}_1, \mathbf{x}_2) \in \mathbb{R}_+^{d_1} \times \mathbb{R}_+^{d_2} = \mathbb{R}_+^d.$$

As we mentioned above, the case $d_1 \geq 1$ and $d_2 = 0$, that is the classical case, is well known and slightly different from the context involving the exotic Bessel operator. Therefore we will consider $d_1 \geq 0$ and $d_2 \geq 1$.

Denote by $\mathbb{W}_{t,v} = \exp(-t\mathbb{B}_v)$ the semigroup of operators generated by $\mathbb{B}_v$. Clearly, the semigroup $\mathbb{W}_{t,v}$ has the integral representation

$$\mathbb{W}_{t,v} f(\mathbf{x}) = \int_{\mathbb{R}_+^d} \mathbb{W}_{t,v}(\mathbf{x}, \mathbf{y}) f(\mathbf{y}) d\mu_v(\mathbf{y}), \quad \mathbf{x} \in \mathbb{R}_+^d, \quad t > 0,$$

where

$$\begin{aligned} \mathbb{W}_{t,v}(\mathbf{x}, \mathbf{y}) &= \mathbb{W}_{t,v_c}^{\text{cls}}(\mathbf{x}_1, \mathbf{y}_1) \mathbb{W}_{t,-v_e}^{\text{exo}}(\mathbf{x}_2, \mathbf{y}_2), \\ \mathbf{x} &= (\mathbf{x}_1, \mathbf{x}_2), \mathbf{y} = (\mathbf{y}_1, \mathbf{y}_2) \in \mathbb{R}_+^{d_1} \times \mathbb{R}_+^{d_2}. \end{aligned} \quad (1.6)$$

We define the Hardy space for the operator $\mathbb{B}_v$ by means of the maximal operator associated with the semigroup of operators $\{\mathbb{W}_{t,v}\}$,

$$H^1(\mathbb{B}_v) = \left\{ f \in L^1(\mathbb{R}_+^d, d\mu_v) : \|f\|_{H^1(\mathbb{B}_v)} := \left\| \sup_{t>0} |\mathbb{W}_{t,v} f| \right\|_{L^1(\mathbb{R}_+^d, d\mu_v)} < \infty \right\}.$$

We will prove that elements of $H^1(\mathbb{B}_v)$ have an atomic decomposition, where atoms are either classical atoms (1.4) or some additional atoms of the form $\mu_v(Q)^{-1} \mathbb{1}_Q$ for some cube $Q \subset \mathbb{R}_+^d$. Such atoms $\mu_v(Q)^{-1} \mathbb{1}_Q$ are called "local atoms" and notice that they do not satisfy cancellation condition. More precisely, we will show that every function $f \in H^1(\mathbb{B}_v)$ belongs also to the local atomic Hardy space $H_{\text{at}}^1(Q, \mu_v)$ associated with some family of cubes Q in $\mathbb{R}_+^d$. That means that f can be decomposed into a sum $f = \sum_k \lambda_k a_k$, where $\sum_k |\lambda_k| < \infty$ and a_k are either classical atoms described in (1.4) and supported in cubes $Q \in \mathcal{Q}$ or atoms of the form $a_k = \mu_v(Q)^{-1} \mathbb{1}_Q$, $Q \in \mathcal{Q}$.

For the general Bessel operator $\mathbb{B}_v$ let us define the family of cubes $\mathcal{Q}_{\mathbb{B}}$ which arises as follows. Let $\mathcal{D} = \{[2^n, 2^{n+1}] : n \in \mathbb{Z}\}$ be the collection of all closed dyadic intervals in $\mathbb{R}_+$. Consider a family $\{\mathbb{R}_+^{d_1} \times Q_1 \times \dots \times Q_{d_2} : Q_i \in \mathcal{D}\}$. Then tile each cylinder $\mathbb{R}_+^{d_1} \times Q_1 \times \dots \times Q_{d_2}$ with countably many cubes having diameters equal to the smallest of the diameters of $Q_1, \dots, Q_{d_2}$. The family $\mathcal{Q}_{\mathbb{B}}$ consists of all of these smaller cubes. For a rigorous definition of $\mathcal{Q}_{\mathbb{B}}$ see Sect. 4.

The main result of this paper is the following.

Theorem 1.1 *Let $d_1 \geq 0$, $d_2 \geq 1$. Assume that $v_c \in (-1, \infty)^{d_1}$ and $v_e \in (0, \infty)^{d_2}$. Then $H^1(\mathbb{B}_{v_c}^{\text{cls}} + \mathbb{B}_{-v_e}^{\text{exo}})$ and $H_{\text{at}}^1(\mathcal{Q}_{\mathbb{B}}, \mu_v)$ are isomorphic as Banach spaces.*

In the proof we will use only general properties of $\mathbb{B}_{v_c}^{\text{cls}}$ and $\mathbb{B}_{-v_e}^{\text{exo}}$, therefore some auxiliary results in Sects. 3 and 4 will be formulated in a more general context. Moreover, we introduce universal conditions on a semigroup kernel so that the product case can be deduced from a lower-dimensional information. The methods we use have roots in [15], however in that paper the Lebesgue measure case is considered, and here we need to adapt them to our situation, which requires some effort.

1.4 Organization of the paper

Section 2 contains definitions and notation used in the paper. Also, some auxiliary results are proved there. In Sect. 3 we consider a general self-adjoint and nonnegative operator L , which generates a semigroup of operators possessing an integral representation. We present general assumptions on the semigroup integral kernel and some family of cubes that are sufficient to prove an atomic decomposition of local type for $H^1(L)$. Section 4 is devoted to a similar atomic characterization in a product situation. More precisely, we slightly generalize conditions from Sect. 3 and show that if (lower-dimensional) component operators L_1, L_2 satisfy them, then the operator $L = L_1 + L_2$ also does. Hence, one may deduce the result for the sum of operators by checking “one-dimensional” conditions. Finally, in Sect. 5, we prove Theorem 1.1.

Throughout the paper we use standard notation. In particular, C and c at each occurrence denote some positive constants independent of relevant quantities. Values of C and c may change from line to line. Further, we write $\alpha \simeq \beta$ if there exists a positive constant C , independent of significant quantities, such that $C^{-1}\alpha \leq \beta \leq C\alpha$.

2 Preliminaries

2.1 Notation and terminology

We consider the metric measure space $(\mathbb{R}_+^d, |\cdot|, d\mu_v)$, where $|\cdot|$ stands for the standard Euclidean metric. It is well known that if $v \in (-1, \infty)^d$, then μ_v possesses the doubling property, i.e. there exists $C > 0$ such that

$$\mu_v(B(x, 2r)) \leq C\mu_v(B(x, r)), \quad x \in \mathbb{R}_+^d, \quad r > 0, \quad (2.1)$$

where $B(x, r) = \{y \in \mathbb{R}_+^d : |x - y| < r\}$.

Since both the space and the measure have product structure, it is convenient to use cubes and cuboids rather than balls. Thus denote by

$$Q(z; r_1, \dots, r_d) = \{x \in \mathbb{R}_+^d : |x_i - z_i| \leq r_i \text{ for } i = 1, \dots, d\}$$

the cuboid centered at $z \in \mathbb{R}_+^d$ having axial radii $r_1, \dots, r_d > 0$. When $r_1 = \dots = r_d = r$ the cuboid becomes a cube which we denote briefly by $Q(z, r)$. We denote by d_Q the Euclidean diameter of a cuboid Q .

Definition 2.1 We call a family $\mathcal{Q}$ of cuboids in $\mathbb{R}^d$ an *admissible covering* if there exist $C_1, C_2 > 0$ such that:

1. $\mathbb{R}_+^d = \bigcup_{Q \in \mathcal{Q}} Q$,
2. if $Q_1, Q_2 \in \mathcal{Q}$ and $Q_1 \neq Q_2$, then $\mu_v(Q_1 \cap Q_2) = 0$,
3. if $Q = Q(z; r_1, \dots, r_d) \in \mathcal{Q}$, then $r_i \leq C_1 r_j$ for $i, j \in \{1, \dots, d\}$,
4. if $Q_1, Q_2 \in \mathcal{Q}$ and $Q_1 \cap Q_2 \neq \emptyset$, then $C_2^{-1} d_{Q_1} \leq d_{Q_2} \leq C_2 d_{Q_1}$.

Observe that item **3** means that admissible cuboids are uniformly bounded deformations of cubes. In fact, we shall often use only cubes. From now on we always assume that $\mathcal{Q}$ is an admissible covering of $\mathbb{R}_+^d$.

Given a cuboid Q , by Q^* we denote a (slight) enlargement of Q . More precisely, if $Q = (z; r_1, \dots, r_d)$, then $Q^* := Q(z; \kappa r_1, \dots, \kappa r_d)$, where $\kappa > 1$ is a fixed constant. Let $\mathcal{Q}$ be a given admissible covering of $\mathbb{R}_+^d$. We fix $\kappa = \kappa(\mathcal{Q})$ sufficiently close to 1, so that for any $Q_1, Q_2 \in \mathcal{Q}$,

$$Q_1^{***} \cap Q_2^{***} \neq \emptyset \iff Q_1 \cap Q_2 \neq \emptyset. \quad (2.2)$$

The family $\{Q^{***}\}_{Q \in \mathcal{Q}}$ is a finite covering of $\mathbb{R}_+^d$,

$$\sum_{Q \in \mathcal{Q}} \mathbb{1}_{Q^{***}}(x) \leq C, \quad x \in \mathbb{R}_+^d.$$

For $Q \in \mathcal{Q}$ denote $N(Q) = \{\tilde{Q} \in \mathcal{Q} : \tilde{Q} \cap Q \neq \emptyset, \tilde{Q} \neq Q\}$ (all neighbors of the cuboid Q), and $\tilde{N}(Q) = \bigcup_{\tilde{Q} \in N(Q)} \tilde{Q}$.

For the covering $\mathcal{Q}$ as above consider functions $\psi_Q \in C^1(\mathbb{R}_+^d)$ satisfying

$$0 \leq \psi_Q(x) \leq \mathbb{1}_{Q^*}(x), \quad \|\psi_Q'\|_\infty \leq C d_Q^{-1}, \quad \sum_{Q \in \mathcal{Q}} \psi_Q(x) = \mathbb{1}_{\mathbb{R}_+^d}(x), \quad x \in \mathbb{R}_+^d. \quad (2.3)$$

It is straightforward to see that such a family $\{\psi_Q\}_{Q \in \mathcal{Q}}$ exists. We call it a *partition of unity* related to $\mathcal{Q}$.

We now define suitable atoms and a local atomic Hardy space $H_{\text{at}}^1(Q, \mu_v)$ related to $\mathcal{Q}$.

Definition 2.2 A function $a: \mathbb{R}_+^d \rightarrow \mathbb{C}$ is a (Q, μ_v) -atom if:

- (i) either there is $Q \in \mathcal{Q}$ and a cube $K \subset Q^*$, such that

$$\text{supp } a \subseteq K, \quad \|a\|_\infty \leq \mu_v(K)^{-1}, \quad \int a d\mu_v = 0;$$

(ii) or there exists $Q \in \mathcal{Q}$ such that

$$a = \mu_v(Q)^{-1} \mathbb{1}_Q.$$

The atoms as in (ii) are called *local atoms*.

Having (Q, μ_v) -atoms at our disposal, we define the *local atomic Hardy space* $H_{\text{at}}^1(Q, \mu_v)$ related to Q in the standard way. Namely, a function f belongs to $H_{\text{at}}^1(Q, \mu_v)$ if it has an atomic decomposition $f = \sum_k \lambda_k a_k$ with $\sum_k |\lambda_k| < \infty$ and a_k being (Q, μ_v) -atoms. The norm in $H_{\text{at}}^1(Q, \mu_v)$ is given by

$$\|f\|_{H_{\text{at}}^1(Q, \mu_v)} = \inf \sum_k |\lambda_k|,$$

where the infimum is taken over all possible representations of f as above. Note that $H_{\text{at}}^1(Q, \mu_v)$ is a Banach space.

2.2 Auxiliary results

Lemma 2.3 *Let $d = 1$ and $v \in \mathbb{R}$. The following estimates hold.*

(a) *If $v > -1$, then*

$$\mu_v(B(x, r)) \simeq \left(1 \wedge \frac{r}{x}\right) (x+r)^{2v+2}, \quad x, r > 0.$$

(b) *If $v = -1$, then*

$$\mu_v(B(x, r)) \simeq \log \frac{x+r}{x-r}, \quad x > r > 0.$$

(c) *If $v < -1$, then*

$$\mu_v(B(x, r)) \simeq \left(1 \wedge \frac{r}{x}\right) (x-r)^{2v+2}, \quad x > r > 0.$$

Proof (a) *If $r > x$, then*

$$\mu_v(B(x, r)) = (2v+2)^{-1} (x+r)^{2v+2}.$$

When $x \geq r > x/2$, we have

$$\mu_v(B(x, r)) = (2v+2)^{-1} \left((x+r)^{2v+2} - (x-r)^{2v+2} \right) \simeq x^{2v+2} \simeq (x+r)^{2v+2}.$$

Finally, if $r \leq x/2$, then $x+r \simeq x \simeq x-r$. Therefore, applying the Mean Value Theorem,

$$\mu_v(B(x, r)) = (2v+2)^{-1} \left((x+r)^{2v+2} - (x-r)^{2v+2} \right) \simeq r x^{2v+1} \simeq \frac{r}{x} (x+r)^{2v+2}.$$

The proof of **(b)** is straightforward. We pass to proving **(c)**. If $x > r > x/2$, then

$$\mu_v(B(x, r)) = (-2v - 2)^{-1} \left((x - r)^{2v+2} - (x + r)^{2v+2} \right) \simeq (x - r)^{2v+2},$$

since now $2v + 2 < 0$.

When $r \leq x/2$, we proceed similarly as in the corresponding part of the proof of **(a)** above. The conclusion follows. $\square$

Corollary 2.4 *Let $d = 1$. Then*

(a) *for each $v \in \mathbb{R}$,*

$$\mu_v(B(x, r)) \simeq rx^{2v+1}, \quad 0 < r < x/2;$$

(b) *for each $v > -1$,*

$$\mu_v(B(x, r)) \simeq r(x + r)^{2v+1}, \quad x, r > 0.$$

Proof **(a)** Since $r < x/2$, we have that $x + r \simeq x - r \simeq x$ and for $v \neq -1$ the claim readily follows from (a) and (c) of Lemma 2.3. To cover $v = -1$, use Lemma 2.3(b) and then observe that by the Mean Value Theorem $\log(x + r) - \log(x - r) \simeq r/x$. Part **(b)** is a simple reformulation of Lemma 2.3(a).

As a consequence of the above results we obtain the following.

Corollary 2.5 *Let $d \geq 1$ and $v \in \mathbb{R}^d$. Then*

$$\begin{aligned} \mu_v(B(x, r)) &\simeq \prod_{j=1}^d \mu_{v_j}(B(x_j, r)) \simeq r^d \prod_{j=1}^d x_j^{2v_j+1}, \\ x &\in \mathbb{R}_+^d, \quad 0 < r < \min(x_1, \dots, x_d)/2. \end{aligned}$$

Moreover, if $v \in (-1, \infty)^d$, then

$$\mu_v(B(x, r)) \simeq \prod_{j=1}^d \mu_{v_j}(B(x_j, r)) \simeq r^d \prod_{j=1}^d (x_j + r)^{2v_j+1}, \quad x \in \mathbb{R}_+^d, \quad r > 0.$$

Lemma 2.6 *Let $\mathcal{Q}$ be an admissible covering of $\mathbb{R}_+^d$. Assume that $v \in (-1, \infty)^d$ and $Q \in \mathcal{Q}$. Let $y \in Q^*$. Then for each $\delta \in (0, \min(1/2, v_1 + 1, \dots, v_d + 1))$ and each $c > 0$*

$$\int_{Q^{**c}} \sup_{t>0} t^\delta \mu_v(B(x, \sqrt{t}))^{-1} \exp\left(-\frac{|x-y|^2}{ct}\right) d\mu_v(x) \leq Cd_Q^{2\delta}, \quad (2.4)$$

and

$$\int_{(Q^{**})^c} \sup_{t>0} t^{-\delta} \mu_v(B(x, \sqrt{t}))^{-1} \exp\left(-\frac{|x-y|^2}{ct}\right) d\mu_v(x) \leq Cd_Q^{-2\delta}, \quad (2.5)$$

where the constant C is independent of Q and y .

