Several results on compact metrizable spaces in $\mathbf{ZF}$

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Abstract

In the absence of the axiom of choice, the set-theoretic status of many natural statements about metrizable compact spaces is investigated. Some of the statements are provable in $\mathbf{ZF}$, some are shown to be independent of $\mathbf{ZF}$. For independence results, distinct models of $\mathbf{ZF}$ and permutation models of $\mathbf{ZFA}$ with transfer theorems of Pinus are applied. New symmetric models of $\mathbf{ZF}$ are constructed in each of which the power set of $\mathbb{R}$ is well-orderable, the Continuum Hypothesis is satisfied but a denumerable family of non-empty finite sets can fail to have a choice function, and a compact metrizable space need not be embeddable into the Tychonoff cube $[0, 1]^{\mathbb{R}}$.

Keywords Weak forms of the Axiom of Choice · Metrizable space · Totally bounded metric · Compact space · Permutation model · Symmetric model

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1 Preliminaries

1.1 The set-theoretic framework

In this paper, the intended context for reasoning and statements of theorems is the Zermelo–Fraenkel set theory $\mathbf{ZF}$ without the axiom of choice $\mathbf{AC}$. The system $\mathbf{ZF} + \mathbf{AC}$ is denoted by $\mathbf{ZFC}$. We recommend [32,33] as a good introduction to $\mathbf{ZF}$. To stress the fact that a result is proved in $\mathbf{ZF}$ or in $\mathbf{ZF} + \mathbf{A}$ (where $\mathbf{A}$ is a statement independent of $\mathbf{ZF}$), we shall write at the beginning of the statements of the theorems and propositions $(\mathbf{ZF})$ or $(\mathbf{ZF} + \mathbf{A})$, respectively. Apart from models of $\mathbf{ZF}$, we refer to some models of $\mathbf{ZFA}$ (or $\mathbf{ZF}^0$ in [15]), that is, we refer also to $\mathbf{ZF}$ with an infinite set of atoms (see [15,20,21]). Our theorems proved here in $\mathbf{ZF}$ are also provable in $\mathbf{ZFA}$; however, we also mention some theorems of $\mathbf{ZF}$ that are not theorems of $\mathbf{ZFA}$.

A *well-ordered cardinal number* is an initial ordinal number, i.e., an ordinal which is not equipotent to any of its elements. Every well-orderable set is equipotent to a unique well-ordered cardinal number, called the cardinality of the well-orderable set. By transfinite recursion over ordinals α , we define:

$$\omega_0 = \omega \text{ (the set of all finite ordinal numbers);}$$

$$\omega_{\alpha+1} = H(\omega_\alpha);$$

$$\omega_\alpha = \sup\{\omega_\beta : \beta < \alpha\} \left(= \bigcup\{\omega_\beta : \beta < \alpha\} \right) \text{ if } \alpha \text{ is a non-zero limit ordinal,}$$

where, for a set A , $H(A)$ is the Hartogs' number of A , i.e., the least ordinal α which is not equipotent to a subset of A . For each ordinal number α , ω_α is an infinite well-ordered cardinal number and, as it is customary, it is denoted by $\aleph_\alpha$. One usually uses $\aleph_\alpha$ when referring to the cardinality of an infinite well-orderable set, and ω_α when referring to the order-type of an infinite well-ordered set. Every well-ordered cardinal number is either a finite ordinal number or an $\aleph_\alpha$ for some ordinal α .

As usual, if $n \in \omega$, then $n + 1 = n \cup \{n\}$. Members of the set $\mathbb{N} = \omega \setminus \{0\}$ are called natural numbers. The power set of a set X is denoted by $\mathcal{P}(X)$. A set X is called *countable* if X is equipotent to a subset of ω . A set X is called *uncountable* if X is not countable. A set X is *finite* if X is equipotent to an element of ω . An *infinite* set is a set which is not finite. An infinite countable set is called *denumerable*. If X is a set and κ is a non-zero well-ordered cardinal number, then $[X]^\kappa$ is the family of all subsets of X equipotent to κ , $[X]^{\leq \kappa}$ is the collection of all subsets of X equipotent to subsets of κ , and $[X]^{< \kappa}$ is the family of all subsets of X equipotent to a (well-ordered) cardinal number in κ .

For sets X and Y ,

- $|X| \leq |Y|$ means that X is equipotent to a subset of Y ;
- $|X| = |Y|$ means that X is equipotent to Y ; and
- $|X| < |Y|$ means that $|X| \leq |Y|$ and $|X| \neq |Y|$.

The set of all real numbers is denoted by $\mathbb{R}$ and, if it is not stated otherwise, $\mathbb{R}$ and every subspace of $\mathbb{R}$ are considered with the usual topology and with the metric induced by the standard absolute value on $\mathbb{R}$.

1.2 Notation and basic definitions

In this subsection, we establish notation and recall several basic definitions.

Let $\mathbf{X} = \langle X, d \rangle$ be a metric space. The d -ball with centre $x \in X$ and radius $r \in (0, +\infty)$ is the set

$$B_d(x, r) = \{y \in X : d(x, y) < r\}.$$

The collection

$$\tau(d) = \{V \subseteq X : (\forall x \in V)(\exists \varepsilon > 0) B_d(x, \varepsilon) \subseteq V\}$$

is the topology on X induced by d . For a set $A \subseteq X$, let $\delta_d(A) = 0$ if $A = \emptyset$, and let $\delta_d(A) = \sup\{d(x, y) : x, y \in A\}$ if $A \neq \emptyset$. Then $\delta_d(A)$ is the diameter of A in $\mathbf{X}$.

Definition 1 Let $\mathbf{X} = \langle X, d \rangle$ be a metric space.

- (i) Given a real number $\varepsilon > 0$, a subset D of X is called ε -dense or an ε -net in $\mathbf{X}$ if $X = \bigcup_{x \in D} B_d(x, \varepsilon)$.
- (ii) $\mathbf{X}$ is called *totally bounded* if, for every real number $\varepsilon > 0$, there exists a finite ε -net in $\mathbf{X}$.
- (iii) $\mathbf{X}$ is called *strongly totally bounded* if it admits a sequence $(D_n)_{n \in \mathbb{N}}$ such that, for every $n \in \mathbb{N}$, D_n is a finite $\frac{1}{n}$ -net in $\mathbf{X}$.
- (iv) (Cf. [24].) d is called *strongly totally bounded* if $\mathbf{X}$ is strongly totally bounded.

Remark 1 Every strongly totally bounded metric space is evidently totally bounded. However, it was shown in [24, Proposition 8] that the sentence “Every totally bounded metric space is strongly totally bounded” is not a theorem of ZF.

Definition 2 Let $\mathbf{X} = \langle X, \tau \rangle$ be a topological space and let $Y \subseteq X$. Suppose that $\mathcal{B}$ is a base of $\mathbf{X}$.

- (i) The closure of Y in $\mathbf{X}$ is denoted by $\text{cl}_\tau(Y)$ or $\text{cl}_\mathbf{X}(Y)$.
- (ii) $\tau|_Y = \{U \cap Y : U \in \tau\}$. $\mathbf{Y} = \langle Y, \tau|_Y \rangle$ is the subspace of $\mathbf{X}$ with the underlying set Y .
- (iii) $\mathcal{B}_Y = \{U \cap Y : U \in \mathcal{B}\}$.

Clearly, in Definition 2 (iii), $\mathcal{B}_Y$ is a base of $\mathbf{Y}$. In Sect. 5, it is shown that $\mathcal{B}_Y$ need not be equipotent to a subset of $\mathcal{B}$.

In the sequel, boldface letters will denote metric or topological spaces (called spaces in abbreviation) and lightface letters will denote their underlying sets.

Definition 3 A collection $\mathcal{U}$ of subsets of a space $\mathbf{X}$ is called:

- (i) *locally finite* if every point of X has a neighbourhood meeting only finitely many members of $\mathcal{U}$;
- (ii) *point-finite* if every point of X belongs to at most finitely many members of $\mathcal{U}$;
- (iii) σ -*locally finite* (respectively, σ -*point-finite*) if $\mathcal{U}$ is a countable union of locally finite (respectively, point-finite) subfamilies.

Definition 4 A space $\mathbf{X}$ is called:

- (i) *first-countable* if every point of X has a countable base of neighbourhoods;
- (ii) *second-countable* if $\mathbf{X}$ has a countable base.

Given a collection $\{X_j : j \in J\}$ of sets, for every $i \in J$, we denote by π_i the projection $\pi_i : \prod_{j \in J} X_j \rightarrow X_i$ defined by $\pi_i(x) = x(i)$ for each $x \in \prod_{j \in J} X_j$. If τ_j is a topology on X_j , then $\mathbf{X} = \prod_{j \in J} \mathbf{X}_j$ denotes the Tychonoff product of the topological spaces $\mathbf{X}_j = \langle X_j, \tau_j \rangle$ with $j \in J$. If $\mathbf{X}_j = \mathbf{X}$ for every $j \in J$, then $\mathbf{X}^J = \prod_{j \in J} \mathbf{X}_j$. As in [8], for an infinite set J and the unit interval $[0, 1]$ of $\mathbb{R}$, the cube $[0, 1]^J$ is called the *Tychonoff cube*. If J is denumerable, then the Tychonoff cube $[0, 1]^J$ is called the *Hilbert cube*. In [12], all Tychonoff cubes are called Hilbert cubes. In [42], Tychonoff cubes are called cubes.

We recall that if $\prod_{j \in J} X_j \neq \emptyset$, then it is said that the family $\{X_j : j \in J\}$ has a choice function, and every element of $\prod_{j \in J} X_j$ is called a *choice function* of the family $\{X_j : j \in J\}$. A *multiple choice function* of $\{X_j : j \in J\}$ is a function $f \in \prod_{j \in J} \mathcal{P}(X_j)$ such that, for every $j \in J$, $f(j)$ is a non-empty finite subset of X_j . A set f is called a *partial (multiple) choice function* of $\{X_j : j \in J\}$ if there exists an infinite subset I of J such that f is a (multiple) choice function of $\{X_j : j \in I\}$. Given a non-indexed family $\mathcal{A}$, we treat $\mathcal{A}$ as an indexed family $\mathcal{A} = \{x : x \in \mathcal{A}\}$ to speak about a (partial) choice function and a (partial) multiple choice function of $\mathcal{A}$.

Let $\{X_j : j \in J\}$ be a disjoint family of sets, that is, $X_i \cap X_j = \emptyset$ for each pair i, j of distinct elements of J . If τ_j is a topology on X_j for every $j \in J$, then $\bigoplus_{j \in J} \mathbf{X}_j$ denotes the direct sum of the spaces $\mathbf{X}_j = \langle X_j, \tau_j \rangle$ with $j \in J$.

Definition 5 (Cf. [2,26,34].)

- (i) A space $\mathbf{X}$ is said to be *Loeb* (respectively, *weakly Loeb*) if the family of all non-empty closed subsets of $\mathbf{X}$ has a choice function (respectively, a multiple choice function).
- (ii) If $\mathbf{X}$ is a (weakly) Loeb space, then every (multiple) choice function of the family of all non-empty closed subsets of $\mathbf{X}$ is called a (*weak*) *Loeb function* of $\mathbf{X}$.

Other topological notions used in this article but not defined here are standard. They can be found, for instance, in [8,42].

Definition 6 A set X is called:

- (i) a *cuf set* if X is expressible as a countable union of finite sets (cf. [5,6,19] and [16, Form 419]);
- (ii) *Dedekind-finite* if X is not equipotent to a proper subset of itself (cf. [15, Note 94], [12, Definition 4.1] and [20, Definition 2.6]); *Dedekind-infinite* if X is not Dedekind-finite (equivalently, if there exists an injection $f : \omega \rightarrow X$) (cf. [15, Note 94] and [12, Definition 2.13]);
- (iii) *amorphous* if X is infinite and there does not exist a partition of X into two infinite sets (cf. [15, Note 57], [20, p. 52] and [12, E. 11 in Section 4.1]).

Definition 7 (Cf. [31].) A topological space $\langle X, \tau \rangle$ is called a *cuf space* if X is a cuf set.

1.3 The list of weaker forms of AC

In this subsection, for readers' convenience, we define and denote most of the weaker forms of **AC** used directly in this paper. If a form is not defined in the forthcoming sections, its definition can be found in this subsection. For the known forms given in [15,16] or [12], we quote in their statements the form number under which they are recorded in [15] (or in [16] if they do not appear in [15]) and, if possible, we refer to their definitions in [12].

- Definition 8**
1. **AC_{fin}** ([15, Form 62]): Every non-empty family of non-empty finite sets has a choice function.
 2. **AC_{WO}** ([15, Form 60]): Every non-empty family of non-empty well-orderable sets has a choice function.
 3. **CAC** ([15, Form 8], [12, Definition 2.5]): Every denumerable family of non-empty sets has a choice function.
 4. **CAC($\mathbb{R}$)** ([15, Form 94], [12, Definition 2.9(1)]): Every denumerable family of non-empty subsets of $\mathbb{R}$ has a choice function.
 5. **CAC _{$\Delta\omega$} ($\mathbb{R}$)** (Cf. [29]): For every family $\mathcal{A} = \{A_n : n \in \omega\}$ such that, for every $n \in \omega$ and all $x, y \in A_n$, $\emptyset \neq A_n \subseteq \mathcal{P}(\omega) \setminus \{\emptyset\}$ and $x \Delta y \in [\omega]^{<\omega}$ (Δ denotes the operation of symmetric difference between sets), there exists a choice function of $\mathcal{A}$.
 6. **IDI** ([15, Form 9], [12, Definition 2.13(ii)]): Every Dedekind-finite set is finite.
 7. **IDI($\mathbb{R}$)** ([15, Form 13], [12, Definition 2.13(2)]): Every Dedekind-finite subset of $\mathbb{R}$ is finite.
 8. **WoAm** ([15, Form 133]): Every set is either well-orderable or has an amorphous subset.
 9. **Part($\mathbb{R}$)**: Every partition of $\mathbb{R}$ is of size $\leq |\mathbb{R}|$.
 10. **WO($\mathbb{R}$)** ([15, Form 79]): $\mathbb{R}$ is well-orderable.
 11. **WO($\mathcal{P}(\mathbb{R})$)** ([15, Form 130]): $\mathcal{P}(\mathbb{R})$ is well-orderable.
 12. **CAC_{fin}** ([15, Form 10], [12, Definition 2.9(3)]): Every denumerable family of non-empty finite sets has a choice function.
 13. For a fixed $n \in \omega \setminus \{0, 1\}$, **CAC_n** ([15, Form 288(n)]): Every denumerable family of n -element sets has a choice function.
 14. **CAC_{WO}**: Every denumerable family of non-empty well-orderable sets has a choice function.
 15. **CMC** ([15, Form 126], [12, Definition 2.10]): Every denumerable family of non-empty sets has a multiple choice function.
 16. **CMC _{ω}** ([15, Form 350]): Every denumerable family of denumerable sets has a multiple choice function.
 17. **CUC** ([15, Form 31], [12, Definition 3.2(1)]): Every countable union of countable sets is countable.
 18. **CUC_{fin}** (Form [10 A] of [15], [12, Definition 3.2(3)]): Every countable union of finite sets is countable.
 19. **UT($\aleph_0$, *cuf*, *cuf*)** ([16, Form 419]): Every countable union of cuf sets is a cuf set. (Cf. also [6].)

20. **UT**($\aleph_0, \aleph_0, cuf$) ([16, Form 420]): Every countable union of countable sets is a cuf set. (Cf. also [6].)
21. **BPI** ([15, Form 14], [12, Definition 2.15(1)]): Every Boolean algebra has a prime ideal.
22. **DC** ([15, Form 43], [12, Definition 2.11(1)]): For every non-empty set X and every binary relation ρ on X if, for each $x \in X$ there exists $y \in X$ such that $x \rho y$, then there exists a sequence $(x_n)_{n \in \mathbb{N}}$ of points of X such that $x_n \rho x_{n+1}$ for each $n \in \mathbb{N}$.

Remark 2 The following are well-known facts in **ZF**:

- (i) **CAC**_{fin} and **CUC**_{fin} are both equivalent to the sentence: Every infinite well-ordered family of non-empty finite sets has a partial choice function (see Form [10 O] of [15] and [12, Diagram 3.4, p. 23]). Moreover, **CAC**_{fin} is equivalent to Form [10 E] of [15], that is, to the sentence: Every denumerable family of non-empty finite sets has a partial choice function. It is known that **IDI** implies **CAC**_{fin} and this implication is not reversible in **ZF** (cf. [12, pp. 324–324]).
- (ii) **CAC** is equivalent to the sentence: Every denumerable family of non-empty sets has a partial choice function (see Form [8 A] of [15]).
- (iii) **BPI** is equivalent to the statement that all products of compact Hausdorff spaces are compact (see Form [14 J] of [15] and [12, Theorem 4.37]).
- (iv) **CMC** _{ω} is equivalent to the following sentence: Every denumerable family of denumerable sets has a multiple choice function.

Remark 3 (a) It was proved in [19] that the following implications are true in **ZF** and none of the implications are reversible in **ZF**:

$$\mathbf{CMC} \rightarrow \mathbf{UT}(\aleph_0, cuf, cuf) \rightarrow \mathbf{CMC}_\omega \rightarrow \mathbf{vDCP}(\aleph_0),$$

where **vDCP**($\aleph_0$) is van Douwen's choice principle: "Every denumerable family $\{ \langle A_n, \leq_n \rangle : n \in \omega \}$ of linearly ordered sets, each of which is order-isomorphic to the set $\langle \mathbb{Z}, \leq \rangle$ of integers with the standard linear order $\leq$, has a choice function" (cf. [7], [12, p. 79], [15, Form 119]).

