Korovkin-type theorems on $B(\mathcal{H})$ and their applications to function spaces

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Received: 7 March 2021 / Accepted: 25 March 2021 / Published online: 26 April 2021
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Abstract

We prove Korovkin-type theorems in the setting of infinite dimensional Hilbert space operators. The classical Korovkin theorem unified several approximation processes. Also, the non-commutative versions of the theorem were obtained in various settings such as Banach algebras, C^* -algebras and lattices etc. The Korovkin-type theorem in the context of preconditioning large linear systems with Toeplitz structure can be found in the recent literature. In this article, we obtain a Korovkin-type theorem on $B(\mathcal{H})$ which generalizes all such results in the recent literature. As an application of this result, we obtain Korovkin-type approximation for Toeplitz operators acting on various function spaces including Bergman space $A^2(\mathbb{D})$, Fock space $F^2(\mathbb{C})$ etc. These results are closely related to the preconditioning problem for operator equations with Toeplitz structure on the unit disk $\mathbb{D}$ and on the whole complex plane $\mathbb{C}$. It is worthwhile to notice that so far such results are available for Toeplitz operators on circle only. This also establishes the role of Korovkin-type approximation techniques on function spaces with certain oscillation property. To address the function theoretic questions using these operator theory tools will be an interesting area of further research.

Keywords Korovkin-type theorems · Toeplitz operators · Bergman space · Fock space · Convergence in cluster sense · Preconditioner

Mathematics Subject Classification Primary 47A58 · 47B35; Secondary 15A18

Communicated by Adrian Constantin.

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1 Introduction

The classical approximation theorem due to P. P. Korovkin is as follows:

Theorem 1.1 [13] *Let $\{\varphi_n\}$ be a sequence of positive linear maps on $C[0, 1]$ such that $\varphi_n(f) \rightarrow f$ uniformly for $f \in \{1, x, x^2\}$, then $\varphi_n(f) \rightarrow f$ uniformly for all $f \in C[0, 1]$.*

The set $\{1, x, x^2\}$ is called the **test set** and we know that C^* -algebra generated by $\{1, x, x^2\}$ is $C[0, 1]$. There are several generalizations of this result in commutative and non-commutative setting with appropriate convergence notions. The idea is to replace $C[0, 1]$ by an appropriate C^* -algebra/Banach algebra and test set by some finite subset. In this article, we deal with Korovkin-type theorems in the setting of preconditioners.

Let $\mathcal{H}$ be a complex separable Hilbert space and consider an orthonormal basis $\{e_n : n = 0, 1, 2, \dots\}$. Let $\{P_n\}$ be a sequence of orthogonal projections on $\mathcal{H}$ such that $P_n(x_0, x_1, \dots) = (x_0, x_1, \dots, x_{n-1}, 0, 0, \dots)$. Then $\dim(P_n(\mathcal{H})) = n$, $n = 1, 2, 3, \dots$ and $\lim_{n \rightarrow \infty} P_n(x) = x$, for every x in $\mathcal{H}$. Let $\{U_n\}$ be a sequence of unitary matrices over $\mathbb{C}$, where U_n is of order n for each n . For each $A \in \mathcal{B}(\mathcal{H})$, consider the following truncations $A_n = P_n A P_n$, which can be regarded as $n \times n$ matrices, by restricting the domain to the range of P_n . For each n , we define the commutative algebra M_{U_n} of matrices as follows:

$$M_{U_n} = \{A \in M_n(\mathbb{C}) : U_n^* A U_n \text{ is complex diagonal}\}.$$

Recall that $M_n(\mathbb{C})$ is a Hilbert space with the Frobenius scalar product, $\langle A, B \rangle = \text{trace}(B^* A)$. Observe that M_{U_n} is a closed subspace of $M_n(\mathbb{C})$. Let $P_{U_n}(\cdot)$ be the projection of $M_n(\mathbb{C})$ onto M_{U_n} . We call $P_{U_n}(A)$ the **preconditioner** for A . It can be shown that $P_{U_n}(A) = U_n \sigma(U_n^* A U_n) U_n^*$ where $\sigma(X)$ is the diagonal matrix having $X_{i,i}$ as the diagonal entries.

Here, mainly the Hilbert spaces under consideration will be function spaces like Hardy space $H^2(\mathbb{T})$, Bergman space $A^2(\mathbb{D})$ and Fock space $F^2(\mathbb{C})$. For $A \in \mathcal{B}(\mathcal{H})$, we consider Korovkin-type theorems for the convergence of $\{P_{U_n}(A_n) - A_n\}$ to $\{O_n\}$ (O_n is the $n \times n$ zero matrix) in Type 2 strong/weak cluster sense (this convergence notion will be defined later). This convergence notion arises from the preconditioning problem of large linear systems with Toeplitz structure.

The following Korovkin-type theorem for Toeplitz operators on Hardy space was obtained in [10].

Theorem 1.2 *Let $\{g_1, g_2, \dots, g_m\} \subseteq C(\mathbb{T})$ such that $\{P_{U_n}(T_n^H(g)) - T_n^H(g)\}$ converges to $\{O_n\}$ in Type 2 strong cluster sense for every g in the set $\{g_1, g_2, \dots, g_m, \sum_{k=1}^m g_k g_k^*\}$. Then $\{P_{U_n}(T_n^H(g)) - T_n^H(g)\}$ converges to $\{O_n\}$ in strong cluster sense (Type 2) for every g in the C^* -algebra generated by $\{g_1, g_2, \dots, g_m\}$.*

Here T_f^H is the Toeplitz operator on $H^2(\mathbb{T})$ and $T_n^H(f) = P_n T_f^H P_n$.

The study of such type of Korovkin theorems was initiated by Stefano Serra-Capizzano in 1999 [14]. The details of recent developments in this setting can be found

in [11]. These results were strengthened and a Korovkin-type theorem for bounded self-adjoint operators acting on a separable Hilbert space was obtained in [12].

Theorem 1.3 (Theorem 4.8 in [12]) *Let $\{A_1, \dots, A_m\}$ be a finite set of self-adjoint operators on $\mathcal{H}$ and Φ_n be a sequence of contractive positive maps on $\mathcal{B}(\mathcal{H})$ such that $\{\Phi_n(A)\}$ converges to $\{A\}$ in the strong (weak respectively) distribution sense for A in $\{A_1, \dots, A_m, A_1^2, \dots, A_m^2\}$. In addition, assume that the difference $\{P_n(A_k^2)P_n - (P_n(A_k)P_n)^2\}$ converges to $\{O_n\}$ in the strong (weak respectively) cluster sense for each k . Then $\{\Phi_n(A)\}$ converges to $\{A\}$ in the strong (weak respectively) distribution sense for all A in the J^* -subalgebra $\mathbb{A}$ generated by $\{A_1, \dots, A_m\}$.*

The convergence notion mentioned in the above theorem will be defined later. J^* -subalgebra is the norm-closed, $*$ -closed subspace of a C^* -algebra which is also closed under the Jordan product $a \circ b = \frac{1}{2}(ab + ba)$.

Aim of this article is to obtain a general result which unifies all the approximation processes obtained in [10,12,14]. We generalize Theorem 1.3 to the case of arbitrary bounded operators and obtain convergence for operators in the C^* -algebra generated by the test set. The advantage is that we are able to go beyond self-adjoint class, which includes a lot of important examples. As an application of this result, we obtain Korovkin-type theorems for Toeplitz operators on Bergman space, Fock space etc. In the case of Bergman space, we obtain Korovkin-type theorems for Toeplitz operators with symbols from a C^* -subalgebra of $VMO \cap L^\infty(\mathbb{D})$ which properly contains $C(\mathbb{D})$ and hence we could go beyond the class of continuous symbols. $VMO \cap L^\infty(\mathbb{D})$ is an important class of functions with Vanishing Mean Oscillation property and this class contains some functions which are nowhere continuous. Using the idea of tensor products, we could also obtain Korovkin-type theorems for Toeplitz operators acting on Hardy space on 2-torus and Fock space on $\mathbb{C}^2$. We could prove Korovkin-type theorems for Toeplitz operators on Bergman space over the unit ball of higher dimensions (we have obtained it for the unit ball of dimension 2). Also as an application of our main result, we obtain the convergence for operators in the C^* -algebra generated by Toeplitz operators associated with the symbols in the test set. This operator algebra will contain operators which are not Toeplitz. In Theorem 1.2, we obtain convergence only for Toeplitz operators with symbols from the C^* -algebra generated by the test set. Also, the test set and the C^* -algebra that we obtain in Theorem 1.2 are commutative. In the Korovkin-type theorems obtained here, the test set and the C^* -algebra are non-commutative, in general.

The major achievement of this article is that we could identify some of the operator theoretic tools that will be useful in several questions regarding complex function spaces. We give a list of open problems at the end.

1.1 Definitions and preliminary results

First we recall some basic definitions and state some important facts from [10], which will be useful in proving our main result.

Definition 1.4 (Type 1) Let $\{A_n\}$ and $\{B_n\}$ be two sequences of matrices, where A_n and B_n are of order $n \times n$ for each n . We say that $\{A_n - B_n\}$ converges to constant

sequence $\{O_n\}$ in **Type 1 strong cluster sense** if for any $\epsilon > 0$, there exist integers $N_{1,\epsilon}, N_{2,\epsilon}$ such that for all $n > N_{2,\epsilon}$, except at most possibly $N_{1,\epsilon}$ (independent of size n) eigenvalues, all eigenvalues of $A_n - B_n$ lie in the ϵ -neighbourhood of 0.

Definition 1.5 (Type 2) Let $\{A_n\}$ and $\{B_n\}$ be two sequences of matrices, where A_n and B_n are of order $n \times n$ for each n . We say that $\{A_n - B_n\}$ converges to constant sequence $\{O_n\}$ in **Type 2 strong cluster sense** if for any $\epsilon > 0$, there exist integers $N_{1,\epsilon}, N_{2,\epsilon}$ such that $A_n - B_n = R_n + N_n$, where for $n > N_{2,\epsilon}$, $\text{rank } R_n \leq N_{1,\epsilon}$ and $\|N_n\| < \epsilon$ ($N_{1,\epsilon}$ is independent of n).

Lemma 1.6 $\{A_n - B_n\}$ converges to $\{O_n\}$ in Type 2 strong cluster sense if and only if for each $\epsilon > 0$, there exist two positive integers $N_{1,\epsilon}$ and $N_{2,\epsilon}$ ($N_{1,\epsilon}$ is independent of n) such that for $n > N_{2,\epsilon}$, except $N_{1,\epsilon}$ singular values, all other singular values of $A_n - B_n$ are in $[0, \epsilon)$. Thus if $A_n - B_n$ is normal for all n , then $\{A_n - B_n\}$ converges to $\{O_n\}$ in Type 2 strong cluster sense if and only if $\{A_n - B_n\}$ converges to $\{O_n\}$ in Type 1 strong cluster sense.

Definition 1.7 Let $\{\Phi_n\}$ be a sequence of positive linear maps on $\mathcal{B}(\mathcal{H})$, where $\mathcal{H}$ is a separable Hilbert space and $\{P_n\}$ be a sequence of projections on $\mathcal{H}$ with rank n that converges strongly to the identity. For a bounded operator A on $\mathcal{H}$, we say that $\{\Phi_n(A)\}$ converges to $\{A\}$ in the **Type 1 strong distribution sense** if the sequence of matrices $\{P_n \Phi_n(A) P_n - P_n A P_n\}$ converges to $\{O_n\}$ in Type 1 strong cluster sense and we say that $\{\Phi_n(A)\}$ converges to $\{A\}$ in the **Type 2 strong distribution sense** if the sequence of matrices $\{P_n \Phi_n(A) P_n - P_n A P_n\}$ converges to $\{O_n\}$ in Type 2 strong cluster sense.