Proof Let $Q = Q_1 \times \dots \times Q_d$, with $d_Q \simeq d_{Q_1} \simeq \dots \simeq d_{Q_d}$. First we prove (2.4). To do that we need some auxiliary estimates.

Case 1: $v_i \geq -1/2$. Applying Corollary 2.4(b) we have

$$\begin{aligned} I_i &:= \int_{Q_i^{**}} \sup_{t>0} t^{\delta/d} \mu_{v_i}(B(x_i, \sqrt{t}))^{-1} \exp\left(-\frac{|x_i - y_i|^2}{ct}\right) d\mu_{v_i}(x_i) \\ &\leq C \int_{Q_i^{**}} \sup_{t>0} t^{\delta/d-1/2} (x_i + \sqrt{t})^{-2v_i-1} \exp\left(-\frac{|x_i - y_i|^2}{ct}\right) x_i^{2v_i+1} dx_i \\ &\leq C \int_{Q_i^{**}} \sup_{t>0} t^{\delta/d-1/2} \exp\left(-\frac{|x_i - y_i|^2}{ct}\right) dx_i \\ &\leq C \int_{Q_i^{**}} |x_i - y_i|^{2\delta/d-1} dx_i \leq C d_{Q_i}^{2\delta/d}. \end{aligned} \quad (2.6)$$

Case 2: $v_i \in (-1, -1/2)$. In this case, since $\delta/d - 1 - v_i < 0$, using Corollary 2.4(b) we obtain

$$\begin{aligned} I_i &\leq C \int_{Q_i^{**}} \sup_{0<t\leq x_i^2} t^{\delta/d-1/2} (x_i + \sqrt{t})^{-2v_i-1} x_i^{2v_i+1} \exp\left(-\frac{|x_i - y_i|^2}{ct}\right) dx_i \\ &\quad + C \int_{Q_i^{**}} \sup_{t>x_i^2} t^{\delta/d-1/2} (x_i + \sqrt{t})^{-2v_i-1} x_i^{2v_i+1} \exp\left(-\frac{|x_i - y_i|^2}{ct}\right) dx_i \\ &\leq C \int_{Q_i^{**}} |x_i - y_i|^{2\delta/d-1} dx_i \\ &\quad + C \int_{Q_i^{**} \cap \{x_i : x_i/2 \leq |x_i - y_i|\}} \sup_{t>0} t^{\delta/d-1-v_i} x_i^{2v_i+1} \exp\left(-\frac{|x_i - y_i|^2}{ct}\right) dx_i \\ &\quad + C \int_{Q_i^{**} \cap \{x_i : x_i/2 > |x_i - y_i|\}} \sup_{t>0} t^{\delta/d-1-v_i} x_i^{2v_i+1} \exp\left(-\frac{|x_i - y_i|^2}{ct}\right) dx_i \\ &\leq C d_{Q_i}^{2\delta/d} + C \int_{\{x_i : x_i \leq c d_{Q_i}\}} x_i^{2\delta/d-1} dx_i + C \int_{Q_i^{**}} |x_i - y_i|^{2\delta/d-1} dx_i \\ &\leq C d_{Q_i}^{2\delta/d}. \end{aligned} \quad (2.7)$$

To get (2.4) we use Corollary 2.5, (2.6) and (2.7) obtaining

$$\int_{Q^{**}} \sup_{t>0} t^{\delta} \mu_v(B(x, \sqrt{t}))^{-1} \exp\left(-\frac{|x - y|^2}{ct}\right) d\mu_v(x) \leq C \prod_{i=1}^d I_i \leq C \prod_{i=1}^d d_{Q_i}^{2\delta/d} \leq C d_Q^{2\delta}.$$

Next, we prove (2.5) by induction with respect to the dimension d .

Step 1. Suppose $d = 1$. We consider two cases.

Case 1: $\nu \geq -1/2$. Using Corollary 2.4(b) we see that the left hand side of (2.5) is controlled by

$$\begin{aligned} & C \int_{(Q^{**})^c} \sup_{t>0} t^{-\delta-1/2} (x + \sqrt{t})^{-2\nu-1} \exp\left(-\frac{|x-y|^2}{ct}\right) x^{2\nu+1} dx \\ & \leq C \int_{(Q^{**})^c} \sup_{t>0} t^{-\delta-1/2} \exp\left(-\frac{|x-y|^2}{ct}\right) dx \\ & \leq C \int_{(Q^{**})^c} |x-y|^{-2\delta-1} dx \leq C d_Q^{-2\delta}. \end{aligned}$$

The last inequality follows from the relation $|x-y| \geq cd_Q$, since $x \in (Q^{**})^c$ and $y \in Q^*$.

Case 2: $\nu \in (-1, -1/2)$. In this case we again apply Corollary 2.4(b) to bound the left hand side of (2.5) by

$$\begin{aligned} & C \int_{(Q^{**})^c} \sup_{t>0} t^{-\delta-1/2} (x + \sqrt{t})^{-2\nu-1} \exp\left(-\frac{|x-y|^2}{ct}\right) x^{2\nu+1} dx \\ & \leq C \int_{(Q^{**})^c} \sup_{0<t\leq x^2} t^{-\delta-1/2} \exp\left(-\frac{|x-y|^2}{ct}\right) dx \\ & \quad + C \int_{(Q^{**})^c \cap \{x : x \leq 2d_Q\}} \sup_{t>x^2} t^{-\delta-1-\nu} \exp\left(-\frac{|x-y|^2}{ct}\right) x^{2\nu+1} dx \\ & \quad + C \int_{(Q^{**})^c \cap \{x : x > 2d_Q\}} \sup_{t>x^2} t^{-\delta-1-\nu} x^{2\nu+1} dx \\ & \leq C \int_{(Q^{**})^c} |x-y|^{-2\delta-1} dx + C d_Q^{-2\delta-2-2\nu} \int_0^{2d_Q} x^{2\nu+1} dx + C \int_{2d_Q}^\infty x^{-2\delta-1} dx \\ & \leq C d_Q^{-2\delta}. \end{aligned}$$

Step 2. Let $d \geq 1$ be fixed. Suppose $\nu = (\nu_1, \tilde{\nu}) = (\nu_1, \nu_2, \dots, \nu_d) \in (-1, \infty)^d$ and $Q = Q_1 \times \tilde{Q}$, where $d_Q \simeq d_{Q_1} \simeq d_{\tilde{Q}}$. We use the notation $\mathfrak{x} = (x_1, \tilde{x})$, $\mathfrak{y} = (y_1, \tilde{y}) \in \mathbb{R}_+ \times \mathbb{R}_+^{d-1}$. Assume that (2.5) holds for $\tilde{Q}$, that is for every $0 < \delta' < \min(1/2, \nu_2 + 1, \dots, \nu_d + 1)$ and $c > 0$ there exists $C > 0$ such that

$$\int_{(\tilde{Q}^{**})^c} \sup_{t>0} t^{-\delta'} \mu_{\tilde{\nu}}(B(\tilde{x}, \sqrt{t}))^{-1} \exp\left(-\frac{|\tilde{x}-\tilde{y}|^2}{ct}\right) d\mu_{\tilde{\nu}}(\tilde{x}) \leq C d_{\tilde{Q}}^{-2\delta'}, \quad \tilde{y} \in \tilde{Q}^*. \quad (2.8)$$

Here C does not depend on $\tilde{Q}$. Take $\delta \in (0, \min(1/2, \nu_1 + 1, \dots, \nu_d + 1))$. We split the set $(Q^{**})^c = S_1 \cup S_2 \cup S_3$, where

$$S_1 = Q_1^{**} \times (\tilde{Q}^{**})^c, \quad S_2 = (Q_1^{**})^c \times \tilde{Q}^{**}, \quad S_3 = (Q_1^{**})^c \times (\tilde{Q}^{**})^c,$$

and write

$$\begin{aligned} & \int_{(Q^{**})^c} \sup_{t>0} t^{-\delta} \mu_v(B(\mathbb{X}, \sqrt{t}))^{-1} \exp\left(-\frac{|\mathbb{X}-\mathbb{Y}|^2}{ct}\right) d\mu_v(x) \\ &= \int_{S_1} \dots + \int_{S_1} \dots + \int_{S_3} \dots =: J_1 + J_2 + J_3. \end{aligned}$$

To estimate J_1 we apply Corollary 2.5(b), (2.8) and (2.4) with some small $\varepsilon > 0$ such that $\delta + \varepsilon < \min(1/2, v_1 + 1, \dots, v_d + 1)$. We get

$$\begin{aligned} J_1 &\leq C \int_{Q_1^{**}} \sup_{t>0} t^\varepsilon \mu_{v_1}(B(x_1, \sqrt{t}))^{-1} \exp\left(-\frac{|x_1 - y_1|^2}{ct}\right) d\mu_{v_1}(x_1) \\ &\quad \times \int_{(\tilde{Q}^{**})^c} \sup_{t>0} t^{-\delta-\varepsilon} \mu_{\tilde{v}}(B(\tilde{x}, \sqrt{t}))^{-1} \exp\left(-\frac{|\tilde{x} - \tilde{y}|^2}{ct}\right) d\mu_{\tilde{v}}(\tilde{x}) \\ &\leq C d_{Q_1}^{2\varepsilon} d_{\tilde{Q}}^{-2\delta-2\varepsilon} \leq C d_{\tilde{Q}}^{-2\delta}. \end{aligned}$$

We deal with J_2 similarly, but this time using (2.4) for the integral over the cube $\tilde{Q}$ and **Step 1** for the integral over Q_1 . This gives

$$\begin{aligned} J_2 &\leq C \int_{(Q_1^{**})^c} \sup_{t>0} t^{-\delta-\varepsilon} \mu_{v_1}(B(x_1, \sqrt{t}))^{-1} \exp\left(-\frac{|x_1 - y_1|^2}{ct}\right) d\mu_{v_1}(x_1) \\ &\quad \times \int_{\tilde{Q}^{**}} \sup_{t>0} t^\varepsilon \mu_{\tilde{v}}(B(\tilde{x}, \sqrt{t}))^{-1} \exp\left(-\frac{|\tilde{x} - \tilde{y}|^2}{ct}\right) d\mu_{\tilde{v}}(\tilde{x}) \\ &\leq C d_{Q_1}^{-2\delta-2\varepsilon} d_{\tilde{Q}}^{2\varepsilon} \leq C d_{\tilde{Q}}^{-2\delta}. \end{aligned}$$

Treating J_3 we apply (2.8) and **Step 1** and obtain

$$\begin{aligned} J_3 &\leq C \int_{(Q_1^{**})^c} \sup_{t>0} t^{-\delta/2} \mu_{v_1}(B(x_1, \sqrt{t}))^{-1} \exp\left(-\frac{|x_1 - y_1|^2}{ct}\right) d\mu_{v_1}(x_1) \\ &\quad \times \int_{(\tilde{Q}^{**})^c} \sup_{t>0} t^{-\delta/2} \mu_{\tilde{v}}(B(\tilde{x}, \sqrt{t}))^{-1} \exp\left(-\frac{|\tilde{x} - \tilde{y}|^2}{ct}\right) d\mu_{\tilde{v}}(\tilde{x}) \\ &\leq C d_{Q_1}^{-\delta} d_{\tilde{Q}}^{-\delta} \leq C d_{\tilde{Q}}^{-2\delta}. \end{aligned}$$

This completes the proof.

2.3 Local Hardy space

In this subsection we consider $\nu \in (-1, \infty)^d$. Let $\tau > 0$ be fixed. We are interested in decomposing into atoms a function f such that

$$\left\| \sup_{t \leq \tau^2} \left| \mathbb{W}_{t,\nu}^{\text{cls}} f \right| \right\|_{L^1(\mathbb{R}_+^d, d\mu_\nu)} < \infty. \quad (2.9)$$

In the classical case of the Laplace operator on $\mathbb{R}^d$ equipped with Lebesgue measure, if one restricts the supremum to $0 < t \leq \tau^2$ in the maximal operator, then one obtains an atomic space with the classical atoms complemented with atoms of the form $|B|^{-1} \mathbb{1}_B$, where the ball B has radius τ , c.f. [13].

It turns out that a similar phenomenon occurs in case of the classical Bessel operator. More precisely, (2.9) holds if and only if $f = \sum_k \lambda_k a_k$, where $\sum_k |\lambda_k| < \infty$ and a_k are either the classical atoms or *local atoms at scale τ* . The latter are atoms a supported in a cube Q of diameter comparable to τ such that $\|a\|_\infty \leq \mu_\nu(Q)^{-1}$ but we do not impose cancellation condition. In other words one could say that these atoms built the space $H_{\text{at}}^1(Q^{(\tau)}, \mu_\nu)$ introduced in Sect. 2.1, where $Q^{(\tau)}$ is a covering of $\mathbb{R}_+^d$ by cubes with diameter τ . The next proposition states a local atomic decomposition theorem that will be suitable for the proof of our main result. This proposition can be obtained by known methods based on Uchiyama [25, Cor. 1']. We refer the reader also to [12, Sec. 4], where the authors check assumptions of Uchiyama's Theorem in the classical Bessel framework for the whole range of the parameter $\nu \in (-1, \infty)^d$. For the sake of completeness, below we present a sketch of the proof.

Let $Q^{(\tau)}$ be a section of $\mathbb{R}_+^d$ consisting of cuboids, which have diameter uniformly comparable to τ , i.e. there exists a positive constant C such that $C^{-1}\tau \leq d_Q \leq C\tau$ for every $Q \in Q^{(\tau)}$.

Proposition 2.7 *There exists $C > 0$ independent of τ such that:*

- (a) *For every classical atom a supported in $K \subset Q^*$ or atom of the form $a = \mu_\nu(Q)^{-1} \mathbb{1}_Q$, where $Q \in Q^{(\tau)}$, we have*

$$\left\| \sup_{t \leq \tau^2} \left| \mathbb{W}_{t,\nu}^{\text{cls}} a \right| \right\|_{L^1(\mathbb{R}_+^d, d\mu_\nu)} \leq C.$$

- (b) *If $f \in L_{\text{loc}}^1(\mathbb{R}_+^d, d\mu_\nu)$ is such that $\text{supp } f \subseteq Q^*$, $Q \in Q^{(\tau)}$, and*

$$\left\| \sup_{t \leq \tau^2} \left| \mathbb{W}_{t,\nu}^{\text{cls}} f \right| \right\|_{L^1(Q^*, d\mu_\nu)} = M < \infty,$$

then there exist a sequence λ_k and $(Q^{(\tau)}, \mu_\nu)$ -atoms a_k , such that $f = \sum_k \lambda_k a_k$, $\sum_k |\lambda_k| \leq CM$, and a_k are either the classical atoms supported in Q^ or $a_k = \mu_\nu(Q)^{-1} \mathbb{1}_Q$. The sum above converges in L^1 norm.*

Proof (a) If a is a classical atom satisfying cancellation condition, then the statement follows from [12, Prop. 4.1]. So assume that $a = \mu_\nu(Q)^{-1} \mathbb{1}_Q$. The maximal operator associated with $\mathbb{W}_{t,\nu}^{\text{cls}}$ is bounded on $L^p(\mathbb{R}_+^d, d\mu_\nu)$ for $p > 1$, by Stein's general maximal theorem for semigroups of operators [22, p. 73]. Hence, using this fact and the Schwarz inequality,

$$\begin{aligned} \left\| \sup_{t \leq \tau^2} \left| \mathbb{W}_{t,\nu}^{\text{cls}} a \right| \right\|_{L^1(Q^{**}, d\mu_\nu)} &\leq C \mu_\nu(Q)^{1/2} \left\| \sup_{t > 0} \left| \mathbb{W}_{t,\nu}^{\text{cls}} a \right| \right\|_{L^2(\mathbb{R}_+^d, d\mu_\nu)} \\ &\leq C \mu_\nu(Q)^{1/2} \|a\|_{L^2(\mathbb{R}_+^d, d\mu_\nu)} \leq C. \end{aligned}$$

To estimate the norm on $(Q^{**})^c$ observe that $|x - y| \gtrsim \tau$ when $y \in Q$ and $x \in (Q^{**})^c$. Applying (1.2) and (2.5) with sufficiently small $\varepsilon > 0$ we obtain

$$\begin{aligned} \left\| \sup_{t \leq \tau^2} \left| \mathbb{W}_{t,\nu}^{\text{cls}} a \right| \right\|_{L^1((Q^{**})^c, d\mu_\nu)} &\leq C \int_{(Q^{**})^c} \sup_{t \leq \tau^2} \int_Q \mu_\nu(Q)^{-1} \mu_\nu(B(x, \sqrt{t}))^{-1} e^{-\frac{|x-y|^2}{ct}} d\mu_\nu(y) d\mu_\nu(x) \\ &\leq C \mu_\nu(Q)^{-1} \tau^{2\varepsilon} \int_Q \int_{(Q^{**})^c} \sup_{t \leq \tau^2} t^{-\varepsilon} \mu_\nu(B(x, \sqrt{t}))^{-1} e^{-\frac{|x-y|^2}{ct}} d\mu_\nu(x) d\mu_\nu(y) \\ &\leq C \mu_\nu(Q)^{-1} \tau^{2\varepsilon} d_Q^{-2\varepsilon} \int_Q d\mu_\nu(y) \leq C, \end{aligned}$$

since $d_Q \simeq \tau$.