- (b) Clearly, **UT**($\aleph_0, cuf, cuf$) implies **UT**($\aleph_0, \aleph_0, cuf$). In [6, proof to Theorem 3.3] a model of **ZFA** was shown in which **UT**($\aleph_0, \aleph_0, cuf$) is true and **UT**($\aleph_0, cuf, cuf$) is false.
- (c) It was proved in [31] that the following equivalences hold in **ZF**:

- (i) **UT**($\aleph_0, cuf, cuf$) is equivalent to the sentence: Every countable product of one-point Hausdorff compactifications of infinite discrete cuf spaces is metrizable (equivalently, first-countable).
- (ii) **UT**($\aleph_0, \aleph_0, cuf$) is equivalent to the sentence: Every countable product of one-point Hausdorff compactifications of denumerable discrete spaces is metrizable (equivalently, first-countable).

Let us pass to definitions of forms concerning metric and metrizable spaces.

Definition 9 1. **CAC**($\mathbb{R}, C$): For every disjoint family $\mathcal{A} = \{A_n : n \in \mathbb{N}\}$ of non-empty subsets of $\mathbb{R}$, if there exists a family $\{d_n : n \in \mathbb{N}\}$ of metrics such that, for every $n \in \mathbb{N}$, $\langle A_n, d_n \rangle$ is a compact metric space, then $\mathcal{A}$ has a choice function.

2. **CAC**(C, M): If $\{\langle X_n, d_n \rangle : n \in \omega\}$ is a family of non-empty compact metric spaces, then the family $\{X_n : n \in \omega\}$ has a choice function.
3. **M**(TB, WO): For every totally bounded metric space $\langle X, d \rangle$, the set X is well-orderable.
4. **M**(TB, S): Every totally bounded metric space is separable.
5. **M**(TB, STB): Every totally bounded metric space is strongly totally bounded.
6. **M**(IC, DI): Every infinite compact metrizable space is Dedekind-infinite.
7. **MP** ([15, Form 383]): Every metrizable space is paracompact.
8. **M**($\sigma - p.f.$) ([15, Form 233]): Every metrizable space has a σ -point-finite base.
9. **M**($\sigma - l.f.$) (Form [232 B] of [15]): Every metrizable space has a σ -locally finite base.

Definition 10 The following forms will be called *forms of type* $\mathbf{M}(C, \square)$.

1. **M**(C, S): Every compact metrizable space is separable.
2. **M**($C, 2$): Every compact metrizable space is second-countable.
3. **M**(C, STB): Every compact metric space is strongly totally bounded.
4. **M**(C, L): Every compact metrizable space is Loeb.
5. **M**(C, WO): Every compact metrizable space is well-orderable.
6. **M**($C, \hookrightarrow [0, 1]^{\mathbb{N}}$): Every compact metrizable space is embeddable in the Hilbert cube $[0, 1]^{\mathbb{N}}$.
7. **M**($C, \hookrightarrow [0, 1]^{\mathbb{R}}$): Every compact metrizable space is embeddable in the Tychonoff cube $[0, 1]^{\mathbb{R}}$.
8. **M**($C, \leq |\mathbb{R}|$): Every compact metrizable space is of size $\leq |\mathbb{R}|$.
9. **M**($C, W(\mathbb{R})$): For every infinite compact metrizable space $\langle X, \tau \rangle$, τ and $\mathbb{R}$ are equipotent.
10. **M**($C, B(\mathbb{R})$): Every compact metrizable space has a base of size $\leq |\mathbb{R}|$.
11. **M**($C, |\mathcal{B}_Y| \leq |\mathcal{B}|$): For every compact metrizable space $\mathbf{X}$, every base $\mathcal{B}$ of $\mathbf{X}$ and every compact subspace $\mathbf{Y}$ of $\mathbf{X}$, $|\mathcal{B}_Y| \leq |\mathcal{B}|$.
12. **M**($[0, 1], |\mathcal{B}_Y| \leq |\mathcal{B}|$): For every base $\mathcal{B}$ of the interval $[0, 1]$ with the usual topology and every compact subspace $\mathbf{Y}$ of $[0, 1]$, $|\mathcal{B}_Y| \leq |\mathcal{B}|$.
13. **M**($C, \sigma - l.f.$): Every compact metrizable space has a σ -locally finite base.
14. **M**($C, \sigma - p.f.$): Every compact metrizable space has a σ -point-finite base.

The notation of type $\mathbf{M}(C, \square)$ was introduced in [22] and was also used in [23], but not all forms from the definition above were defined in [22, 23]. The forms $\mathbf{M}(C, L)$ and $\mathbf{M}(C, WO)$ were denoted by CML and $CMWO$ in [27]. Most forms from Definition 10 are new here. That the new forms $\mathbf{M}(C, \hookrightarrow [0, 1]^{\mathbb{N}})$, $\mathbf{M}(C, \hookrightarrow [0, 1]^{\mathbb{R}})$, $\mathbf{M}(C, W(\mathbb{R}))$, $\mathbf{M}(C, B(\mathbb{R}))$, $\mathbf{M}(C, |\mathcal{B}_Y| \leq |\mathcal{B}|)$ and $\mathbf{M}([0, 1], |\mathcal{B}_Y| \leq |\mathcal{B}|)$ are all important is shown in Sect. 4.

Apart from the forms defined above, we also refer to the following forms that are not weaker than **AC** in **ZF**:

- Definition 11**
1. **LW** ([15, Form 90]): For every linearly ordered set $\langle X, \leq \rangle$, the set X is well-orderable.
 2. **CH** (the *Continuum Hypothesis*): $2^{\aleph_0} = \aleph_1$.

Remark 4 It is known that **AC** and **LW** are equivalent in **ZF**; however, **LW** does not imply **AC** in **ZFA** (see [20, Theorems 9.1 and 9.2]).

2 Introduction

2.1 The content of the article in brief

Although mathematicians are aware that a lot of theorems of **ZFC** that are included in standard textbooks on general topology (e.g., in [8,42]) may fail in **ZF** and many amazing disasters in topology in **ZF** have been discovered, new non-trivial results showing significant differences between truth values in **ZFC** and in **ZF** of some given propositions can be still surprising. In this paper, we show new results concerning forms of type $\mathbf{M}(C, \square)$ in **ZF**. The main aim of our work is to establish in **ZF** the set-theoretic strength of the forms of type $\mathbf{M}(C, \square)$, as well as to clarify possible relationships between those forms and relevant ones. Taking care of the readability of the article, in the forthcoming Sects. 2.2–2.4, we include some known facts and few definitions for future references. In particular, in Sect. 2.4, we give definitions of permutation models (also called Fraenkel–Mostowski models) and formulate a version of a transfer theorem due to Pincus [38], called here the *Pincus Transfer Theorem* (cf. Theorem 7), which will be useful for the transfer of certain **ZFA**-independence results to **ZF**. The main new results of the paper are included in Sects. 3–5. Section 6 contains a list of open problems that suggest a direction for future research in this field.

In Sect. 3, we construct a (infinite) class of new symmetric **ZF**-models in each of which the conjunction $\mathbf{CH} \wedge \mathbf{WO}(\mathcal{P}(\mathbb{R})) \wedge \neg \mathbf{CAC}_{fin}$ is true (see Theorem 8).

In Sect. 4, we investigate relationships between the forms $\mathbf{M}(TB, WO)$, $\mathbf{M}(TB, S)$, $\mathbf{M}(IC, DI)$ and $\mathbf{M}(C, S)$. Among other results of Sect. 4, by using appropriate permutation models and the Pincus Transfer Theorem, we prove that the conjunctions $\mathbf{BPI} \wedge \mathbf{M}(IC, DI) \wedge \neg \mathbf{IDI}$, $(\neg \mathbf{BPI}) \wedge \mathbf{M}(IC, DI) \wedge \neg \mathbf{IDI}$ and $\mathbf{UT}(\aleph_0, \aleph_0, cuf) \wedge \neg \mathbf{M}(IC, DI)$ have **ZF**-models (see Theorems 11, 12 and 13, respectively). We deduce that $\mathbf{UT}(\aleph_0, \aleph_0, cuf) \wedge \neg \mathbf{M}(C, S)$ has a **ZF**-model (see Corollary 5). The latter result provides a *partial answer* to the open problem of whether or not **CUC** implies $\mathbf{M}(C, S)$ in **ZF**. The status of the reverse implication is also unknown (see the discussion in Remark 14(a)). Taking the opportunity, we also fill a gap in [15,16] by proving that **WoAm** implies **CUC** (see Proposition 8).

In Sect. 5, among a plethora of results, we show that $\mathbf{CAC}_{fin}$ implies neither $\mathbf{M}(C, S) \leq \mathbf{M}(C, \leq |\mathbb{R}|)$ in **ZF** (see Proposition 10), and $\mathbf{M}(C, S)$ is equivalent to each one of the conjunctions: $\mathbf{CAC}_{fin} \wedge \mathbf{M}(C, \sigma - l.f.)$, $\mathbf{CAC}_{fin} \wedge \mathbf{M}(C, STB)$ and $\mathbf{CAC}(\mathbb{R}, C) \wedge \mathbf{M}(C, \leq |\mathbb{R}|)$ (see Theorems 14 and 15, respectively). We deduce that $\mathbf{M}(C, \sigma - l.f.)$ is unprovable in **ZF** (see Remark 16). We also deduce that $\mathbf{M}(C, S)$ and $\mathbf{M}(C, \leq |\mathbb{R}|)$ are equivalent in every permutation model (see Corollary 6). Furthermore, we prove that $\mathbf{M}(C, S)$ and $\mathbf{M}(C, \leftrightarrow [0, 1]^{\mathbb{N}})$ are equivalent (see Theorem 16). We show that, surprisingly, $\mathbf{M}(C, |\mathcal{B}_Y| \leq |\mathcal{B}|)$ and $\mathbf{M}(C, B(\mathbb{R}))$ are independent of **ZF** (see Theorem 17). In Theorem 18, we show that $\mathbf{M}(C, B(\mathbb{R}))$ is equivalent to the conjunction $\mathbf{M}(C, \leftrightarrow [0, 1]^{\mathbb{R}}) \wedge \mathbf{Part}(\mathbb{R})$ and that, under the assumption of $\mathbf{CAC}(\mathbb{R})$, $\mathbf{M}(C, S)$, $\mathbf{M}(C, W(\mathbb{R}))$, and $\mathbf{M}(C, B(\mathbb{R}))$, are pairwise equivalent. The symmetric models of **ZF** + $\mathbf{WO}(\mathcal{P}(\mathbb{R}))$ + $\neg \mathbf{CAC}_{fin}$ constructed in the proof of Theorem 8 of Sect. 3 are applied to a proof that $\mathbf{Part}(\mathbb{R})$ does not imply $\mathbf{M}(C, \leftrightarrow [0, 1]^{\mathbb{R}})$ in **ZF** (see Theorem 19).

2.2 A list of several known theorems

We list below some known theorems for future references.

Theorem 1 (Cf. [36].) **CAC** implies $\mathbf{M}(TB, S)$ in **ZF**.

Theorem 2 (Cf. [23].) (**ZF**)

- (i) Let $\mathbf{X} = \langle X, d \rangle$ be an uncountable compact separable metric space. Then $|X| = |\mathbb{R}|$.
- (ii) $\mathbf{CAC}_{fin}$ follows from each of the statements: $\mathbf{M}(C, S)$, $\mathbf{M}(C, \leq |\mathbb{R}|)$ and “For every compact metric space $\langle X, d \rangle$, either $|X| \leq |\mathbb{R}|$ or $|\mathbb{R}| \leq |X|$ ”.

Theorem 3 (**ZF**)

- (a) ([28, Theorem 8].) The statements $\mathbf{M}(C, S)$, $\mathbf{CAC}(C, M)$ are equivalent.
- (b) ([28, Corollary 1(a)].) $\mathbf{CAC}(C, M)$ implies $\mathbf{CAC}_{fin}$.

Theorem 4 ([9, Corollary 4.8], Urysohn’s Metrization Theorem.) (**ZF**) If $\mathbf{X}$ is a second-countable T_3 -space, then $\mathbf{X}$ is metrizable.

Theorem 5 (Cf. [1,3,35,39].)

- (i) (**ZFC**) Every metrizable space has a σ -locally finite base.
- (ii) (**ZF**) If a T_1 -space $\mathbf{X}$ is regular and has a σ -locally finite base, then $\mathbf{X}$ is metrizable.

Remark 5 The fact that, in **ZFC**, a T_1 -space is metrizable if and only if it is regular and has a σ -locally finite base was originally proved by Nagata in [35], Smirnov in [39] and Bing in [1]. It was shown in [3] that it is provable in **ZF** that every regular T_1 -space which admits a σ -locally finite base is metrizable. It was established in [14] that $\mathbf{M}(\sigma - l.f.)$ is equivalent to $\mathbf{M}(\sigma - p.f.)$ and implies **MP**. Using similar arguments, one can prove that $\mathbf{M}(C, \sigma - l.f.)$ and $\mathbf{M}(C, \sigma - p.f.)$ are also equivalent in **ZF**. In [10], a model of **ZF** + **DC** was shown in which **MP** fails. In [4], a model of **ZF** + **BPI** was shown in which **MP** fails. This implies that, in each of the above-mentioned **ZF**-models constructed in [4,10], there exists a metrizable space which fails to have a σ -point-finite base. This means that $\mathbf{M}(\sigma - l.f.)$ is unprovable in **ZF**. In Sect. 4, it is clearly explained that $\mathbf{M}(C, \sigma - f.l.)$ is also unprovable in **ZF**.

Theorem 6 (**ZF**)

- (i) (Cf. [27].) A compact metrizable space is Loeb iff it is second-countable iff it is separable. In consequence, the statements $\mathbf{M}(C, L)$, $\mathbf{M}(C, S)$ and $\mathbf{M}(C, 2)$ are all equivalent.
- (ii) (Cf. [30].) If $\mathbf{X}$ is a compact second-countable and metrizable space, then $\mathbf{X}^\omega$ is compact and separable. In particular, the Hilbert cube $[0, 1]^\mathbb{N}$ is a compact, separable metrizable space.
- (iii) (Cf. [23].) **BPI** implies $\mathbf{M}(C, S)$ and $\mathbf{M}(C, \leq |\mathbb{R}|)$.

2.3 Frequently used metrics

Similarly to [31], we make use of the following idea several times in the sequel.

Suppose that $\mathcal{A} = \{A_n : n \in \mathbb{N}\}$ is a disjoint family of non-empty sets, $A = \bigcup \mathcal{A}$ and $\infty \notin A$. Let $X = A \cup \{\infty\}$. Suppose that $(\rho_n)_{n \in \mathbb{N}}$ is a sequence such that, for each $n \in \mathbb{N}$, ρ_n is a metric on A_n . Let $d_n(x, y) = \min\{\rho_n(x, y), \frac{1}{n}\}$ for all $x, y \in A_n$. We define a function $d : X \times X \rightarrow \mathbb{R}$ as follows:

$$(*) \quad d(x, y) = \begin{cases} 0 & \text{if } x = y; \\ \max\{\frac{1}{n}, \frac{1}{m}\} & \text{if } x \in A_n, y \in A_m \text{ and } n \neq m; \\ d_n(x, y) & \text{if } x, y \in A_n; \\ \frac{1}{n} & \text{if } x \in A_n \text{ and } y = \infty \text{ or } x = \infty \text{ and } y \in A_n. \end{cases}$$

Proposition 1 *The function d , defined by $(*)$, has the following properties:*

- (i) d is a metric on X (cf. [31]);
- (ii) if, for every $n \in \mathbb{N}$, the space $\langle A_n, \tau(\rho_n) \rangle$ is compact, then so is the space $\langle X, \tau(d) \rangle$ (cf. [31]);
- (iii) the space $\langle X, \tau(d) \rangle$ has a σ -locally finite base;
- (iv) if $\mathcal{A}$ does not have a choice function, the space $\langle X, \tau(d) \rangle$ is not separable.

Metrics defined by $(*)$ were used, for instance, in [26,27,31], as well as in several other papers not cited here.

2.4 Permutation models and the Pincus Transfer Theorem

Let us clarify definitions of the permutation models we deal with. We refer to [20, Chapter 4] and [21, Chapter 15, p. 251] for the basic terminology and facts concerning permutation models.

Suppose we are given a model $\mathcal{M}$ of $\mathbf{ZFA} + \mathbf{AC}$ with an infinite set A of all atoms of $\mathcal{M}$, and a group $\mathcal{G}$ of permutations of A . For a set $x \in \mathcal{M}$, we denote by $\text{TC}(x)$ the transitive closure of x in $\mathcal{M}$. Every permutation ϕ of A extends uniquely to an ϵ -automorphism (usually denoted also by ϕ) of $\mathcal{M}$. For $x \in \mathcal{M}$, we put:

$$\text{fix}_{\mathcal{G}}(x) = \{\phi \in \mathcal{G} : (\forall t \in x)\phi(t) = t\} \text{ and } \text{sym}_{\mathcal{G}}(x) = \{\phi \in \mathcal{G} : \phi(x) = x\}.$$

We refer the readers to [20, Chapter 4, pp. 46–47] for the definitions of the concepts of a *normal filter* and a *normal ideal*.

Definition 12 (i) The permutation model $\mathcal{N}$ determined by $\mathcal{M}$, $\mathcal{G}$ and a normal filter $\mathcal{F}$ of subgroups of $\mathcal{G}$ is defined by the equality:

$$\mathcal{N} = \{x \in \mathcal{M} : (\forall t \in \text{TC}(\{x\}))(\text{sym}_{\mathcal{G}}(t) \in \mathcal{F})\}.$$

- (ii) The permutation model $\mathcal{N}$ determined by $\mathcal{M}$, $\mathcal{G}$ and a normal ideal $\mathcal{I}$ of subsets of the set of all atoms of $\mathcal{M}$ is defined by the equality:

$$\mathcal{N} = \{x \in \mathcal{M} : (\forall t \in \text{TC}(\{x\}))(\exists E \in \mathcal{I})(\text{fix}_{\mathcal{G}}(E) \subseteq \text{sym}_{\mathcal{G}}(t))\}.$$

- (iii) (Cf. [20, p. 46] and [21, p. 251].) A permutation model (or, equivalently, a Fraenkel–Mostowski model) is a class $\mathcal{N}$ which can be defined by (i).