Remark 1.8 If $N_{1,\epsilon}$ is of $o(n)$, then we say that the convergence is in Type 1 weak cluster sense and Type 2 weak cluster sense, respectively. If $N_{1,\epsilon}$ is independent of ϵ , then we say that the convergence is in Type 1 and Type 2 uniform cluster sense, respectively.

Remark 1.9 For Hermitian matrices, Type 1 and Type 2 convergences are equivalent. Therefore, for positive linear maps between classes of bounded self-adjoint operators on $\mathcal{H}$, the notions in Definition 1.7 are equivalent (see [10]).

The particular positive linear maps that we are interested in are defined as follows. Consider the map $\Phi_n : \mathcal{B}(\mathcal{H}) \rightarrow M_{U_n}$ defined as $\Phi_n(A) = P_{U_n}(A_n)$, where $A_n = P_n A P_n$.

Theorem 1.10 [12] *The maps $\{\Phi_n\}$ defined above form a sequence of completely positive maps on $\mathcal{B}(\mathcal{H})$ such that*

- $\|\Phi_n\| = 1$, for each n .
- Φ_n is continuous in the strong topology of operators for each n .
- $\Phi_n(I) = I_n$ for each n , where I is the identity operator on $\mathcal{H}$ and I_n is the $n \times n$ identity matrix.

Remark 1.11 For special choices of U_n , we get important matrix algebras M_{U_n} which are useful in obtaining optimal preconditioners with better convergence rate in finding approximate solutions. We give concrete examples later.

Remark 1.12 For the maps Φ_n defined as in Theorem 1.10, the convergence in strong distribution sense and the convergence in strong cluster sense are the same.

The following lemmas are needed to prove our main results.

Lemma 1.13 [10] *Let $\{A_n\}$ and $\{B_n\}$ be two sequences of positive matrices, where for each n , A_n and B_n are of order $n \times n$ such that $\{A_n + B_n\}$ converges to $\{O_n\}$ in Type 1 strong cluster sense. Then $\{A_n\}$ and $\{B_n\}$ converge to $\{O_n\}$ in Type 1 strong cluster sense.*

Lemma 1.14 [10] *Let $\{A_n\}$ and $\{B_n\}$ be two sequences of matrices, where for each n , A_n and B_n are of order $n \times n$ such that $\{A_n - B_n\}$ converges to $\{O_n\}$ in Type 2 strong cluster sense. Let $\|B_n\| \leq \beta < \infty$, for some $\beta > 0$ and for all n . Then $\{A_n A_n^* - B_n B_n^*\}$ converges to $\{O_n\}$ in Type 2 strong cluster.*

Remark 1.15 Lemma 1.13 and 1.14 hold for weak cluster sense also.

In [9], an equivalent condition for self-adjoint compact operators in terms of strong cluster sense convergence is obtained.

Lemma 1.16 [9] *Let A, B be bounded self-adjoint operators. $A - B$ is compact if and only if $\{A_n - B_n\}$ converges to $\{O_n\}$ in Type 1 and Type 2 strong cluster sense.*

Lemma 1.17 *Let $A \in B(\mathcal{H})$. A is compact if and only if $\{A_n\}$ converges to $\{O_n\}$ in Type 2 strong cluster sense.*

Proof $A = A_R + iA_I$, where A_R and A_I are the real and imaginary parts of A . A is compact if and only if A_R and A_I are compact. By Lemma 1.16, A_R and A_I are compact if and only if $\{(A_R)_n\}$ and $\{(A_I)_n\}$ converge to $\{O_n\}$ in Type 2 strong cluster sense. Thus A is compact if and only if $\{A_n\}$ converge to $\{O_n\}$ in Type 2 strong cluster sense. $\square$

The main tool we used in proving the Korovkin-type theorems is a very important inequality due to Uchiyama [18]. We state the inequality below in its general form.

A *Schwarz map* on a C^* -algebra $\mathbb{A}$ is a positive linear map ψ satisfying the condition $\psi(a^*a) \geq \psi(a^*)\psi(a)$, for all $a \in \mathbb{A}$. A *generalized Schwarz map* on $\mathbb{A}$ with respect to the binary operation $\circ$ is a map Φ which satisfies $\Phi(x^*) = \Phi(x)^*$ and $\Phi(x^*) \circ \Phi(x) \leq \Phi(x^* \circ x)$ for every $x \in \mathbb{A}$. Here ' $\circ$ ' denotes a binary operation on $\mathbb{A}$ with certain properties (see [18]). We avoid listing these properties, instead we mention that these properties are satisfied by composition of operators, Jordan product and pointwise product of functions.

Theorem 1.18 (Uchiyama's inequality) [18] *Let Φ be a generalized Schwarz map with respect to a binary operation $\circ$ on a C^* -algebra $\mathbb{A}$. For $f, g \in \mathbb{A}$, let*

$$\begin{aligned} X &= \Phi(f^* \circ f) - \Phi(f)^* \circ \Phi(f) \geq 0, \\ Y &= \Phi(g^* \circ g) - \Phi(g)^* \circ \Phi(g) \geq 0, \\ Z &= \Phi(f^* \circ g) - \Phi(f)^* \circ \Phi(g). \end{aligned}$$

Then

$$|\phi(Z)| \leq |\phi(X)|^{\frac{1}{2}} |\phi(Y)|^{\frac{1}{2}} \text{ for all states } \phi \text{ on } \mathbb{A}.$$

Remark 1.19 Note that the above inequality holds for Schwarz maps with respect to the C^* -product. It can be shown that completely positive maps with norm less than or equal to 1 are Schwarz maps.

2 Main results

Now we obtain Korovkin-type theorems for arbitrary bounded operators and with conclusion valid in the C^* -algebra generated by the test set. Therefore we are able to go beyond self-adjoint case and to much larger space. The proof techniques are almost similar to that of [10, 12]. However we give all the details for the sake of completion.

Theorem 2.1 *Let $\{A_1, \dots, A_m\}$ be a finite set of bounded operators on $\mathcal{H}$ and $\{\Phi_n\}$ be a sequence of contractive completely positive maps on $\mathcal{B}(\mathcal{H})$ such that $\{\Phi_n(A)\}$ converges to $\{A\}$ in Type 2 strong (weak respectively) distribution sense for A in $\{A_1, \dots, A_m, \sum_{k=1}^m A_k^* A_k\}$. In addition, assume that the difference $\{P_n(A_k^* A_k) P_n - (P_n(A_k^*) P_n)(P_n(A_k) P_n)\}$ converges to $\{O_n\}$ in Type 2 strong (weak respectively) cluster sense for each k . Then $\{\Phi_n(A)\}$ converges to $\{A\}$ in Type 2 strong (weak respectively) distribution sense for all A in the C^* -algebra $\mathbb{A}$ generated by $\{A_1, \dots, A_m\}$.*

Proof First we consider the following sequences of matrices:

$$X_n = P_n \Phi_n(A_k^* A_k) P_n - (P_n \Phi_n(A_k^*) P_n)(P_n \Phi_n(A_k) P_n) \geq 0, \text{ (Due to Schwarz}$$

$$Y_n = P_n \Phi_n(A_l^* A_l) P_n - (P_n \Phi_n(A_l^*) P_n)(P_n \Phi_n(A_l) P_n) \geq 0, \text{ inequality)}$$

$$Z_n = P_n \Phi_n(A_k^* A_l) P_n - (P_n \Phi_n(A_k^*) P_n)(P_n \Phi_n(A_l) P_n).$$

Since all Φ_n' s are CP maps of norm less than or equal to 1, Uchiyama's inequality is applicable here. These sequences are norm bounded, in particular, we have, $\|Y_n\| < \gamma < \infty$, for all $n \in \mathbb{N}$ and for some $\gamma > 0$. Consider,

$$\begin{aligned} & \sum_{k=1}^m [P_n \Phi_n(A_k^* A_k) P_n - (P_n \Phi_n(A_k^*) P_n)(P_n \Phi_n(A_k) P_n)] \\ &= \sum_{k=1}^m [P_n \Phi_n(A_k^* A_k) P_n - P_n(A_k^* A_k) P_n] \\ & \quad + \sum_{k=1}^m [P_n(A_k^* A_k) P_n - (P_n(A_k^*) P_n)(P_n(A_k) P_n)] \\ & \quad + \sum_{k=1}^m [(P_n(A_k^*) P_n)(P_n(A_k) P_n) - (P_n \Phi_n(A_k^*) P_n)(P_n \Phi_n(A_k) P_n)]. \end{aligned}$$

By assumption and Lemma 1.14, we conclude that the whole sum converges to $\{O_n\}$ in Type 2 strong cluster sense. Hence by Lemma 1.13, we obtain that the sequence of

positive matrices, $\{X_n\}$ converges to $\{O_n\}$ in Type 2 strong cluster sense and hence in Type 1 strong cluster sense also. Now for each fixed $x \in \mathbb{C}^n$ with $\|x\| = 1$, consider the state ϕ_x on $B(\mathbb{C}^n)$ defined as $\phi_x(A) = \langle Ax, x \rangle$. By applying Uchiyama's inequality (Theorem 1.18), we get

$$|\langle Z_n x, x \rangle| \leq |\langle X_n x, x \rangle|^{1/2} |\langle Y_n x, x \rangle|^{1/2} = \langle X_n x, x \rangle^{1/2} \langle Y_n x, x \rangle^{1/2}$$

(since X_n and Y_n are positive, $\langle X_n x, x \rangle, \langle Y_n x, x \rangle \geq 0$ for all x). Let $Z_n = B_n + iC_n$, where B_n and C_n are self-adjoint for all n . Then,

$$|\langle B_n x, x \rangle| \leq \langle X_n x, x \rangle^{1/2} \langle Y_n x, x \rangle^{1/2} \text{ and } |\langle C_n x, x \rangle| \leq \langle X_n x, x \rangle^{1/2} \langle Y_n x, x \rangle^{1/2},$$

for all $x \in \mathbb{C}^n$. We will show that both $\{B_n\}$ and $\{C_n\}$ converge to $\{O_n\}$ in Type 1 strong cluster sense.