(b) Assume that $f \in L_{\text{loc}}^1(\mathbb{R}_+^d, d\mu_\nu)$ and $\text{supp } f \subset Q^*$. Consider $X = Q^*$ as a space. Define $\lambda_0 = \int f d\mu_\nu$ and $g = f - \lambda_0 \mu_\nu(Q)^{-1} \mathbb{1}_Q$. Notice that

$$|\lambda_0| \leq \|f\|_{L^1(Q^*, d\mu_\nu)} \leq M, \quad (2.10)$$

and

$$\int g d\mu_\nu = 0. \quad (2.11)$$

Therefore $\|g\|_{L^1(Q^*, d\mu_\nu)} \leq 2\|f\|_{L^1(Q^*, d\mu_\nu)} \leq 2M$. Moreover, $\left\| \sup_{t \leq \tau^2} \left| \mathbb{W}_{t,\nu}^{\text{cls}} g \right| \right\|_{L^1(Q^*, d\mu_\nu)} \leq CM$. Indeed, using (a) and (2.10) we have

$$\begin{aligned} \left\| \sup_{t \leq \tau^2} \left| \mathbb{W}_{t,\nu}^{\text{cls}} g \right| \right\|_{L^1(Q^*, d\mu_\nu)} &\leq \left\| \sup_{t \leq \tau^2} \left| \mathbb{W}_{t,\nu}^{\text{cls}} f \right| \right\|_{L^1(Q^*, d\mu_\nu)} \\ &\quad + |\lambda_0| \left\| \sup_{t \leq \tau^2} \left| \mathbb{W}_{t,\nu}^{\text{cls}} \left(\mu_\nu(Q)^{-1} \mathbb{1}_Q \right) \right| \right\|_{L^1(Q^*, d\mu_\nu)} \leq CM. \end{aligned}$$

For $t > \tau^2$, by (1.2) we obtain

$$\begin{aligned}
 \left\| \sup_{t > \tau^2} |\mathbb{W}_{t,v}^{\text{cls}} g| \right\|_{L^1(Q^*, d\mu_v)} &\leq C \int_{Q^*} \sup_{t > \tau^2} \int_{Q^*} \mu_v(B(x, \sqrt{t}))^{-1} |g(y)| d\mu_v(y) d\mu_v(x) \\
 &\leq C \int_{Q^*} |g(y)| \int_{Q^*} \sup_{t > \tau^2} \mu_v(B(x, \sqrt{t}))^{-1} d\mu_v(x) d\mu_v(y) \\
 &\leq C \int_{Q^*} |g(y)| \int_{Q^*} \mu_v(B(x, \tau))^{-1} d\mu_v(x) d\mu_v(y) \\
 &\leq C \|g\|_{L^1(Q^*, d\mu_v)} \\
 &\leq CM.
 \end{aligned}$$

We used the fact that by the doubling property of μ_v we have $\mu_v(B(x, \tau)) \simeq \mu_v(Q^*)$ for $x \in Q^*$.

Repeating the proof of [12, Prop. 4.1], where the authors use Uchiyama's Theorem [25, Cor. 1'], we conclude that the function g may be decomposed as

$$g = \tilde{\lambda} \mu_v(Q^*)^{-1} \mathbb{1}_{Q^*} + \sum_k \lambda_k a_k,$$

where a_k are atoms supported in $X = Q^*$ and satisfy cancellation condition. The sum above converges in L^1 norm, since $\|a_k\|_{L^1(\mathbb{R}_+^d, d\mu_v)} \leq 1$. The constant atom $\mu_v(Q^*)^{-1} \mathbb{1}_{Q^*}$ appears since $\mu_v(X) = \mu_v(Q^*) < \infty$, see [25, Sec. 2]. However, by (2.11) we get $\tilde{\lambda} = 0$. Further, $\inf \sum_k |\lambda_k| \simeq \left\| \sup_{t > 0} |\mathbb{W}_{t,v}^{\text{cls}} g| \right\|_{L^1(Q^*, d\mu_v)} \leq CM$, which combined with (2.10) gives the desired conclusion.

3 Local atomic decomposition

3.1 Local atomic decomposition theorem

We now consider an operator $L : D(L) \subset L^2(\mathbb{R}_+^d, d\mu_v) \rightarrow L^2(\mathbb{R}_+^d, d\mu_v)$, where $D(L)$ is the domain of L . Assume that L is self-adjoint and nonnegative. Denote by $T_t = \exp(-tL)$ the semigroup generated by L and suppose there exists a nonnegative integral kernel $T_t(x, y)$, such that for every $f \in L^1(\mathbb{R}_+^d, d\mu_v)$, $T_t f(x) = \int_{\mathbb{R}_+^d} T_t(x, y) f(y) d\mu_v(y)$, $t > 0$. We consider the maximal Hardy space for the operator L ¹.

$$H^1(L) = \left\{ f \in L^1(\mathbb{R}_+^d, d\mu_v) : \|f\|_{H^1(L)} := \left\| \sup_{t > 0} |T_t f| \right\|_{L^1(\mathbb{R}_+^d, d\mu_v)} + \|f\|_{L^1(\mathbb{R}_+^d, d\mu_v)} < \infty \right\}.$$

¹ Notice that for many operators, i.e. which satisfy the upper gaussian estimates, the property $\|f\|_{L^1(\mathbb{R}_+^d, d\mu_v)} \leq \left\| \sup_{t > 0} |T_t f| \right\|_{L^1(\mathbb{R}_+^d, d\mu_v)}$ hold, therefore in these cases one may put $\|f\|_{H^1(L)} := \left\| \sup_{t > 0} |T_t f| \right\|_{L^1(\mathbb{R}_+^d, d\mu_v)}$

For $v \in (-1, \infty)^d$, an admissible covering $\mathcal{Q}$, and the semigroup $\{T_t\}$ we consider the following conditions.

$$0 \leq T_t(x, y) \leq C\mu_v(B(x, \sqrt{t}))^{-1} \exp\left(-\frac{|x-y|^2}{ct}\right), \quad (A_0)$$

$$t > 0, \quad Q \in \mathcal{Q}, \quad x \in \tilde{N}(Q), \quad y \in Q^*.$$

$$\sup_{y \in Q^*} \int_{(Q^{**})^c} \sup_{t>0} T_t(x, y) d\mu_v(x) \leq C, \quad Q \in \mathcal{Q}. \quad (A'_1)$$

$$\sup_{y \in Q^*} \int_{Q^{**}} \sup_{t \leq d_Q^2} |T_t(x, y) - \mathbb{W}_{t,v}^{\text{cls}}(x, y)| d\mu_v(x) \leq C, \quad Q \in \mathcal{Q}. \quad (A'_2)$$

Let us emphasize that the constants C, c in these above conditions are independent of $Q \in \mathcal{Q}$.

The main result of this section is the following.

Theorem 3.1 *Let $v \in (-1, \infty)^d$. Assume that $L, \{T_t\}$ with an admissible covering $\mathcal{Q}$ satisfy (A_0) , (A'_1) and (A'_2) . Then $H^1(L)$ and $H_{\text{at}}^1(\mathcal{Q}, \mu_v)$ are isomorphic as Banach spaces.*

To prove Theorem 3.1 we need one more lemma.

Lemma 3.2 *Let $v \in (-1, \infty)^d$. Assume that $\{T_t\}$ and $\mathcal{Q}$ satisfy conditions (A_0) , (A'_1) and a family $\{\psi_Q\}_{Q \in \mathcal{Q}}$ satisfies (2.3). Then*

$$\sup_{y \in Q^*} \int_{Q^{**}} \sup_{t>d_Q^2} T_t(x, y) d\mu_v(x) \leq C, \quad Q \in \mathcal{Q}, \quad (3.1)$$

$$\sup_{y \in \mathbb{R}_+^d} \sum_{Q \in \mathcal{Q}} \int_{Q^{**}} \sup_{t \leq d_Q^2} T_t(x, y) |\psi_Q(x) - \psi_Q(y)| d\mu_v(x) < \infty. \quad (3.2)$$

Proof First, we prove (3.1). Let $Q \in \mathcal{Q}$ and take $y \in Q^*$. Notice that if $x \in Q^{**}$ then in particular $x \in \tilde{N}(Q)$, hence we may use (A_0) . We have

$$\begin{aligned} \int_{Q^{**}} \sup_{t>d_Q^2} T_t(x, y) d\mu_v(x) &\leq C \int_{Q^{**}} \sup_{t>d_Q^2} \mu_v(B(x, \sqrt{t}))^{-1} d\mu_v(x) \\ &\leq C \int_{Q^{**}} \mu_v(B(x, d_Q))^{-1} d\mu_v(x) \\ &\leq C\mu_v(Q)^{-1} \int_{Q^{**}} d\mu_v(x) \leq C, \end{aligned}$$

since $\mu_v(B(x, d_Q)) \simeq \mu_v(Q)$, $x \in Q^*$, by the doubling property of μ_v .

Next we show (3.2). Let $y \in \mathbb{R}_+^d$. There exists $Q_0 \in \mathcal{Q}$ such that $y \in Q_0$. Recall that $N(Q_0) = \{Q \in \mathcal{Q} : Q \cap Q_0 \neq \emptyset\}$. We write

$$\begin{aligned} & \sum_{Q \in \mathcal{Q}} \int_{Q^{**}} \sup_{t \leq d_Q^2} T_t(x, y) |\psi_Q(x) - \psi_Q(y)| d\mu_v(x) \\ &= \sum_{Q \in N(Q_0)} \cdots + \sum_{Q \in \mathcal{Q} \setminus N(Q_0)} \cdots =: S_1 + S_2. \end{aligned}$$

To estimate S_2 we use $0 \leq \psi_Q \leq 1$ and (A'_1) , getting

$$\begin{aligned} S_2 &\leq \sum_{Q \in \mathcal{Q} \setminus N(Q_0)} \int_{Q^{**}} \sup_{t > 0} T_t(x, y) d\mu_v(x) \\ &\leq C \int_{(Q_0^{**})^c} \sup_{t > 0} T_t(x, y) d\mu_v(x) \leq C. \end{aligned}$$

To treat S_1 , write $Q = Q_1 \times \cdots \times Q_d$. We claim that, given $j = 1, \dots, d$, and $\varepsilon_j \in (0, 2v_j + 2)$, for $x_j \in \tilde{N}(Q_j)$, $y_j \in Q_j^*$ one has the bound

$$\begin{aligned} P &:= \sup_{t \leq d_Q^2} \mu_{v_j}(B(x_j, \sqrt{t}))^{-1} \exp\left(-\frac{|x - y|^2}{ct}\right) |x - y|^{\varepsilon_j} x_j^{2v_j+1} \\ &\leq C \min(x_j, |x_j - y_j|)^{-1+\varepsilon_j}, \end{aligned} \quad (3.3)$$

that holds with C independent of $Q \in \mathcal{Q}$, x_j and y_j . This will be verified in a moment. Now choose ε_j , $j = 1, \dots, d$, such that $\varepsilon_j \in (0, 2v_j + 2)$ and $\varepsilon := \sum_j \varepsilon_j < 1$. Using (A_0) , the Mean Value Theorem for ψ_Q , (2.3), Corollary 2.5 and then (3.3) we obtain

$$\begin{aligned} S_1 &\leq C \sum_{Q \in N(Q_0)} \int_{Q^{**}} \sup_{t \leq d_Q^2} \mu_v(B(x, \sqrt{t}))^{-1} \exp\left(-\frac{|x - y|^2}{ct}\right) \frac{|x - y|}{d_Q} d\mu_v(x) \\ &\leq C \sum_{Q \in N(Q_0)} d_Q^{-\varepsilon} \int_{Q^{**}} \prod_{j=1}^d \left(\sup_{t \leq d_Q^2} \mu_{v_j}(B(x_j, \sqrt{t}))^{-1} \exp\left(-\frac{|x - y|^2}{cdt}\right) |x - y|^{\varepsilon_j} x_j^{2v_j+1} \right) dx \\ &\leq C \sum_{Q \in N(Q_0)} d_Q^{-\varepsilon} \prod_{j=1}^d \int_{Q_j^{**}} \min(x_j, |x_j - y_j|)^{-1+\varepsilon_j} dx_j \\ &\leq C \sum_{Q \in N(Q_0)} d_Q^{-\varepsilon} \prod_{j=1}^d \left(\int_{\{x_j : x_j \leq |x_j - y_j| \leq cd_Q\}} x_j^{-1+\varepsilon_j} dx_j + \int_{\{x_j : x_j > |x_j - y_j|\} \cap Q_j^{**}} |x_j - y_j|^{-1+\varepsilon_j} dx_j \right) \\ &\leq C \sum_{Q \in N(Q_0)} d_Q^{-\varepsilon} \prod_{j=1}^d d_{Q_j}^{\varepsilon_j} \leq C, \end{aligned}$$

since $d_{Q_j} \simeq d_Q$ for $j = 1, \dots, d$.

It remains to show (3.3). We will consider several cases and subcases.