Remark 6 (a) Let $\mathcal{F}$ be a normal filter of subgroups of $\mathcal{G}$ and let $x \in \mathcal{M}$. If $\text{sym}_{\mathcal{G}}(x) \in \mathcal{F}$, then x is called *symmetric*. If every element of $\text{TC}(\{x\})$ is symmetric, then x is called *hereditarily symmetric* (cf. [20, p. 46] and [21, p. 251]).

- (b) Given a normal ideal $\mathcal{I}$ of subsets of the set A of atoms of $\mathcal{M}$, the filter $\mathcal{F}_{\mathcal{I}}$ of subgroups of $\mathcal{G}$ generated by $\{\text{fix}_{\mathcal{G}}(E) : E \in \mathcal{I}\}$ is a normal filter such that the permutation model determined by $\mathcal{M}$, $\mathcal{G}$ and $\mathcal{F}_{\mathcal{I}}$ coincides with the permutation model determined by $\mathcal{M}$, $\mathcal{G}$ and $\mathcal{I}$ (see [20, p. 47]). For $x \in \mathcal{M}$, a set $E \in \mathcal{I}$ such that $\text{fix}_{\mathcal{G}}(E) \subseteq \text{sym}_{\mathcal{G}}(x)$ is called a *support* of x .

In the forthcoming sections, we describe and apply several permutation models. For example, we apply the permutation model which appeared in [26, the proof of Theorem 2.5] and was also used in [27], the *Basic Fraenkel Model* (labeled as $\mathcal{N}1$ in [15]) and the *Mostowski Linearly Ordered Model* (labeled as $\mathcal{N}3$ in [15]). Let us give definitions of these models and recall some of their properties for future references.

Definition 13 (Cf. [26].) Let $\mathcal{M}$ be a model of $\mathbf{ZFA} + \mathbf{AC}$. Let A be the set of all atoms of $\mathcal{M}$ and let $\mathcal{I} = [A]^{<\omega}$. Assume that:

- (i) A is expressed as $\bigcup_{n \in \mathbb{N}} A_n$ where $\{A_n : n \in \mathbb{N}\}$ is a disjoint family such that, for every $n \in \mathbb{N}$,

$$A_n = \left\{ a_{n,x} : x \in S \left(0, \frac{1}{n} \right) \right\}$$

- and $S(0, \frac{1}{n})$ is the circle of the Euclidean plane $\langle \mathbb{R}^2, \rho_e \rangle$ of radius $\frac{1}{n}$, centered at 0;
(ii) $\mathcal{G}$ is the group of all permutations of A that rotate the A_n 's by an angle $\theta_n \in \mathbb{R}$.

Then the permutation model $\mathcal{N}_{cr}$ determined by $\mathcal{M}$, $\mathcal{G}$ and the normal ideal $\mathcal{I}$ will be called the *concentric circles permutation model*.

Remark 7 We need to recall some properties of $\mathcal{N}_{cr}$ for applications in this paper. Let us use the notation from Definition 13. In [26, proof of Theorem 2.6], it was proved that $\{A_n : n \in \mathbb{N}\}$ does not have a multiple choice function in $\mathcal{N}_{cr}$. In [27, proof of Theorem 3.5], it was proved that **IDI** holds in $\mathcal{N}_{cr}$, so **CAC**_{fin} also holds in $\mathcal{N}_{cr}$ (see Remark 2(i)).

Definition 14 (Cf. [15, p. 176] and [20, Section 4.3].) Let $\mathcal{M}$ be a model of $\mathbf{ZFA} + \mathbf{AC}$. Let A be the set of all atoms of $\mathcal{M}$ and let $\mathcal{I} = [A]^{<\omega}$. Assume that:

- (i) A is a denumerable set;
(ii) $\mathcal{G}$ is the group of all permutations of A .

Then the *Basic Fraenkel Model* $\mathcal{N}1$ is the permutation model determined by $\mathcal{M}$, $\mathcal{G}$ and $\mathcal{I}$.

Remark 8 It is known that, in $\mathcal{N}1$, the set A of all atoms is amorphous, so **IDI** fails (see [20, p. 52] and [15, pp. 176–177]). It is also known that **BPI** is false in $\mathcal{N}1$ but **CAC**_{fin} is true in $\mathcal{N}1$ (see [15, p. 177]).

Definition 15 (Cf. [15, p. 182] and [20, Section 4.6].) Let $\mathcal{M}$ be a model of **ZFA** + **AC**. Let A be the set of all atoms of $\mathcal{M}$ and let $\mathcal{I} = [A]^{<\omega}$. Assume that:

- (i) the set A is denumerable and there is a fixed ordering $\leq$ in A such that $\langle A, \leq \rangle$ is order isomorphic to the set of all rational numbers equipped with the standard linear order;
- (ii) $\mathcal{G}$ is the group of all order-automorphisms of $\langle A, \leq \rangle$.

Then the *Mostowski Linearly Ordered Model* $\mathcal{N}3$ is the permutation model determined by $\mathcal{M}$, $\mathcal{G}$ and $\mathcal{I}$.

Remark 9 It is known that the power set of the set of all atoms is Dedekind-finite in $\mathcal{N}3$, so **IDI** fails in $\mathcal{N}3$ (see [15, pp. 182–183]). However, **BPI** and **CAC**_{fin} are true in $\mathcal{N}3$ (see [15, p. 183]).

It is well known that, in any permutation model, the power set of any pure set (that is, a set with no atoms in its transitive closure) is well-orderable (see, e.g., [15, p. 176]). This can be deduced from the following helpful proposition:

Proposition 2 (Cf. [20, Item (4.2), p. 47].) *Let $\mathcal{N}$ be the permutation model determined by $\mathcal{M}$, $\mathcal{G}$ and a normal filter $\mathcal{F}$. For every $x \in \mathcal{N}$, x is well-orderable in $\mathcal{N}$ iff $\text{fix}_{\mathcal{G}}(x) \in \mathcal{F}$.*

Remark 10 If a statement **A** is satisfied in a permutation model, then to show that there exists a **ZF**-model in which **A** is satisfied, we use transfer theorems due to Pincus (cf. [37,38]). Pincus transfer theorems, together with definitions of a *boundable formula* and an *injectively boundable formula* that are involved in the theorems, are included in [15, Note 103].

To our transfer results, we apply mainly the following fragment of the third theorem from [15, p. 286]:

Theorem 7 (The Pincus Transfer Theorem.) (Cf. [37,38] and [15, p. 286].) *Let Φ be a conjunction of statements that are either injectively boundable or **BPI**. If Φ has a permutation model, then Φ has a **ZF**-model.*

In the definition of an injectively boundable formula, the notion of injective cardinality is involved (see [37], [15, Item (3), p. 284]). Let us recall the definition of the latter notion.

Definition 16 For a set x , the *injective cardinality* of x , denoted by $|x|_-$, is the (well-ordered) cardinal number defined as follows:

$$|x|_- = \sup\{\kappa : \kappa \text{ is a well-ordered cardinal equipotent to a subset of } x\}.$$

Now, we are in a position to pass to the main body of the paper.

3 New symmetric models of ZF

Suppose that Φ is a form that is satisfied in a **ZFA**-model. Even if Φ fulfills the assumptions of the Pincus Transfer Theorem, it might be complicated to check it and to see well a **ZF**-model in which Φ is satisfied. This is why it is good to give a direct relatively simple description of a **ZF**-model satisfying Φ . By the proof of Theorem 8 below, we shall obtain an infinite class of symmetric models, each satisfying $\mathbf{CH} \wedge \mathbf{WO}(\mathcal{P}(\mathbb{R})) \wedge \neg \mathbf{CAC}_{fin}$. In Sect. 5, models of this class are applied to a proof that the conjunction $\mathbf{Part}(\mathbb{R}) \wedge \neg \mathbf{M}(C, \hookrightarrow [0, 1]^{\mathbb{R}})$ has a **ZF**-model (see the forthcoming Theorem 19).

For the convenience of readers, before embarking on the proof of Theorem 8, let us recall in brief the construction of symmetric extension models. Assume that M is a countable transitive model of **ZFC** and that $\langle \mathbb{P}, \leq \rangle \in M$ is a poset with a maximum element denoted by $\mathbf{1}_{\mathbb{P}}$; such a poset $\langle \mathbb{P}, \leq \rangle$ in M is said to be a notion of forcing. Let $M^{\mathbb{P}}$ be the (proper) class of all $\mathbb{P}$ -names, which are defined by transfinite recursion within M (cf. [32, Definitions 2.5, 2.6, pp. 188–189]). We will denote a $\mathbb{P}$ -name by $\dot{x}$ and, following the notation of [32, Definition 2.10, p. 190], for $x \in M$, we will denote by $\check{x}$ the canonical name $\{\langle \check{y}, \mathbf{1}_{\mathbb{P}} \rangle : y \in x\}$ for x .

If ϕ is an order-automorphism of $\langle \mathbb{P}, \leq \rangle$, then ϕ can be extended to an automorphism $\tilde{\phi}$ of $M^{\mathbb{P}}$ defined by recursion,

$$\tilde{\phi}(\dot{x}) = \{\langle \tilde{\phi}(\dot{y}), \phi(p) \rangle : \langle \dot{y}, p \rangle \in \dot{x}\}.$$

For every $x \in M$, $\tilde{\phi}(\check{x}) = \check{x}$. We shall henceforth use ϕ to denote both the automorphism of $\langle \mathbb{P}, \leq \rangle$ and the automorphism $\tilde{\phi}$ of the $\mathbb{P}$ -names.

Let $\mathcal{G}$ be a group of order-automorphisms of $\langle \mathbb{P}, \leq \rangle$, and also let Γ be a normal filter on $\mathcal{G}$, that is, Γ is a filter of subgroups of $\mathcal{G}$ closed under conjugation (i.e., for all $\phi \in \mathcal{G}$ and $H \in \Gamma$, $\phi H \phi^{-1} \in \Gamma$). A $\mathbb{P}$ -name $\dot{x}$ is called Γ -symmetric if $\text{sym}_{\mathcal{G}}(\dot{x}) \in \Gamma$, where $\text{sym}_{\mathcal{G}}(\dot{x})$ is the stabilizer of the name $\dot{x}$, i.e. the subgroup $\{\phi \in \mathcal{G} : \phi(\dot{x}) = \dot{x}\}$ of $\mathcal{G}$. $\dot{x}$ is called *hereditarily* Γ -symmetric if $\dot{x}$ is Γ -symmetric and, for every $\langle \dot{y}, p \rangle \in \dot{x}$, $\dot{y}$ is hereditarily Γ -symmetric. The class of hereditarily Γ -symmetric $\mathbb{P}$ -names (which, in view of the above, is defined by transfinite recursion over the rank of $\dot{x}$) is denoted by HS^{Γ} .

Let G be a $\mathbb{P}$ -generic filter over M , and also let

$$N = \{\dot{x}_G : \dot{x} \in \text{HS}^{\Gamma}\},$$

where $\dot{x}_G$ denotes the value of the name $\dot{x}$ by G , i.e. $\dot{x}_G = \{\dot{y}_G : \exists p \in G (\langle \dot{y}, p \rangle \in \dot{x})\}$. Then N is a transitive model of **ZF** and $M \subseteq N \subseteq M[G]$ (cf. [20, Section 5.2, p. 64]), where $M[G] = \{\dot{x}_G : \dot{x} \in M^{\mathbb{P}}\}$ is the generic extension model of M . N is called a *symmetric extension* of M generated by Γ .

In the proof of Theorem 8 below, we will write “**1**” instead of “ $\mathbf{1}_{\mathbb{P}}$ ”, “(hereditarily) symmetric” instead of “(hereditarily) Γ -symmetric”, and “HS” instead of “ HS^{Γ} ”.

Theorem 8 *Let $n, \ell \in \omega \setminus \{0, 1\}$. There is a symmetric model $N_{n, \ell}$ of **ZF** such that*

$$N_{n, \ell} \models \forall m \in n (2^{\aleph_m} = \aleph_{m+1}) \wedge \neg \mathbf{CAC}_{\ell}.$$

Hence, it is also the case that

$$N_{n,\ell} \models \mathbf{CH} \wedge \mathbf{WO}(\mathcal{P}(\mathbb{R})) \wedge \neg \mathbf{CAC}_{fin}.$$

Proof By [32, Theorem 6.18, p. 216], we may fix a countable transitive model M of $\mathbf{ZFC} + \forall m \in n(2^{\aleph_m} = \aleph_{m+1})$. Our plan is to construct a symmetric extension model $N_{n,\ell}$ of M with the required properties.

We use as our notion of forcing the set $\mathbb{P} = \text{Fn}(\omega \times \ell \times \omega_n \times \omega_n, 2, \omega_n)$ of all partial functions p with $|p| < \aleph_n$, $\text{dom}(p) \subseteq \omega \times \ell \times \omega_n \times \omega_n$ and $\text{ran}(p) \subseteq 2 = \{0, 1\}$, partially ordered by reverse inclusion, i.e., for $p, q \in \mathbb{P}$, $p \leq q$ if and only if $p \supseteq q$. The poset $\langle \mathbb{P}, \leq \rangle$ has the empty function as its maximum element, which we denote by $\mathbf{1}$. Furthermore, since ω_n is a regular cardinal, it follows from [32, Lemma 6.13, p. 214] that $\langle \mathbb{P}, \leq \rangle$ is an ω_n -closed poset. Therefore, by [32, Theorem 6.14, p. 214], forcing with $\langle \mathbb{P}, \leq \rangle$ adds no new subsets of ω_m for $m \in n$, and hence it adds no new reals or sets of reals, but it does add new subsets of ω_n . Moreover, by [32, Corollary 6.15, p. 215], we have that $\langle \mathbb{P}, \leq \rangle$ preserves cofinalities $\leq \omega_n$, and hence cardinals $\leq \omega_n$.

Let G be a $\mathbb{P}$ -generic filter over M , and let $M[G]$ be the corresponding generic extension model of M . By [32, Theorem 4.2, p. 201], $\mathbf{AC}$ is true $M[G]$. In view of the observations of the previous paragraph, for every model N with $M \subseteq N \subseteq M[G]$, we have the following:

$$N \models \forall m \in n(2^{\aleph_m} = \aleph_{m+1}).$$

In $M[G]$, for $k \in \omega$, $t \in \ell$, and $i \in \omega_n$, we define the following sets, together with their canonical names:

1. $a_{k,t,i} = \{j \in \omega_n : \exists p \in G(p(k, t, i, j) = 1)\}$,
 $\dot{a}_{k,t,i} = \{\langle \check{j}, p \rangle : j \in \omega_n \wedge p \in \mathbb{P} \wedge p(k, t, i, j) = 1\}$.
2. $A_{k,t} = \{a_{k,t,i} : i \in \omega_n\}$,
 $\dot{A}_{k,t} = \{\langle \dot{a}_{k,t,i}, \mathbf{1} \rangle : i \in \omega_n\}$.
3. $A_k = \{A_{k,s} : s \in \ell\}$,
 $\dot{A}_k = \{\langle \dot{A}_{k,s}, \mathbf{1} \rangle : s \in \ell\}$.
4. $\mathcal{A} = \{A_k : k \in \omega\}$,
 $\dot{\mathcal{A}} = \{\langle \dot{A}_k, \mathbf{1} \rangle : k \in \omega\}$.

Now, every permutation ϕ of $\omega \times \ell \times \omega_n$ induces an order-automorphism of $\langle \mathbb{P}, \leq \rangle$ as follows: for every $p \in \mathbb{P}$,

$$\begin{aligned} \text{dom } \phi(p) &= \{\langle \phi(k, t, i), j \rangle : \langle k, t, i, j \rangle \in \text{dom}(p)\}, \\ \phi(p)(\phi(k, t, i), j) &= p(k, t, i, j). \end{aligned} \tag{1}$$

Let $\mathcal{G}$ be the group of all order-automorphisms of $\langle \mathbb{P}, \leq \rangle$ induced (as in (1)) by all those permutations ϕ of $\omega \times \ell \times \omega_n$ which are defined as follows:

For every $k \in \omega$, let σ_k be a permutation of ℓ and also let η_k be a permutation of ω_n . We define

$$\phi(k, t, i) = \langle k, \sigma_k(t), \eta_k(i) \rangle \quad (2)$$

for all $\langle k, t, i \rangle \in \omega \times \ell \times \omega_n$. By (2), it follows that, for all $\phi \in \mathcal{G}$, $k \in \omega$, and $t \in \ell$,

$$\phi(\dot{A}_{k,t}) = \dot{A}_{k,\sigma_k(t)}, \quad (3)$$

and thus, for all $\phi \in \mathcal{G}$ and $k \in \omega$,

$$\phi(\dot{A}_k) = \dot{A}_k. \quad (4)$$

Hence, for every $\phi \in \mathcal{G}$,

$$\phi(\dot{\mathcal{A}}) = \dot{\mathcal{A}}. \quad (5)$$

For every $E \in [\omega \times \ell \times \omega_n]^{<\omega}$, we let $\text{fix}_{\mathcal{G}}(E) = \{\phi \in \mathcal{G} : \forall e \in E (\phi(e) = e)\}$ and we also let Γ be the filter of subgroups of $\mathcal{G}$ generated by the filter base $\{\text{fix}_{\mathcal{G}}(E) : E \in [\omega \times \ell \times \omega_n]^{<\omega}\}$. It is not hard to verify that Γ is a normal filter on $\mathcal{G}$, so we leave this to interested readers. If $\dot{x}$ is a $\mathbb{P}$ -name, $E \in [\omega \times \ell \times \omega_n]^{<\omega}$, and $\text{fix}_{\mathcal{G}}(E) \subseteq \text{sym}_{\mathcal{G}}(\dot{x})$, then we call E a *support* of $\dot{x}$. Let

$$N_{n,\ell} = \{\dot{x}_G : \dot{x} \in \text{HS}\}$$

be the symmetric extension model of M . By the definitions of Γ and HS, it is clear that every $\dot{x} \in \text{HS}$ has a support in the above sense.