Consider B_n . Since $\{X_n\}$ converges to $\{O_n\}$ in Type 1 strong cluster sense, for $\epsilon > 0$, there exist natural numbers $N_{1,\epsilon}$ and $N_{2,\epsilon}$ such that for all $n > N_{2,\epsilon}$, except at most $N_{1,\epsilon}$ eigenvalues, all other eigenvalues are in $[0, \epsilon)$. For any Hermitian matrix A , let $\lambda_k(A)$ denote the k^{th} eigenvalue of A when arranged in descending order. Then $\lambda_k(X_n) \in [0, \epsilon)$, for $k > N_{1,\epsilon}$. Also by Min Max principle for eigenvalues of Hermitian matrices we have,

$$\begin{aligned} \lambda_k(X_n) &= \max_{\substack{M \subseteq \mathbb{C}^n \\ \dim M = k}} \min_{\substack{\|x\|=1 \\ x \in M}} \langle X_n x, x \rangle, \\ [\lambda_k(X_n)]^{1/2} &= \max_{\substack{M \subseteq \mathbb{C}^n \\ \dim M = k}} \min_{\substack{\|x\|=1 \\ x \in M}} [\langle X_n x, x \rangle]^{1/2}. \end{aligned}$$

Now,

$$\min_{\substack{\|x\|=1 \\ x \in M}} \langle B_n x, x \rangle \leq \min_{\substack{\|x\|=1 \\ x \in M}} [\langle X_n x, x \rangle]^{1/2} \sqrt{\gamma},$$

for any subspace M of $\mathbb{C}^n$. So we obtain,

$$\max_{\substack{M \subseteq H \\ \dim M = k}} \min_{\substack{\|x\|=1 \\ x \in M}} \langle B_n x, x \rangle \leq \sqrt{\epsilon \gamma}.$$

Let $\delta > 0$ and put $\epsilon = \frac{\delta^2}{\gamma}$, then $\lambda_k(B_n) \leq \delta$ for $k > N_{1,\epsilon}$. Similarly, $\lambda_k(-B_n) \leq \delta$, for $k > N_{1,\epsilon}$. But $\lambda_k(-B_n) = -\lambda_{n-k+1}(B_n)$. Therefore $\lambda_k(B_n) \leq \delta$ for $k = N_{1,\epsilon} + 1, \dots, n$ and $\lambda_k(B_n) \geq -\delta$ except at most for $k = n - N_{1,\epsilon} + 1, \dots, n$. This will imply that $-\delta \leq \lambda_k(B_n) \leq \delta$, for $k = N_{1,\epsilon} + 1, \dots, n - N_{1,\epsilon}$. Hence except at most $2N_{1,\epsilon}$ eigenvalues, all other eigenvalues of B_n are in $[-\delta, \delta]$ and note that $N_{1,\epsilon} = O(1)$ ($o(n)$ respectively). Thus $\{B_n\}$ converges to $\{O_n\}$ in Type 1 strong cluster sense. Similarly $\{C_n\}$ converges to $\{O_n\}$ in Type 1 strong cluster sense. Since both are self-adjoint, they converge to $\{O_n\}$ in Type 2 strong cluster sense also. Hence, $\{Z_n\}$ converges to $\{O_n\}$ in Type 2 strong cluster sense.

Now consider,

$$\begin{aligned} P_n \Phi_n(A_k^* A_l) P_n - P_n(A_k^* A_l) P_n \\ = [P_n \Phi_n(A_k^* A_l) P_n - (P_n \Phi_n(A_k^*) P_n)(P_n \Phi_n(A_l) P_n)] \\ + [(P_n \Phi_n(A_k^*) P_n)(P_n \Phi_n(A_l) P_n) - (P_n(A_k^*) P_n)(P_n(A_l) P_n)] \\ + [(P_n(A_k^*) P_n)(P_n(A_l) P_n) - P_n(A_k^* A_l) P_n]. \end{aligned}$$

The first term on the right hand side is Z_n and the last term is also in the form of Z_n for the contractive completely positive maps $P_n(\cdot)P_n$ on $\mathcal{B}(\mathcal{H})$. Therefore both terms converge to $\{O_n\}$ in Type 2 strong (weak respectively) cluster sense. By simple computation, the same can be proved for the middle term. Hence the conclusion is proved for operators of the form $A_k^* A_l$.

The same proof can be repeated for operators of the form $A_j^*(A_k A_l)$, using the boundedness of $A_k A_l$ and the convergence assumption on A_j in the strong (weak respectively) cluster sense. Hence the assertion is true for any operator from the $*$ -algebra generated by $\{A_1, \dots, A_m\}$.

Now for $A \in \mathbb{A}$ and $\epsilon > 0$, let T be an operator in the $*$ -algebra generated by $\{A_1, \dots, A_m\}$, such that $\|A - T\| < \epsilon/3$, $\|\Phi_n(A) - \Phi_n(T)\| < \epsilon/3$. Write

$$\begin{aligned} P_n \Phi_n(A) P_n - P_n A P_n &= [P_n \Phi_n(A) P_n - P_n \Phi_n(T) P_n] + [P_n \Phi_n(T) P_n - P_n T P_n] \\ &\quad + [P_n T P_n - P_n A P_n]. \end{aligned}$$

The norm of the sum of the first and third term is less than $2\epsilon/3$. Clearly the middle term $P_n \Phi_n(T) P_n - P_n T P_n$ can be split into a term with norm less than $\epsilon/3$ and a term with constant rank independent of n (or $o(n)$ respectively). Thus the sequence of matrices $\{P_n \Phi_n(A) P_n - P_n A P_n\}$ converges to $\{O_n\}$ in Type 2 strong (weak respectively) cluster sense. $\square$

As an application of Theorem 2.1, we prove Korovkin-type results for Toeplitz operators in the next section.

3 Korovkin-type theorems for Toeplitz operators

In this section, we obtain Korovkin-type results for Toeplitz operators on various function spaces like Bergman space $A^2(\mathbb{D})$ and Fock space $F^2(\mathbb{C})$. For each function space, we obtain two Korovkin-type results. In one of them, the convergence is for Toeplitz operators with symbols from the function algebra generated by the test set. In the other one, the convergence is for the operators in the operator algebra generated by the Toeplitz operators associated with the symbols in the test set. This operator algebra will also contain operators which are not Toeplitz. The latter result is obtained as an application of Theorem 2.1. In case of the Bergman space, we obtain Korovkin-type theorems which hold for continuous functions on the closed disk. Also, we obtained a class of functions which properly contains $C(\overline{\mathbb{D}})$ and for which Korovkin-type theorems hold. Similarly, for Fock space, we obtain a class of bounded functions for

which Korovkin-type theorems hold. In addition, we present an independent proof of the Bergman space result in Sect. 5.

3.1 Toeplitz operators on Bergman spaces

Let $\mathbb{D}$ be the unit disk. The Bergman space $A^2(\mathbb{D})$ is defined as the space of all analytic functions in $L^2(\mathbb{D}, dA)$, where dA is the normalized area measure given by $dA = \frac{1}{\pi} r dr d\theta$. Let P_A denote the orthogonal projection onto $A^2(\mathbb{D})$. Let $f \in L^\infty(\mathbb{D}, dA)$ and M_f denote the corresponding multiplication operator on $L^2(\mathbb{D}, dA)$. The Toeplitz operator T_f^B , defined on $A^2(\mathbb{D})$ is given by $T_f^B = P_A M_f$. Several properties of Toeplitz operators on $A^2(\mathbb{D})$ can be found in [20, 22].

3.1.1 Symbols from $C(\overline{\mathbb{D}})$

Let $\overline{\mathbb{D}}$ denote the closure of $\mathbb{D}$. Let $f, g \in C(\overline{\mathbb{D}})$ and $f' = f|_{\partial\mathbb{D}}$ and $g' = g|_{\partial\mathbb{D}}$ be their restrictions to the boundary, respectively. Let $\{e_j^B = \sqrt{j+1}z^j : j \in \mathbb{N}_0\}$ be the standard orthonormal basis of the Bergman space $A^2(\mathbb{D})$ and $T_n^B(f) = P_n T_f^B P_n$.

Consider the unitary map, $V : A^2(\mathbb{D}) \mapsto H^2(\mathbb{T})$, which maps e_j^B to e_j^H , where $\{e_j^H = z^j : j \in \mathbb{N}_0\}$ denotes the standard orthonormal basis for $H^2(\mathbb{T})$. Let T_f^H denote the Toeplitz operator on the Hardy space with symbol $f \in L^\infty(\mathbb{T})$. Then from [7], we have

Theorem 3.1 $V^* T_{f'}^H V = T_f^B + K$, where K is a compact operator and $f \in C(\overline{\mathbb{D}})$.

3.2 Symbols from VMO class

VMO denotes the class of functions having vanishing mean oscillation near the boundary. This class of functions properly contains $C(\overline{\mathbb{D}})$ [3].

For $z \in \mathbb{D}$, let $K_z(w) = \frac{1}{(1-w\bar{z})^2}$, the Bergman kernel and $k_z(w) = (1-|z|^2)K_z(w)$, its normalized version. For a bounded operator S on $A^2(\mathbb{D})$, the Berezin transform of S is the function $B(S)$ on $\mathbb{D}$ defined by $B(S)(z) = \langle S k_z, k_z \rangle$. Similarly, for a function $f \in L^1(\mathbb{D})$, the Berezin transform is

$$B(f)(z) = \int_{\mathbb{D}} f(w) |k_z(w)|^2 dw.$$

Note that for a Toeplitz operator T_f^B , $B(T_f^B)(z) = B(f)(z)$.

$$VMO = \{f \in L^2(\mathbb{D}) : \lim_{|z| \rightarrow 1^-} (B(|f|^2)(z) - |B(f)|^2(z)) = 0\}.$$

Since B fixes the boundary values of a continuous function on $\overline{\mathbb{D}}$, it follows that $C(\overline{\mathbb{D}}) \subseteq VMO$. Note that $VMO \cap L^\infty(\mathbb{D})$ is the maximal C^* -subalgebra of $L^\infty(\mathbb{D})$ such that whenever $f, g \in VMO \cap L^\infty(\mathbb{D})$, $T_{fg}^B - T_f^B T_g^B$ is compact [21]. We denote $\mathcal{Q} = VMO \cap L^\infty(\mathbb{D})$. The properties of $\mathcal{Q}$, described below can be found in [21].

Definition 3.2 [21] Let $f \in L^\infty(\mathbb{D})$. We say f is in ESV if for $\epsilon > 0$ and $\sigma \in (0, 1)$, there is $\delta_0 > 0$ such that $|f(z) - f(w)| < \epsilon$ whenever $|z|, |w| \in [1 - \delta, 1 - \sigma\delta]$, $\delta < \delta_0$ and $|\arg(z) - \arg(w)| \leq \max(1 - |z|, 1 - |w|)$.

ESV is a C^* -subalgebra of $\mathcal{Q}$ and $C(\overline{\mathbb{D}}) \subsetneq ESV$. Also, it is easy to see that all bounded functions in $\mathbb{D}$ which are continuous in a neighbourhood of $\partial\mathbb{D}$ belong to ESV .

Consider the set $\mathbb{B}$ defined as $\mathbb{B} = \{f \in L^\infty(\mathbb{D}) : B(f) \rightarrow 0 \text{ as } |z| \rightarrow 1^-\}$. As is known, for $f \in L^\infty(\mathbb{D})$, the Toeplitz operator T_f^B is compact if and only if $f \in \mathbb{B}$. It is easy to see that all compactly supported bounded functions on $\mathbb{D}$ belong to $\mathcal{Q} \cap \mathbb{B}$.

The following is a theorem proved by Kehe Zhu in [21]. It states that $\mathcal{Q}$ can be decomposed as sum of ESV and a closed self-adjoint ideal of $\mathcal{Q}$.

Theorem 3.3 1. $\mathcal{Q} \cap \mathbb{B}$ is a closed self adjoint ideal of $\mathcal{Q}$.
2. $\mathcal{Q} = ESV + \mathcal{Q} \cap \mathbb{B}$.

Now we give a flowchart depicting the classes on which Korovkin-type theorems have been obtained.

Classes on which Korovkin-type theorems hold in the case of $A^2(\mathbb{D})$

- $A \rightarrow B$ means $A \subseteq B$
- Green colour indicates that we obtained the Korovkin-type results and red colour indicates that we have not obtained the results.

Now we obtain the Korovkin-type theorems in the case of Toeplitz operators on Bergman space. The result is as follows.