Case 1: $v_j \geq -1/2$. In this case, by Corollary 2.4(b) we have

$$P \leq C \sup_{t \leq d_Q^2} t^{-1/2} (x_j + \sqrt{t})^{-2v_j-1} \exp\left(-\frac{|x-y|^2}{c_j t}\right) |x-y|^{\varepsilon_j} x_j^{2v_j+1} =: P_1.$$

Case 1a: $x_j < |x_j - y_j|/2$. We have

$$\begin{aligned} P_1 &\leq C \sup_{t \leq d_Q^2} t^{-1-v_j} \exp\left(-\frac{|x-y|^2}{c_j t}\right) |x-y|^{\varepsilon_j} x_j^{2v_j+1} \\ &\leq C x_j^{2v_j+1} |x-y|^{-2v_j-2+\varepsilon_j} \\ &\leq C x_j^{2v_j+1} |x_j - y_j|^{-2v_j-2+\varepsilon_j} \\ &\leq C x_j^{-1+\varepsilon_j}. \end{aligned}$$

Case 1b: $x_j \geq |x_j - y_j|/2$. Here we obtain

$$\begin{aligned} P_1 &\leq C \sup_{t \leq d_Q^2} t^{-1/2} \exp\left(-\frac{|x-y|^2}{c_j t}\right) |x-y|^{\varepsilon_j} \\ &\leq C |x-y|^{-1+\varepsilon_j} \\ &\leq C |x_j - y_j|^{-1+\varepsilon_j}. \end{aligned}$$

Case 2: $v_j \in (-1, -1/2)$ and $Cd_Q \geq x_j$. Using Corollary 2.4(b) we get

$$\begin{aligned} P &\leq C \sup_{t \leq x_j^2} t^{-1/2} (x_j + \sqrt{t})^{-2v_j-1} \exp\left(-\frac{|x-y|^2}{c_j t}\right) |x-y|^{\varepsilon_j} x_j^{2v_j+1} \\ &\quad + C \sup_{x_j^2 < t \leq d_Q^2} t^{-1/2} (x_j + \sqrt{t})^{-2v_j-1} \exp\left(-\frac{|x-y|^2}{c_j t}\right) |x-y|^{\varepsilon_j} x_j^{2v_j+1} =: P_2. \end{aligned}$$

Case 2a: $x_j < |x_j - y_j|/2$. We write

$$\begin{aligned} P_2 &\leq C \sup_{t>0} t^{-1/2} \exp\left(-\frac{|x-y|^2}{c_j t}\right) |x-y|^{\varepsilon_j} \\ &\quad + C \sup_{x_j^2 < t \leq d_Q^2} t^{-v_i-1} x_j^{2v_j+1} \exp\left(-\frac{|x-y|^2}{c_j t}\right) |x-y|^{\varepsilon_j} \\ &\leq C |x-y|^{-1+\varepsilon_j} + C x_j^{2v_j+1} |x-y|^{-2v_j-2+\varepsilon_j} \\ &\leq C |x_j - y_j|^{-1+\varepsilon_j} + C x_j^{-1+\varepsilon_j} \\ &\leq C x_j^{-1+\varepsilon_j}. \end{aligned}$$

Case 2b: $|x_j - y_j|/2 < x_j$. In this case

$$\begin{aligned} P_2 &\leq C \sup_{t>0} t^{-1/2} \exp\left(-\frac{|x-y|^2}{c_j t}\right) |x-y|^{\varepsilon_j} \\ &\quad + C \sup_{x_j^2 < t \leq d_Q^2} t^{-\nu_j-1} \exp\left(-\frac{|x-y|^2}{c_j t}\right) |x-y|^{\varepsilon_j} |x_j-y_j|^{2\nu_j+1} \\ &\leq C|x-y|^{-1+\varepsilon_j} + C|x-y|^{-2\nu_j-2+\varepsilon_j} |x_j-y_j|^{2\nu_j+1} \\ &\leq C|x_j-y_j|^{-1+\varepsilon_j}. \end{aligned}$$

Case 3: $\nu_j \in (-1, -1/2)$ and $Cd_Q < x_j$. Here we have

$$\begin{aligned} P &\leq C \sup_{t \leq d_Q^2} t^{-1/2} (x_j + \sqrt{t})^{-2\nu_j-1} \exp\left(-\frac{|x-y|^2}{c_j t}\right) |x-y|^{\varepsilon_j} x_j^{2\nu_j+1} \\ &\leq C \sup_{t \leq d_Q^2} t^{-1/2} \exp\left(-\frac{|x-y|^2}{c_j t}\right) |x-y|^{\varepsilon_j} \\ &\leq C|x-y|^{-1+\varepsilon_j} \leq C|x_j-y_j|^{-1+\varepsilon_j}. \end{aligned}$$

This finishes proving (3.3).

3.2 Proof of Theorem 3.1

We shall prove the two inclusions.

First inclusion. Let $f \in H_{\text{at}}^1(\mathcal{Q}, \mu_\nu)$. Obviously $\|f\|_{L^1(X, d\mu_\nu)} \leq \|f\|_{H_{\text{at}}^1(\mathcal{Q}, \mu_\nu)}$. We need to show $\left\| \sup_{t>0} |T_t f| \right\|_{L^1(\mathbb{R}_+^d, d\mu_\nu)} \leq C \|f\|_{H_{\text{at}}^1(\mathcal{Q}, \mu_\nu)}$. It is enough to check that $\left\| \sup_{t>0} |T_t a| \right\|_{L^1(\mathbb{R}_+^d, d\mu_\nu)} \leq C$ for every $(\mathcal{Q}, \mu_\nu)$ -atom a , where C does not depend on a . Indeed, by (A_0) and (A'_1) we know that linear operator T_t , for fixed $t > 0$, is bounded on $L^1(X, d\mu_\nu)$, hence it is continuous. Moreover, taking the supremum is sub-additive and $\|a\|_{L^1(X, d\mu_\nu)} \leq 1$.

Take an atom a associated with a cuboid $Q \in \mathcal{Q}$, see Definition 2.2. Using (A'_1) , (3.1), (A'_2) and Proposition 2.7(a) with $\tau = d_Q$ we get

$$\begin{aligned} \left\| \sup_{t>0} |T_t a| \right\|_{L^1(\mathbb{R}_+^d, d\mu_\nu)} &\leq \left\| \sup_{t>0} |T_t a| \right\|_{L^1((Q^{**})^c, d\mu_\nu)} + \left\| \sup_{t>d_Q^2} |T_t a| \right\|_{L^1(Q^{**}, d\mu_\nu)} \\ &\quad + \left\| \sup_{t \leq d_Q^2} |(T_t - \mathbb{W}_{t,\nu}^{\text{cls}})a| \right\|_{L^1(Q^{**}, d\mu_\nu)} \\ &\quad + \left\| \sup_{t \leq d_Q^2} |\mathbb{W}_{t,\nu}^{\text{cls}} a| \right\|_{L^1(Q^{**}, d\mu_\nu)} \leq C. \end{aligned}$$

Second inclusion. Let $f \in H^1(L)$. We shall prove $\|f\|_{H_{\text{at}}^1(Q, \mu_\nu)} \leq C \left\| \sup_{t>0} |T_t f| \right\|_{L^1(\mathbb{R}_+^d, d\mu_\nu)}$. Let ψ_Q be a partition of unity related to Q , see (2.3). Then $f = \sum_{Q \in \mathcal{Q}} \psi_Q f$. Denote $f_Q = \psi_Q f$ and notice that $\text{supp } f_Q \subset Q^*$. We have

$$\mathbb{W}_{t,v}^{\text{cls}} f_Q = (\mathbb{W}_{t,v}^{\text{cls}} - T_t) f_Q + (T_t f_Q - \psi_Q T_t f) + \psi_Q T_t f. \quad (3.4)$$

Clearly,

$$\sum_{Q \in \mathcal{Q}} \left\| \sup_{t \leq d_Q^2} |\psi_Q T_t f| \right\|_{L^1(Q^{**}, d\mu_\nu)} \leq C \left\| \sup_{t>0} |T_t f| \right\|_{L^1(\mathbb{R}_+^d, d\mu_\nu)}. \quad (3.5)$$

Using (A'_2),

$$\sum_{Q \in \mathcal{Q}} \left\| \sup_{t \leq d_Q^2} |(\mathbb{W}_{t,v}^{\text{cls}} - T_t) f_Q| \right\|_{L^1(Q^{**}, d\mu_\nu)} \leq C \sum_{Q \in \mathcal{Q}} \|f_Q\|_{L^1(\mathbb{R}_+^d, d\mu_\nu)} \leq C \|f\|_{L^1(\mathbb{R}_+^d, d\mu_\nu)}. \quad (3.6)$$

By (3.2),

$$\begin{aligned} & \sum_{Q \in \mathcal{Q}} \left\| \sup_{t \leq d_Q^2} |T_t f_Q - \psi_Q T_t f| \right\|_{L^1(Q^{**}, d\mu_\nu)} \\ & \leq \sum_{Q \in \mathcal{Q}} \int_{\mathbb{R}_+^d} |f(y)| \int_{Q^{**}} \sup_{t \leq d_Q^2} T_t(x, y) |\psi_Q(y) - \psi_Q(x)| d\mu_\nu(x) d\mu_\nu(y) \\ & \leq C \|f\|_{L^1(\mathbb{R}_+^d, d\mu_\nu)}. \end{aligned} \quad (3.7)$$

Combining (3.5) – (3.7) with (3.4) we arrive at

$$\sum_{Q \in \mathcal{Q}} \left\| \sup_{t \leq d_Q^2} |\mathbb{W}_{t,v}^{\text{cls}} f_Q| \right\|_{L^1(Q^*, d\mu_\nu)} \leq C \left(\left\| \sup_{t>0} |T_t f| \right\|_{L^1(\mathbb{R}_+^d, d\mu_\nu)} + \|f\|_{L^1(\mathbb{R}_+^d, d\mu_\nu)} \right). \quad (3.8)$$

Now we apply Proposition 2.7(b) for each f_Q separately with $\tau = d_Q$. For each f_Q we obtain a sequence λ_k^Q and atoms a_k^Q such that

$$f_Q = \sum_k \lambda_k^Q a_k^Q.$$

Observe that for each Q the atoms a_k^Q are either local atoms of the form $\mu_\nu(Q)^{-1} \mathbb{1}_Q$ or classical atoms supported in Q^* , therefore they are indeed (Q, μ_ν) -atoms. Finally we have that

$$f = \sum_Q f_Q = \sum_Q \sum_k \lambda_k^Q a_k^Q,$$

where the series converges absolutely in L^1 , since $\|a_k^Q\|_{L^1(\mathbb{R}_+^d, d\mu_\nu)} \leq 1$ and by Proposition 2.7(b) and (3.8),

$$\begin{aligned} \sum_Q \sum_k |\lambda_k^Q| &\leq C \sum_{Q \in \mathcal{Q}} \left\| \sup_{t \leq d_Q^2} \left| \mathbb{W}_{t,v}^{\text{cls}} f_Q \right| \right\|_{L^1(Q^*, d\mu_\nu)} \\ &\leq C \left(\left\| \sup_{t>0} |T_t f| \right\|_{L^1(\mathbb{R}_+^d, d\mu_\nu)} + \|f\|_{L^1(\mathbb{R}_+^d, d\mu_\nu)} \right). \end{aligned}$$

4 Local atomic decomposition in the product case

4.1 Product of local atomic Hardy spaces

In this section we consider operators of the form $L = L_1 + \cdots + L_N$, where each L_i acts nontrivially only on the variable $x_i \in X_i = \mathbb{R}_+^{d_i}$. We introduce conditions on the kernels of the semigroups generated by L_i , and admissible coverings $\mathcal{Q}_i$ of X_i , which are sufficient to prove the local atomic decomposition theorem for the Hardy space $H^1(L)$.

More precisely, we consider $X = X_1 \times \cdots \times X_N = \mathbb{R}_+^{d_1} \times \cdots \times \mathbb{R}_+^{d_N} = \mathbb{R}_+^d$. We equip the space X with the Euclidean metric and the measure $d\mu_\nu(x) = x^{2\nu+1} dx$, where $\nu = (\nu_1, \dots, \nu_N)$ and $\nu_i = (\nu_{i,1}, \dots, \nu_{i,d_i}) \in (-1, \infty)^{d_i}$ for $i = 1, \dots, N$. Assume that $L_i : D(L_i) \subset L^2(X_i, d\mu_{\nu_i}) \rightarrow L^2(X_i, d\mu_{\nu_i})$ is a nonnegative and self-adjoint operator, as in Sect. 3.1. Slightly abusing the notation we keep the symbol

$$L_i = \underbrace{I \otimes \cdots \otimes I}_{i-1 \text{ times}} \otimes L_i \otimes \underbrace{I \otimes \cdots \otimes I}_{N-i \text{ times}}$$

for the operator on $L^2(X, d\mu_\nu)$, where I denotes the identity operator on the corresponding subspace, and we define

$$Lf(x) = L_1 f(x) + \cdots + L_N f(x), \quad x = (x_1, \dots, x_N) \in X. \quad (4.1)$$

Since the operators L_i are self-adjoint, the operator L is well defined and essentially self-adjoint, see e.g. [21, Thm. 7.23].

We denote by $T_t^{[i]}(x_i, y_i)$, $x_i, y_i \in X_i$, the integral kernel of $T_t^{[i]} = \exp(-tL_i)$.

Given two admissible coverings $\mathcal{Q}_1$ and $\mathcal{Q}_2$ of X_1 and X_2 , respectively, we would like to produce an admissible covering on $X_1 \times X_2$. However, the products $\{Q_1 \times Q_2 : Q_1 \in \mathcal{Q}_1, Q_2 \in \mathcal{Q}_2\}$ do not form an admissible covering (in general item 3 of Definition 2.1 fails). Therefore, we need a finer construction. We split each $Q = Q_1 \times Q_2$ (without loss of generality let us assume that $d_{Q_1} > d_{Q_2}$) into cuboids Q^j , $j = 1, \dots, M$, such that all of them have diameters comparable to d_{Q_2} . Then the cuboids $Q^j = Q_1^j \times Q_2$, $j = 1, \dots, M$, satisfy:

- $Q = \bigcup_{j=1}^M Q^j$,
- $\mu_v(Q^i \cap Q^j) = 0$ for $i, j \in \{1, \dots, M\}, i \neq j$,
- each Q^j satisfies condition 3 of Definition 2.1.

Definition 4.1 The admissible covering of $X_1 \times X_2$ described in the above construction will be called the product admissible covering and denoted by $Q_1 \boxtimes Q_2$.

As an example, consider the family $\mathcal{D} = \{[2^n, 2^{n+1}] : n \in \mathbb{Z}\}$, which is an admissible covering of $\mathbb{R}_+$. The product admissible covering $\mathcal{D} \boxtimes \mathcal{D}$ of $\mathbb{R}_+^2$ is illustrated by Fig. 1 below.

For a product admissible covering $Q_1 \boxtimes Q_2$ we also fix a new κ such that (2.2) is fulfilled. The covering $Q_1 \boxtimes \dots \boxtimes Q_N$ of $X_1 \times \dots \times X_N$ is constructed as above by induction.

Remark 4.2 The collection of all the smaller cuboids Q_1^j emerging from splitting the cuboid Q_1 and described before Definition 4.1 forms an admissible covering of $\mathbb{R}_+^{d_1}$.

We introduce modifications of conditions $(A'_1), (A'_2)$. For the semigroup of operators $\{S_t\}$ and an admissible covering Q we consider the following conditions. There exists $\gamma \in (0, 1/3)$ such that:

- for each $\delta \in (-\gamma, \gamma)$

$$\sup_{y \in Q^*} \int_{(Q^{**})^c} \sup_{t>0} t^\delta S_t(x, y) d\mu_v(x) \leq C d_Q^{2\delta}, \quad Q \in \mathcal{Q}, \quad (A_1)$$

Fig. 1 Product admissible covering $\mathcal{D} \boxtimes \mathcal{D}$ of $\mathbb{R}_+^2$

and

- for each $\delta \in [0, \gamma)$

$$\sup_{y \in Q^*} \int_{Q^{**}} \sup_{t \leq d_Q^2} t^{-\delta} |S_t(x, y) - \mathbb{W}_{t,v}^{\text{cls}}(x, y)| d\mu_v(x) \leq C d_Q^{-2\delta}, \quad Q \in \mathcal{Q}. \quad (A_2)$$

Notice that if $\{S_t\}$ and $\mathcal{Q}$ satisfy (A_1) and (A_2) , then (A'_1) and (A'_2) also hold. We shall assume that each $T_t^{[i]}(x_i, y_i)$, $i = 1, \dots, N$, and the corresponding admissible covering $\mathcal{Q}_i$ satisfy conditions (A_0) – (A_2) .

Remark 4.3 Condition (A_1) for negative $\delta \in (-\gamma, 0)$ is automatically satisfied if the kernels $T_t^{[i]}(x_i, y_i)$ satisfy the upper Gaussian estimates, that is the bound of condition (A_0) for all $x_i, y_i \in X_i$. This is a direct consequence of (2.5).

Our main result in this section is the following.

Theorem 4.4 *Let L be as in (4.1) and assume that for each $i = 1, \dots, N$, the kernel $T_t^{[i]}(x_i, y_i)$ together with the corresponding admissible covering $\mathcal{Q}_i$ satisfy conditions (A_0) – (A_2) . Then*

$$H^1(L) = H_{\text{at}}^1(\mathcal{Q}_1 \boxtimes \dots \boxtimes \mathcal{Q}_N, \mu_v)$$

and the corresponding norms are equivalent.

Proof We will show the following claim. If conditions (A_0) – (A_2) hold for $T_t^{[i]}(x_i, y_i)$ together with admissible coverings $\mathcal{Q}_i$ for $i = 1, 2$, then (A_0) – (A_2) also hold for $T_t(x, y) = T_t^{[1]}(x_1, y_1)T_t^{[2]}(x_2, y_2)$, together with $\mathcal{Q} = \mathcal{Q}_1 \boxtimes \mathcal{Q}_2$. This is enough, since by simple induction we shall get that in the general case $T_t(x, y) = T_t^{[1]}(x_1, y_1) \dots T_t^{[N]}(x_N, y_N)$ with $\mathcal{Q}_1 \boxtimes \dots \boxtimes \mathcal{Q}_N$ satisfy (A_0) – (A_2) , and consequently, the assumptions of Theorem 3.1 will be fulfilled.