In view of the observations at the beginning of the proof, we have

$$N_{n,\ell} \models \forall m \in n (2^{\aleph_m} = \aleph_{m+1}),$$

and thus

$$N_{n,\ell} \models \mathbf{CH} \wedge \mathbf{WO}(\mathcal{P}(\mathbb{R})).$$

□

Claim For $k \in \omega$, $t \in \ell$, and $i \in \omega_n$, the sets $a_{k,t,i}$, $A_{k,t}$, A_k , and $\mathcal{A}$, are all elements of $N_{n,\ell}$. Moreover, $\mathcal{A}$ is denumerable in $N_{n,\ell}$.

Proof Fix $k \in \omega$, $t \in \ell$, and $i \in \omega_n$. By the definition of $\mathcal{G}$, it easily follows that $E = \{\langle k, t, i \rangle\}$ is a support of $\dot{a}_{k,t,i}$ and $\dot{A}_{k,t}$, and since (by (4) and (5)) $\phi(\dot{A}_k) = \dot{A}_k$ and $\phi(\dot{\mathcal{A}}) = \dot{\mathcal{A}}$ for all $\phi \in \mathcal{G}$, we conclude that $a_{k,t,i}$, $A_{k,t}$, A_k , and $\mathcal{A}$ all belong to the model $N_{n,\ell}$.

Furthermore, $\dot{f} = \{\langle \text{op}(\check{m}, \dot{A}_m), \mathbf{1} \rangle : m \in \omega\}$, where $\text{op}(\sigma, \tau)$ is the name for the ordered pair $\langle \sigma_G, \tau_G \rangle$ given in [32, Definition 2.16, p. 191], is an HS-name for the mapping $f = \{\langle m, A_m \rangle : m \in \omega\}$ since, for every $\phi \in \mathcal{G}$, ϕ fixes $\dot{f}$ (pointwise), and all names in $\dot{f}$ are hereditarily symmetric. Thus, $\mathcal{A}$ is denumerable in $N_{n,\ell}$. □

Claim The denumerable family $\mathcal{A} = \{A_k : k \in \omega\}$ has no partial choice function in $N_{n,\ell}$. Hence,

$$N_{n,\ell} \models \neg \text{CAC}_\ell.$$

Proof By way of contradiction, we assume that $\mathcal{A}$ has an infinite subfamily in $N_{n,\ell}$, $\mathcal{B} = \{A_m : m \in W\}$ for some infinite set $W \subseteq \omega$, which has a choice function in $N_{n,\ell}$, f say. Clearly, $\mathcal{B}$ has a canonical HS-name, namely $\dot{\mathcal{B}} = \{\langle \dot{A}_m, 1 \rangle : m \in W\}$. We let $\dot{f}$ be an HS-name for f . There exists $p \in G$ such that

$$p \Vdash \text{“}\dot{f} \text{ is a choice function for } \dot{\mathcal{B}}\text{”}. \quad (6)$$

Let $E \in [\omega \times \ell \times \omega_n]^{<\omega}$ be a support of $\dot{f}$. Since W is infinite and E is finite, there exists $m_0 \in W$ such that $E \cap (\{m_0\} \times \ell \times \omega_n) = \emptyset$. Let t_0 be the unique element of ℓ such that $f(A_{m_0}) = A_{m_0,t_0}$.

Let $q \in G$ be such that $q \leq p$ and

$$q \Vdash \dot{f}(\dot{A}_{m_0}) = \dot{A}_{m_0,t_0}. \quad (7)$$

Since $|q| < \aleph_n$, there exists $k \in \omega_n$ such that, for all $i \in \omega_n$ with $i \geq k$ and for all $t \in \ell$ and $j \in \omega_n$, $\langle m_0, t, i, j \rangle \notin \text{dom}(q)$. We let σ_{m_0} be the following ℓ -cycle:

$$\sigma_{m_0} : 0 \mapsto 1 \mapsto \dots \mapsto \ell - 1 \mapsto 0,$$

and we also let $\eta : [0, k) \rightarrow [k, 2k)$ be an order-isomorphism. Then η induces a permutation η_{m_0} of ω_n defined by

$$\eta_{m_0}(i) = \begin{cases} \eta(i), & \text{if } i \in [0, k); \\ \eta^{-1}(i), & \text{if } i \in [k, 2k); \\ i, & \text{if } 2k \leq i. \end{cases}$$

We define a $\psi \in \mathcal{G}$ by stipulating, for all $\langle m, t, i \rangle \in \omega \times \ell \times \omega_n$,

$$\psi(m, t, i) = \begin{cases} \langle m_0, \sigma_{m_0}(t), \eta_{m_0}(i) \rangle, & \text{if } m = m_0; \\ \langle m, t, i \rangle, & \text{if } m \neq m_0. \end{cases}$$

(So, for $m \neq m_0$, σ_m and η_m are the identity permutations of ℓ and ω_n , respectively—recall the definition of $\mathcal{G}$.) Then the following hold:

- (a) $\psi \in \text{fix}_{\mathcal{G}}(E)$, and hence $\psi(\dot{f}) = \dot{f}$ (since E is a support of $\dot{f}$),
- (b) $\psi(\dot{A}_{m_0,t_0}) = \dot{A}_{m_0,\sigma_{m_0}(t_0)}$,
- (c) q and $\psi(q)$ are compatible conditions. Thus, $q \cup \psi(q)$ is a well-defined extension of q , $\psi(q)$, and p .

By (4), (a), (b), and (7), we obtain that

$$\psi(q) \Vdash \dot{f}(\dot{A}_{m_0}) = \dot{A}_{m_0, \sigma_{m_0}(t_0)}, \quad (8)$$

and from (c), together with Eqs. (6), (7) and (8), we conclude that

$$\begin{aligned} q \cup \psi(q) \Vdash & \text{“}\dot{f} \text{ is a choice function for } \dot{B}'' \wedge \dot{f}(\dot{A}_{m_0}) = \dot{A}_{m_0, t_0} \\ & \wedge \dot{f}(\dot{A}_{m_0}) = \dot{A}_{m_0, \sigma_{m_0}(t_0)}. \end{aligned} \quad (9)$$

But then, (9) yields a contradiction. Indeed, using **DC** in $M[G]$ (recall that $M[G]$ satisfies the full **AC**) and the proof of [32, Lemma 2.3, pp. 186–187], we may construct a $\mathbb{P}$ -generic filter H over M such that $q \cup \psi(q) \in H$. Then, by (9) and [32, Theorem 3.6(2), p. 200], we deduce that, in $M[H]$, the following hold:

- (i) $\dot{f}_H$ is a choice function for $\dot{B}_H$,
- (ii) $\dot{f}_H((\dot{A}_{m_0})_H) = (\dot{A}_{m_0, t_0})_H$, and
- (iii) $\dot{f}_H((\dot{A}_{m_0})_H) = (\dot{A}_{m_0, \sigma_{m_0}(t_0)})_H$,

where $\dot{B}_H = \{(\dot{A}_m)_H : m \in W\}$, $(\dot{A}_{m_0})_H = \{(\dot{A}_{m_0, t})_H : t \in \ell\}$, $(\dot{A}_{m_0, t})_H = \{(\dot{a}_{m_0, t, i})_H : i \in \omega_n\}$ ($t \in \ell$), and $(\dot{a}_{m_0, t, i})_H = \{j \in \omega_n : \exists r \in H(r(m_0, t, i, j) = 1)\}$ ($t \in \ell, i \in \omega_n$).

However, since $t_0 \neq \sigma_{m_0}(t_0)$ (recall that σ_{m_0} is the cycle $(0, 1, \dots, l-1)$, so σ_{m_0} moves every element of ℓ), we have that, for any $\mathbb{P}$ -generic filter Q over M , $(\dot{A}_{m_0, t_0})_Q \cap (\dot{A}_{m_0, \sigma_{m_0}(t_0)})_Q = \emptyset$. If not, then there exist a $\mathbb{P}$ -generic filter Q over M and an $x \in (\dot{A}_{m_0, t_0})_Q \cap (\dot{A}_{m_0, \sigma_{m_0}(t_0)})_Q$. Then $x = (\dot{a}_{m_0, t_0, i})_Q$ and $x = (\dot{a}_{m_0, \sigma_{m_0}(t_0), i'})_Q$ for some ordinals $i, i' \in \omega_n$. Let

$$D = \{r \in \mathbb{P} : \exists j \in \omega_n(r(m_0, t_0, i, j) \neq r(m_0, \sigma_{m_0}(t_0), i', j))\}.$$

Then $D \in M$ and it is fairly easy to verify that D is dense in $\mathbb{P}$. Hence, $Q \cap D \neq \emptyset$. Letting $r \in Q \cap D$, we obtain a $j \in (\dot{a}_{m_0, t_0, i})_Q \triangle (\dot{a}_{m_0, \sigma_{m_0}(t_0), i'})_Q$, contradicting $(\dot{a}_{m_0, t_0, i})_Q = (\dot{a}_{m_0, \sigma_{m_0}(t_0), i'})_Q$. Thus, $(\dot{A}_{m_0, t_0})_Q \cap (\dot{A}_{m_0, \sigma_{m_0}(t_0)})_Q = \emptyset$, as required.¹

In particular, for the $\mathbb{P}$ -generic filter H over M , we have

$$(\dot{A}_{m_0, t_0})_H \cap (\dot{A}_{m_0, \sigma_{m_0}(t_0)})_H = \emptyset,$$

so $(\dot{A}_{m_0, t_0})_H \neq (\dot{A}_{m_0, \sigma_{m_0}(t_0)})_H$. This, together with (ii) and (iii), yields that $\dot{f}_H$ is not a function, contradicting property (i) of $\dot{f}_H$.

Hence, $\mathcal{A}$ has no partial choice function in the model $N_{n, \ell}$, finishing the proof of the claim.

The above arguments complete the proof of the theorem.

¹ Note that the above argument yields that, for every $s \in \mathbb{P}$ (so also for $s = q \cup \psi(q)$), $s \Vdash \dot{A}_{m_0, t_0} \cap \dot{A}_{m_0, \sigma_{m_0}(t_0)} = \emptyset$ (cf. [32, Definition 3.1, p. 194]).

4 $\mathbf{M(IC, DI)}$ and $\mathbf{M(TB, WO)}$

Since every compact metric space is totally bounded and every infinite separable Hausdorff space is Dedekind-infinite, let us begin our investigations of the forms of type $\mathbf{M(C, \square)}$ with a deeper look at the forms $\mathbf{M(TB, WO)}$, $\mathbf{M(TB, S)}$ and $\mathbf{M(IC, DI)}$. We include a simple proof of the following proposition for completeness.

Proposition 3 (ZF) *Let $\mathbf{X} = \langle X, d \rangle$ is a totally bounded metric space such that X is well-orderable. Then $\mathbf{X}$ is separable.*

Proof Since X is well-orderable, so is the set $Y = \bigcup_{n \in \mathbb{N}} (X^n \times \{n\})$. Let $\leq$ be a fixed well-ordering in Y . For every $m \in \mathbb{N}$, let $y_m = \langle x_m, k_m \rangle \in X^{k_m} \times \{k_m\}$ be the first element of $\langle Y, \leq \rangle$ such that $X = \bigcup \{B_d(x_m(i), \frac{1}{m}) : i \in k_m\}$. The set $D = \bigcup_{m \in \mathbb{N}} \{x_m(i) : i \in k_m\}$ is countable and dense in $\mathbf{X}$. $\square$

Theorem 9 (ZF)

- (i) $\mathbf{M(TB, WO)} \rightarrow \mathbf{M(TB, S)}$ and $\mathbf{M(C, WO)} \rightarrow \mathbf{M(C, S)}$. None of these implications are reversible.
- (ii) $\mathbf{M(TB, WO)} \rightarrow \mathbf{M(C, WO)} \rightarrow \mathbf{M(C, S)} \rightarrow \mathbf{M(IC, DI)}$.
- (iii) (Cf. [23, Theorem 7 (i)].) $\mathbf{CAC} \rightarrow \mathbf{M(TB, S)} \rightarrow \mathbf{M(C, S)}$.
- (iv) Neither $\mathbf{M(TB, WO)}$ nor $\mathbf{M(TB, S)}$ imply $\mathbf{CAC}$.

Proof It follows from Proposition 3 that the implications from (i) are both true. It is known that, in Feferman's model $\mathcal{M}2$ in [15], $\mathbf{CAC}$ is true but $\mathbb{R}$ is not well-orderable (see [15, p. 140]). Then $[0, 1]$ is a compact, metrizable but not well-orderable space in $\mathcal{M}2$. Hence $\mathbf{M(TB, S)} \wedge \neg \mathbf{M(TB, WO)}$ and $\mathbf{M(C, S)} \wedge \neg \mathbf{M(C, WO)}$ are both true in $\mathcal{M}2$. This completes the proof to (i). In view of (i), it is obvious that (ii) holds. It is known from [23] that (iii) also holds. It follows from the first implication of (i) that to prove (iv), it suffices to show that $\mathbf{M(TB, WO)}$ does not imply $\mathbf{CAC}$.

It was shown in [23, proof of Theorem 15] that there exists a model $\mathcal{M}$ of $\mathbf{ZF} + \neg \mathbf{CAC}$ in which it is true that if a metric space $\mathbf{X} = \langle X, d \rangle$ is sequentially bounded (i.e., every sequence of points of $\mathbf{X}$ has a Cauchy's subsequence), then $\mathbf{X}$ is well-orderable and separable. By [23, Theorem 7 (vii)], every totally bounded metric space is sequentially bounded. This shows that there exists a model $\mathcal{M}$ of $\mathbf{ZF}$ in which $\mathbf{M(TB, WO)} \wedge \neg \mathbf{CAC}$ is true. Hence (iv) holds. $\square$

That $\mathbf{M(IC, DI)}$ does not imply $\mathbf{M(C, S)}$ is shown in Proposition 10(iv). It is unknown whether $\mathbf{M(C, WO)}$ is equivalent to or weaker than $\mathbf{M(TB, WO)}$ in $\mathbf{ZF}$.

To compare $\mathbf{M(C, S)}$ with $\mathbf{M(TB, S)}$, we recall that it was proved in [24] that the implication $\mathbf{M(TB, S)} \rightarrow \mathbf{CAC}(\mathbb{R})$ holds in $\mathbf{ZF}$; however, the implication $\mathbf{CAC} \rightarrow \mathbf{M(TB, S)}$ of Theorem 1 is not reversible in $\mathbf{ZF}$. On the other hand, it is known that $\mathbf{CAC}(\mathbb{R})$ and $\mathbf{M(C, S)}$ are independent of each other in $\mathbf{ZF}$ (see, e.g., [23]). The following proposition, together with the fact that $\mathbf{M(TB, S)}$ implies $\mathbf{M(C, S)}$, shows that $\mathbf{M(TB, S)}$ is essentially stronger than $\mathbf{M(C, S)}$ in $\mathbf{ZF}$.

Proposition 4 (ZF)

- (i) (Cf. [24, Proposition 8].) $\mathbf{M(TB, STB)}$ implies $\mathbf{CAC}(\mathbb{R})$.

- (ii) In Cohen's Original Model $\mathcal{M}1$ of [15] the following hold: $\mathbf{M}(C, S)$ is true but both $\mathbf{M}(TB, S)$ and $\mathbf{M}(TB, STB)$ are false.
 (iii) $\mathbf{M}(C, S)$ implies neither $\mathbf{M}(TB, S)$ nor $\mathbf{M}(TB, STB)$.

Proof (i) has been established in [24].

(ii)-(iii) It is known that **BPI** is true $\mathcal{M}1$ (see [15, p. 147]). It follows from Theorem 6(iii) that $\mathbf{M}(C, S)$ holds in $\mathcal{M}1$. On the other hand, it is known that **CAC**($\mathbb{R}$) fails in $\mathcal{M}1$ (see [15, p. 147]). Therefore, by the observation in the paragraph preceding this theorem and (i), $\mathbf{M}(TB, S)$ and $\mathbf{M}(TB, STB)$ are both false in $\mathcal{M}1$. $\square$

We recall that a topological space $\mathbf{X}$ is called *limit point compact* if every infinite subset of $\mathbf{X}$ has an accumulation point in $\mathbf{X}$ (see, e.g., [23]).

Proposition 5 (ZFA) **WoAm** implies both $\mathbf{M}(TB, WO)$ and “every limit point compact, first-countable T_1 -space is well-orderable”.

Proof Let us assume **WoAm**. Consider an arbitrary metric space $\langle X, d \rangle$. Suppose that X is not well-orderable. By **WoAm**, there exists an amorphous subset B of X . Let $\rho = d \upharpoonright B \times B$. Lemma 1 of [5] states that every metric on an amorphous set has a finite range. Therefore, the set $\text{ran}(\rho) = \{\rho(x, y) : x, y \in B\}$ is finite. Since B is infinite, the set $\text{ran}(\rho) \setminus \{0\}$ is non-empty. If $\varepsilon = \min(\text{ran}(\rho) \setminus \{0\})$, then there does not exist a finite ε -net in $\langle B, \rho \rangle$ because B is infinite and, for every $x \in B$, $B_\rho(x, \varepsilon) = \{x\}$. This implies that ρ is not totally bounded. Hence d is not totally bounded.