Theorem 3.4 Let $\{g_1, \dots, g_m\} \subseteq C(\overline{\mathbb{D}})$ such that $\{P_{U_n}(T_n^B(g)) - T_n^B(g)\}$ converges to $\{O_n\}$ in Type 2 strong cluster sense for all $g \in \{g_1, \dots, g_m, \sum_{k=1}^m g_k g_k^*\}$. Then $\{P_{U_n}(T_n^B(g)) - T_n^B(g)\}$ converges to $\{O_n\}$ in Type 2 strong cluster sense for all g in the C^* -algebra generated by $\{g_1, \dots, g_m\}$.

Proof The proof techniques are similar to that of Theorem 1.2. The same technique is adapted to the proof of Theorem 2.1 also. The crucial part is to obtain the convergence

of $\{T_n^B(g_i g_j) - T_n^B(g_i)T_n^B(g_j)\}$ to $\{O_n\}$ in Type 2 strong cluster sense whenever $g_i, g_j \in C(\overline{\mathbb{D}})$. Here we establish this convergence first and then sketch the remaining part of the proof.

Let $f, g \in C(\overline{\mathbb{D}})$. By Theorem 3.1, we obtain, $V^*T_{f'}^H V = T_f^B + K_1$, $V^*T_{g'}^H V = T_g^B + K_2$, and $V^*T_{f'g'}^H V = T_{fg}^B + K_3$. Where K_1, K_2 and K_3 are compact operators. Let Q_n be the projection on $H^2(\mathbb{T})$ defined by

$Q_n(x_0, x_1, \dots) = (x_{n-1}, x_{n-2}, \dots, x_0, 0, 0, \dots)$. Let $P'_n = V^*P_n V$ and $Q'_n = V^*Q_n V$ be the corresponding projections on $A^2(\mathbb{D})$.

$$\begin{aligned} T_n^B(f) &= P'_n T_f^B P'_n \\ &= P'_n V^* T_{f'}^H V P'_n + P'_n K_1 P'_n \\ &= V^* [P_n T_{f'}^H P_n] V + K_{1,n}, \text{ where } K_{1,n} = P'_n K_1 P'_n \\ &= V_n^* T_n^H(f') V_n + K_{1,n}, \text{ where } V_n = P_n V P'_n \\ &\quad (\text{using } P_n^2 = P_n \text{ and } (P'_n)^2 = P'_n). \end{aligned}$$

Similarly we obtain,

$$\begin{aligned} T_n^B(g) &= V_n^* T_n^H(g') V_n + K_{2,n}, \\ T_n^B(fg) &= V_n^* T_n^H(f'g') V_n + K_{3,n}. \end{aligned}$$

Consider

$$T_n^B(fg) - T_n^B(f)T_n^B(g) = V_n^* (T_n^H(f'g') - T_n^H(f')T_n^H(g')) V_n + K_n,$$

where $f, g \in C(\overline{\mathbb{D}})$ and K_n is given by

$$K_n = V_n^* T_n^H(f') V_n K_{2,n} + K_{1,n} V_n^* T_n^H(g') V_n + K_{3,n} + K_{1,n} K_{2,n}.$$

By Lemma 3.1 in [10], we have $\{T_n^H(f'g') - T_n^H(f')T_n^H(g')\}$ converges to $\{O_n\}$ in Type 2 strong cluster sense. Now it suffices to show that $\{K_n\}$ converges to $\{O_n\}$ in Type 2 strong cluster sense. $K_{1,n}, K_{2,n}$ and $K_{3,n}$ are truncations of compact operators K_1, K_2 and K_3 , respectively. Hence by Lemma 1.17, they converge to $\{O_n\}$ in Type 2 strong cluster sense. Also, $\|V_n^* T_n^H(h) V_n\| \leq \|h\|_\infty < \infty$, for any $h \in L^\infty(\mathbb{T})$. Hence $\{K_n\}$ converges to $\{O_n\}$ in Type 2 strong cluster sense.

Now we outline the remaining part of the proof.

With the help of Uchiyama's inequality, we obtain that the sequence

$\{P_{U_n}(T_n^B(g_i^* g_j)) - P_{U_n}(T_n^B(g_i^*))P_{U_n}(T_n^B(g_j))\}$ converges to $\{O_n\}$ in Type 2 strong cluster sense, for $i, j = 1, 2, \dots, n$. Along with this, using the fact that $\{T_n^B(g_i^* g_j) - T_n^B(g_i^*)T_n^B(g_j)\}$ converges to $\{O_n\}$ in Type 2 strong cluster sense whenever $g_i, g_j \in C(\overline{\mathbb{D}})$, we obtain that $\{P_{U_n}(T_n^B(g_i^* g_j)) - T_n^B(g_i^* g_j)\}$ converges to $\{O_n\}$ in Type 2 strong cluster sense, for $i, j = 1, 2, \dots, n$. Thus we obtain that the convergence holds for Toeplitz operators with symbols from the $*$ -closed algebra S , generated by $\{g_1, \dots, g_m\}$. Now for $f \in C^*$ -algebra generated by $\{g_1, \dots, g_m\}$, the result follows by an approximation as in the proof of Theorem 2.1. $\square$

Now we address the question whether the convergence mentioned in Theorem 3.4 holds for the operators in the C^* -algebra generated by $\{T_g^B : g \in \{g_1, \dots, g_m\}\}$. We obtain this as an application of our main result, Theorem 2.1. Before that, we need to check whether $\{P_{U_n}(K_n) - K_n\}$ converges to $\{O_n\}$ in Type 2 strong cluster sense, whenever K is compact.

Lemma 3.5 [17] *If $\{A_n\}$ and $\{B_n\}$ are sequences of $n \times n$ Hermitian matrices such that $\|A_n - B_n\|_F^2 = O(1)$, then $\{A_n - B_n\}$ converges to $\{O_n\}$ in Type 1 strong cluster sense.*

Lemma 3.6 *$\{P_{U_n}(K_n)\}$ converges to $\{O_n\}$ in Type 2 strong cluster sense, whenever K is compact.*

Proof Let K be a compact self-adjoint operator. By Lemma 1.17, we know that $\{K_n\}$ converges to $\{O_n\}$ in Type 1 and Type 2 strong cluster sense. We need to show that $\{P_{U_n}(K_n)\}$ converges to $\{O_n\}$ in Type 2 strong cluster sense. Fix $\epsilon > 0$. Then there exist positive integers $N_{1,\epsilon}, N_{2,\epsilon}$ such that for all $n > N_{2,\epsilon}$, except at most $N_{1,\epsilon}$ eigenvalues, all other eigenvalues of K_n are in $(-\epsilon, \epsilon)$. Since K_n is self-adjoint, there exists a unitary matrix W_n such that $W_n^* K_n W_n$ is a diagonal matrix and the diagonal entries are eigenvalues of K_n . Note that every eigenvalue of K_n has absolute value at most $\|K\|$. Then we can decompose as $W_n^* K_n W_n = X_n + Y_n$, where X_n and Y_n are diagonal matrices. The non zero diagonal entries of X_n are k_i 's such that $|k_i| \geq \epsilon$ and that of Y_n are k_i 's with $|k_i| < \epsilon$. In particular, $\text{rank } X_n \leq N_{1,\epsilon}$ and $\|Y_n\| < \epsilon$. Let $R_n = W_n X_n W_n^*$ and $N_n = W_n Y_n W_n^*$. Then for $n > N_{2,\epsilon}$, $K_n = R_n + N_n$, with $\text{rank}(R_n) \leq N_{1,\epsilon}$ and $\|N_n\| < \epsilon$. Thus $P_{U_n}(K_n) = P_{U_n}(R_n) + P_{U_n}(N_n)$, for $n > N_{2,\epsilon}$. Note that $\|P_{U_n}(N_n)\| \leq \|N_n\| < \epsilon$.

We will show that $\{P_{U_n}(R_n)\}$ converges to $\{O_n\}$ in Type 2 strong cluster sense. Write $R_n = R_n^+ - R_n^-$, where R_n^+ and R_n^- are positive matrices. We choose R_n^+ and R_n^- in such a way that it has the following property. Let a_i, b_i and c_i be the i^{th} eigenvalues of R_n, R_n^+ and R_n^- , respectively, such that $\{a_i\}$ and $\{b_i\}$ are arranged in decreasing order with $a_i = b_i - c_i$.

Consider $P_{U_n}(R_n^+)$. It is clear that $\|R_n^+\|_F^2 \leq N_{1,\epsilon} \|K\|^2$ and $\text{trace}(R_n^+) \leq N_{1,\epsilon} \|K\|$. $P_{U_n}(R_n^+)$ is positive and note that $\text{trace}(P_{U_n}(R_n^+)) = \text{trace}(R_n^+)$. Then $\|P_{U_n}(R_n^+)\|_F^2 \leq (N_{1,\epsilon} \|K\|)^2$ and hence by Lemma 3.5, $\{P_{U_n}(R_n^+)\}$ converges to $\{O_n\}$ in Type 1 strong cluster sense. Similarly $\{P_{U_n}(R_n^-)\}$ converges to $\{O_n\}$ in Type 1 strong cluster sense. Hence $\{P_{U_n}(R_n)\}$ converges to $\{O_n\}$ in Type 2 strong cluster sense. So for fixed $\epsilon > 0$, there exist positive integers $M_{1,\epsilon}$ and $M_{2,\epsilon}$ ($M_{2,\epsilon} > N_{2,\epsilon}$) such that $P_{U_n}(R_n) = T_n + S_n$, where for $n > M_{2,\epsilon}$, $\text{rank } T_n \leq M_{1,\epsilon}$ and $\|S_n\| < \epsilon$. Thus for $\epsilon > 0$, there exist positive integers $M_{1,\epsilon}$ and $M_{2,\epsilon}$ such that $P_{U_n}(K_n) = T_n + S_n + P_{U_n}(N_n)$, where for $n > M_{2,\epsilon}$, $\|S_n + P_{U_n}(N_n)\| < 2\epsilon$.

Repeating the above process for any ϵ , it follows that $\{P_{U_n}(K_n)\}$ converges to $\{O_n\}$ in Type 1 strong cluster sense. By Lemma 1.6, $\{P_{U_n}(K_n)\}$ converges to $\{O_n\}$ in Type 2 strong cluster sense also. For an arbitrary compact operator K , apply the above argument to its real and imaginary part. Hence $\{P_{U_n}(K_n)\}$ converges to $\{O_n\}$ in Type 2 strong cluster sense whenever K is compact. $\square$

Corollary 3.7 $\{P_{U_n}(K_n) - K_n\}$ converges to $\{O_n\}$ in Type 2 strong cluster sense whenever K is compact.

Remark 3.8 For $f, g \in C(\overline{\mathbb{D}})$, $T_{fg}^B - T_f^B T_g^B$ is compact. Hence $\{(T_{fg}^B - T_f^B T_g^B)_n\}$ converges to $\{O_n\}$ in Type 2 strong cluster sense.

Theorem 3.9 Let $\{g_1, \dots, g_m\} \subseteq C(\overline{\mathbb{D}})$ such that $\{P_{U_n}(T_n^B(g)) - T_n^B(g)\}$ converges to $\{O_n\}$ in Type 2 strong cluster sense for all $g \in \{g_1, \dots, g_m, \sum_{k=1}^m g_k g_k^*\}$. Then $\{P_{U_n}(T_n) - T_n\}$ converges to $\{O_n\}$ in Type 2 strong cluster sense for all T in the C^* -algebra C generated by $\{T_g^B : g \in \{g_1, \dots, g_m\}\}$.