To prove the claim let $T_t^{[i]}(x_i, y_i)$ and $\mathcal{Q}_i$ satisfy (A_0) – (A_2) with γ_i for $i = 1, 2$. Let $0 < \gamma < \min(\gamma_1, \gamma_2, v_{1,1}+1, \dots, v_{1,d_1}+1, v_{2,1}+1, \dots, v_{2,d_2}+1)$. Suppose that $\mathcal{Q} \ni Q \subseteq Q_1 \times Q_2$, where $Q_1 \in \mathcal{Q}_1$, $Q_2 \in \mathcal{Q}_2$. Without loss of generality we may assume that $d_{Q_1} \geq d_{Q_2}$. Hence $Q = K \times Q_2$, where $K \subseteq Q_1$ is a cuboid emerging from the very construction of $\mathcal{Q}_1 \boxtimes \mathcal{Q}_2$, see the description preceding Definition 4.1 and Fig. 2 below. Denote by $z = (z_1, z_2)$ the center of $Q = K \times Q_2$.

Verification of (A_0) . Notice that if $x = (x_1, x_2) \in N(Q)$, then in particular $x_1 \in \tilde{N}(Q_1)$ and $x_2 \in \tilde{N}(Q_2)$. Similarly, if $y = (y_1, y_2) \in Q^*$, then $y_1 \in Q_1^*$ and $y_2 \in Q_2^*$.

Fig. 2 Splitting $Q_1 \times Q_2$ into admissible cuboids

Therefore, for such x, y we may use (A₀) for each of $T_t^{[1]}(x_1, y_1)$ and $T_t^{[2]}(x_2, y_2)$ obtaining (A₀) for $T_t(x, y) = T_t^{[1]}(x_1, y_1)T_t^{[2]}(x_2, y_2)$.

Verification of (A₁). Fix $\delta \in (-\gamma, \gamma)$. Let $y \in Q^*$. Recall that $d_Q \simeq d_K \simeq d_{Q_2} \simeq d_{Q_1}$. Decompose $(Q^{**})^c = S_1 \cup S_2 \cup S_3 \cup S_4 \cup S_5$, where

$$\begin{aligned} S_1 &= (Q_1^{**})^c \times Q_2^{**}, \quad S_2 = K^{**} \times (Q_2^{**})^c, \quad S_3 = (Q_1^{**} \setminus K^{**}) \times (Q_2^{**})^c, \\ S_4 &= (Q_1^{**})^c \times (Q_2^{**})^c, \quad S_5 = (Q_1^{**} \setminus K^{**}) \times Q_2^{**}. \end{aligned}$$

We start with estimating the integral over S_1 . We shall consider two cases.

Case 1: $\delta \in (-\gamma, 0]$. In this case we take a small $\varepsilon > 0$ such that $\delta - \varepsilon > -\gamma$. Using (A₁) for $T_t^{[1]}(x_1, y_1)$, (A₀) for $T_t^{[2]}(x_2, y_2)$ and then (2.4), we have

$$\begin{aligned} &\int_{S_1} \sup_{t>0} t^\delta T_t(x, y) d\mu_\nu(x) \\ &\leq C \int_{(Q_1^{**})^c} \sup_{t>0} t^{\delta-\varepsilon} T_t^{[1]}(x_1, y_1) d\mu_{\nu_1}(x_1) \\ &\quad \times \int_{Q_2^{**}} \sup_{t>0} t^\varepsilon \mu_{\nu_2}(B(x_2, \sqrt{t}))^{-1} \exp\left(-\frac{|x_2 - y_2|^2}{ct}\right) d\mu_{\nu_2}(x_2) \\ &\leq C d_{Q_1}^{2\delta-2\varepsilon} d_{Q_2}^{2\varepsilon} \leq C d_{Q_1}^{2\delta} \leq C d_Q^{2\delta}, \end{aligned}$$

since $d_{Q_1} \geq d_{Q_2} \simeq d_Q$.

Case 2: $\delta \in (0, \gamma)$. Then, again using (A₁) for $T_t^{[1]}(x_1, y_1)$, (A₀) for $T_t^{[2]}(x_2, y_2)$ and then (2.4) we have

$$\begin{aligned} &\int_{S_1} \sup_{t>0} t^\delta T_t(x, y) d\mu_\nu(x) \\ &\leq C \int_{(Q_1^{**})^c} \sup_{t>0} T_t^{[1]}(x_1, y_1) d\mu_{\nu_1}(x_1) \\ &\quad \times \int_{Q_2^{**}} \sup_{t>0} t^\delta \mu_{\nu_2}(B(x_2, \sqrt{t}))^{-1} \exp\left(-\frac{|x_2 - y_2|^2}{ct}\right) d\mu_{\nu_2}(x_2) \\ &\leq C d_{Q_2}^{2\delta} \leq C d_Q^{2\delta}. \end{aligned}$$

The integral over S_2 is treated similarly. Indeed we take a small $\varepsilon > 0$ such that $\delta - \varepsilon > -\gamma$. We apply (A₀) for $T_t^{[1]}(x_1, y_1)$ and then (2.4), and (A₁) for $T_t^{[2]}(x_2, y_2)$. We get

$$\begin{aligned} \int_{S_2} \sup_{t>0} t^\delta T_t(x, y) d\mu_\nu(x) &\leq C \int_{K^{**}} \sup_{t>0} t^\varepsilon \mu_{\nu_1}(B(x_1, \sqrt{t}))^{-1} \exp\left(-\frac{|x_1 - y_1|^2}{ct}\right) d\mu_{\nu_1}(x_1) \\ &\quad \times \int_{(Q_2^{**})^c} \sup_{t>0} t^{\delta-\varepsilon} T_t^{[2]}(x_2, y_2) d\mu_{\nu_2}(x_2) \\ &\leq C d_K^{2\varepsilon} d_{Q_2}^{2\delta-2\varepsilon} \leq C d_Q^{2\delta}, \end{aligned}$$

since $d_K \simeq d_{Q_2} \simeq d_Q$. By Remark 4.2 we were allowed to use Lemma 2.6 for the cuboid K .

To estimate the integral over S_3 notice that since $x_1 \in Q_1^{**}$ we may use (A₀) for $T_t^{[1]}(x_1, y_1)$. Let $\varepsilon > 0$ be such that $\delta + \varepsilon < \gamma$. Then we use (A₁) for $T_t^{[2]}(x_2, y_2)$ and (2.5) and arrive at

$$\begin{aligned} \int_{S_3} \sup_{t>0} t^\delta T_t(x, y) d\mu_v(x) &\leq C \int_{Q_1^{**} \setminus K^{**}} \sup_{t>0} t^{-\varepsilon} \mu_{v_1}(B(x_1, \sqrt{t}))^{-1} \exp\left(-\frac{|x_1 - y_1|^2}{ct}\right) d\mu_{v_1}(x_1) \\ &\quad \times \int_{(Q_2^{**})^c} \sup_{t>0} t^{\delta+\varepsilon} T_t^{[2]}(x_2, y_2) d\mu_{v_2}(x_2) \\ &\leq C d_K^{-2\varepsilon} d_{Q_2}^{2\delta+2\varepsilon} \leq C d_Q^{2\delta}. \end{aligned}$$

Treating the integral over S_4 we make use of (A₁) for both $T_t^{[1]}(x_1, y_1)$ and $T_t^{[2]}(x_2, y_2)$. We get

$$\begin{aligned} \int_{S_4} \sup_{t>0} t^\delta T_t(x, y) d\mu_v(x) &\leq C \int_{(Q_1^{**})^c} \sup_{t>0} T_t^{[1]}(x_1, y_1) d\mu_{v_1}(x_1) \\ &\quad \times \int_{(Q_2^{**})^c} \sup_{t>0} t^\delta T_t^{[2]}(x_2, y_2) d\mu_{v_2}(x_2) \\ &\leq C d_{Q_2}^{2\delta} \leq C d_Q^{2\delta}. \end{aligned}$$

Turning to the last integral over S_5 we consider two cases.

Case 1: $\delta \in (-\gamma, 0]$. Take $\varepsilon > 0$ such that $\delta - \varepsilon > -\gamma$. We use (A₀) for $T_t^{[1]}(x_1, y_1)$ and $T_t^{[2]}(x_2, y_2)$, and then Lemma 2.6 getting

$$\begin{aligned} \int_{S_5} \sup_{t>0} t^\delta T_t(x, y) d\mu_v(x) &\leq C \int_{Q_1^{**} \setminus K^{**}} \sup_{t>0} t^{\delta-\varepsilon} \mu_{v_1}(B(x_1, \sqrt{t}))^{-1} \exp\left(-\frac{|x_1 - y_1|^2}{ct}\right) d\mu_{v_1}(x_1) \\ &\quad \times \int_{Q_2^{**}} \sup_{t>0} t^\varepsilon \mu_{v_2}(B(x_2, \sqrt{t}))^{-1} \exp\left(-\frac{|x_2 - y_2|^2}{ct}\right) d\mu_{v_2}(x_2) \\ &\leq C d_{Q_2}^{2\varepsilon} \int_{(K^{**})^c} \sup_{t>0} t^{\delta-\varepsilon} \mu_{v_1}(B(x_1, \sqrt{t}))^{-1} \exp\left(-\frac{|x_1 - y_1|^2}{ct}\right) d\mu_{v_1}(x_1) \\ &\leq C d_{Q_2}^{2\varepsilon} d_K^{2\delta-2\varepsilon} \leq C d_Q^{2\delta}, \end{aligned}$$

since $d_K \simeq d_{Q_2} \simeq d_Q$.

Case 2: $\delta \in (0, \gamma)$. Let $\varepsilon > 0$ be such that $\delta + \varepsilon < \gamma$. Using (A₀) for $T_t^{[1]}(x_1, y_1)$, $T_t^{[2]}(x_2, y_2)$ and then Lemma 2.6 we obtain

$$\begin{aligned} & \int_{S_5} \sup_{t>0} t^\delta T_t(x, y) d\mu_v(x) \\ & \leq C \int_{Q_1^{**} \setminus K^{**}} \sup_{t>0} t^{-\varepsilon} \mu_{v_1}(B(x_1, \sqrt{t}))^{-1} \exp\left(-\frac{|x_1 - y_1|^2}{ct}\right) d\mu_{v_1}(x_1) \\ & \quad \times \int_{Q_2^{**}} \sup_{t>0} t^{\delta+\varepsilon} \mu_{v_2}(B(x_2, \sqrt{t}))^{-1} \exp\left(-\frac{|x_2 - y_2|^2}{ct}\right) d\mu_{v_2}(x_2) \\ & \leq C d_{Q_2}^{2\delta+2\varepsilon} \int_{(K^{**})^c} \sup_{t>0} t^{-\varepsilon} \mu_{v_1}(B(x_1, \sqrt{t}))^{-1} \exp\left(-\frac{|x_1 - y_1|^2}{ct}\right) d\mu_{v_1}(x_1) \\ & \leq C d_{Q_2}^{2\delta+2\varepsilon} d_K^{-2\varepsilon} \leq C d_Q^{2\delta}. \end{aligned}$$

Condition (A₁) is now verified.

Verification of (A₂). Fix $\delta \in [0, \gamma)$. Let $y \in Q^*$. In this proof $\mathbb{W}_{t,v}^{\text{cls}}$ will be the classical Bessel semigroup on $\mathbb{R}_+^{d_1}$, $\mathbb{R}_+^{d_2}$ or on $\mathbb{R}_+^{d_1+d_2}$, depending on the context. First, notice that by (A₀) and (2.4), for any given constant $c > 1$ and $i = 1, 2$, we have

$$\begin{aligned} & \int_{Q_i^{**}} \sup_{c^{-1}d_{Q_i}^2 \leq t \leq cd_{Q_i}^2} t^{-\gamma} \left| T_t^{[i]}(x_i, y_i) - \mathbb{W}_{t,v_i}^{\text{cls}}(x_i, y_i) \right| d\mu_{v_i}(x_i) \\ & \leq C d_{Q_i}^{-2\gamma-2\varepsilon} \int_{Q_i^{**}} \sup_{c^{-1}d_{Q_i}^2 \leq t \leq cd_{Q_i}^2} t^\varepsilon \mu_{v_i}(B(x_i, \sqrt{t}))^{-1} \exp\left(-\frac{|x_i - y_i|^2}{\tilde{c}t}\right) d\mu_{v_i}(x_i) \quad (4.2) \\ & \leq C d_{Q_i}^{-2\gamma}. \end{aligned}$$

Since $\mathbb{W}_{t,v}^{\text{cls}}(x, y) = \mathbb{W}_{t,v_1}^{\text{cls}}(x_1, y_1) \mathbb{W}_{t,v_2}^{\text{cls}}(x_2, y_2)$, by the triangle inequality we have

$$\int_{Q^{**}} \sup_{t \leq d_Q^2} t^{-\delta} \left| T_t(x, y) - \mathbb{W}_{t,v}^{\text{cls}}(x, y) \right| d\mu_v(x) \leq I_1 + I_2,$$

where

$$\begin{aligned} I_1 &= \int_{Q^{**}} \sup_{t \leq d_Q^2} t^{-\delta} T_t^{[1]}(x_1, y_1) \left| T_t^{[2]}(x_2, y_2) - \mathbb{W}_{t,v_2}^{\text{cls}}(x_2, y_2) \right| d\mu_v(x), \\ I_2 &= \int_{Q^{**}} \sup_{t \leq d_Q^2} t^{-\delta} \mathbb{W}_{t,v_2}^{\text{cls}}(x_2, y_2) \left| T_t^{[1]}(x_1, y_1) - \mathbb{W}_{t,v_1}^{\text{cls}}(x_1, y_1) \right| d\mu_v(x). \end{aligned}$$

Applying (A₀) for $T_t^{[1]}(x_1, y_1)$ and then (2.4), and (A₂) together with (4.2) for $T_t^{[2]}(x_2, y_2)$, gives

$$\begin{aligned} I_1 &\leq C \int_{K^{**}} \sup_{t \leq d_Q^2} t^{\gamma-\delta} T_t^{[1]}(x_1, y_1) d\mu_{v_1}(x_1) \\ &\quad \times \int_{Q_2^{**}} \sup_{t \leq cd_{Q_2}^2} t^{-\gamma} \left| T_t^{[2]}(x_2, y_2) - \mathbb{W}_{t,v_2}^{\text{cls}}(x_2, y_2) \right| d\mu_{v_2}(x_2) \\ &\leq C d_K^{2\gamma-2\delta} d_{Q_2}^{-2\gamma} \leq C d_Q^{-2\delta}, \end{aligned}$$

since $0 \leq \delta < \gamma < \min(\gamma_1, \gamma_2)$.

Similarly, by (1.2), (2.4), (A₂) and (4.2), we have

$$\begin{aligned} I_2 &\leq C \int_{Q_2^{**}} \sup_{t \leq d_Q^2} t^{\gamma-\delta} \mathbb{W}_{t,v_2}^{\text{cls}}(x_2, y_2) d\mu_{v_2}(x_2) \\ &\quad \times \int_{Q_1^{**}} \sup_{t \leq cd_{Q_1}^2} t^{-\gamma} \left| T_t^{[1]}(x_1, y_1) - \mathbb{W}_{t,v_1}^{\text{cls}}(x_1, y_1) \right| d\mu_{v_1}(x_1) \\ &\leq C d_{Q_2}^{2\gamma-2\delta} d_{Q_1}^{-2\gamma} \leq C d_Q^{-2\delta}, \end{aligned}$$

since $d_{Q_1} \geq d_{Q_2} \simeq d_Q$. This finishes verification of (A₂).

The proof of Theorem 4.4 is complete.

4.2 Product of local and nonlocal atomic Hardy space

As we have mentioned, all atoms of the Hardy space $H^1(\mathbb{B}_{v_1}^{\text{cls}})$ satisfy cancellation condition, i.e. they are nonlocal atoms. However, if we consider the product $\mathbb{R}_+^d = \mathbb{R}_+^{d_1} \times \mathbb{R}_+^{d_2}$ with the measure $\mu_v = \mu_{v_1} \otimes \mu_{v_2}$ and the operator $L = \mathbb{B}_{v_1}^{\text{cls}} + L_2$, where the semigroup $\mathbb{K}_t := \exp(-tL_2)$ generated by L_2 , together with an admissible covering $\mathcal{Q}_2$ satisfy conditions (A₀)–(A₂) on $\mathbb{R}_+^{d_2}$, then the resulting Hardy space $H^1(L)$ shall have local character.