Now, suppose that $\mathbf{Y} = \langle Y, \tau \rangle$ is a first-countable, limit point compact T_1 -space. Let C be an infinite subset of Y . Since $\mathbf{Y}$ is limit point compact, the set C has an accumulation point in $\mathbf{Y}$. Let y_0 be an accumulation point of C and let $\{U_n : n \in \mathbb{N}\}$ be a base of neighborhoods of y_0 in $\mathbf{Y}$. Since $\mathbf{Y}$ is a T_1 -space, we can inductively define an increasing sequence $(n_k)_{k \in \mathbb{N}}$ of natural numbers such that, for every $k \in \mathbb{N}$, $C \cap (U_{n_k} \setminus U_{n_{k+1}}) \neq \emptyset$. This implies that C is not amorphous. Hence, no infinite subset of Y is amorphous, so Y is well-orderable by **WoAm**. $\square$

Corollary 1 $\mathcal{N}1 \models \mathbf{M}(TB, WO)$.

Proof This follows from Proposition 5 and the known fact that **WoAm** is true in $\mathcal{N}1$ (see p. 177 in [15]). $\square$

To prove that $\mathbf{M}(TB, WO)$ does not imply **WoAm** in **ZFA**, let us use the model $\mathcal{N}3$. In what follows, the notation concerning $\mathcal{N}3$ is the same as in Definition 15. For $a, b \in A$ with $a < b$ (where A is the set of atoms of $\mathcal{N}3$ and $\leq$ is the fixed linear order on A), we denote by (a, b) the open interval in the linearly ordered set $\langle A, \leq \rangle$; that is, $(a, b) = \{x \in A : a < x < b\}$. A proof of the following lemma can be found in [18].

Lemma 1 (Cf. [18, Lemma 3.17 and its proof].) *If $X \in \mathcal{N}3$, E is a support of X and there is $x \in X$ for which E is not a support, then there exist a subset Y of X and atoms $a, b \in A$ with $a < b$, such that $Y \in \mathcal{N}3$, $E \cap (a, b) = \emptyset$ and, in $\mathcal{N}3$, there exists a bijection $f : (a, b) \rightarrow Y$ having a support E' such that $E \cup \{a, b\} \subseteq E'$ and $E' \cap (a, b) = \emptyset$.*

Theorem 10 $\mathcal{N}3 \models \mathbf{M}(TB, WO)$.

Proof We use the notation from Definition 15. Suppose that $\langle X, d \rangle$ is a metric space in $\mathcal{N}3$ such that X is not well-orderable in $\mathcal{N}3$. Then X is infinite. Let $E \in [A]^{<\omega}$ be a support of both X and d . By Proposition 2, there exists $x \in X$ such that E is not a support of x .

By Lemma 1, there exist $a, b \in A$ with $a < b$ and $(a, b) \cap E = \emptyset$, such that there exists in $\mathcal{N}3$ an injection $f : (a, b) \rightarrow X$ which has a support E' such that $E \cup \{a, b\} \subseteq E'$ and $E' \cap (a, b) = \emptyset$. We put $B = (a, b)$ and $\rho(x, y) = d(f(x), f(y))$ for all $x, y \in B$. Let us notice that $\rho \in \mathcal{N}3$ because E' is also a support of ρ . We prove that $\text{ran}(\rho) = \{\rho(x, y) : x, y \in B\}$ is a two-element set. To this aim, we fix $b_1, b_2 \in B$ with $b_1 < b_2$ and put $r = \rho(b_1, b_2)$. Let $u, v \in B$ and $u \neq v$. To show that $\rho(u, v) = r$, we must consider several cases regarding the ordering of the elements b_1, b_2, u, v . We consider only one of the possible cases since all the other cases can be treated in much the same way as the chosen one. So, assume, for example, that $b_2 < u < v$. Let ϕ be an order-automorphism of $\langle A, \leq \rangle$ such that $\phi(b_1) = u$, $\phi(b_2) = v$, and ϕ is the identity mapping on $A \setminus B$. Then $\phi \in \text{fix}_{\mathcal{G}}(E')$, so $\phi(\rho) = \rho$. This implies that $\phi(r) = \rho(\phi(b_1), \phi(b_2))$. Since, in addition $\phi(r) = r$, we have $r = \rho(b_1, b_2) = \rho(\phi(b_1), \phi(b_2)) = \rho(u, v)$. Therefore, $\text{ran}(\rho) = \{0, r\}$. Since the range of ρ is finite, in much the same way, as in the proof to Proposition 5, we deduce that $\langle B, \rho \rangle$ is not totally bounded. Hence d is not totally bounded. $\square$

Corollary 2 $\mathbf{M}(TB, WO)$ does not imply **WoAm** in **ZFA**.

Proof It is known that **WoAm** is false in $\mathcal{N}3$ (see [15, p.183]). Therefore, the conjunction $\mathbf{M}(TB, WO) \wedge \neg \mathbf{WoAm}$ is true in $\mathcal{N}3$ by Theorem 10. $\square$

In contrast to Corollary 1 and Theorem 10, we have the following proposition:

Proposition 6 $\mathcal{N}_{cr} \models \neg \mathbf{M}(C, WO)$.

Proof It was shown in [27, proof of Theorem 3.5] that, in $\mathcal{N}_{cr}$, there exists a compact metric space $\mathbf{X} = \langle X, d \rangle$ which is not weakly Loeb. Then X cannot be well-orderable in $\mathcal{N}_{cr}$. $\square$

Remark 11 In [40, proof of Theorem 2.1], a symmetric model $\mathcal{N}$ of **ZF** was constructed such that, in $\mathcal{N}$, there exists a compact metric space $\langle X, d \rangle$ which is not weakly Loeb; thus, $\mathbf{M}(C, WO)$ fails in $\mathcal{N}$.

It is obvious that **IDI** implies $\mathbf{M}(IC, DI)$ in **ZF**; however, it seems to be still an open problem of whether this implication is not reversible in **ZF**. To solve this problem, first of all, let us notice that the following corollary follows directly from Corollary 1 and Theorem 10:

Corollary 3 (i) $\mathcal{N}1 \models (\mathbf{M}(IC, DI) \wedge \neg \mathbf{IDI})$.
(ii) $\mathcal{N}3 \models (\mathbf{BPI} \wedge \mathbf{M}(IC, DI) \wedge \neg \mathbf{IDI})$.

To transfer $\mathbf{BPI} \wedge \mathbf{M}(IC, DI) \wedge \neg \mathbf{IDI}$ to a model of **ZF**, let us prove the following lemma:

Lemma 2 $\mathbf{M}(IC, DI)$ is injectively boundable.

Proof First, we put

$$\Phi(x) = \neg(\exists y)(y \subseteq x \wedge |y| = \omega),$$

and

$$\Psi(x) = \text{“}x \text{ is finite”}.$$

Note that the formula $\Psi(x)$ is boundable (see [37] or [15, Note 103, p. 284]). Now it is not hard to verify that $\mathbf{M}(IC, DI)$ is logically equivalent to the statement Ω , where

$$\begin{aligned} \Omega = & (\forall x)(|x|_{-} \leq \omega \rightarrow (\forall \rho \in \mathcal{P}(x \times x \times \mathbb{R})) \\ & [(\Phi(x) \wedge (\rho \text{ is a metric on } x \text{ such that } \langle x, \rho \rangle \text{ is compact})) \rightarrow \Psi(x)]) \end{aligned}$$

Since Ω is obviously injectively boundable, so is $\mathbf{M}(IC, DI)$. $\square$

Theorem 11 *The conjunction $\mathbf{BPI} \wedge \mathbf{M}(IC, DI) \wedge \neg\mathbf{IDI}$ has a ZF-model.*

Proof It is known that $\neg\mathbf{IDI}$ is boundable, and hence injectively boundable (see [37, 2A5, p. 772] or [15, p. 285]). By Theorem 7, if a conjunction of $\mathbf{BPI}$ with injectively boundable statements has a Fraenkel–Mostowski model, then it has a ZF-model. This, together with Corollary 3(ii) and Lemma 2, completes the proof. $\square$

Theorem 12 *The conjunction $(\neg\mathbf{BPI}) \wedge \mathbf{M}(IC, DI) \wedge \neg\mathbf{IDI}$ has a ZF-model.*

Proof Let Φ be the conjunction $(\neg\mathbf{BPI}) \wedge \mathbf{M}(IC, DI) \wedge \neg\mathbf{IDI}$. It is known that $\mathbf{BPI}$ is false in $\mathcal{N}1$ (see [15, p. 177]). Hence, Φ has a permutation model by Corollary 3(i). Since the statements $\neg\mathbf{BPI}$, $\mathbf{M}(IC, DI)$ and $\neg\mathbf{IDI}$ are all injectively boundable, Φ has a ZF-model by Theorem 7. $\square$

Corollary 4 *$\mathbf{M}(IC, DI)$ does not imply $\mathbf{IDI}$ in ZF. Furthermore, $\mathbf{BPI}$ is independent of $\mathbf{ZF} + \mathbf{M}(IC, DI) + \neg\mathbf{IDI}$.*

Proposition 7 *$\mathbf{M}(IC, DI)$ implies $\mathbf{CAC}_{fin}$ in ZF.*

Proof It suffices to apply Corollary 2.2 (iii) in [27] which states that if every infinite compact metrizable space has an infinite well-orderable subset, then $\mathbf{CAC}_{fin}$ holds. $\square$

At this moment, it is unknown whether there is a model of ZF in which the conjunction $\mathbf{CAC}_{fin} \wedge \neg\mathbf{M}(IC, DI)$ is true.

Remark 12 In Cohen’s original model $\mathcal{M}1$ in [15], $\mathbf{CUC}$ holds and there exists a dense Dedekind-finite subset X of the interval $[0, 1]$ of $\mathbb{R}$. The metric space $\mathbf{X} = \langle X, d \rangle$, where $d(x, y) = |x - y|$ for all $x, y \in X$, is totally bounded but X is not well-orderable in $\mathcal{M}1$. It was remarked in [24] that, since $\langle X, d \rangle$ is not separable, $\mathbf{CUC}$ does not imply $\mathbf{M}(TB, S)$ in ZF. Now, it is easily seen that $\mathbf{CUC}$ does not imply $\mathbf{M}(TB, WO)$ in ZF.

It is stated neither in [15] nor in [16] that **WoAm** implies **CUC**. Since we have not seen a solution to the problem of whether this implication is true in other sources, let us notice that it follows from the following proposition that this implication holds in **ZF**:

Proposition 8 (i) **(ZFA) WoAm** $\rightarrow$ **CUC**;

(ii) If $\mathcal{N}$ is a model of **ZFA** in which $\mathbb{R}$ is well-orderable (in particular, if $\mathcal{N}$ is a permutation model), then:

$$\mathcal{N} \models (\mathbf{CAC}_{WO} \rightarrow \mathbf{CUC}).$$

(iii) If $\mathcal{N}$ is a permutation model, then:

$$\mathcal{N} \models (\mathbf{AC}_{fin} \rightarrow \mathbf{CUC}).$$

Proof Let $\mathcal{A} = \{A_n : n \in \omega\}$ be a disjoint family of non-empty countable sets and let $A = \bigcup \mathcal{A}$. Clearly, if $\bigcup_{n \in \omega} (A_n \times \omega)$ is countable, then A is countable. Therefore, to show that A is countable, we may assume that, for every $n \in \omega$, the set A_n is denumerable. For every $n \in \omega$, let B_n be the set of all bijections from ω onto A_n . Since $|\omega^\omega| = |\mathbb{R}^\omega| = |\mathbb{R}|$ and the sets A_n are all denumerable, for every $n \in \omega$, the set B_n is equipotent to $\mathbb{R}$.

- (i) Let $B = \bigcup_{n \in \omega} B_n$. If B is well-orderable, then there exists a sequence $(f_n)_{n \in \omega}$ of bijections $f_n : \omega \rightarrow A_n$, so A is countable. Suppose that B is not well-orderable. Then it follows from **WoAm** that there exists an amorphous subset C of B . Since C cannot be partitioned into two infinite subsets, the set $\{n \in \omega : C \cap B_n \neq \emptyset\}$ is finite. This implies that there exists $m \in \omega$ such that $C \subseteq \bigcup_{n \in m+1} B_n$, so C is equipotent to a subset of $\mathbb{R}$. But this is impossible because $\mathbb{R}$ does not have amorphous subsets. The contradiction obtained completes the proof of (i).
- (ii) Now, assume that $\mathbb{R}$ is well-orderable and **CAC**_{WO} holds. Then the sets B_n , being equipotent to $\mathbb{R}$, are all well-orderable. Hence, it follows from **CAC**_{WO} that there exists $f \in \prod_{n \in \omega} B_n$. Then we have a sequence $(f(n))_{n \in \omega}$ of bijections $f(n) : \omega \rightarrow A_n$, so A is countable.
- (iii) Since **AC**_{fin} implies **AC**_{WO} in every permutation model (cf. [13] and Note 2 in [15]), we infer that if **AC**_{fin} is satisfied in $\mathcal{N}$ and $\mathcal{N}$ is a permutation model, then **CAC**_{WO} holds in $\mathcal{N}$. This, taken together with (ii), implies (iii).

□

Remark 13 (a) In Felgner's Model I (labeled as $\mathcal{M}20$ in [15]), **AC**_{WO} holds and **CUC** fails (cf. [15, p. 159]). We recall that **AC**_{fin} $\wedge$ $\neg$ **AC**_{WO} is true in Sageev's Model I (labeled as model $\mathcal{M}6$ in [15]). Moreover, since **IDI** $\wedge$ $\neg$ **CUC** is true in $\mathcal{M}6$ (cf. [15, p. 152]), **M**(IC, DI) does not imply **CUC** is **ZF**.

- (b) One should not claim that **CAC**_{fin} and **CAC**_{WO} are equivalent in every permutation model. Namely, let $\mathcal{N}$ be the permutation model of **IDI** $\wedge$ $\neg$ **CUC** constructed in [41, the proof to Theorem 4 (iv)]. Since **IDI** implies **CAC**_{fin}, it follows from Proposition 8 that, in this model $\mathcal{N}$, **CAC**_{fin} is true but **CAC**_{WO} is false.

- Remark 14** (a) It is unknown whether $\mathbf{M}(C, S)$ implies $\mathbf{CUC}$ in $\mathbf{ZF}$ or in $\mathbf{ZFA}$. We recall that $\mathbf{BPI}$ implies $\mathbf{M}(C, S)$. The problem of whether $\mathbf{BPI}$ implies $\mathbf{CUC}$ in $\mathbf{ZF}$ or in $\mathbf{ZFA}$ is still unsolved. However, if $\mathcal{N}$ is a permutation model in which $\mathbf{BPI}$ is true, then $\mathbf{AC}_{fin}$ is also true in $\mathcal{N}$ (see, e.g., [12, Proposition 4.39]); hence, by Proposition 8(iii), $\mathbf{BPI}$ implies $\mathbf{CUC}$ in every permutation model.
- (b) Since $\mathbf{M}(C, S)$ implies $\mathbf{CAC}_{fin}$ (see Theorem 2(ii) or Theorem 9 with Proposition 7), it follows from Proposition 8(iii) that $\mathbf{CAC}_{fin} \wedge \neg \mathbf{AC}_{fin}$ is satisfied in every permutation model of $\mathbf{M}(C, S) \wedge \neg \mathbf{CUC}$.

It still eludes us whether or not $\mathbf{CUC}$ implies $\mathbf{M}(C, S)$ in $\mathbf{ZF}$. However, we are able to provide a partial solution to this intriguing open problem by proving (in Corollary 5) that $\mathbf{UT}(\aleph_0, \aleph_0, cuf)$ does not imply $\mathbf{M}(C, S)$ in $\mathbf{ZF}$. To achieve our goal, we will first prove (in Theorem 13) that the statement $\mathbf{UT}(\aleph_0, \aleph_0, cuf) \wedge \neg \mathbf{M}(IC, DI)$ has a permutation model and that it is transferable to $\mathbf{ZF}$. For the transfer of the latter conjunction to $\mathbf{ZF}$, we will need the following two auxiliary results of Lemma 3 and Proposition 9.

Lemma 3 (**ZF**) (Cf. [17, Lemma 3.5].) *For any ordinal α , if $\mathcal{R}$ is a collection of sets such that $|\mathcal{R}| \leq \aleph_{\alpha+1}$ and, for every $x \in \mathcal{R}$, $|x| \leq \aleph_\alpha$, then $|\bigcup \mathcal{R}| \not\leq \aleph_{\alpha+2}$.*

Proposition 9 *The statement $\mathbf{UT}(\aleph_0, \aleph_0, cuf)$ is injectively boundable.*

Proof In the light of Lemma 3, $\mathbf{UT}(\aleph_0, \aleph_0, cuf)$ is equivalent to the statement:

$$(\forall x)(|x| \not\leq \aleph_3 \rightarrow (\forall y) \text{ “if } y \text{ is a countable collection of countable sets whose union is } x, \text{ then } x \text{ is a cuf set”}). \quad (10)$$

Since, for every set x , the statements $|x| \not\leq \aleph_3$ and $|x|_- \leq \aleph_2$ are equivalent, it is obvious that (10) is injectively boundable. Thus, $\mathbf{UT}(\aleph_0, \aleph_0, cuf)$ is also injectively boundable. $\square$

Theorem 13 (i) *The statement $\mathbf{LW} \wedge \mathbf{UT}(\aleph_0, \aleph_0, cuf) \wedge \neg \mathbf{M}(IC, DI)$ has a permutation model.*

(ii) *The statement $\mathbf{UT}(\aleph_0, \aleph_0, cuf) \wedge \neg \mathbf{M}(IC, DI)$ has a **ZF**-model.*

Proof (i) We will use the permutation model $\mathcal{N}$ which was constructed in [6, proof of Theorem 3.3]. To describe $\mathcal{N}$, we start with a model $\mathcal{M}$ of $\mathbf{ZFA} + \mathbf{AC}$ with a set A of atoms such that A has a denumerable partition $\{A_i : i \in \omega\}$ into denumerable sets, and for each $i \in \omega$, A_i has a denumerable partition $P_i = \{A_{i,j} : j \in \mathbb{N}\}$ into finite sets such that, for every $j \in \mathbb{N}$, $|A_{i,j}| = j$. Let $\text{sym}(A)$ be the group of all permutations of A and let

$$\mathcal{G} = \{\phi \in \text{sym}(A) : (\forall i \in \omega)(\phi(A_i) = A_i) \text{ and } |\{x \in A : \phi(x) \neq x\}| < \aleph_0\}.$$

Let $\mathbf{P}_i = \{\phi(P_i) : \phi \in \mathcal{G}\}$ and also let $\mathbf{P} = \bigcup \{\mathbf{P}_i : i \in \omega\}$. Let $\mathcal{F}$ be the normal filter of subgroups of $\mathcal{G}$ generated by the filter base $\{\text{fix}_{\mathcal{G}}(E) : E \in [\mathbf{P}]^{<\omega}\}$. Then $\mathcal{N}$ is the permutation model determined by $\mathcal{M}$, $\mathcal{G}$ and $\mathcal{F}$. We say that a finite subset E of $\mathbf{P}$ is a *support* of an element x of $\mathcal{N}$ if $\text{fix}_{\mathcal{G}}(E) \subseteq \text{sym}_{\mathcal{G}}(x)$.