Proof Theorem 3.9 can be proved by applying Theorem 2.1 and Lemma 3.6. Here, $\Phi_n(A) = P_{U_n}(A_n)$ where $A_n = P_n A P_n$. The assumption that the difference

$$\{P_n(A_k^* A_k) P_n - (P_n(A_k^*) P_n)(P_n(A_k) P_n)\} \text{ converges to } \{O_n\}$$

in the Type 2 strong cluster sense holds automatically (here $A_k = T_{g_k}^B$).

In fact, for any $g \in C(\overline{\mathbb{D}})$,

$$\begin{aligned} P_n(T_{g^* g}^B T_g^B) P_n - (P_n(T_{g^*}^B) P_n)(P_n(T_g^B) P_n) &= [P_n(T_{g^* g}^B T_g^B) P_n - P_n(T_{g^* g}^B) P_n] \\ &\quad + [P_n(T_{g^* g}^B) P_n - (P_n(T_{g^*}^B) P_n)(P_n(T_g^B) P_n)]. \end{aligned}$$

The first term on right hand side is of the form $P_n K P_n$, where $K = T_{g^* g}^B - T_{g^*}^B T_g^B$ is compact and hence it converges to $\{O_n\}$ in Type 2 strong cluster sense. Second term is $T_n^B(g^* g) - T_n^B(g^*) T_n^B(g)$, which converges to $\{O_n\}$ in Type 2 strong cluster sense whenever g is continuous on $\mathbb{D}$ (see the proof of Theorem 3.4). Also, $\{P_{U_n}(\sum_{k=1}^m T_n^B(g_k^*) T_n^B(g_k)) - \sum_{k=1}^m T_n^B(g_k^*) T_n^B(g_k)\}$ converges to $\{O_n\}$ in Type 2 strong cluster sense. For if, $T_n^B(g_k^*) T_n^B(g_k) = T_n^B(g_k^* g_k) + K_n^k$, where K^k is a compact operator for each $k = 1, 2, 3, \dots, n$. By our assumption, $\{P_{U_n}(\sum_{k=1}^m T_n^B(g_k^* g_k)) - \sum_{k=1}^m T_n^B(g_k^* g_k)\}$ converges to $\{O_n\}$ in Type 2 strong cluster sense and due to Corollary 3.7, $\{P_{U_n}(K_n^k) - K_n^k\}$ converges to $\{O_n\}$ in Type 2 strong cluster sense, for each $k = 1, 2, \dots, n$. Hence Theorem 2.1 can be applied to obtain the result. $\square$

In Theorems 3.4 and 3.9, the result holds for Toeplitz operators with symbols from $C(\overline{\mathbb{D}})$. We shall check whether these theorems extend to Toeplitz operators with symbols from $\mathcal{Q} = VMO \cap L^\infty(\mathbb{D})$. If we apply the same proof techniques of Theorems 3.4 and 3.9, then the problem reduces to checking the convergence of $\{T_n^B(fg) - T_n^B(f) T_n^B(g)\}$ and $\{(T_f^B T_g^B)_n - T_n^B(f) T_n^B(g)\}$ to $\{O_n\}$ in Type 2 strong cluster sense, whenever $f, g \in \mathcal{Q}$. We could not extend the result to whole of $\mathcal{Q}$. But we could extend to a C^* -subalgebra of $\mathcal{Q}$ which properly contains $C(\overline{\mathbb{D}})$.

Consider the set $\mathcal{D} = C(\overline{\mathbb{D}}) + \mathcal{Q} \cap \mathbb{B}$. It can be shown that $\mathcal{D}$ is a C^* -subalgebra containing all bounded functions on $\mathbb{D}$ which are continuous in a neighbourhood of $\partial\mathbb{D}$.

Now consider the following result.

Lemma 3.10 Let $f, g \in \mathcal{D}$, then $\{T_n^B(fg) - T_n^B(f) T_n^B(g)\}$ converges to $\{O_n\}$ in Type 2 strong cluster sense.

Proof Let $f, g \in \mathcal{D}$ then $f = f^1 + f^2$ and $g = g^1 + g^2$, with $f^1, g^1 \in C(\overline{\mathbb{D}})$ and $f^2, g^2 \in Q \cap \mathbb{B}$. Then $fg = f^1g^1 + f^1g^2 + f^2g^1 + f^2g^2$. By Theorem 3.3, f^1g^2, f^2g^1 and $f^2g^2 \in Q \cap \mathbb{B}$. Hence by Lemma 1.17, $\{T_n^B(f^1g^2)\}$, $\{T_n^B(f^2g^1)\}$ and $\{T_n^B(f^2g^2)\}$ converge to $\{O_n\}$ in Type 2 strong cluster sense.

$$\begin{aligned} T_n^B(fg) - T_n^B(f)T_n^B(g) &= T_n^B(f^1g^1) - T_n^B(f^1)T_n^B(g^1) \\ &\quad + T_n^B(f^1g^2) - T_n^B(f^1)T_n^B(g^2) \\ &\quad + T_n^B(f^2g^1) - T_n^B(f^2)T_n^B(g^1) \\ &\quad + T_n^B(f^2g^2) - T_n^B(f^2)T_n^B(g^2). \end{aligned}$$

Since $f^1, g^1 \in C(\overline{\mathbb{D}})$, the first term on the right hand side converges to $\{O_n\}$ in Type 2 strong cluster sense. $\{T_n^B(g^2)\}$ converges to $\{O_n\}$ in Type 2 strong cluster sense and the sequence $\{\|T_n^B(f^1)\|\}$ is bounded. Hence the second term converges to $\{O_n\}$ in Type 2 strong cluster sense. Similarly the third and fourth term also converge to $\{O_n\}$ in Type 2 strong cluster sense. Thus $\{T_n^B(fg) - T_n^B(f)T_n^B(g)\}$ converges to $\{O_n\}$ in Type 2 strong cluster sense for $f, g \in \mathcal{D}$. $\square$

We obtained a C^* -subalgebra, $\mathcal{D}$ of Q such that $\{T_n^B(fg) - T_n^B(f)T_n^B(g)\}$ converges to $\{O_n\}$ in Type 2 strong cluster sense whenever $f, g \in \mathcal{D}$. Thus we obtained a C^* -subalgebra $\mathcal{D}$ of $Q = VMO \cap L^\infty(\mathbb{D})$ for which the Korovkin-type theorem holds.

Remark 3.11 The algebra $\mathcal{D}$ does not need to be the maximal C^* -subalgebra of Q with this property. The problem to find the maximal C^* -subalgebra of Q , which we denote by $\mathcal{N}$ such that for $f, g \in \mathcal{N}$, $\{T_n^B(fg) - T_n^B(f)T_n^B(g)\}$ converges to $\{O_n\}$ in Type 2 strong cluster sense is open.

Remark 3.12 T_f^B is a diagonal operator with respect to the standard orthonormal basis under consideration whenever f is a radial bounded function. Hence it can be shown that whenever f and g are radial bounded functions in Q , then $\{T_n^B(fg) - T_n^B(f)T_n^B(g)\}$ converges to $\{O_n\}$ in Type 2 strong cluster sense.

Now we give the Korovkin-type results for $\mathcal{D}$.

Theorem 3.13 Let $\{g_1, \dots, g_m\} \subseteq \mathcal{D}$ such that $\{P_{U_n}(T_n^B(g)) - T_n^B(g)\}$ converges to $\{O_n\}$ in Type 2 strong cluster sense for all $g \in \{g_1, \dots, g_m, \sum_{k=1}^m g_k g_k^*\}$. Then $\{P_{U_n}(T_n) - T_n\}$ converges to $\{O_n\}$ in Type 2 strong cluster sense for all T in the C^* -algebra generated by $\{T_g^B : g \in \{g_1, \dots, g_m\}\}$. Also, $\{P_{U_n}(T_n^B(g)) - T_n^B(g)\}$ converges to $\{O_n\}$ in Type 2 strong cluster sense for all g in the C^* -algebra generated by $\{g_1, \dots, g_m\}$.

3.3 Examples

In this section, we provide examples of sequences $\{U_n\}$ such that $\{P_{U_n}(T_n^B(f)) - T_n^B(f)\}$ converges to $\{O_n\}$ in strong cluster sense whenever $f \in C(\overline{\mathbb{D}})$.

In [6], R. Chan and M.C. Yeung proved that whenever U_n is the $n \times n$ Fourier matrix F_n and f is a real valued continuous function on $\mathbb{T}$, then $\{P_{U_n}(T_n^H(f)) - T_n^H(f)\}$ converges to $\{O_n\}$ in Type 1 strong cluster sense ($F_n = (\frac{1}{\sqrt{n}} e^{\frac{i2\pi jk}{n}})_{j,k=0}^{n-1}$ and the corresponding M_{U_n} is known as Circulant algebra). Similarly in [15], S. Serra-Capizzano showed that if

$$U_n = S_n = \left(\sqrt{\frac{2}{n+1}} \sin((j+1)x_i^{(n)}) \right)_{i,j=0}^{n-1}, \quad x_i^n = \frac{i+1}{n+1}, \quad i = 0, 1, \dots, n-1,$$

then $\{P_{S_n}(T_n^H(f)) - T_n^H(f)\}$ converges to $\{O_n\}$ in Type 1 strong cluster sense, for real valued continuous even functions f on $\mathbb{T}$ (M_{U_n} is called τ -algebra). Also, in [8], Jin showed that if $U_n = H_n = (\frac{1}{\sqrt{n}} [\sin(jx_i^{(n)}) + \cos(jx_i^{(n)})])_{i,j=0}^{n-1}$, where $x_i^n = \frac{2i\pi}{n}$, $i = 0, \dots, n-1$ (M_{U_n} is the Hartley algebra), then $\{P_{H_n}(T_n^H(f)) - T_n^H(f)\}$ converges to $\{O_n\}$ in Type 1 strong cluster sense, for real valued continuous even functions f on $\mathbb{T}$.

Here we construct examples in the Bergman space case using the above mentioned popular algebras. Let $f \in C(\mathbb{D})$ and $f' = f|_{\partial\mathbb{D}}$. By decomposition of f' into its real and imaginary part and applying the result in [6] we see that the sequence $\{P_{F_n}(T_n^H(f')) - T_n^H(f')\}$ converges to $\{O_n\}$ in Type 2 strong cluster sense. By an application of Theorem 2.1, we obtain that the sequence $\{P_{F_n}(T_n) - T_n\}$ converges to $\{O_n\}$ in Type 2 strong cluster sense for every T in the C^* -algebra generated by $\{T_f^H : f \in C(\mathbb{T})\}$. This C^* -algebra contains all the compact operators on $H^2(\mathbb{T})$ [19].

Let $U_n = V_n^* F_n$ and $f \in C(\overline{\mathbb{D}})$. We have already obtained that, $T_n^B(f) = V_n^* T_n^H(f') V_n + V_n^* K'_n V_n$ where K' is a compact operator on $H^2(\mathbb{T})$. Then we have,

$$P_{U_n}(T_n^B(f)) - T_n^B(f) = V_n^* [P_{V_n U_n}(T_n^H(f')) - T_n^H(f') + P_{V_n U_n}(K'_n) - K'_n] V_n.$$

Since $V_n U_n = F_n$ and $f' \in C(\mathbb{T})$, $\{P_{V_n U_n}(T_n^H(f')) - T_n^H(f')\}$ converges to $\{O_n\}$ in Type 2 strong cluster sense. By Corollary 3.7, $\{P_{V_n U_n}(K'_n) - K'_n\}$ converges to $\{O_n\}$ in Type 2 strong cluster sense. Hence $\{P_{U_n}(T_n^B(f)) - T_n^B(f)\}$ converges to $\{O_n\}$ in Type 2 strong cluster sense, whenever $f \in C(\mathbb{D})$.