Let $\mathbb{R}_+^{d_1} \boxtimes \mathcal{Q}_2$ be the admissible covering that arises by splitting all the cylinders $\mathbb{R}_+^{d_1} \times Q_2$, $Q_2 \in \mathcal{Q}_2$, into countably many cubes $Q_{1,n} \times Q_2$, where $Q_{1,n} = Q(z_n, d_{Q_2}/2)$. Denote by $\mathbb{T}_t = \exp(-tL)$ the semigroup of operators generated by the operator L . There exists an integral kernel associated with $\{\mathbb{T}_t\}$ satisfying

$$\mathbb{T}_t(x, y) = \mathbb{W}_{t,v_1}^{\text{cls}}(x_1, y_1) \mathbb{K}_t(x_2, y_2), \quad x = (x_1, x_2), y = (y_1, y_2) \in \mathbb{R}_+^{d_1} \times \mathbb{R}_+^{d_2}.$$

We will prove the following.

Proposition 4.5 *Let $v = (v_1, v_2) \in (-1, \infty)^{d_1} \times (-1, \infty)^{d_2}$ and let L_2 and L be as above. Assume that for $\mathbb{K}_t(x_2, y_2)$ and an admissible covering $\mathcal{Q}_2$ of $\mathbb{R}_+^{d_2}$ conditions (A₀)–(A₂) hold. Then (A₀)–(A₂) are satisfied for $\mathbb{T}_t(x, y)$ and $\mathcal{Q}_L := \mathbb{R}_+^{d_1} \boxtimes \mathcal{Q}_2$.*

Combining Proposition 4.5 with Theorem 4.4 we get the following.

Corollary 4.6 *Let $v = (v_1, v_2) \in (-1, \infty)^{d_1} \times (-1, \infty)^{d_2}$ and let L_2, L be as above. Assume that $\mathbb{K}_t(x_2, y_2)$ with an admissible covering $\mathcal{Q}_2$ of $\mathbb{R}_+^{d_2}$ satisfy (A₀) – (A₂). Then the spaces $H^1(L)$ and $H_{\text{at}}^1(\mathbb{R}^{d_1} \boxtimes \mathcal{Q}_2, \mu_v)$ are isomorphic as Banach spaces.*

Now we pass to proving Proposition 4.5.

Proof Verification of (A₀). Let $\mathcal{Q}_L \ni Q \subset \mathbb{R}_+^{d_1} \times \mathcal{Q}_2$, where $\mathcal{Q}_2 \in \mathcal{Q}_2$. We take $y \in Q^*$ and $x \in \tilde{N}(Q)$. Since $x_2 \in \tilde{N}(\mathcal{Q}_2)$, $y_2 \in \mathcal{Q}_2^*$, we know that (A₀) holds for $\mathbb{K}_t(x_2, y_2)$. Moreover, $\mathbb{W}_{t,v_1}^{\text{cls}}(x_1, y_1)$ satisfies (1.2), hence (A₀) for $\mathbb{T}_t(x, y)$ follows.

To verify (A₁) and (A₂) suppose that γ is as in (A₁) for $\mathbb{K}_t(x_2, y_2)$ and $\mathcal{Q}_2$, and let $0 < \gamma' < \min(\gamma, v_{1,1} + 1, \dots, v_{1,d_1} + 1, v_{2,1} + 1, \dots, v_{2,d_2} + 1)$. Let $\mathcal{Q}_L \ni Q = Q_1 \times \mathcal{Q}_2 \subset \mathbb{R}_+^{d_1} \times \mathcal{Q}_2$ and take $y = (y_1, y_2) \in Q^*$. Recall that $d_{Q_1} \simeq d_{Q_2} \simeq d_Q$.

Verification of (A₁). Let $\delta \in (-\gamma', \gamma')$. In what follows $\varepsilon > 0$ will be always a fixed constant such that $\delta - \varepsilon > -\gamma'$ and $\delta + \varepsilon < \gamma'$. We decompose $(Q^{**})^c = S_1 \cup S_2 \cup S_3$, where

$$S_1 = Q_1^{**} \times (Q_2^{**})^c, \quad S_2 = (Q_1^{**})^c \times Q_2^{**}, \quad S_3 = (Q_1^{**})^c \times (Q_2^{**})^c.$$

To deal with the integral over S_1 we consider two cases.

Case 1: $\delta \in (-\gamma', 0]$. We use (A₁) for $\mathbb{K}_t(x_2, y_2)$, (1.2) for $\mathbb{W}_{t,v_1}^{\text{cls}}(x_1, y_1)$ and (2.4), getting

$$\begin{aligned} \int_{S_1} \sup_{t>0} t^\delta \mathbb{T}_t(x, y) d\mu_v(x) &\leq C \int_{Q_1^{**}} \sup_{t>0} t^\varepsilon \mu_{v_1}(B(x_1, \sqrt{t}))^{-1} \exp\left(-\frac{|x_1 - y_1|^2}{ct}\right) d\mu_{v_1}(x_1) \\ &\quad \times \int_{(Q_2^{**})^c} \sup_{t>0} t^{\delta-\varepsilon} \mathbb{K}_t(x_2, y_2) d\mu_{v_2}(x_2) \\ &\leq Cd_{Q_1}^{2\varepsilon} d_{Q_2}^{2\delta-2\varepsilon} \leq Cd_Q^{2\delta}. \end{aligned}$$

Case 2: $\delta \in (0, \gamma')$. By (A₁) for $\mathbb{K}_t(x_2, y_2)$, (1.2) for $\mathbb{W}_{t,v_1}^{\text{cls}}(x_1, y_1)$ and (2.4) we obtain

$$\begin{aligned} \int_{S_1} \sup_{t>0} t^\delta \mathbb{T}_t(x, y) d\mu_v(x) &\leq C \int_{Q_1^{**}} \sup_{t>0} t^\delta \mu_{v_1}(B(x_1, \sqrt{t}))^{-1} \exp\left(-\frac{|x_1 - y_1|^2}{ct}\right) d\mu_{v_1}(x_1) \\ &\quad \times \int_{(Q_2^{**})^c} \sup_{t>0} \mathbb{K}_t(x_2, y_2) d\mu_{v_2}(x_2) \\ &\leq Cd_{Q_1}^{2\delta} \leq Cd_Q^{2\delta}. \end{aligned}$$

To estimate the integral over S_2 we again consider the two cases.

Case 1: $\delta \in (-\gamma, 0]$. Using (A₀) for $\mathbb{K}_t(x_2, y_2)$ and (1.2) for the classical Bessel semigroup kernel, and then applying (2.5) and (2.4) we arrive at

$$\begin{aligned} \int_{S_2} \sup_{t>0} t^\delta \mathbb{T}_t(x, y) d\mu_\nu(x) &\leq C \int_{(Q_1^{**})^c} \sup_{t>0} t^{\delta-\varepsilon} \mu_{v_1}(B(x_1, \sqrt{t}))^{-1} \exp\left(-\frac{|x_1 - y_1|^2}{ct}\right) d\mu_{v_1}(x_1) \\ &\quad \times \int_{Q_2^{**}} \sup_{t>0} t^\varepsilon \mu_{v_2}(B(x_2, \sqrt{t}))^{-1} \exp\left(-\frac{|x_2 - y_2|^2}{ct}\right) d\mu_{v_2}(x_2) \\ &\leq C d_{Q_1}^{2\delta-2\varepsilon} d_{Q_2}^{-2\varepsilon} \leq C d_Q^{2\delta}. \end{aligned}$$

Case 2: $\delta \in (0, \gamma')$. In this case we use (A₀), (1.2) and Lemma 2.6, getting

$$\begin{aligned} \int_{S_2} \sup_{t>0} t^\delta \mathbb{T}_t(x, y) d\mu_\nu(x) &\leq C \int_{(Q_1^{**})^c} \sup_{t>0} t^{-\varepsilon} \mu_{v_1}(B(x_1, \sqrt{t}))^{-1} \exp\left(-\frac{|x_1 - y_1|^2}{ct}\right) d\mu_{v_1}(x_1) \\ &\quad \times \int_{Q_2^{**}} \sup_{t>0} t^{\delta+\varepsilon} \mu_{v_2}(B(x_2, \sqrt{t}))^{-1} \exp\left(-\frac{|x_2 - y_2|^2}{ct}\right) d\mu_{v_2}(x_2) \\ &\leq C d_{Q_1}^{-2\varepsilon} d_{Q_2}^{2\delta+2\varepsilon} \leq C d_Q^{2\delta}. \end{aligned}$$

Finally, to treat the integral over S_3 we use (A₁) for $\mathbb{K}_t(x_2, y_2)$, (1.2) for $\mathbb{W}_{t,v_1}^{\text{cls}}(x_1, y_1)$ and (2.5) obtaining

$$\begin{aligned} \int_{S_3} \sup_{t>0} t^\delta \mathbb{T}_t(x, y) d\mu_\nu(x) &\leq C \int_{(Q_1^{**})^c} \sup_{t>0} t^{-\varepsilon} \mu_{v_1}(B(x_1, \sqrt{t}))^{-1} \exp\left(-\frac{|x_1 - y_1|^2}{ct}\right) d\mu_{v_1}(x_1) \\ &\quad \times \int_{(Q_2^{**})^c} \sup_{t>0} t^{\delta+\varepsilon} \mathbb{K}_t(x_2, y_2) d\mu_{v_2}(x_2) \\ &\leq C d_{Q_1}^{-2\varepsilon} d_{Q_2}^{2\delta+2\varepsilon} \leq C d_Q^{2\delta}. \end{aligned}$$

Verification of (A₂). Let $\delta \in [0, \gamma')$ and fix $\varepsilon > 0$ such that $\delta + \varepsilon < \gamma'$. We write $\mathbb{W}_{t,v}^{\text{cls}}(x, y) = \mathbb{W}_{t,v_1}^{\text{cls}}(x_1, y_1) \mathbb{W}_{t,v_2}^{\text{cls}}(x_2, y_2)$. By (A₂) for $\mathbb{K}_t(x_2, y_2)$ and (1.2) for $\mathbb{W}_{t,v_1}^{\text{cls}}(x_1, y_1)$ and (2.4), we have

$$\begin{aligned} &\int_{Q^{**}} \sup_{t \leq d_Q^2} t^{-\delta} \left| \mathbb{T}_t(x, y) - \mathbb{W}_{t,v}^{\text{cls}}(x, y) \right| d\mu_\nu(x) \\ &= \int_{Q^{**}} \sup_{t \leq d_Q^2} t^{-\delta} \mathbb{W}_{t,v_1}^{\text{cls}}(x_1, y_1) \left| \mathbb{W}_{t,v_2}^{\text{cls}}(x_2, y_2) - \mathbb{K}_t(x_2, y_2) \right| d\mu_\nu(x) \\ &\leq C \int_{Q_1^{**}} \sup_{t \leq d_Q^2} t^\varepsilon \mu_{v_1}(B(x_1, \sqrt{t}))^{-1} \exp\left(-\frac{|x_1 - y_1|^2}{ct}\right) d\mu_{v_1}(x_1) \\ &\quad \times \int_{Q_2^{**}} \sup_{t \leq d_Q^2} t^{-\delta-\varepsilon} \left| \mathbb{W}_{t,v_2}^{\text{cls}}(x_2, y_2) - \mathbb{K}_t(x_2, y_2) \right| d\mu_{v_2}(x_2) \\ &\leq C d_{Q_1}^{2\varepsilon} d_{Q_2}^{-2\delta-2\varepsilon} \leq C d_Q^{-2\delta}. \end{aligned}$$

The conclusion follows.

5 Atomic Hardy space for the general Bessel operator

In this section we prove Theorem 1.1. Recall that we consider the operator $\mathbb{B}_v = \mathbb{B}_{v_c}^{\text{cls}} + \mathbb{B}_{-v_e}^{\text{exo}}$ acting on functions defined on the space $X = \mathbb{R}_+^{d_1} \times \mathbb{R}_+^{d_2} = \mathbb{R}_+^d$ equipped with the measure $\mu_v = \mu_{(v_c, -v_e)} := \mu_{v_c} \otimes \mu_{-v_e}$, where $v_c \in (-1, \infty)^{d_1}$, $v_e \in (0, \infty)^{d_2}$ and $d_1 \geq 0$, $d_2 \geq 1$. We use the notation $v_c = (v_1, \dots, v_{d_1})$, $v_e = (v_{d_1+1}, \dots, v_{d_1+d_2})$ and $\mathbf{x} = (\mathbf{x}_1, \mathbf{x}_2)$, $\mathbf{y} = (\mathbf{y}_1, \mathbf{y}_2) \in \mathbb{R}_+^{d_1} \times \mathbb{R}_+^{d_2}$. Moreover, we write $\mathbf{x}^{2v+1} = \mathbf{x}_1^{2v_c+1} \mathbf{x}_2^{-2v_e+1} := x_1^{2v_1+1} \dots x_{d_1}^{2v_{d_1}+1} x_{d_1+1}^{-2v_{d_1+1}+1} \dots x_{d_1+d_2}^{-2v_{d_1+d_2}+1}$.

Observe that

$$\begin{aligned} \|f\|_{H^1(\mathbb{B}_v)} &= \int_X \sup_{t>0} \left| \int_X \mathbb{W}_{t,v_c}^{\text{cls}}(\mathbf{x}_1, \mathbf{y}_1) \mathbb{W}_{t,-v_e}^{\text{exo}}(\mathbf{x}_2, \mathbf{y}_2) f(\mathbf{y}) d\mu_{(v_c, -v_e)}(\mathbf{y}) \right| d\mu_{(v_c, -v_e)}(\mathbf{x}) \\ &= \int_X \sup_{t>0} \left| \int_X \mathbb{W}_{t,v_c}^{\text{cls}}(\mathbf{x}_1, \mathbf{y}_1) \mathbf{x}_2^{-4v_e} \mathbb{W}_{t,-v_e}^{\text{exo}}(\mathbf{x}_2, \mathbf{y}_2) \mathbf{y}_2^{-4v_e} f(\mathbf{y}) d\mu_{(v_c, v_e)}(\mathbf{y}) \right| d\mu_{(v_c, v_e)}(\mathbf{x}) \\ &= \int_X \sup_{t>0} \left| \int_X \mathbb{W}_{t,v_c}^{\text{cls}}(\mathbf{x}_1, \mathbf{y}_1) \mathbb{K}_t(\mathbf{x}_2, \mathbf{y}_2) \tilde{f}(\mathbf{y}) d\mu_{(v_c, v_e)}(\mathbf{y}) \right| d\mu_{(v_c, v_e)}(\mathbf{x}) \\ &= \int_X \sup_{t>0} \left| \int_X \mathbb{T}_t(\mathbf{x}, \mathbf{y}) \tilde{f}(\mathbf{y}) d\mu_{(v_c, v_e)}(\mathbf{y}) \right| d\mu_{(v_c, v_e)}(\mathbf{x}) \\ &= \left\| \sup_{t>0} |\mathbb{T}_t \tilde{f}| \right\|_{L^1(X, d\mu_{(v_c, v_e)})}, \end{aligned} \quad (5.1)$$

where $\tilde{f}(\mathbf{y}) = \mathbf{y}_2^{-4v_e} f(\mathbf{y})$, $\mathbb{T}_t(\mathbf{x}, \mathbf{y}) = \mathbb{W}_{t,v_c}^{\text{cls}}(\mathbf{x}_1, \mathbf{y}_1) \mathbb{K}_t(\mathbf{x}_2, \mathbf{y}_2)$, and

$$\begin{aligned} \mathbb{K}_t(\mathbf{x}_2, \mathbf{y}_2) &= \mathbf{x}_2^{-4v_e} \mathbb{W}_{t,-v_e}^{\text{exo}}(\mathbf{x}_2, \mathbf{y}_2) \\ &= \prod_{i=d_1+1}^{d_1+d_2} (2t)^{-1} y_i^{v_i} x_i^{-3v_i} I_{v_i} \left(\frac{x_i y_i}{2t} \right) \exp \left(-\frac{x_i^2 + y_i^2}{4t} \right) \\ &=: \prod_{i=d_1+1}^{d_1+d_2} K_{t,v_i}(x_i, y_i). \end{aligned} \quad (5.2)$$

From the above we see that $f \in H^1(\mathbb{B}_v)$ if and only if $\sup_{t>0} |\mathbb{T}_t \tilde{f}| \in L^1(X, d\mu_{(v_c, v_e)})$. To prove Theorem 1.1 we need the following.