It was observed in [6, proof of Theorem 3.3] that, for every $i \in \omega$ and every $Q \in \mathbf{P}_i$, the following hold:

- (a) for any $\phi \in \mathcal{G}$, ϕ fixes Q if and only if ϕ fixes Q pointwise;
- (b) $(\exists j_Q \in \omega)(Q \supseteq \{A_{i,j} : j > j_Q\})$.

To prove that **LW** is true in $\mathcal{N}$, we fix a linearly ordered set $\langle Y, \leq \rangle$ in $\mathcal{N}$ and prove that $\text{fix}_{\mathcal{G}}(Y) \in \mathcal{F}$. To this aim, we choose a set $E \in [\mathbf{P}]^{<\omega}$ such that E is a support of both Y and $\leq$. To show that $\text{fix}_{\mathcal{G}}(E) \subseteq \text{fix}_{\mathcal{G}}(Y)$, let us consider any element $y \in Y$ and a permutation $\phi \in \text{fix}_{\mathcal{G}}(E)$. Suppose that $\phi(y) \neq y$. Then either $y < \phi(y)$ or $\phi(y) < y$. Since every element of $\mathcal{G}$ moves only finitely many atoms, there exists $k \in \mathbb{N}$ such that ϕ^k is the identity mapping on A . Assuming that $y < \phi(y)$, for such a k , we obtain the following:

$$y < \phi(y) < \phi^2(y) < \dots < \phi^{k-1}(y) < \phi^k(y) = y,$$

and thus $y < y$. Arguing similarly, we deduce that if $\phi(y) < y$, then $y < y$. The contradiction obtained shows that $\phi(y) = y$ for every $y \in Y$ and every $\phi \in \text{fix}_{\mathcal{G}}(E)$. Hence $\text{fix}_{\mathcal{G}}(E) \subseteq \text{fix}_{\mathcal{G}}(Y)$. Since $\mathcal{F}$ is a filter and $\text{fix}_{\mathcal{G}}(E) \in \mathcal{F}$, we infer that $\text{fix}_{\mathcal{G}}(Y) \in \mathcal{F}$. This, together with Proposition 2, implies that the set Y is well-orderable in $\mathcal{N}$. Hence, $\mathcal{N} \models \mathbf{LW}$.

Now, let us prove that **M(IC, DI)** fails in $\mathcal{N}$. First, to find a metric d on A_0 such that $\langle A_0, d \rangle$ is a compact metric space in $\mathcal{N}$, we denote by ∞ the unique element of $A_{0,1}$ and, for every $n \in \mathbb{N}$, we denote by ρ_n the discrete metric on $A_{0,n+1}$. Then, making obvious adjustments in notation, we let d be the metric on A_0 defined by (*) in Sect. 2.3. By Proposition 1, the metric space $\langle A_0, d \rangle$ is compact. Using (a), one can check that $\{P_0\}$ is a support $\langle A_0, d \rangle$ and, therefore, $\langle A_0, d \rangle \in \mathcal{N}$. Moreover, for every $n \in \mathbb{N}$, $\{P_0\}$ is a support of $A_{0,n}$. Hence the family $\mathcal{A} = \{A_{0,n+1} : n \in \mathbb{N}\}$ is denumerable in $\mathcal{N}$. We note that if $M \subseteq \mathbb{N}$, then $\{A_{0,n+1} : n \in M\} \in \mathcal{N}$ because $\{P_0\}$ is a support of $A_{0,n+1}$ for every $n \in M$. Suppose that $\mathcal{A}$ has a partial choice function in $\mathcal{N}$. Then there exists an infinite set $M \subseteq \mathbb{N}$ such that the family $\mathcal{B} = \{A_{0,n+1} : n \in M\}$ has a choice function in $\mathcal{N}$. Let f be a choice function of $\mathcal{B}$ such that $f \in \mathcal{N}$. Let $D \in [\mathbf{P}]^{<\omega}$ be a support of f . Then $D' = D \cap \mathbf{P}_0$ is also a support of f . Let $n \in \omega$ and $\phi_i \in \mathcal{G}$ with $i \in n+1$ be such that $D' = \{\phi_i(P_0) : i \in n+1\}$. Since every permutation from $\mathcal{G}$ moves only finitely many atoms, there exists $n_0 \in M$ such that $n_0 \geq 2$ and $A_{0,n_0} \in \phi_i(P_0)$ for all $i \in n+1$.

Assume that $f(A_{0,n_0}) = x_0$. Since $|A_{0,n_0}| = n_0 \geq 2$, there exists $y_0 \in A_{0,n_0}$ such that $y_0 \neq x_0$. Let $\eta = (x_0, y_0)$, i.e., η is the permutation of A which interchanges x_0 and y_0 , and fixes all other atoms of $\mathcal{N}$. Then $\eta(A_{0,n_0}) = A_{n_0}$ and $\eta \in \text{fix}_{\mathcal{G}}(D')$. Since D' is a support of f , we have $\eta(f) = f$. Therefore, since $\langle A_{0,n_0}, x_0 \rangle \in f$, we infer that $\langle \eta(A_{0,n_0}), \eta(x_0) \rangle \in \eta(f) = f$, so $\langle A_{0,n_0}, y_0 \rangle \in f$ and, in consequence, $x_0 = y_0$. The contradiction obtained shows that $\mathcal{A}$ does not have a partial choice function in $\mathcal{N}$. This implies that the set A_0 is Dedekind-finite, and thus **M(IC, DI)** is false in $\mathcal{N}$.

- (ii) Let Φ be the statement $\mathbf{UT}(\aleph_0, \aleph_0, \text{cuf}) \wedge \neg \mathbf{M(IC, DI)}$. Since the statement $\neg \mathbf{M(IC, DI)}$ is boundable, it is also injectively boundable, so this, together with Proposition 9, implies that Φ is a conjunction of injectively boundable statements. Therefore, (ii) follows from (i) and from Theorem 7. $\square$

Clearly, every **ZF**-model for $\mathbf{UT}(\aleph_0, \aleph_0, cuf) \wedge \neg \mathbf{M}(IC, DI)$ is also a model for $\mathbf{UT}(\aleph_0, \aleph_0, cuf) \wedge \neg \mathbf{M}(C, S)$. This, together with Theorem 13(ii), implies the following corollary:

Corollary 5 *The conjunction $\mathbf{UT}(\aleph_0, \aleph_0, cuf) \wedge \neg \mathbf{M}(C, S)$ has a **ZF**-model.*

Remark 15 Let $\mathcal{N}$ be the permutation model of the proof of Theorem 13(i). The proof of Theorem 3.3 in [6] shows that $\mathbf{UT}(\aleph_0, cuf, cuf)$ is false in $\mathcal{N}$ (and hence **CUC** is also false in $\mathcal{N}$). Since $\mathbf{UT}(\aleph_0, \aleph_0, cuf) \wedge \neg \mathbf{UT}(\aleph_0, cuf, cuf)$ has a permutation model (for instance, $\mathcal{N}$), it also has a **ZF**-model by Proposition 9 and Theorem 7.

5 The forms of type $\mathbf{M}(C, \square)$

It is known that every separable metrizable space is second-countable in **ZF**. It is also known, for instance, from Theorem 4.54 of [12] or from [9] that, in **ZF**, the statement “every second-countable metrizable space is separable” is equivalent to $\mathbf{CAC}(\mathbb{R})$. The negation of $\mathbf{CAC}(\mathbb{R})$ is relatively consistent with **ZF**, so it is relatively consistent with **ZF** that there are non-separable second-countable metrizable spaces. On the other hand, by Theorem 6(i), it holds in **ZF** that separability and second-countability are equivalent in the class of compact metrizable spaces. Theorem 1 shows that totally bounded metric spaces are second-countable in $\mathbf{ZF} + \mathbf{CAC}$; in particular, it holds in $\mathbf{ZF} + \mathbf{CAC}$ that all compact metrizable spaces are second-countable. However, the situation is completely different in **ZF**. There exist **ZF**-models including compact non-separable metric spaces. Namely, it follows from Theorem 2 that in every **ZF**-model satisfying the negation of $\mathbf{CAC}_{fin}$, there exists an uncountable, non-separable compact metric space whose size is incomparable to $|\mathbb{R}|$. The following proposition shows (among other facts) that $\mathbf{M}(C, \leq |\mathbb{R}|)$ and $\mathbf{M}(C, S)$ are essentially stronger than $\mathbf{CAC}_{fin}$ in **ZF** and, furthermore, **IDI** is independent of both $\mathbf{ZF} + \mathbf{M}(C, \leq |\mathbb{R}|)$ and $\mathbf{ZF} + \mathbf{M}(C, S)$.

Proposition 10 (i) $(\mathbf{ZFA}) \mathbf{M}(C, S) \rightarrow \mathbf{M}(C, \leq |\mathbb{R}|) \rightarrow \mathbf{CAC}_{fin}$.

(ii) $\mathcal{N}_{cr} \models (\neg \mathbf{M}(C, S)) \wedge \neg \mathbf{M}(C, \leq |\mathbb{R}|)$.

(iii) $\mathbf{CAC}_{fin}$ implies neither $\mathbf{M}(C, \leq |\mathbb{R}|)$ nor $\mathbf{M}(C, S)$ in **ZF**.

(iv) **IDI** implies neither $\mathbf{M}(C, \leq |\mathbb{R}|)$ nor $\mathbf{M}(C, S)$ in **ZF**.

(v) Neither $\mathbf{M}(C, \leq |\mathbb{R}|)$ nor $\mathbf{M}(C, S)$ imply **IDI** in **ZF**.

Proof (i) It follows directly from Theorem 2 that the implications given in (i) are true in **ZF**; however, the arguments from [23] are sufficient to show that these implications are also true in **ZFA**.

(ii) By Proposition 6, there exists a compact metric space $\mathbf{X} = \langle X, d \rangle$ in $\mathcal{N}_{cr}$ such that the set X is not well-orderable in $\mathcal{N}_{cr}$. Since $\mathbb{R}$ is well-orderable in $\mathcal{N}_{cr}$, the set $\mathbf{X}$ is not equipotent to a subset of $\mathbb{R}$ in $\mathcal{N}_{cr}$. Hence $\mathbf{M}(C, \leq |\mathbb{R}|)$ fails in $\mathcal{N}_{cr}$. This, together with (i), implies (ii).

(iii)–(iv) Let Φ be either $\mathbf{CAC}_{fin}$ or **IDI**. In the light of (i), to prove (iii) and (iv), it suffices to show that the conjunction $\Phi \wedge \neg \mathbf{M}(C, \leq |\mathbb{R}|)$ has a **ZF**-model. It follows from (ii) that the conjunction $\Phi \wedge \neg \mathbf{M}(C, \leq |\mathbb{R}|)$ has a permutation

model (for instance, $\mathcal{N}_{cr}$). Therefore, since the statements $\mathbf{CAC}_{fin}$, $\mathbf{IDI}$ and $\neg\mathbf{M}(C, \leq |\mathbb{R}|)$ are all injectively boundable, $\Phi \wedge \neg\mathbf{M}(C, \leq |\mathbb{R}|)$ has a **ZF**-model by Theorem 7.

- (v) Let Ψ be either $\mathbf{M}(C, \leq |\mathbb{R}|)$ or $\mathbf{M}(C, S)$. Since **BPI** is true in $\mathcal{N}3$, it follows from Theorem 6 (iii) that Ψ is true in $\mathcal{N}3$. It is known that **IDI** is false in $\mathcal{N}3$. Hence, the conjunction $\Psi \wedge \neg\mathbf{IDI}$ has a permutation model. To complete the proof, it suffices to apply Theorem 7.

□

Theorem 14 (ZF)

- (i) $(\mathbf{CAC}_{fin} \wedge \mathbf{M}(C, \sigma - l.f)) \leftrightarrow \mathbf{M}(C, S)$.
(ii) $(\mathbf{CAC}_{fin} \wedge \mathbf{M}(C, STB)) \leftrightarrow \mathbf{M}(C, S)$.

Proof Let $\mathbf{X} = \langle X, d \rangle$ be a compact metric space.

($\rightarrow$) We assume both $\mathbf{CAC}_{fin}$ and $\mathbf{M}(C, \sigma - l.f)$. By our hypothesis, $\mathbf{X}$ has a base $\mathcal{B} = \bigcup \{\mathcal{B}_n : n \in \mathbb{N}\}$ such that, for every $n \in \mathbb{N}$, the family $\mathcal{B}_n$ is locally finite. In **ZF**, to check that if $\mathcal{A}$ is a locally finite family in $\mathbf{X}$, then it follows from the compactness of $\mathbf{X}$ that $\mathcal{A}$ is finite, we notice that the collection $\mathcal{V}$ of all open sets V of $\mathbf{X}$ such that V meets only finitely many members of $\mathcal{A}$ is an open cover of $\mathbf{X}$, so $\mathcal{V}$ has a finite subcover. In consequence, X meets only finitely many members of $\mathcal{A}$, so $\mathcal{A}$ is finite. Therefore, for every $n \in \mathbb{N}$, the family $\mathcal{B}_n$ is finite. This, together with $\mathbf{CAC}_{fin}$, implies that the family $\mathcal{B}$ is countable, so $\mathbf{X}$ is second-countable. Hence, by Theorem 6(i), $\mathbf{X}$ is separable as required.

($\leftarrow$) By Proposition 10, $\mathbf{M}(C, S)$ implies $\mathbf{CAC}_{fin}$. To conclude the proof to (i), it suffices to notice that $\mathbf{M}(C, S)$ implies $\mathbf{M}(C, 2)$ and $\mathbf{M}(C, 2)$ trivially implies that every compact metric space has a σ -locally finite base.

(ii) ($\rightarrow$) Now, we assume both $\mathbf{CAC}_{fin}$ and $\mathbf{M}(C, STB)$. Since $\mathbf{X}$ is strongly totally bounded, it follows that it admits a sequence $(D_n)_{n \in \mathbb{N}}$ such that, for every $n \in \mathbb{N}$, D_n is a $\frac{1}{n}$ -net of $\mathbf{X}$. By $\mathbf{CAC}_{fin}$, the set $D = \bigcup \{D_n : n \in \mathbb{N}\}$ is countable. Since, D is dense in $\mathbf{X}$, it follows that $\mathbf{X}$ is separable.

($\leftarrow$) It is straightforward to check that every separable compact metric space is strongly totally bounded. Hence $\mathbf{M}(C, S)$ implies $\mathbf{M}(C, STB)$. Proposition 10 completes the proof. □

Remark 16 In the light of Proposition 10, there exists a model $\mathcal{M}$ of **ZF** in which $\mathbf{CAC}_{fin}$ holds and $\mathbf{M}(C, S)$ fails. By Theorem 14(i), $\mathbf{M}(C, \sigma - l.f)$ fails in $\mathcal{M}$. This, together with Theorem 5(i), implies that $\mathbf{M}(C, \sigma - l.f)$ independent of **ZF**.

Theorem 15 (ZF)

- (i) $(\mathbf{CAC}(\mathbb{R}, C) \wedge \mathbf{M}(C, \leq |\mathbb{R}|)) \leftrightarrow \mathbf{M}(C, S)$.
(ii) $\mathbf{CAC}(\mathbb{R})$ does not imply $\mathbf{M}(C, \leq |\mathbb{R}|)$.