Taking $U_n = V_n^* S_n$ or $U_n = V_n^* H_n$, with a similar argument as above, we conclude that $P_{U_n}(T_n^B(f)) - T_n^B(f)$ converges to O_n in Type 2 strong cluster sense, whenever $f \in C(\mathbb{D})$ for which f' is even.

Remark 3.14 Let $f \in C(\overline{\mathbb{D}})$, then we already obtained that whenever $U_n = V_n^* F_n$, $\{P_{U_n}(T_n^B(f)) - T_n^B(f)\}$ converges to $\{O_n\}$ in Type 2 strong cluster sense. Now consider $f \in \mathcal{D}$, then $f = f^1 + f^2$ where $f^1 \in C(\overline{\mathbb{D}})$ and $f^2 \in \mathcal{Q} \cap \mathbb{B}$. Since $T_{f^2}^B$ is compact, by Corollary 3.7 the sequence $\{P_{U_n}(T_n^B(f^2)) - T_n^B(f^2)\}$ converges to $\{O_n\}$ in Type 2 strong cluster sense. Thus $\{P_{U_n}(T_n^B(f)) - T_n^B(f)\}$ converges to $\{O_n\}$ in Type 2 strong cluster sense whenever $f \in \mathcal{D}$.

3.4 Toeplitz operators on Fock space

Consider $\mathbb{C}$ with Gaussian measure $d\mu(z) = \frac{1}{\pi} e^{-|z|^2} dv(z)$, where $dv(z)$ denotes the Lebesgue measure on $\mathbb{C}$. $F^2(\mathbb{C})$ denote the Gaussian square integrable entire functions on $\mathbb{C}$. $F^2(\mathbb{C})$ is a closed subspace of $L^2(\mathbb{C}, d\mu)$. We shall prove Korovkin-type theorems for Toeplitz operators on Fock space (several properties of Toeplitz operators on Fock space can be found in [23]). Let T_f^F denote the Toeplitz operator on $F^2(\mathbb{C})$ with bounded symbols f . As in the previous cases, we need to check the convergence of $\{T_n^F(fg) - T_n^F(f)T_n^F(g)\}$ to $\{O_n\}$ ($T_n^F(f) = P_n T_f^F P_n$) in Type 2 strong cluster sense. First we need to identify a class of symbols for which this convergence holds. The idea is to find a relation of the form $U^* T_f^H U + K_f = T_f^F$, where U is a unitary map between Fock space and Hardy space and $f'(e^{i\theta}) = \lim_{r \rightarrow \infty} f(re^{i\theta})$ (provided the limit exists). The map U is given as follows. Consider the standard orthonormal basis of $F^2(\mathbb{C})$, $\{e_k^F : k \in \mathbb{N}_0\}$, where $e_k^F(z) = \frac{z^k}{\sqrt{k!}}$. Define U from $F^2(\mathbb{C})$ to $H^2(\mathbb{T})$ as $U(e_k^F) = e_k^H$.

As in the case of Bergman space, we have a maximal C^* -subalgebra, W of $L^\infty(\mathbb{C})$ for which $T_{fg}^F - T_f^F T_g^F$ is compact for all $f, g \in W$ [4]. Here, we find a C^* -subalgebra of W for which $\{T_n^F(fg) - T_n^F(f)T_n^F(g)\}$ converges to $\{O_n\}$ in Type 2 strong cluster sense.

Define

$$ESV_F = \left\{ f \in L^\infty(\mathbb{C}) : \lim_{R \rightarrow \infty} \sup_{\substack{|z-w| \leq 1 \\ |z| \geq R}} |f(z) - f(w)| = 0 \right\} \text{ and} \\ C = \left\{ f \in L^\infty(\mathbb{C}) : \lim_{z \rightarrow \infty} B_F(f)(z) = 0 \right\},$$

where $B_F(f)$ is the Berezin transform of f defined by

$$B_F(f)(z) = \frac{1}{\pi} \int_{\mathbb{C}} f(w+z) e^{-|w|^2} dv(w) \quad \forall f \in L^\infty(\mathbb{C}).$$

Note that for $f \in L^\infty(\mathbb{C})$, T_f^F is compact if and only if $f \in C$ [2,23]. Consider the set

$$V = \{f : \mathbb{C} \rightarrow \mathbb{C} \mid f \text{ is bounded measurable and } \lim_{|z| \rightarrow \infty} f(z) = 0\}.$$

$V \subseteq W \cap C$ [4]. So $f \in V$ implies T_f^F is compact.

Consider the following result.

Theorem 3.15 [4]

1. $W \cap C$ is a closed self adjoint ideal of W .
2. $W = ESV_F + W \cap C$.

ESV_F is a C^* -subalgebra of W . Also, $\mathcal{E} = \{\tilde{g} : g \in C(\mathbb{T})\} \subseteq ESV_F$, where $\tilde{g}(z) = g(\frac{z}{|z|})$ [4]. Note that $\mathcal{E}$ is a C^* -subalgebra of ESV_F and is generated by $\{f_m(z) = \frac{z^m}{|z|^m} : m \in \mathbb{N}_0\}$. For $\tilde{g} \in \mathcal{E}$, $(\tilde{g})' = g$.

Using the notation above, we give a flowchart depicting the classes on which Korovkin-type theorems are known to us to hold.

Classes on which Korovkin-type theorems hold in the case of $F^2(\mathbb{C})$

Lemma 3.16 $T_g^F - U^*T_{f_m}^H U$ is compact for $g \in \mathcal{E}$.

Proof We will compute the difference of operators for $g = f_m$. A standard calculation shows:

$$\begin{aligned} T_{f_m}^F(e_k^F) &= \frac{1}{\sqrt{k!(m+k)!}} \left(\int_0^\infty t^{(2k+m)/2} e^{-t} dt \right) e_{m+k}^F \\ &= \begin{cases} \frac{1}{\sqrt{k!(m+k)!}} \Gamma(k+n+1) e_{m+k}^F, & \text{if } m = 2n \\ \frac{1}{\sqrt{k!(m+k)!}} \Gamma(k+n+\frac{3}{2}) e_{m+k}^F, & \text{if } m = 2n+1. \end{cases} \end{aligned}$$

By using Stirling's formula, it can be observed that for each m , as $k \rightarrow \infty$, $T_{f_m}^F(e_k^F) - U^*T_{f_m}^H U(e_k^F) \rightarrow 0$. Hence $T_{f_m}^F - U^*T_{f_m}^H U$ is compact.

We can do the same for $\overline{f_m}$ and for any g in the $*$ -algebra E generated by $\{f_m : m \in \mathbb{N}_0\}$. Thus it can be shown that the difference is compact for all g in $\mathcal{E}$. $\square$

Remark 3.17 As we did in the case of the Bergman space, we can show that $\{T_n^F(fg) - T_n^F(f)T_n^F(g)\}$ converges to $\{O_n\}$ in Type 2 strong cluster sense whenever $f, g \in \mathcal{E}$.

Remark 3.18 It can be shown that $\mathcal{S}$ is a C^* -subalgebra of W .

We know that $W \cap C$ is a closed self adjoint ideal and also $f \in W \cap C$ implies that T_f^F is compact. Then as we proved in the case of the Bergman space for $\mathcal{D} = C(\overline{\mathbb{D}}) + Q \cap B$, we can show the following.

Theorem 3.19 Let g, h belong to $\mathcal{S}$. Then $\{T_n^F(gh) - T_n^F(g)T_n^F(h)\}$ converges to $\{O_n\}$ in Type 2 strong cluster sense.

Remark 3.20 The algebra $\mathcal{S}$ does not need to be the maximal C^* -subalgebra of W with this property. The problem to find the maximal C^* -subalgebra of W , which we denote by $\mathcal{M}$ such that for $g, h \in \mathcal{M}$, $\{T_n^F(gh) - T_n^F(g)T_n^F(h)\}$ converges to $\{O_n\}$ in Type 2 strong cluster sense is open.

Thus we obtain the following Korovkin-type theorems for the Fock space.

Theorem 3.21 *Let $\{g_1, \dots, g_m\} \subseteq \mathcal{S}$ such that $\{P_{U_n}(T_n^F(g)) - T_n^F(g)\}$ converges to $\{O_n\}$ in Type 2 strong cluster sense for all $g \in \{g_1, \dots, g_m, \sum_{k=1}^m g_k g_k^*\}$. Then $\{P_{U_n}(T_n) - T_n\}$ converges to $\{O_n\}$ in Type 2 strong cluster sense for all T in the C^* -algebra generated by $\{T_g^F : g \in \{g_1, \dots, g_m\}\}$. Also, $P_{U_n}(T_n^F(g)) - T_n^F(g)$ converges to $\{O_n\}$ in Type 2 strong cluster sense for all $g \in C^*$ -algebra generated by $\{g_1, \dots, g_m\}$.*

Remark 3.22 Consider $\mathbb{C}$ with the Gaussian measure $d\mu_\alpha(z) = \frac{\alpha}{\pi} e^{-\alpha|z|^2} dv(z)$, $\alpha > 0$. Let $F^2(\mathbb{C}, \alpha)$ be the space of all entire functions which are square integrable with respect to this measure. One can consider the Toeplitz operator $T_f^{F, \alpha}$ acting on $F^2(\mathbb{C}, \alpha)$ and the maximal C^* -subalgebra W^α of $L^\infty(\mathbb{C})$ consisting of functions f, g with the difference $T_{fg}^{F, \alpha} - T_f^{F, \alpha} T_g^{F, \alpha}$ being compact. All the results we proved will work for any $\alpha > 0$. In fact $W^\alpha = W$ for all $\alpha > 0$ (see [1, 2, 4]). We considered the case $\alpha = 1$ in this section.

4 Toeplitz operators in higher dimensions

Here we obtain Korovkin-type results for Toeplitz operators acting on function spaces with 2-dimensional domains such as $H^2(\mathbb{T}^2)$, $A^2(\mathbb{B}^2)$ and $F^2(\mathbb{C}^2)$. We mostly obtain weak cluster sense convergence. The sequence of finite rank orthogonal projections that we use to define the truncations are of rank n^2 and we consider a sequence of matrices of order n^2 for each n . Also, the unitary matrices U_n that we consider here will be of order n^2 , for each n .

Before going into further details, we obtain some convergence results related to sequences of tensor products of matrices. These are important tools for obtaining Korovkin-type theorems in the case of $H^2(\mathbb{T}^2)$ and $F^2(\mathbb{C}^2)$.