Proposition 5.1 *Let $\mathcal{D} = \{[2^n, 2^{n+1}] : n \in \mathbb{Z}\}$ be the admissible covering of $\mathbb{R}_+$. Assume $v \in (0, \infty)$ and let*

$$K_{t,v}(x, y) = (2t)^{-1} y^v x^{-3v} I_v \left(\frac{xy}{2t} \right) \exp \left(-\frac{x^2 + y^2}{4t} \right). \quad (5.3)$$

Then $K_{t,v}$ with $\mathcal{D}$ satisfy conditions (A₀) – (A₂).

In the proof we will need the following standard asymptotics of the modified Bessel function I_τ , see e.g. [26, p. 203–204],

$$I_\tau(x) = \Gamma(\tau + 1)^{-1} (x/2)^\tau + O(x^{\tau+2}), \quad \text{for } x \sim 0, \quad (5.4)$$

$$I_\tau(x) = (2\pi x)^{-1/2} e^x + O(x^{-3/2} e^x), \quad \text{for } x \sim \infty. \quad (5.5)$$

Proof To verify (A₀) – (A₂) for $K_{t,v}$ with $\mathcal{D}$, let $\mathcal{D} \ni Q = [2^n, 2^{n+1}]$ for some $n \in \mathbb{Z}$ and take $y \in Q^*$.

Verification of (A₀). We consider $x \in \tilde{N}(Q)$. Notice that then $x^{-1}y \simeq 1$, hence $K_{t,v}(x, y) = (x^{-1}y)^{2\nu} W_{t,v}^{\text{cls}}(x, y) \simeq W_{t,v}^{\text{cls}}(x, y)$. Therefore

$$0 \leq K_{t,v}(x, y) \leq C\mu_v(B(x, \sqrt{t}))^{-1} \exp\left(-\frac{|x-y|^2}{ct}\right),$$

since the kernel $W_{t,v}^{\text{cls}}(x, y)$ is non-negative and satisfies the Gaussian estimate (1.2).

Verification of (A₁). Suppose $0 < \gamma < \min(1/2, \nu)$ and take $\delta \in (-\gamma, \gamma)$. We write

$$\begin{aligned} \int_{(Q^{**})^c} \sup_{t>0} t^\delta K_{t,v}(x, y) d\mu_v(x) &\leq \int_{(Q^{**})^c} \sup_{t>xy/2} t^\delta K_{t,v}(x, y) d\mu_v(x) \\ &\quad + \int_{(Q^{**})^c \cap (0, 2d_Q)} \sup_{0<t\leq xy/2} t^\delta K_{t,v}(x, y) d\mu_v(x) \\ &\quad + \int_{(Q^{**})^c \cap (2d_Q, \infty)} \sup_{0<t\leq xy/2} t^\delta K_{t,v}(x, y) d\mu_v(x) \\ &=: I_1 + I_2 + I_3. \end{aligned}$$

To treat I_1 we use (5.3) and (5.4).

$$\begin{aligned} I_1 &\leq C \int_0^\infty \sup_{t>xy/2} t^\delta (2t)^{-1} y^\nu x^{-3\nu} \left(\frac{xy}{2t}\right)^\nu \exp\left(-\frac{x^2+y^2}{4t}\right) x^{2\nu+1} dx \\ &\leq Cy^{2\nu} \int_0^\infty (x^2+y^2)^{\delta-\nu-1} x dx \\ &= Cy^{2\nu} \lim_{\varepsilon \rightarrow 0} \left| (\varepsilon^{-2} + y^2)^{\delta-\nu} - (\varepsilon^2 + y^2)^{\delta-\nu} \right| \\ &= Cy^{2\nu} y^{2\delta-2\nu} \leq Cd_Q^{2\delta}, \end{aligned}$$

since $\delta - \nu < 0$ and $y \simeq d_Q$.

To deal with I_2 we use (5.3) and (5.5). Observing that $xy \leq cd_Q^2 \leq c|x-y|^2$, we get

$$\begin{aligned}
 I_2 &\leq C \int_{(Q^{**})^c \cap (0, 2d_Q)} \sup_{0 < t \leq xy/2} t^{\delta-1/2} y^{\nu-1/2} x^{-\nu+1/2} \exp\left(-\frac{|x-y|^2}{4t}\right) dx \\
 &\leq Cy^{\delta+\nu-1} \int_{(Q^{**})^c \cap (0, 2d_Q)} x^{\delta-\nu} \exp\left(-\frac{|x-y|^2}{cxy}\right) dx \\
 &\leq Cy^{\delta+\nu-1} \int_{(Q^{**})^c \cap (0, 2d_Q)} x^{\delta-\nu} \left(\frac{|x-y|^2}{cxy}\right)^{-N} dx \\
 &\leq Cy^{\delta+\nu-1+N} d_Q^{\delta-\nu+N} \int_{(Q^{**})^c} |x-y|^{-2N} dx \\
 &\leq Cy^{\delta+\nu-1+N} d_Q^{\delta-\nu-N+1} \leq Cd_Q^{2\delta},
 \end{aligned}$$

since $y \simeq d_Q$, and we choose N large enough so that $\delta - \nu + N \geq 0$.

Treating I_3 we observe that $x \geq 2d_Q$ implies $|x-y| \simeq x$. Then using (5.3) and (5.5) we obtain

$$\begin{aligned}
 I_3 &\leq C \int_{(Q^{**})^c \cap (2d_Q, \infty)} \sup_{0 < t \leq xy/2} t^{\delta-1/2} y^{\nu-1/2} x^{-\nu+1/2} \exp\left(-\frac{|x-y|^2}{4t}\right) dx \\
 &\leq Cy^{\nu-1/2} \int_{(Q^{**})^c \cap (2d_Q, \infty)} \sup_{0 < t \leq xy/2} t^{\delta-1/2} x^{-\nu+1/2} \left(\frac{|x-y|^2}{4t}\right)^{-N} dx \\
 &\leq Cy^{\nu-1+\delta+N} \int_{(Q^{**})^c \cap (2d_Q, \infty)} x^{\delta+N-\nu} |x-y|^{-2N} dx \\
 &\leq Cy^{\nu-1+\delta+N} \int_{(Q^{**})^c} |x-y|^{\delta-N-\nu} dx \\
 &\leq Cy^{\nu-1+\delta+N} d_Q^{\delta-N-\nu+1} \leq Cd_Q^{2\delta},
 \end{aligned}$$

by taking N sufficiently large and since $y \simeq d_Q$.

Verification of (A₂). Let $0 < \gamma < \min(1/2, \nu)$ and take $\delta \in [0, \gamma)$. Notice that here $x \simeq y \simeq d_Q$. Moreover, $t \leq d_Q^2 \leq cxy$. Using (5.3), (1.1), (5.5) and the Mean Value Theorem, we obtain

$$\begin{aligned}
& \int_{Q^{**}} \sup_{t \leq d_Q^2} t^{-\delta} \left| K_{t,v}(x, y) - W_{t,v}^{\text{cls}}(x, y) \right| d\mu_v(x) \\
& \leq \int_{Q^{**}} \sup_{t \leq d_Q^2} t^{-\delta} (2t)^{-1} I_v \left(\frac{xy}{2t} \right) \exp \left(-\frac{x^2 + y^2}{4t} \right) x^{-\nu} y^{\nu} \left| x^{-2\nu} - y^{-2\nu} \right| x^{2\nu+1} dx \\
& \leq C \int_{Q^{**}} \sup_{t > 0} t^{-1-\delta} \left(\frac{xy}{2t} \right)^{-1/2} \exp \left(-\frac{|x-y|^2}{4t} \right) x^{\nu+1} y^{\nu} |x-y| \xi_{x,y}^{-2\nu-1} dx \\
& \leq C y^{2\nu} \int_{Q^{**}} |x-y|^{-1-2\delta} |x-y| d_Q^{-2\nu-1} dx \\
& \leq C d_Q^{-1} \int_{Q^{**}} |x-y|^{-2\delta} dx \leq C d_Q^{-2\delta},
\end{aligned}$$

since $y \simeq d_Q$ and $\delta < 1/2$; here $\xi_{x,y}$ denotes a point between x and y .

Lemma 5.2 *Let $v = (v_c, -v_e)$, where $v_c \in (-1, \infty)^{d_1}$, and $v_e \in (0, \infty)^{d_2}$. Let $f \in L^1(X, d\mu_v)$. Then*

$$\|f\|_{L^1(X, d\mu_v)} \leq \left\| \sup_{t>0} |\mathbb{W}_{t,v} f| \right\|_{L^1(X, d\mu_v)}.$$

The proof of Lemma 5.2 is presented in Appendix.

Proof of Theorem 1.1. Recall that $Q_{\mathbb{B}} = \mathbb{R}_+^{d_1} \boxtimes \underbrace{\mathcal{D} \boxtimes \dots \boxtimes \mathcal{D}}_{d_2 \text{ times}}$. In view of (5.1) we

have

$$\|f\|_{H^1(\mathbb{B}_v)} = \left\| \sup_{t>0} |\mathbb{T}_t \tilde{f}| \right\|_{L^1(X, d\mu_{(v_c, v_e)})}, \quad (5.6)$$

where $\tilde{f}(\mathbb{y}) = y_2^{-4v_e} f(\mathbb{y})$. Observe that

$$\|f\|_{L^1(X, d\mu_{(v_c, -v_e)})} = \|\tilde{f}\|_{L^1(X, d\mu_{(v_c, v_e)})}. \quad (5.7)$$

Combining Proposition 5.1 with Proposition 4.5 and Theorem 4.4 we infer that $\sup_{t>0} |\mathbb{T}_t \tilde{f}| \in L^1(X, d\mu_{(v_c, v_e)})$ if and only if there exist a sequence λ_k and $(Q_{\mathbb{B}}, \mu_{(v_c, v_e)})$ -atoms $\tilde{a}_k$ such that

$$\tilde{f} = \sum_k \lambda_k \tilde{a}_k,$$

and

$$\left(\left\| \sup_{t>0} |\mathbb{T}_t \tilde{f}| \right\|_{L^1(X, d\mu_{(v_c, v_e)})} + \|\tilde{f}\|_{L^1(X, d\mu_{(v_c, v_e)})} \right) \simeq \inf \sum_k |\lambda_k|. \quad (5.8)$$

Hence, f belongs to $H^1(\mathbb{B}_v)$ if and only if it has a decomposition

$$f(\mathbb{y}) = \sum_k \lambda_k \tilde{a}_k(\mathbb{y}) \mathbb{y}_2^{4v_e} =: \sum_k \lambda_k a_k(\mathbb{y}).$$

Notice, that since $\tilde{f} \in H^1_{\text{at}}(\mathcal{Q}_{\mathbb{B}}, \mu_{(v_c, v_e)})$ and $\mathbb{T}_t$ satisfies (A₀)–(A₂), we have

$$\left\| \sup_{t>0} |\mathbb{T}_t \tilde{f}| \right\|_{L^1(X, d\mu_{(v_c, v_e)})} \leq \|\tilde{f}\|_{L^1(X, d\mu_{(v_c, v_e)})}, \quad (5.9)$$

(see the first inequality in the proof of Theorem 3.1). Moreover, in the view of Lemma 5.2 we have that

$$\|f\|_{L^1(X, d\mu_{(v_c, -v_e)})} \leq \left\| \sup_{t>0} |\mathbb{W}_{t,v} f| \right\|_{L^1(X, d\mu_{(v_c, -v_e)})} = \|f\|_{H^1(\mathbb{B}_v)}. \quad (5.10)$$

By (5.6) – (5.10) we get that $\|f\|_{H^1(\mathbb{B}_v)} \simeq \inf \sum_k |\lambda_k|$. Thus it suffices to show that functions of the form $a_k(\mathbb{y}) = \tilde{a}_k(\mathbb{y}) \mathbb{y}_2^{4v_e}$ are $(\mathcal{Q}_{\mathbb{B}}, \mu_{(v_c, -v_e)})$ -atoms after a modification by multiplying by some function which is comparable to a constant.

Suppose first that $\tilde{a}_k = \mu_{(v_c, v_e)}(Q)^{-1} \mathbb{1}_Q$ for some cube $\mathcal{Q}_{\mathbb{B}} \ni Q = Q_1 \times Q_2 \subset \mathbb{R}_+^{d_1} \times [2^{n_1}, 2^{n_1+1}] \times \dots \times [2^{n_{d_2}}, 2^{n_{d_2}+1}]$. Without loss of generality we may assume that $n_1 = \min_i n_i$ and that $d_Q \simeq 2^{n_1} \simeq d_{Q_1} \simeq d_{Q_2}$. Denote by $\mathbb{z}$ the center of Q . Then

$$\begin{aligned} a_k(\mathbb{y}) &= \mathbb{y}_2^{4v_e} \tilde{a}_k(\mathbb{y}) = \mathbb{y}_2^{4v_e} \frac{\mu_{-v_e}(Q_2)}{\mu_{v_e}(Q_2)} \mu_{v_c}(Q_1)^{-1} \mu_{-v_e}^{-1}(Q_2) \mathbb{1}_Q(\mathbb{y}) \\ &= \mathbb{y}_2^{4v_e} \frac{\mu_{-v_e}(Q_2)}{\mu_{v_e}(Q_2)} \mu_{(v_c, -v_e)}(Q)^{-1} \mathbb{1}_Q(\mathbb{y}), \end{aligned}$$

where $\mu_{(v_c, -v_e)}(Q)^{-1} \mathbb{1}_Q(\mathbb{y})$ is $(\mathcal{Q}_{\mathbb{B}}, \mu_{(v_c, -v_e)})$ -atom. Using Corollary 2.4(a), for $\mathbb{y}_2 \in Q_2$ we obtain

$$\begin{aligned} \mathbb{y}_2^{4v_e} \frac{\mu_{-v_e}(Q_2)}{\mu_{v_e}(Q_2)} &\simeq \mathbb{y}_2^{4v_e} \frac{\prod_{i=d_1+1}^{d_1+d_2} \mu_{-v_i}(B(z_i, 2^{n_1}))}{\prod_{i=d_1+1}^{d_1+d_2} \mu_{v_i}(B(z_i, 2^{n_1}))} \simeq \prod_{i=d_1+1}^{d_1+d_2} y_i^{4v_i} \frac{z_i^{-2v_i+1} \cdot 2^{n_1}}{z_i^{2v_i+1} \cdot 2^{n_1}} \\ &= \prod_{i=d_1+1}^{d_1+d_2} y_i^{4v_i} z_i^{-4v_i} \simeq 1, \end{aligned} \quad (5.11)$$

since $y_i \simeq z_i$ for $i = d_1 + 1, \dots, d_1 + d_2$. Notice that the above estimate is uniform in Q_2 .

On the other hand, if $\tilde{a}_k$ is a $(\mathcal{Q}_{\mathbb{B}}, \mu_{(v_c, v_e)})$ -atom of the form (i) from Definition 2.2 associated with a cube $\mathcal{Q}_{\mathbb{B}} \ni Q = Q_1 \times Q_2 \subset \mathbb{R}_+^{d_1} \times [2^{n_1}, 2^{n_1+1}] \times \dots \times [2^{n_{d_2}}, 2^{n_{d_2}+1}]$ as above, then the function $a_k(\mathbb{y}) = \mathbb{y}_2^{4v_e} \tilde{a}_k(\mathbb{y})$ satisfies:

1. $\text{supp } a_k \subseteq K = K_1 \times K_2 \subseteq Q^*$, (Suppose that $K_2 = Q(\mathbb{W}, r_{d_1+1}, \dots, r_{d_1+d_2}) \subset [2^{n_1}, 2^{n_1+1}] \times \dots \times [2^{n_{d_2}}, 2^{n_{d_2}+1}]$.)
2. similarly as in (5.11), since $r_i < w_i/2$ for $i = d_1 + 1, \dots, d_1 + d_2$, we use Corollary 2.4(a) and have

$$\begin{aligned}
 \|a_k\|_\infty &= \text{ess sup}_{\mathbb{Y} \in K} \mathbb{Y}_2^{4v_e} |\tilde{a}_k(\mathbb{Y})| \leq \text{ess sup}_{\mathbb{Y} \in K} \mathbb{Y}_2^{4v_e} \|\tilde{a}_k\|_\infty \\
 &\leq \mu_{(v_c, -v_e)}(K)^{-1} \text{ess sup}_{\mathbb{Y}_2 \in K_2} \mathbb{Y}_2^{4v_e} \frac{\mu_{-v_e}(K_2)}{\mu_{v_e}(K_2)} \\
 &\simeq \mu_{(v_c, -v_e)}(K)^{-1} \text{ess sup}_{\mathbb{Y}_2 \in K_2} \prod_{i=d_1+1}^{d_1+d_2} y_i^{4v_i} \frac{w_i^{-2v_i+1} \cdot r_i}{w_i^{2v_i+1} \cdot r_i} \\
 &\simeq \mu_{(v_c, -v_e)}(K)^{-1} \text{ess sup}_{\mathbb{Y}_2 \in K_2} \prod_{i=d_1+1}^{d_1+d_2} y_i^{4v_i} w_i^{-4v_i} \leq C \mu_{(v_c, -v_e)}(K)^{-1},
 \end{aligned}$$

where the constant C does not depend on K_2 . In the last inequality we used the fact that $y_i \simeq w_i$ for $i = d_1 + 1, \dots, d_1 + d_2$, when $y_i, w_i \in [2^{n_i}, 2^{n_i+1}]$.