Proof (i) ($\rightarrow$) We assume $\mathbf{CAC}(\mathbb{R}, C)$ and $\mathbf{M}(C, \leq |\mathbb{R}|)$. We fix a compact metric space $\mathbf{X} = \langle X, d \rangle$ and prove that $\mathbf{X}$ is separable. For every $n \in \mathbb{N}$, let $\mathbf{X}^n = \langle X^n, d_n \rangle$ where d_n is the metric on X^n defined by:

$$d_n(x, y) = \max\{d(x(i), y(i)) : i \in n\}.$$

Then $\mathbf{X}^n$ is compact for every $n \in \mathbb{N}$. By $\mathbf{M}(C, \leq |\mathbb{R}|)$, $|X| \leq |\mathbb{R}|$. Therefore, since $|X^{\mathbb{N}}| \leq |\mathbb{R}|$, there exists a family $\{\psi_n : n \in \mathbb{N}\}$ such that, for every $n \in \mathbb{N}$, $\psi_n : X^n \rightarrow \mathbb{R}$ is an injection. The metric d is totally bounded, so, for every $n \in \mathbb{N}$, the set

$$M_n = \left\{ m \in \mathbb{N} : \exists y \in X^m, \forall x \in X, d(x, \{y(i) : i \in n\}) < \frac{1}{n} \right\}$$

is non-empty. Let $k_n = \min M_n$ for every $n \in \mathbb{N}$. To prove that $\mathbf{X}$ is strongly totally bounded, for every $n \in \mathbb{N}$, we consider the set C_n defined as follows:

$$C_n = \left\{ y \in X^{k_n} : \forall x \in X \left(d(x, \{y(i) : i \in k_n\}) < \frac{1}{n} \right) \right\}.$$

We claim that for every $n \in \mathbb{N}$, C_n is a closed subset of $\mathbf{X}^{k_n}$. To this end, we fix $y_0 \in X^{k_n} \setminus C_n$. Then, since X is infinite, there exists $x_0 \in X$ such that $B_d(x_0, \frac{1}{n}) \cap \{y_0(i) : i \in n\} = \emptyset$. Let $r = d(x_0, \{y_0(i) : i \in n\})$ and $\varepsilon = r - \frac{1}{n}$. Then $\varepsilon > 0$. To show that $B_{d_{k_n}}(y_0, \varepsilon) \cap C_n = \emptyset$, suppose that $z_0 \in B_{d_{k_n}}(y_0, \varepsilon) \cap C_n$. Then

$$d(x_0, \{z_0(i) : i \in k_n\}) = \max\{d(x_0, z_0(i)) : i \in k_n\} < \frac{1}{n}$$

and

$$d_{k_n}(y_0, z_0) = \max\{d(y_0(i), z_0(i)) : i \in k_n\} < \varepsilon.$$

For every $i \in k_n$, we have:

$$r \leq d(x_0, y_0(i)) \leq d(x_0, z_0(i)) + d(z_0(i), y_0(i)) \leq d(x_0, z_0(i)) + d_{k_n}(z_0, y_0).$$

Hence, for every $i \in k_n$, the following inequalities hold:

$$r - d_{k_n}(z_0, y_0) \leq d(x_0, z_0(i)) < \frac{1}{n}.$$

Therefore, $\varepsilon < d_{k_n}(z_0, y_0)$. The contradiction obtained shows that $B_{d_{k_n}}(y_0, \varepsilon) \cap C_n = \emptyset$. Hence, for every $n \in \mathbb{N}$, the non-empty set C_n is compact in the metric space $\mathbf{X}^{k_n}$. Therefore, it follows from $\mathbf{CAC}(\mathbb{R}, C)$ that the family $\{\psi_n(C_n) : n \in \mathbb{N}\}$ has a choice function. This implies that $\{C_n : n \in \mathbb{N}\}$ has a choice function, so we can fix $f \in \prod_{n \in \mathbb{N}} C_n$. Then, for every $n \in \mathbb{N}$, the set $D_n = \{f(n)(i) : i \in k_n\}$ is a $\frac{1}{n}$ -net in $\mathbf{X}$. This shows that $\mathbf{X}$ is strongly totally bounded. It is easily seen that the set $D = \bigcup_{n \in \mathbb{N}} D_n$ is countable and dense in $\mathbf{X}$. Hence $\mathbf{CAC}(\mathbb{R}, C) \wedge \mathbf{M}(C, \leq |\mathbb{R}|)$ implies $\mathbf{M}(C, S)$.

($\leftarrow$) By Proposition 10(i), $\mathbf{M}(C, S)$ implies $\mathbf{M}(C, \leq |\mathbb{R}|)$. Now, we assume $\mathbf{M}(C, S)$ and prove that $\mathbf{CAC}(\mathbb{R}, C)$ holds. To this aim, we fix a disjoint family $\mathcal{A} = \{A_n : n \in \mathbb{N}\}$ of non-empty subsets of $\mathbb{R}$ such that there exists a family

$\{\rho_n : n \in \mathbb{N}\}$ of metrics such that, for every $n \in \mathbb{N}$, $\langle A_n, \rho_n \rangle$ is a compact metric space. Let $A = \bigcup \mathcal{A}$, let $\infty \notin A$ and $X = A \cup \{\infty\}$. Let d be the metric on X defined by $(*)$ in Sect. 2.3. Then, by Proposition 1, $\mathbf{X} = \langle X, d \rangle$ is a compact metric space. It follows from $\mathbf{M}(C, S)$ that $\mathbf{X}$ is separable. Let $H = \{x_n : n \in \mathbb{N}\}$ be a dense set in $\mathbf{X}$. For every $n \in \mathbb{N}$, let $m_n = \min\{m \in \mathbb{N} : x_m \in A_n\}$ and let $h(n) = x_{m_n}$. Then h is a choice function of $\mathcal{A}$. Hence $\mathbf{M}(C, S)$ implies $\mathbf{CAC}(\mathbb{R}, C)$.

- (ii) It was shown in [23] that $\mathbf{CAC}(\mathbb{R})$ and $\mathbf{M}(C, S)$ are independent of each other. Since $\mathbf{M}(C, S)$ implies $\mathbf{M}(C, \leq |\mathbb{R}|)$ (see Proposition 10(i)), while $\mathbf{CAC}(\mathbb{R})$ implies $\mathbf{CAC}(\mathbb{R}, C)$ but not $\mathbf{M}(C, S)$ the conclusion follows from (i). $\square$

Corollary 6 *In every permutation model, the statements $\mathbf{M}(C, \leq |\mathbb{R}|)$ and $\mathbf{M}(C, S)$ are equivalent.*

Proof Let $\mathcal{N}$ be a permutation model. Since $\mathbb{R}$ is well-orderable in $\mathcal{N}$ (see Sect. 2.4), $\mathbf{CAC}(\mathbb{R})$ is true in $\mathcal{N}$. This, together with Theorem 15(i), completes the proof. $\square$

Remark 17 (i) The proof to Corollary 6 shows that $\mathbf{M}(C, \leq |\mathbb{R}|)$ and $\mathbf{M}(C, S)$ are equivalent in every model of $\mathbf{ZFA}$ in which $\mathbb{R}$ is well-orderable.

- (ii) In much the same way, as in the proof to Theorem 15(ii)($\leftarrow$), one can show that, for every family $\{\mathbf{X}_n : n \in \omega\}$ of pairwise disjoint compact spaces, if the direct sum $\mathbf{X} = \bigoplus_{n \in \omega} \mathbf{X}_n$ is metrizable, then it is separable.

We include a sketch of a $\mathbf{ZF}$ -proof to the following lemma for completeness. We use this lemma in our $\mathbf{ZF}$ -proof that $\mathbf{M}(C, \hookrightarrow [0, 1]^{\mathbb{N}})$ and $\mathbf{M}(C, S)$ are equivalent.

Lemma 4 (ZF) *Suppose that $\mathcal{B}$ is a base of a non-empty metrizable space $\mathbf{X} = \langle X, \tau \rangle$. Then there exists a homeomorphic embedding of $\mathbf{X}$ into the cube $[0, 1]^{\mathcal{B} \times \mathcal{B}}$.*

Proof We may assume that X consists of at least two points. Let d be a metric on X such that $\tau = \tau(d)$ and let

$$\mathcal{W} = \{\langle U, V \rangle \in \mathcal{B} \times \mathcal{B} : \emptyset \neq \text{cl}_{\mathbf{X}} U \subseteq V \neq X\}.$$

For every $W = \langle U, V \rangle \in \mathcal{W}$, by defining

$$f_W(x) = \frac{d(x, \text{cl}_{\mathbf{X}}(U))}{d(x, \text{cl}_{\mathbf{X}}(U)) + d(x, X \setminus V)} \text{ whenever } x \in X,$$

we obtain a continuous function from $\mathbf{X}$ into $[0, 1]$. Let $h : X \rightarrow [0, 1]^{\mathcal{W}}$ be the evaluation mapping defined by: $h(x)(W) = f_W(x)$ for all $x \in X$ and $W \in \mathcal{W}$. Then h is a homeomorphic embedding of $\mathbf{X}$ into $[0, 1]^{\mathcal{W}}$. To complete the proof, it suffices to notice that $[0, 1]^{\mathcal{W}}$ is homeomorphic to a subspace of $[0, 1]^{\mathcal{B} \times \mathcal{B}}$. $\square$

Theorem 16 (ZF)

- (i) $\mathbf{M}(C, \hookrightarrow [0, 1]^{\mathbb{N}}) \leftrightarrow \mathbf{M}(C, S)$.
(ii) $\mathbf{M}(C, \leq |\mathbb{R}|) \rightarrow \mathbf{M}(C, \hookrightarrow [0, 1]^{\mathbb{R}})$.

Proof Let $\mathbf{X} = \langle X, \tau \rangle$ be an infinite compact metrizable space and let d be a metric on X such that $\tau = \tau(d)$.

- (i) $(\rightarrow)$ We assume $\mathbf{M}(C, \hookrightarrow [0, 1]^{\mathbb{N}})$ and show that $\mathbf{X}$ is separable. By our hypothesis, $\mathbf{X}$ is homeomorphic to a compact subspace $\mathbf{Y}$ of the Hilbert cube $[0, 1]^{\mathbb{N}}$. Since $[0, 1]^{\mathbb{N}}$ is second-countable, it follows from Theorem 6(i) that $\mathbf{Y}$ is separable. Hence $\mathbf{X}$ is separable. In consequence, $\mathbf{M}(C, \hookrightarrow [0, 1]^{\mathbb{N}})$ implies $\mathbf{M}(C, S)$.
 $(\leftarrow)$ If $\mathbf{M}(C, S)$ holds, then every compact metrizable space is second-countable. Since, by Lemma 4, every second-countable metrizable space is embeddable in the Hilbert cube $[0, 1]^{\mathbb{N}}$, $\mathbf{M}(C, S)$ implies $\mathbf{M}(C, \hookrightarrow [0, 1]^{\mathbb{N}})$.
(ii) Now, suppose that $\mathbf{M}(C, \leq |\mathbb{R}|)$ holds. Then $|X| \leq |\mathbb{R}|$. Since $|\mathbb{R}|^{<\omega} = |\mathbb{R}|$, we infer that $|[X]^{<\omega}| \leq |\mathbb{R}|$. For every $n \in \mathbb{N}$, let

$$k_n = \min \left\{ m \in \mathbb{N} : \bigcup_{x \in A} B_d \left(x, \frac{1}{n} \right) = X \text{ for some } A \in [X]^m \right\}$$

and

$$E_n = \left\{ A \in [X]^{k_n} : \bigcup_{x \in A} B_d \left(x, \frac{1}{n} \right) = X \right\}.$$

Let $\mathcal{B} = \{B_d(x, \frac{1}{n}) : x \in A, A \in E_n, n \in \mathbb{N}\}$. It is straightforward to verify that $\mathcal{B}$ is a base for $\mathbf{X}$ of size $|\mathcal{B}| \leq |\mathbb{R}| \times \mathbb{N} \leq |\mathbb{R}|$, so $|\mathcal{B}| \leq |\mathbb{R}|$. This, together with Lemma 4, implies that $\mathbf{X}$ is embeddable into $[0, 1]^{\mathbb{R}}$. Hence $\mathbf{M}(C, \leq |\mathbb{R}|)$ implies $\mathbf{M}(C, \hookrightarrow [0, 1]^{\mathbb{R}})$. $\square$

In view of Theorem 16, one may ask the following questions:

- Question 1** (i) Does $\mathbf{M}(C, \hookrightarrow [0, 1]^{\mathbb{R}})$ imply $\mathbf{M}(C, \hookrightarrow [0, 1]^{\mathbb{N}})$?
(ii) Does $\mathbf{M}(C, \hookrightarrow [0, 1]^{\mathbb{R}})$ imply $\mathbf{M}(C, \leq |\mathbb{R}|)$?
(iii) Does $\mathbf{CAC}_{fin}$ imply $\mathbf{M}(C, \hookrightarrow [0, 1]^{\mathbb{R}})$?
(iv) Does $\mathbf{M}(C, \hookrightarrow [0, 1]^{\mathbb{R}})$ imply $\mathbf{CAC}_{fin}$?

Remark 18 (a) Regarding Question 1 (i)–(ii), we notice that the answer is in the affirmative in permutation models. Indeed, let $\mathcal{N}$ be a permutation model. It is known that $\mathbb{R}$ and $\mathcal{P}(\mathbb{R})$ are well-orderable in $\mathcal{N}$ (see Sect. 2.4). Therefore, assuming that $\mathbf{M}(C, \hookrightarrow [0, 1]^{\mathbb{R}})$ holds in $\mathcal{N}$ and working inside $\mathcal{N}$, we deduce that, given a compact metrizable space $\mathbf{X}$ in $\mathcal{N}$, $\mathbf{X}$ embeds in $[0, 1]^{\mathbb{R}}$. Hence $\mathbf{X}$ is a well-orderable space, so $\mathbf{X}$ is Loeb. Since $\mathbf{X}$ is a compact metrizable Loeb space, by Theorem 6(i), $\mathbf{X}$ is second-countable. Therefore, by Lemma 4, $\mathbf{X}$ embeds in $[0, 1]^{\mathbb{N}}$ and, consequently, $|X| \leq |\mathbb{R}|$.

- (b) Regarding Question 1(iii), we note that $\mathbf{CAC}_{fin}$ holds in the permutation model $\mathcal{N}_{cr}$. To show that $\mathbf{M}(C, \hookrightarrow [0, 1]^{\mathbb{R}})$ fails in $\mathcal{N}_{cr}$, we observe that, by Proposition 6, there exists a compact metric space $\mathbf{X} = \langle X, d \rangle$ in $\mathcal{N}_{cr}$ such that X is not well-orderable in $\mathcal{N}_{cr}$. Since $[0, 1]^{\mathbb{R}}$, being equipotent to the well-orderable set $\mathcal{P}(\mathbb{R})$ of $\mathcal{N}_{cr}$, is well-orderable in $\mathcal{N}_{cr}$, it is true in $\mathcal{N}_{cr}$ that $\mathbf{X}$ is not embeddable in

$[0, 1]^{\mathbb{R}}$. This explains why $\mathbf{M}(C, \hookrightarrow [0, 1]^{\mathbb{R}})$ fails in $\mathcal{N}_{cr}$. Therefore, since the conjunction $\mathbf{CAC}_{fin} \wedge \neg \mathbf{M}(C, \hookrightarrow [0, 1]^{\mathbb{R}})$ has a permutation model, it also has a **ZF**-model by Theorem 7.

To shed more light on Questions 1 (iii)-(iv), let us prove the following Theorems 17 and 18.

Theorem 17 (ZF)

- (i) $\mathbf{M}(C, |\mathcal{B}_Y| \leq |\mathcal{B}|) \rightarrow \mathbf{M}([0, 1], |\mathcal{B}_Y| \leq |\mathcal{B}|) \rightarrow \mathbf{IDI}(\mathbb{R})$.
- (ii) *The following are equivalent:*
 - (a) *every compact (0-dimensional) subspace of the Tychonoff cube $[0, 1]^{\mathbb{R}}$ has a base of size $\leq |\mathbb{R}|$;*
 - (b) *every compact (0-dimensional) subspace of the Tychonoff cube $[0, 1]^{\mathbb{R}}$ with a unique accumulation point has a base of size $\leq |\mathbb{R}|$;*
 - (c) **Part**($\mathbb{R}$).
- (iii) *Every compact metrizable subspace of the Tychonoff cube $[0, 1]^{\mathbb{R}}$ with a unique accumulation point has a base of size $\leq |\mathbb{R}|$ iff for every denumerable family $\mathcal{A}$ of finite subsets of $\mathcal{P}(\mathbb{R})$ such that $\bigcup \mathcal{A}$ is pairwise disjoint, $|\bigcup \mathcal{A}| \leq |\mathbb{R}|$.*

Proof (i) Assume that **IDI**($\mathbb{R}$) is false. By a result of N. Brunner (cf. [13] and [15, Note 2]), there exists a Dedekind-finite dense subset of the interval $(0, 1)$ with its usual topology. Then

$$\mathcal{B} = \{(x, y) : x, y \in D, x < y\} \cup \{[0, x) : x \in D\} \cup \{(x, 1] : x \in D\}$$

is a base for the usual topology of $[0, 1]$. Clearly, the set $\mathcal{B}$ is Dedekind-finite. Let us consider the compact subspace Y of $[0, 1]$ where

$$Y = \{0\} \cup \left\{ \frac{1}{n} : n \in \mathbb{N} \right\}. \quad (11)$$

Since $\{\frac{1}{n} : n \in \mathbb{N}\} \subseteq \mathcal{B}_Y$, the set $\mathcal{B}_Y$ is Dedekind-infinite. Therefore, if $|\mathcal{B}_Y| \leq |\mathcal{B}|$, then $\mathcal{B}$ is Dedekind-infinite but this is impossible. Hence $\mathbf{M}([0, 1], |\mathcal{B}_Y| \leq |\mathcal{B}|)$ implies **IDI**($\mathbb{R}$). It is clear that $\mathbf{M}(C, |\mathcal{B}_Y| \leq |\mathcal{B}|)$ implies $\mathbf{M}([0, 1], |\mathcal{B}_Y| \leq |\mathcal{B}|)$. This completes the proof to (i).

(ii) It is obvious that (a) implies (b).