Lemma 4.1 *Let $\{C_n\}$ and $\{D_n\}$ be sequences of $n \times n$ matrices. Let $\|C_n\| \leq k < \infty$, for some $k > 0$. Let $\{D_n\}$ converge to $\{O_n\}$ in Type 2 strong cluster sense. Then $\{C_n \otimes D_n\}$ and $\{D_n \otimes C_n\}$ converge to $\{O_{n^2}\}$ in Type 2 weak cluster sense.*

Proof Given $\{D_n\}$ converges to $\{O_n\}$ in Type 2 strong cluster sense. For $\epsilon > 0$, there exist positive integers $N_{1, \epsilon}$, $N_{2, \epsilon}$ such that for $n > N_{2, \epsilon}$, $D_n = R_n + N_n$, where $\text{rank } R_n \leq N_{1, \epsilon}$ and $\|N_n\| < \epsilon$. Now for $n > N_{2, \epsilon}$,

$$C_n \otimes D_n = C_n \otimes (R_n + N_n) = C_n \otimes R_n + C_n \otimes N_n.$$

Note that $\text{rank } C_n \otimes R_n \leq n N_{1, \epsilon}$ ($N_{1, \epsilon}$ is independent of n) and $\|C_n \otimes N_n\| < k\epsilon$. Clearly, $\text{rank } C_n \otimes R_n$ is $o(n^2)$. Hence $\{C_n \otimes D_n\}$ converges to $\{O_{n^2}\}$ in Type 2 weak cluster sense. Similarly, we can show that $\{D_n \otimes C_n\}$ converges to $\{O_{n^2}\}$ in Type 2 weak cluster sense. $\square$

Lemma 4.2 A, B be non-zero elements in $B(\mathcal{H})$. $\{A_n \otimes B_n\}$ converges to $\{O_{n^2}\}$ in Type 2 strong cluster sense if and only if both $\{A_n\}$ and $\{B_n\}$ converge to $\{O_n\}$ in Type 2 strong cluster sense.

Proof Assume $\{A_n \otimes B_n\}$ converges to $\{O_{n^2}\}$ in Type 2 strong cluster sense. Then for $\epsilon > 0$, there exist positive integers $N_{1,\epsilon}, N_{2,\epsilon}$ such that for $n > N_{2,\epsilon}$, $A_n \otimes B_n = R_{n^2} + N_{n^2}$, where $\text{rank } R_{n^2} \leq N_{1,\epsilon}$ ($N_{1,\epsilon}$ is independent of n) and $\|N_{n^2}\| < \epsilon$. Let $\alpha_n = \max_{0 \leq i, j \leq n-1} |\langle Ae_j, e_i \rangle|$. $\{\alpha_n\}$ is a monotonically increasing sequence of real numbers. Let α_k be the smallest non zero term of the sequence. $\alpha_n = |(A_n)_{r,s}|$, for some r, s such that $1 \leq r, s \leq n$. Consider the submatrix $(A_n)_{r,s} B_n$ of $A_n \otimes B_n$.

Assume $(A_n)_{r,s} \neq 0$ (for a non-zero operator A , $\alpha_n = 0$ only for finitely many n 's). Note that $(A_n)_{r,s} B_n = V_n(A_n \otimes B_n)W_n$, where V_n and W_n are projections acting on the space of n^2 dimension and whose range is of dimension n . Let $T_n = V_n R_{n^2} W_n$ and $S_n = V_n N_{n^2} W_n$. Then $(A_n)_{r,s} B_n = T_n + S_n$. Since $\text{rank } R_{n^2} \leq N_{1,\epsilon}$, $\text{rank } T_n \leq N_{1,\epsilon}$. Also, $\|\frac{S_n}{(A_n)_{r,s}}\| \leq \frac{\epsilon}{\alpha_k}$. Thus $\{B_n\}$ converges to $\{O_n\}$ in Type 2 strong cluster sense.

Note that $A_n \otimes B_n$ and $B_n \otimes A_n$ have same set of singular values. So by Lemma 1.6, $B_n \otimes A_n$ converges to $\{O_{n^2}\}$ in Type 2 strong cluster sense. Proceeding as above, we get $\{A_n\}$ converges to $\{O_n\}$ in Type 2 strong cluster sense.

Conversely, assume that $\{A_n\}$ and $\{B_n\}$ converge to $\{O_n\}$ in Type 2 strong cluster sense. Then for $\epsilon > 0$, there exist positive integers $N_{1,\epsilon}, N_{2,\epsilon}$ such that for $n > N_{2,\epsilon}$, $A_n = C_n + D_n$ and $B_n = X_n + Y_n$, with $\text{rank } C_n, \text{rank } X_n \leq N_{1,\epsilon}$ and $\|D_n\|, \|Y_n\| < \epsilon$. Note that we can choose C_n and X_n such that for all n , $\|X_n\| \leq \|B\|$ and $\|C_n\| \leq \|A\|$. Then using tensor products, we can show that $\{A_n \otimes B_n\}$ and $\{B_n \otimes A_n\}$ converge to $\{O_{n^2}\}$ in Type 2 strong cluster sense. $\square$

4.1 Toeplitz operators on $H^2(\mathbb{T}^2)$ and $F^2(\mathbb{C}^2)$

Let $L^2(\mathbb{T}^2)$ ($\mathbb{T}^2 = \mathbb{T} \times \mathbb{T}$) denote the set of measurable functions on $\mathbb{T}^2$ which satisfy the condition

$$\|f\|_2^2 = \frac{1}{(2\pi)^2} \int_0^{2\pi} \int_0^{2\pi} |f(e^{i\theta}, e^{i\psi})|^2 d\theta d\psi < \infty.$$

For $f \in L^2(\mathbb{T}^2)$, the Fourier coefficient $f_{m,n}$ is defined as

$$f_{m,n} = \frac{1}{(2\pi)^2} \int_0^{2\pi} \int_0^{2\pi} f(e^{i\theta}, e^{i\psi}) e^{-im\theta} e^{-in\psi} d\theta d\psi.$$

$H^2(\mathbb{T}^2)$ denotes the Hardy space inside $L^2(\mathbb{T}^2)$ consisting of all $f \in L^2(\mathbb{T}^2)$ for which $f_{m,n} = 0$ unless $m \geq 0$ and $n \geq 0$.

It is known that $L^2(\mathbb{T}^2) = L^2(\mathbb{T}) \otimes L^2(\mathbb{T})$ and $H^2(\mathbb{T}^2) = H^2(\mathbb{T}) \otimes H^2(\mathbb{T})$. Here we consider the Toeplitz operator $T_a^{H^2}$ on $H^2(\mathbb{T}^2)$ associated with $a \in L^\infty(\mathbb{T}^2)$. More properties of Toeplitz operators on $H^2(\mathbb{T}^2)$ can be found in [5].

Now we define the finite rank orthogonal projections of rank n^2 . We have already defined P_n on $H^2(\mathbb{T})$. We define the projections to be $\mathbb{P}_n = P_n \otimes P_n$ and it is of rank

n^2 . We define $T_n^{H^2}(a) = \mathbb{P}_n T_a^{H^2} \mathbb{P}_n$. $T_n^{H^2}(a)$ can be considered as an $n^2 \times n^2$ matrix. If $a = b \otimes c$, then $T_a^{H^2} = T_b^H \otimes T_c^H$ (T_b^H denotes the Toeplitz operator on $H^2(\mathbb{T})$). Also, $T_n^{H^2}(a) = T_n^H(b) \otimes T_n^H(c)$.

We obtain results analogous to Theorems 1.2 and 3.9 in the case of $H^2(\mathbb{T}^2)$. From the proofs of these theorems, it is clear that the problem can be reduced to checking the convergence of $\{T_n^{H^2}(fg) - T_n^{H^2}(f)T_n^{H^2}(g)\}$ and $\{(T_f^{H^2}T_g^{H^2})_n - T_n^{H^2}(f)T_n^{H^2}(g)\}$, whenever $f, g \in C(\mathbb{T}^2)$. It can be shown that both differences will converge to $\{O_{n^2}\}$ in Type 2 weak cluster sense.

Lemma 4.3 *For $f, g \in C(\mathbb{T}^2)$, $\{T_n^{H^2}(fg) - T_n^{H^2}(f)T_n^{H^2}(g)\}$ converges to $\{O_{n^2}\}$ in Type 2 weak cluster sense.*

Proof We know that trigonometric polynomials are dense in $C(\mathbb{T}^2)$. Trigonometric polynomials are functions of the form $\sum_{j=-l}^l \sum_{k=-s}^s a_{jk} e_j^H \otimes e_k^H$. We show the convergence when $f = e_{j_1}^H \otimes e_{k_1}^H$ and $g = e_{j_2}^H \otimes e_{k_2}^H$.

$$\begin{aligned} T_n^{H^2}(f) &= T_n^H(e_{j_1}^H) \otimes T_n^H(e_{k_1}^H), \\ T_n^{H^2}(g) &= T_n^H(e_{j_2}^H) \otimes T_n^H(e_{k_2}^H), \\ T_n^{H^2}(fg) &= T_n^H(e_{j_1}^H e_{j_2}^H) \otimes T_n^H(e_{k_1}^H e_{k_2}^H). \end{aligned}$$

$$\begin{aligned} &T_n^{H^2}(fg) - T_n^{H^2}(f)T_n^{H^2}(g) \\ &= T_n^H(e_{j_1}^H e_{j_2}^H) \otimes T_n^H(e_{k_1}^H e_{k_2}^H) - T_n^H(e_{j_1}^H)T_n^H(e_{j_2}^H) \otimes T_n^H(e_{k_1}^H)T_n^H(e_{k_2}^H) \\ &= T_n^H(e_{j_1}^H e_{j_2}^H) \otimes T_n^H(e_{k_1}^H e_{k_2}^H) - T_n^H(e_{j_1}^H e_{j_2}^H) \otimes T_n^H(e_{k_1}^H)T_n^H(e_{k_2}^H) \\ &\quad + T_n^H(e_{j_1}^H e_{j_2}^H) \otimes T_n^H(e_{k_1}^H)T_n^H(e_{k_2}^H) - T_n^H(e_{j_1}^H)T_n^H(e_{j_2}^H) \otimes T_n^H(e_{k_1}^H)T_n^H(e_{k_2}^H) \\ &= T_n^H(e_{j_1}^H e_{j_2}^H) \otimes (T_n^H(e_{k_1}^H e_{k_2}^H) - T_n^H(e_{k_1}^H)T_n^H(e_{k_2}^H)) \\ &\quad + (T_n^H(e_{j_1}^H e_{j_2}^H) - T_n^H(e_{j_1}^H)T_n^H(e_{j_2}^H)) \otimes T_n^H(e_{k_1}^H)T_n^H(e_{k_2}^H). \end{aligned}$$

$\{T_n^H(e_{k_1}^H e_{k_2}^H) - T_n^H(e_{k_1}^H)T_n^H(e_{k_2}^H)\}$ and $\{T_n^H(e_{j_1}^H e_{j_2}^H) - T_n^H(e_{j_1}^H)T_n^H(e_{j_2}^H)\}$ converge to $\{O_n\}$ in Type 2 strong cluster sense.

Also, the sequences $\{T_n^H(e_{j_1}^H e_{j_2}^H)\}$ and $\{T_n^H(e_{k_1}^H)T_n^H(e_{k_2}^H)\}$ are norm bounded. Hence by Lemma 4.1, we obtain the convergence in Type 2 weak cluster sense. Similarly, it is easy to show that $\{T_n^{H^2}(fg) - T_n^{H^2}(f)T_n^{H^2}(g)\}$ converges to $\{O_{n^2}\}$ in Type 2 weak cluster sense, when f and g are trigonometric polynomials. Since polynomials are dense in $C(\mathbb{T}^2)$, we have the result. $\square$

Lemma 4.4 *$\{(T_f^{H^2}T_g^{H^2})_n - T_n^{H^2}(f)T_n^{H^2}(g)\}$ converges to $\{O_{n^2}\}$ in Type 2 weak cluster sense whenever $f, g \in C(\mathbb{T}^2)$.*

Proof Similar to that of Lemma 4.3 □

Example 4.5 The following example shows that in Lemma 4.3, we cannot expect strong cluster convergence. Let $f = e_0^H \otimes e_1^H$ and $g = e_0^H \otimes e_{-1}^H$. Then $T_n^{H^2}(fg) - T_n^{H^2}(f)T_n^{H^2}(g)$ will be

$$I_n \otimes I_n - I_n \otimes T_n^H(e_1^H)T_n^H(e_{-1}^H).$$

The n^2 order matrix that we obtained is diagonal and has the eigenvalue 1 with multiplicity n and all other eigenvalues are 0. So it will not converge to $\{O_{n^2}\}$ in Type 2 strong cluster sense. But it will converge in Type 2 weak cluster sense.