3.

$$\int a_k(\mathbb{Y}) d\mu_{(v_c, -v_e)}(\mathbb{Y}) = \int \tilde{a}_k(\mathbb{Y}) d\mu_{(v_c, v_e)}(\mathbb{Y}) = 0.$$

This finishes the proof. $\square$

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Appendix

In this Appendix we prove Lemma 5.2. The proof goes by standard methods, but for the convenience of the reader we present details. Actually we prove a more general result, Lemma 5.6, from which Lemma 5.2 follows immediately. Recall that $\mathbb{W}_{t,v}(\mathbb{x}, \mathbb{y}) = \mathbb{W}_{t,v_c}^{\text{cls}}(\mathbb{x}_1, \mathbb{y}_1) \cdot \mathbb{W}_{t,-v_e}^{\text{exo}}(\mathbb{x}_2, \mathbb{y}_2)$, where $v = (v_c, -v_e)$, and $v_c \in (-1, \infty)^{d_1}$, and $v_e \in (0, \infty)^{d_2}$. We start with the following lemma.

Lemma 5.3 *Let $v \in (-1, \infty)^d$. There exists a sequence $\{t_n\}_n$ such that $t_n \rightarrow 0$ and for every $r > 0$ we have:*

$$\lim_{n \rightarrow \infty} \int_{|x-y|>r} \mathbb{W}_{t_n,v}^{\text{cls}}(x, y) d\mu_v(y) = 0, \quad (5.12)$$

$$\lim_{n \rightarrow \infty} \int_{|x-y| \leq r} \mathbb{W}_{t_n, v}^{\text{cls}}(x, y) d\mu_v(y) = 1, \quad (5.13)$$

for a.e. $x \in X$.

Proof To prove (5.12) we use (1.2) and the symmetry of $\mathbb{W}_{t_n, v}^{\text{cls}}(x, y)$, having

$$\begin{aligned} & \int_{|x-y| > r} \mathbb{W}_{t_n, v}^{\text{cls}}(x, y) d\mu_v(y) \\ & \leq C \int_{|x-y| > r} \mu(B(y, \sqrt{t}))^{-1} \exp\left(-\frac{|x-y|^2}{ct}\right) y^{2v+1} dy \\ & \leq C \int_{|x-y| > r} t^{-d/2} \left(\prod_{i=1}^d y_i^{-2v_i-1}\right) \exp\left(-\frac{|x-y|^2}{ct}\right) y_1^{2v_1+1} \dots y_d^{2v_d+1} dy_1 \dots dy_d \\ & \leq C \int_{|y| > \frac{r}{\sqrt{t}}} \exp\left(-\frac{|y|^2}{c}\right) dy_1 \dots dy_d \rightarrow 0, \text{ when } t \rightarrow 0. \end{aligned}$$

In order to prove (5.13), observe that for $f \in L^2(X, d\mu_v)$ we have that $\lim_{t \rightarrow 0} \mathbb{W}_{t, v}^{\text{cls}} f$ converges to f in $L^2(X, d\mu_v)$, so we can choose a sequence with a.e. convergence. Applying this to functions $f_n(x) = \chi_{Q(0, n)}(x)$ and using a diagonal argument we obtain a sequence $\{t_n\}_n$, which goes to 0, and such that for a.e. $x \in X$ we have

$$\lim_{n \rightarrow \infty} \int_X \mathbb{W}_{t_n, v}^{\text{cls}}(x, y) d\mu_v(y) = 1 \quad (5.14)$$

and (5.13) follows from (5.14) and (5.12).

Proposition 5.4 Let $v \in (-1, \infty)^d$ and $f \in C_c(X)$. There exists a sequence $\{t_n\}_n$ such that $t_n \rightarrow 0$ and for almost every $x \in X$,

$$\lim_{n \rightarrow \infty} \mathbb{W}_{t_n, v}^{\text{cls}} f(x) = f(x).$$

Proof By the Lebesgue differentiation theorem we have

$$\lim_{s \rightarrow 0} \mu_v(B(x, s))^{-1} \int_{B(x, s)} |f(y) - f(x)| d\mu_v(y) = 0 \quad (5.15)$$

for almost every $x \in X$, since $f \in C_c(X)$. Consider the set A of points $x \in X$ such that we have (5.15), and, additionally, (5.12)–(5.13) hold for all rational $r > 0$. Obviously, such set has full measure. Fix $\varepsilon > 0$ and $x \in A$. We will show that $|\mathbb{W}_{t_n, v}^{\text{cls}} f(x) - f(x)| \leq C\varepsilon$ for large $n \in \mathbb{N}$. Let $r > 0$ be a fixed rational number such that for $s < r$ we have

$$\int_{B(x, s)} |f(y) - f(x)| d\mu_v(y) \leq \varepsilon \mu_v(B(x, s)). \quad (5.16)$$

Assume that $\sqrt{t_n} < r$ for large n . We write

$$\begin{aligned}\mathbb{W}_{t_n, v}^{\text{cls}} f(x) - f(x) &= f(x) \left(\int_{|x-y| \leq r} \mathbb{W}_{t_n, v}^{\text{cls}}(x, y) d\mu_v(y) - 1 \right) \\ &\quad + \int_{|x-y| > r} \mathbb{W}_{t_n, v}^{\text{cls}}(x, y) f(y) d\mu_v(y) \\ &\quad + \int_{|x-y| < \sqrt{t_n}} \mathbb{W}_{t_n, v}^{\text{cls}}(x, y) (f(y) - f(x)) d\mu_v(y) \\ &\quad + \int_{\sqrt{t_n} \leq |x-y| \leq r} \mathbb{W}_{t_n, v}^{\text{cls}}(x, y) (f(y) - f(x)) d\mu_v(y) \\ &=: I_1 + I_2 + I_3 + I_4.\end{aligned}$$

Applying (5.13) we obtain that $|I_1| < \varepsilon$ for t_n small enough.

To treat I_2 we use that $f \in C_c(X)$ and (5.12), having $|I_2| < \varepsilon$ for t_n small enough.

To estimate I_3 we apply (1.2). Hence, by (5.16), since $\sqrt{t_n} < r$, we obtain

$$\begin{aligned}|I_3| &\leq C\mu_v(B(x, \sqrt{t_n}))^{-1} \int_{|x-y| < \sqrt{t_n}} |f(y) - f(x)| d\mu_v(y) \\ &\leq C\mu_v(B(x, \sqrt{t_n}))^{-1} \varepsilon \mu_v(B(x, \sqrt{t_n})) \leq C\varepsilon.\end{aligned}$$

To deal with I_4 let $N = \lceil \log_2(r/\sqrt{t_n}) \rceil$, so that $r \leq \sqrt{t_n} 2^N \leq 2r$. Define

$$S_k = \left\{ x \in X : r2^{-k} < |x - y| < r2^{-k+1} \right\}$$

for $k = 1, \dots, N$. Notice, that by (2.1) we have $\mu_v(B(x, r2^{-k}))/\mu_v(B(x, r2^{-N})) \leq C_0^{N-k+1}$. Using (1.2), (5.16), we get

$$\begin{aligned}|I_4| &\leq C \sum_{k=1}^N \int_{S_k} \mu(B(x, \sqrt{t_n}))^{-1} \exp\left(-\frac{|x-y|^2}{ct_n}\right) |f(y) - f(x)| d\mu_v(y) \\ &\leq C\varepsilon \sum_{k=1}^N \frac{\mu_v(B(x, r2^{-k}))}{\mu_v(B(x, r2^{-N}))} \exp\left(-\frac{r2^{-2k}}{cr2^{-2N}}\right) \\ &\leq C\varepsilon \sum_{k=1}^N C_0^{N-k+1} \left(\frac{2^{-2k}}{2^{-2N}}\right)^{-M} \\ &\leq CC_0^{N+1} \varepsilon 2^{-2NM} \sum_{k=1}^N C_0^{-k} (2^{2M})^k \leq C\varepsilon,\end{aligned}$$

by taking large $M \in \mathbb{N}$.

Lemma 5.5 *Let $f \in C_c(X)$. There exists a sequence $\{t_n\}_n$ such that $t_n \rightarrow 0$ and for almost every $x \in X$,*

$$\lim_{n \rightarrow \infty} \mathbb{W}_{t_n, v} f(x) = f(x).$$

Proof Take $f \in C_c(X)$ and let $\{t_n\}_n$ be as in Lemma 5.4 for $\mathbb{W}_{t, (v_c, v_e)}^{\text{cls}}$. By Lemma 5.4 and (1.5) we obtain

$$\begin{aligned} \mathbb{W}_{t_n, v} f(x) &= \int_X \mathbb{W}_{t, v_c}^{\text{cls}}(x_1, y_1) \mathbb{W}_{t, -v_e}^{\text{exo}}(x_2, y_2) f(y) d\mu_{(v_c, -v_e)}(y) \\ &= x_2^{2v_e} \int_X \mathbb{W}_{t_n, (v_c, v_e)}^{\text{cls}}(x, y) F(y) d\mu_{(v_c, v_e)}(y) \\ &\rightarrow x_2^{2v_e} F(x) = x_2^{2v_e} x_2^{-2v_e} f(x) = f(x), \end{aligned}$$

since $F(x) = x_2^{-2v_e} f(x) \in C_c(X)$.

Lemma 5.6 *Let $f \in L^1(X, d\mu_v)$. There exists a sequence $\{t_n\}_n$ such that $t_n \rightarrow 0$ and for almost every $x \in X$,*

$$\lim_{n \rightarrow \infty} \mathbb{W}_{t_n, v} f(x) = f(x).$$

Proof It was proved in [19, Thm. 4.1] that the maximal operator $\sup_{t>0} |\mathbb{W}_{t, v} f|$ is weak type (1,1). Hence, by [10, Thm. 2.2] we know that the set $\{f \in L^1(X, d\mu_v) : \lim_{t \rightarrow 0} \mathbb{W}_{t, v} f(x) = f(x) \text{ a.e.}\}$ is closed in $L^1(X, d\mu_v)$. Combining this with Lemma 5.5 and the fact that $C_c(X)$ is dense in $L^1(X, d\mu_v)$ we get the desired result.

References

- Betancor, J.J., Buraczewski, D., Fariña, J.C., Martínez, T., Torrea, J.L.: Riesz transforms related to Bessel operators. *Proc. R. Soc. Edinburgh Sect. A* **137**, 701–725 (2007)
- Betancor, J.J., Castro, A.J., Curbelo, J.: Spectral multipliers for multidimensional Bessel operators. *J. Fourier Anal. Appl.* **17**, 932–975 (2011)
- Betancor, J.J., Castro, A.J., Nowak, A.: Calderón–Zygmund operators in the Bessel setting. *Monatsh. Math.* **167**, 375–403 (2012)
- Betancor, J.J., De León-Contreras, M.: Parabolic equations involving Bessel operators and singular integrals. *Int. Equ. Oper. Theory* **90** (2018), Art. 18, 58
- Betancor, J.J., Dziubański, J., Torrea, J.L.: On Hardy spaces associated with Bessel operators. *J. Anal. Math.* **107**, 195–219 (2009)
- Betancor, J.J., Harboure, E., Nowak, A., Viviani, B.: Mapping properties of fundamental operators in harmonic analysis related to Bessel operators. *Studia Math.* **197**, 101–140 (2010)
- Carbonaro, A., Mauceri, G., Meda, S.: H^1 and BMO for certain locally doubling metric measure spaces. *Ann. Sci. Norm. Super. Pisa Cl. Sci. (5)* **8**, 543–582 (2009)
- Castro, A.J., Szarek, T.Z.: Calderón–Zygmund operators in the Bessel setting for all possible type indices. *Acta Math. Sin. (Engl. Ser.)* **30**, 637–648 (2014)
- Coifman, R.R., Weiss, G.: Extensions of Hardy spaces and their use in analysis. *Bull. Am. Math. Soc.* **83**, 569–645 (1977)
- Duoandikoetxea, J.: *Fourier Analysis*, Graduate Studies in Mathematics, vol. 29. American Mathematical Society, Providence, RI (2001). Translated and revised from the 1995 Spanish original by David Cruz-Uribe
- Dziubański, J., Preisner, M.: Hardy spaces for semigroups with Gaussian bounds. *Ann. Mat. Pura Appl. (4)* **197**, 965–987 (2018)

12. Dziubański, J., Preisner, M., Wróbel, B.: Multivariate Hörmander-type multiplier theorem for the Hankel transform. *J. Fourier Anal. Appl.* **19**, 417–437 (2013)
13. Goldberg, D.: A local version of real Hardy spaces. *Duke Math. J.* **46**, 27–42 (1979)
14. Kania, E., Preisner, M.: Sharp multiplier theorem for multidimensional Bessel operators. *J. Fourier Anal. Appl.* **25**, 2419–2446 (2019)
15. Kania-Strojec, E., Plewa, P., Preisner, M.: Local atomic decompositions for multidimensional Hardy spaces. *Rev. Mat. Complut.* **34**, 409–434 (2021)
16. Muckenhoupt, B., Stein, E.M.: Classical expansions and their relation to conjugate harmonic functions. *Trans. Am. Math. Soc.* **118**, 17–92 (1965)
17. Nazarov, F., Treil, S., Volberg, A.: Weak type estimates and Cotlar inequalities for Calderón–Zygmund operators on nonhomogeneous spaces. *Int. Math. Res. Not.* 463–487
18. Nazarov, F., Treil, S., Volberg, A.: The Tb -theorem on non-homogeneous spaces. *Acta Math.* **190**, 151–239 (2003)
19. Nowak, A., Sjögren, P., Szarek, T.Z.: Maximal operators of exotic and non-exotic Laguerre and other semigroups associated with classical orthogonal expansions. *Adv. Math.* **318**, 307–354 (2017)
20. Nowak, A., Stempak, K.: Potential operators associated with Hankel and Hankel–Dunkl transforms. *J. Anal. Math.* **131**, 277–321 (2017)
21. Schmüdgen, K.: Unbounded Self-Adjoint Operators on Hilbert Space, Graduate Texts in Mathematics, vol. 265. Springer, Dordrecht (2012)
22. Stein, E.M.: Topics in Harmonic Analysis Related to the Littlewood–Paley Theory, Annals of Mathematics Studies. Princeton University Press, Princeton (1970)
23. Tolsa, X.: BMO, H^1 , and Calderón–Zygmund operators for non doubling measures. *Math. Ann.* **319**, 89–149 (2001)
24. Tolsa, X.: The space H^1 for nondoubling measures in terms of a grand maximal operator. *Trans. Am. Math. Soc.* **355**, 315–348 (2003)
25. Uchiyama, A.: A maximal function characterization of H^p on the space of homogeneous type. *Trans. Am. Math. Soc.* **262**, 579–592 (1980)
26. Watson, G.N.: A Treatise on the Theory of Bessel Functions. Cambridge Mathematical Library, Cambridge University Press, Cambridge (1995). Reprint of the second (1944) edition
27. Yang, D., Yang, D., Hu, G.: The Hardy Space H^1 with Non-doubling Measures and Their Applications. Lecture Notes in Mathematics, vol. 2084. Springer, Cham (2013)

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