(b) $\rightarrow$ (c) Fix a partition $\mathcal{P}$ of $\mathbb{R}$. That is, $\mathcal{P}$ is a disjoint family of non-empty subsets of $\mathbb{R}$ such that $\mathbb{R} = \bigcup \mathcal{P}$. Assuming (b), we show that $|\mathcal{P}| \leq |\mathbb{R}|$. For $P \in \mathcal{P}$, let $f_P : \mathbb{R} \rightarrow \{0, 1\}$ be the characteristic function of P and let $f(x) = 0$ for each $x \in \mathbb{R}$. We put

$$X = \{f\} \cup \{f_P : P \in \mathcal{P}\}.$$

We claim that the subspace X of $[0, 1]^{\mathbb{R}}$ is compact. To see this, let us consider an arbitrary family $\mathcal{U}$ of open subsets of $[0, 1]^{\mathbb{R}}$ such that $X \subseteq \bigcup \mathcal{U}$. There exists

$U_0 \in \mathcal{U}$ such that $f \in U_0$. There exist $\varepsilon \in (0, 1)$ and a non-empty finite subset J of $\mathbb{R}$ such that the set

$$V = \bigcap \{ \pi_j^{-1}([0, \varepsilon)) : j \in J \}$$

is a subset of U_0 where, for each $j \in \mathbb{R}$ and $x \in [0, 1]^{\mathbb{R}}$, $\pi_j(x) = x(j)$. Since J is finite, there exists a finite set $\mathcal{P}_J \subseteq \mathcal{P}$ such that $J \subseteq \bigcup \mathcal{P}_J$. We notice that, for every $P \in \mathcal{P} \setminus \mathcal{P}_J$ and every $j \in J$, $f_P(j) = 0$. Hence $f_P \in U_0$ for every $P \in \mathcal{P} \setminus \mathcal{P}_J$. This implies that there exists a finite set $\mathcal{W}$ such that $\mathcal{W} \subseteq \mathcal{U}$ and $X \subseteq \bigcup \mathcal{W}$. Hence $\mathbf{X}$ is compact as claimed. If $P \in \mathcal{P}$ and $j \in P$, then $\pi_j^{-1}((\frac{1}{2}, 1]) \cap X = \{f_P\}$, so f_P is an isolated point of $\mathbf{X}$. Hence f is the unique accumulation point of $\mathbf{X}$. The space $\mathbf{X}$ is also 0-dimensional. By (b), $\mathbf{X}$ has a base $\mathcal{B}$ equipotent to a subset of $\mathbb{R}$. Since $\{\{f_P\} : P \in \mathcal{P}\} \subseteq \mathcal{B}$, it follows that $|\mathcal{P}| \leq |\mathbb{R}|$ as required.

(c) $\rightarrow$ (a) We assume **Part**($\mathbb{R}$) and fix a compact subspace $\mathbf{X}$ of the cube $[0, 1]^{\mathbb{R}}$. It is well known that $[0, 1]^{\mathbb{R}}$ is separable in **ZF** (cf., e.g., [25]). Fix a countable dense subset D of $[0, 1]^{\mathbb{R}}$. For every $y \in D$, let

$$\mathcal{V}_y = \left\{ \bigcap \{ \pi_i^{-1}((y(i) - 1/m, y(i) + 1/m)) : i \in F \} : \emptyset \neq F \in [\mathbb{R}]^{<\omega}, m \in \mathbb{N} \right\}.$$

Since $|\mathbb{R}|^{<\omega} = |\mathbb{R} \times \mathbb{N}| = |\mathbb{R}|$ in **ZF**, it follows that $\mathcal{B} = \bigcup \{ \mathcal{V}_y : y \in D \}$ is equipotent to $\mathbb{R}$. It is a routine work to verify that $\mathcal{B}$ is a base for $[0, 1]^{\mathbb{R}}$. Define an equivalence relation $\sim$ on $\mathcal{B}$ by requiring:

$$O \sim Q \text{ iff } O \cap X = Q \cap X. \quad (12)$$

Clearly

$$\mathcal{B}_X = \{ P \cap X : [P] \in \mathcal{B} / \sim \} \quad (13)$$

is a base for $\mathbf{X}$ of size $|\mathcal{B} / \sim|$. Since $|\mathcal{B} / \sim| \leq |\mathbb{R}|$, it follows that $|\mathcal{B}_X| \leq |\mathbb{R}|$ as required.

- (iii) $(\rightarrow)$ Fix family $\mathcal{A} = \{A_n : n \in \mathbb{N}\}$ of finite subsets of $\mathcal{P}(\mathbb{R})$ such that the family $\mathcal{P}_0 = \bigcup \mathcal{A}$ is pairwise disjoint. Let $\mathcal{P}_1 = \mathcal{P}_0 \cup \{\mathbb{R} \setminus \bigcup \mathcal{P}_0\}$ and $\mathcal{P} = \mathcal{P}_1 \setminus \{\emptyset\}$. Then $\mathcal{P}$ is a partition of $\mathbb{R}$. Let f, f_P with $P \in \mathcal{P}$ and X be defined as in the proof of (ii) that (b) implies (c). Since $\mathcal{P}$ is a cuf set, the space $\mathbf{X}$ has a σ -locally finite base. This, together with Theorem 5(ii), implies that $\mathbf{X}$ is metrizable. Suppose $\mathbf{X}$ has a base $\mathcal{B}$ such that $|\mathcal{B}| \leq |\mathbb{R}|$. In much the same way, as in the proof that (b) implies (c) in (ii), we can show that $|\mathcal{P}| \leq |\mathbb{R}|$. Then $|\bigcup \mathcal{A}| \leq |\mathbb{R}|$.

$(\leftarrow)$ Now, we consider an arbitrary compact metrizable subspace $\mathbf{X}$ of the cube $[0, 1]^{\mathbb{R}}$ such that $\mathbf{X}$ has a unique accumulation point. Let x_0 be the accumulation point of $\mathbf{X}$ and let d be a metric on X which induces the topology of $\mathbf{X}$. For every $x \in X \setminus \{x_0\}$ let

$$n_x = \min \left\{ n \in \mathbb{N} : B_d \left(x, \frac{1}{n} \right) = \{x\} \right\}.$$

For every $n \in \mathbb{N}$, let

$$E_n = \{x \in X : n_x = n\}.$$

Without loss of generality, we may assume that, for every $n \in \mathbb{N}$, $E_n \neq \emptyset$. Since $\mathbf{X}$ is compact, it follows easily that, for every $n \in \mathbb{N}$, the set E_n is finite. Let us apply the base $\mathcal{B}$ of $[0, 1]^{\mathbb{R}}$ given in the proof of part (ii) that (c) implies (a). Let $\sim$ be the equivalence relation on $\mathcal{B}$ given by (12). Let

$$\mathcal{C}_X = \{[P] \in \mathcal{B}/\sim : |P \cap (X \setminus \{x_0\})| = 1\}.$$

For every $n \in \mathbb{N}$, let

$$C_n = \{[P] \in \mathcal{C}_X : P \cap E_n \neq \emptyset\}.$$

Clearly, for every $n \in \mathbb{N}$, $|C_n| = |E_n|$. Let $\mathcal{C} = \bigcup \{C_n : n \in \mathbb{N}\}$. There exists a bijection $\psi : \mathcal{B} \rightarrow \mathbb{R}$. For every $n \in \mathbb{N}$, we put $A_n = \{\{\psi(U) : U \in H\} : H \in C_n\}$. Then, for every $n \in \mathbb{N}$, A_n is a finite subset of $\mathcal{P}(\mathbb{R})$. Let $\mathcal{A} = \{A_n : n \in \mathbb{N}\}$. Then $\bigcup \mathcal{A}$ is pairwise disjoint. Suppose that $|\bigcup \mathcal{A}| \leq |\mathbb{R}|$. Then $|\mathcal{C}| \leq |\mathbb{R}|$. This implies that the family $\mathcal{W} = \{P \cap (X \setminus \{x_0\}) : [P] \in \mathcal{C}\}$ is of size $\leq |\mathbb{R}|$. The family $\mathcal{G} = \mathcal{W} \cup \{B_d(x_0, \frac{1}{n}) : n \in \mathbb{N}\}$ is a base of $\mathbf{X}$ such that $|\mathcal{G}| \leq |\mathbb{R}|$. $\square$

The following theorem leads to a partial answer to Question 1(iv).

Theorem 18 (ZF)

- (i) $(\mathbf{M}(C, \hookrightarrow [0, 1]^{\mathbb{R}}) \wedge \mathbf{Part}(\mathbb{R})) \leftrightarrow \mathbf{M}(C, B(\mathbb{R}))$.
- (ii) $\mathbf{M}(C, S) \rightarrow \mathbf{M}(C, W(\mathbb{R})) \rightarrow \mathbf{M}(C, B(\mathbb{R})) \rightarrow \mathbf{CAC}_{fin}$.
- (iii) $(\mathbf{CAC}(\mathbb{R}) \wedge \mathbf{M}(C, B(\mathbb{R}))) \rightarrow \mathbf{M}(C, S)$.
- (iv) $\mathbf{CAC}(\mathbb{R}) \rightarrow (\mathbf{M}(C, S) \leftrightarrow \mathbf{M}(C, W(\mathbb{R})) \leftrightarrow \mathbf{M}(C, B(\mathbb{R})))$.

Proof (i) This follows from Theorem 17(ii) and Lemma 4.

- (ii) It is obvious that $\mathbf{M}(C, W(\mathbb{R}))$ implies $\mathbf{M}(C, B(\mathbb{R}))$. Assume $\mathbf{M}(C, S)$ and let $\mathbf{Y} = \langle Y, \tau \rangle$ be an infinite compact metrizable separable space. Since $\mathbf{Y}$ is second-countable and $|\mathbb{R}^\omega| = |\mathbb{R}|$, it follows that $|\tau| \leq |\mathbb{R}|$. To show that $|\mathbb{R}| \leq |\tau|$, we notice that, since X is infinite and $\mathbf{X}$ is second-countable, there exists a disjoint family $\{U_n : n \in \omega\}$ such that, for each $n \in \omega$, $U_n \in \tau$. For $J \in \mathcal{P}(\omega)$, we put $\psi(J) = \bigcup \{U_n : n \in J\}$ to obtain an injection $\psi : \mathcal{P}(\omega) \rightarrow \tau$. Hence $|\mathbb{R}| = |\mathcal{P}(\omega)| \leq |\tau|$.

To see that $\mathbf{M}(C, B(\mathbb{R})) \rightarrow \mathbf{CAC}_{fin}$, we assume $\mathbf{M}(C, B(\mathbb{R}))$, fix a disjoint family $\mathcal{A} = \{A_n : n \in \mathbb{N}\}$ of non-empty finite sets and show that $\mathcal{A}$ has a choice function. To this aim, we put $A = \bigcup \mathcal{A}$, take an element $\infty \notin A$ and $X = A \cup \{\infty\}$. For each $n \in \mathbb{N}$, let ρ_n be the discrete metric on A_n . Let d be the metric on X defined by $(*)$ in Sect. 2.3. By our hypothesis, the space $\mathbf{X} = \langle X, d \rangle$ has a base $\mathcal{B}$ of size $\leq |\mathbb{R}|$. Let $\psi : \mathcal{B} \rightarrow \mathbb{R}$ be an injection. Since $\{\{x\} : x \in A\} \subseteq \mathcal{B}$ and the sets A_n are finite, for each $n \in \mathbb{N}$, we can define $A_n^* = \{\psi(\{x\}) : x \in A_n\}$ and $a_n^* = \min A_n^*$. For each $n \in \mathbb{N}$, there is a unique $x_n \in A_n$ such that $\psi(\{x_n\}) = a_n^*$. This shows that $\mathcal{A}$ has a choice function.

- (iii) Now, we assume both **CAC**($\mathbb{R}$) and **M**($C, B(\mathbb{R})$). Let us consider an arbitrary compact metric space $\mathbf{X} = \langle X, \rho \rangle$. By our hypothesis, $\mathbf{X}$ has a base $\mathcal{B}$ of size $\leq |\mathbb{R}|$. Since, $|\mathbb{R}|^{<\omega} \leq |\mathbb{R}|$, it follows that $|\mathcal{B}|^{<\omega} \leq |\mathbb{R}|$. For every $n \in \mathbb{N}$, let

$$\mathcal{A}_n = \left\{ \mathcal{F} \in [\mathcal{B}]^{<\omega} : \bigcup \mathcal{F} = X \wedge \forall F \in \mathcal{F} \left(\delta_\rho(F) \leq \frac{1}{n} \right) \right\}.$$

Since $\mathbf{X}$ is compact, ρ is totally bounded. Therefore, $\mathcal{A}_n \neq \emptyset$ for every $n \in \mathbb{N}$. By **CAC**($\mathbb{R}$), we can fix a sequence $(\mathcal{F}_n)_{n \in \mathbb{N}}$ such that, for every $n \in \mathbb{N}$, $\mathcal{F}_n \in \mathcal{A}_n$. Since $|\mathcal{B}|^{<\omega} \leq |\mathbb{R}|$, we can also fix a sequence $(\leq_n)_{n \in \mathbb{N}}$ such that, for every $n \in \mathbb{N}$, $\leq_n$ is a well-ordering on $\mathcal{F}_n$. This implies that the family $\mathcal{F}_0 = \bigcup \{ \mathcal{F}_n : n \in \mathbb{N} \}$ is countable. Furthermore, it is a routine work to verify that $\mathcal{F}_0$ is a base of $\mathbf{X}$. Hence $\mathbf{X}$ is second-countable. By Theorem 6 (i), $\mathbf{X}$ is separable. This completes the proof to (iii).

That (iv) holds follows directly from (ii) and (iii). $\square$

Our proof of the following theorem emphasizes the usefulness of Theorems 8 and 18:

Theorem 19 (a) *The following implications are true in **ZF**:*

$$\mathbf{WO}(\mathcal{P}(\mathbb{R})) \rightarrow \mathbf{WO}(\mathbb{R}) \rightarrow \mathbf{Part}(\mathbb{R}) \rightarrow (\mathbf{M}(C, \hookrightarrow [0, 1]^{\mathbb{R}}) \rightarrow \mathbf{CAC}_{fin}).$$

- (b) *There exists a symmetric model of **ZF** + **CH** + **WO**($\mathcal{P}(\mathbb{R})$) in which the statement **M**($C, \hookrightarrow [0, 1]^{\mathbb{R}}$) is false. Hence, **M**($C, \hookrightarrow [0, 1]^{\mathbb{R}}$) does not follow from **Part**($\mathbb{R}$) in **ZF**.*

Proof It is obvious that the first two implications of (a) are true in **ZF**. Thus, it follows directly from Theorem 18 (i)–(ii) that (a) holds. To prove (b), let us notice that, in the light of Theorem 8, we can fix a symmetric model $\mathcal{M}$ of **ZF** + **CH** + **WO**($\mathcal{P}(\mathbb{R})$) + $\neg \mathbf{CAC}_{fin}$. It follows from (a) that **Part**($\mathbb{R}$) is true in $\mathcal{M}$ but **M**($C, \hookrightarrow [0, 1]^{\mathbb{R}}$) fails in $\mathcal{M}$. $\square$

Remark 19 (i) To show that **Part**($\mathbb{R}$) is not provable in **ZF**, let us recall that, in [11], a **ZF**-model Γ was constructed such that, in Γ , there exists a family $\mathcal{F} = \{F_n : n \in \mathbb{N}\}$ of two-element sets such that $\bigcup \mathcal{F}$ is a partition of $\mathbb{R}$ but $\mathcal{F}$ does not have a choice function. Then, in Γ , there does not exist an injection $\psi : \bigcup \mathcal{F} \rightarrow \mathbb{R}$ (otherwise, $\mathcal{F}$ would have a choice function in Γ). Hence **Part**($\mathbb{R}$) fails in Γ .

- (ii) Since **Part**($\mathbb{R}$) is independent of **ZF**, it follows from Theorem 17(ii) that it is not provable in **ZF** that every compact metrizable subspace of the cube $[0, 1]^{\mathbb{R}}$ has a base of size $\leq |\mathbb{R}|$. We do not know if **M**($C, \hookrightarrow [0, 1]^{\mathbb{R}}$) implies every compact metrizable subspace of the cube $[0, 1]^{\mathbb{R}}$ has a base of size $\leq |\mathbb{R}|$.
- (iii) It is not provable in **ZFA** that every compact metrizable space with a unique accumulation point embeds in $[0, 1]^{\mathbb{R}}$. Indeed, in the Second Fraenkel model $\mathcal{N}_2$ of [15], there exists a disjoint family of two-element sets $\mathcal{A} = \{A_n : n \in \mathbb{N}\}$ whose union has no denumerable subset. Let $A = \bigcup \mathcal{A}$, $\infty \notin A$, $X = A \cup \{\infty\}$ and, for every $n \in \mathbb{N}$, let ρ_n be the discrete metric on A_n . Let d be the metric

on X defined by $(*)$ in Sect. 2.3. Let $\mathbf{X} = \langle X, \tau(d) \rangle$. Then $\mathbf{X}$ is a compact metrizable space having ∞ as its unique accumulation point. Since, in $\mathcal{N}2$, the set $[0, 1]^{\mathbb{R}}$ is well-orderable, while $\mathcal{A}$ has no choice function, it follows that $\mathbf{X}$ does not embed in the Tychonoff cube $[0, 1]^{\mathbb{R}}$. This shows that the statement “There exists a compact metrizable space with a unique accumulation point which is not embeddable in $[0, 1]^{\mathbb{R}}$ ” has a permutation model.

6 The list of open problems

For the convenience of readers, we summarize the open problems mentioned in Sects. 4 and 5.

1. Is $\mathbf{M}(C, WO)$ equivalent to or weaker than $\mathbf{M}(TB, WO)$ in $\mathbf{ZF}$? (Cf. Sect. 4, paragraph following Theorem 9.)
2. Does $\mathbf{M}(C, S)$ imply $\mathbf{CUC}$ in $\mathbf{ZF}$? (Cf. Sect. 4, Remark 14(a).)
3. Does $\mathbf{BPI}$ imply $\mathbf{CUC}$ in $\mathbf{ZF}$? (Cf. Sect. 4, Remark 14(a).)
4. Does $\mathbf{CUC}$ imply $\mathbf{M}(C, S)$ in $\mathbf{ZF}$? (Cf. Sect. 4, paragraph following Remark 14.)
5. Does $\mathbf{M}(C, \hookrightarrow [0, 1]^{\mathbb{R}})$ imply $\mathbf{CAC}_{fin}$ in $\mathbf{ZF}$? (Cf. Sect. 5, Question 1(iv).)
6. Is there a model of $\mathbf{ZF}$ in which $\mathbf{CAC}_{fin} \wedge \neg \mathbf{M}(IC, DI)$ is true? (Cf. Sect. 4, paragraph following Proposition 7.)

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Declarations

Conflict of interest The authors declare that they have no conflict of interest.

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