So we obtain the following results in the case of $H^2(\mathbb{T}^2)$.

Theorem 4.6 Let $\{g_1, g_2, \dots, g_m\} \subseteq C(\mathbb{T}^2)$ such that $\{P_{U_n}(T_n^{H^2}(g)) - T_n^{H^2}(g)\}$ converges to $\{O_{n^2}\}$ in Type 2 weak cluster sense for every g in the set $\{g_1, g_2, \dots, g_m, \sum_{k=1}^m g_k g_k^*\}$. Then $\{P_{U_n}(T_n^{H^2}(g)) - T_n^{H^2}(g)\}$ converges to $\{O_{n^2}\}$ in Type 2 weak cluster sense for every g in the C^* -algebra generated by $\{g_1, g_2, \dots, g_m\}$.

Theorem 4.7 Let $\{g_1, \dots, g_m\} \subseteq C(\mathbb{T}^2)$ such that $\{P_{U_n}(T_n^{H^2}(g)) - T_n^{H^2}(g)\}$ converges to $\{O_{n^2}\}$ in Type 2 weak cluster sense for all functions g belonging to $\{g_1, \dots, g_m, \sum_{k=1}^m g_k g_k^*\}$. Then $\{P_{U_n}(T_n) - T_n\}$ converges to $\{O_{n^2}\}$ in Type 2 weak cluster sense for all T in the C^* -algebra generated by $\{T_g^{H^2} : g \in \{g_1, \dots, g_m\}\}$.

We shall apply the same proof technique as that of Theorems 1.2 and 3.9 for proving the above two theorems.

Now we state a Korovkin-type theorem for operators on $F^2(\mathbb{C}^2)$. Note that $F^2(\mathbb{C}^2) = F^2(\mathbb{C}) \otimes F^2(\mathbb{C})$. For $f \in L^\infty(\mathbb{C}^2)$, let $T_f^{F^2}$ denote the Toeplitz operator on $F^2(\mathbb{C}^2)$. As we did for $H^2(\mathbb{T}^2)$, $\{T_n^{F^2}(fg) - T_n^{F^2}(f)T_n^{F^2}(g)\}$ converges to $\{O_{n^2}\}$ in Type 2 weak cluster sense, for $f, g \in A_{F^2} = \mathcal{S} \otimes \mathcal{S}$ ($T_n^{F^2}(f) = \mathbb{P}_n T_f^{F^2} \mathbb{P}_n$, $\mathbb{P}_n = P_n \otimes P_n$). As in the case of $H^2(\mathbb{T}^2)$, we obtain the following results.

Theorem 4.8 Let $\{g_1, g_2, \dots, g_m\} \subseteq A_{F^2}$ such that $\{P_{U_n}(T_n^{F^2}(g)) - T_n^{F^2}(g)\}$ converges to $\{O_{n^2}\}$ in Type 2 weak cluster sense for every g in the set $\{g_1, g_2, \dots, g_m, \sum_{k=1}^m g_k g_k^*\}$. Then $\{P_{U_n}(T_n^{F^2}(g)) - T_n^{F^2}(g)\}$ converges to $\{O_{n^2}\}$ in weak cluster sense (Type 2) for every g in the C^* -algebra generated by $\{g_1, g_2, \dots, g_m\}$. Also, $\{P_{U_n}(T_n) - T_n\}$ converges to $\{O_{n^2}\}$ in Type 2 weak cluster sense for all T in the C^* -algebra generated by $\{T_g^{F^2} : g \in \{g_1, \dots, g_m\}\}$.

4.2 Toeplitz operators on the Bergman space $A^2(\mathbb{B}^2)$

Consider $A^2(\mathbb{B}^2)$, Bergman space on the open unit ball $\mathbb{B}^2$ of $\mathbb{C}^2$. For $\alpha = (\alpha_1, \alpha_2) \in \mathbb{Z}_+ \times \mathbb{Z}_+$, let $e_\alpha^{B^2}(z) = \frac{1}{\pi} \sqrt{\frac{(2+|\alpha|)!}{\alpha!}} z^\alpha$, where $z^\alpha = z_1^{\alpha_1} z_2^{\alpha_2}$. The set $\{e_\alpha^{B^2}\}_{\alpha \in \mathbb{Z}_+ \times \mathbb{Z}_+}$ forms an orthonormal basis for $A^2(\mathbb{B}^2)$. We prove Korovkin-type theorems for Toeplitz

operators acting on $A^2(\mathbb{B}^2)$ associated with functions from $C(\overline{\mathbb{B}^2})$. Let $T_f^{B^2}$ denote the Toeplitz operator associated with $f \in L^\infty(\mathbb{B}^2)$ (more properties of Toeplitz operators acting on $A^2(\mathbb{B}^2)$ can be found in [19]). We obtain the convergence of $\{T_n^{B^2}(fg) - T_n^{B^2}(f)T_n^{B^2}(g)\}$ to $\{O_{n^2}\}$ in Type 2 weak cluster sense whenever f and g belong to $C(\overline{\mathbb{B}^2})$. Put $T_n^{B^2}(f) = \mathbb{P}_n T_f^{B^2} \mathbb{P}_n$, where $\mathbb{P}_n$ is the orthogonal projection of $A^2(\mathbb{B}^2)$ onto the span of $\{e_\alpha^{B^2}\}_{0 \leq \alpha_1, \alpha_2 \leq n-1}$.

Let $f = z_1^{m_1} z_2^{n_1} \bar{z}_1^{k_1} \bar{z}_2^{i_1}$ and $g = z_1^{m_2} z_2^{n_2} \bar{z}_1^{k_2} \bar{z}_2^{i_2}$.

$T_f^{B^2}(e_\alpha^{B^2}) = A_\alpha^f e_{\alpha_f}^{B^2}$, where $\alpha_f = (\alpha_1 + m_1 - k_1, \alpha_2 + n_1 - i_1)$ and

$$A_\alpha^f = \begin{cases} 0, & \text{if } \alpha_1 + m_1 - k_1 < 0 \text{ or} \\ & \alpha_2 + n_1 - i_1 < 0 \\ \sqrt{\frac{(2+|\alpha|)!}{\alpha!}} \sqrt{\frac{(2+|\alpha_f|)!}{\alpha_f!}} \frac{(\alpha_1+m_1)!(\alpha_2+n_1)!}{(2+\alpha_1+\alpha_2+m_1+n_1)!}, & \text{otherwise.} \end{cases}$$

Similarly we have, $T_g^{B^2}(e_\alpha^{B^2}) = A_\alpha^g e_{\alpha_g}^{B^2}$ and $T_{fg}^{B^2}(e_\alpha^{B^2}) = A_\alpha^{fg} e_{\alpha_{fg}}^{B^2}$. Since $(T_{fg}^{B^2} - T_f^{B^2} T_g^{B^2})$ is compact, $(T_{fg}^{B^2} - T_f^{B^2} T_g^{B^2})(e_\alpha^{B^2}) \rightarrow 0$ as $|\alpha| \rightarrow \infty$. By straight forward computation, we obtain $(T_{fg}^{B^2} - T_f^{B^2} T_g^{B^2})(e_\alpha^{B^2}) = K_\alpha e_{\alpha_{fg}}^{B^2}$, where $\alpha_{fg} = (\alpha_1 + m_1 + m_2 - k_1 - k_2, \alpha_2 + n_1 + n_2 - i_1 - i_2)$ and $K_\alpha \rightarrow 0$ as $|\alpha| \rightarrow \infty$.

Recall that if an operator K is compact, then $\{\mathbb{P}_n K \mathbb{P}_n\}$ converges to $\{O_{n^2}\}$ in Type 2 strong cluster sense.

Lemma 4.9 *The matrix representations of $T_n^{B^2}(fg) - T_n^{B^2}(f)T_n^{B^2}(g)$ and $\mathbb{P}_n(T_{fg}^{B^2} - T_f^{B^2} T_g^{B^2})\mathbb{P}_n$ with respect to the basis $\{e_\alpha^{B^2}\}_{0 \leq \alpha_1, \alpha_2 \leq n-1}$ (we shall fix any order) differ at most at M_n entries where $M_n = o(n^2)$.*

Proof The columns of these matrices are determined using the evaluation of $T_n^{B^2}(fg) - T_n^{B^2}(f)T_n^{B^2}(g)$ and $\mathbb{P}_n(T_{fg}^{B^2} - T_f^{B^2} T_g^{B^2})\mathbb{P}_n$ at $e_\alpha^{B^2}$ for $0 \leq \alpha_1, \alpha_2 \leq n-1$. We can see that each column and each row of these matrices can have at most one non zero entry. In the case of $\mathbb{P}_n(T_{fg}^{B^2} - T_f^{B^2} T_g^{B^2})\mathbb{P}_n$, the non zero entry will be K_α . There are three cases for the evaluation of $T_n^{B^2}(fg) - T_n^{B^2}(f)T_n^{B^2}(g)$ at $e_\alpha^{B^2}$. Before explaining each case, note the following.

$$T_n^{B^2}(g)(e_\alpha^{B^2}) = \begin{cases} T_g^{B^2}(e_\alpha^{B^2}), & \text{if } 0 \leq \alpha_1 + m_2 - k_2, \alpha_2 + n_2 - i_2 \leq n-1. \\ 0, & \text{otherwise} \end{cases}$$

case 1: $0 \leq \alpha_1 + m_2 - k_2, \alpha_2 + n_2 - i_2 \leq n-1$ and $0 \leq \alpha_1 + m_1 + m_2 - k_1 - k_2, \alpha_2 + n_1 + n_2 - i_1 - i_2 \leq n-1$.

$$(T_n^{B^2}(fg) - T_n^{B^2}(f)T_n^{B^2}(g))(e_\alpha^{B^2}) = (T_{fg}^{B^2} - T_f^{B^2} T_g^{B^2})(e_\alpha^{B^2}) = K_\alpha e_{\alpha_{fg}}^{B^2}.$$

case 2: $0 \leq \alpha_1 + m_1 + m_2 - k_1 - k_2, \alpha_2 + n_1 + n_2 - i_1 - i_2 \leq n-1$, and the condition $0 \leq \alpha_1 + m_2 - k_2, \alpha_2 + n_2 - i_2 \leq n-1$ is violated. Then $T_n^{B^2}(g)(e_\alpha^{B^2}) = 0$ and hence $(T_n^{B^2}(fg) - T_n^{B^2}(f)T_n^{B^2}(g))(e_\alpha^{B^2}) = T_f^{B^2}(e_\alpha^{B^2}) = A_\alpha^{fg} e_{\alpha_{fg}}^{B^2}$. Note that this can occur for at most $(|m_2 - k_2| + |n_2 - i_2| + 2)n$ values of α .

case 3: $0 \leq \alpha_1 + m_1 + m_2 - k_1 - k_2, \alpha_2 + n_1 + n_2 - i_1 - i_2 \leq n - 1$ is violated.

Then $(T_n^{B^2}(fg) - T_n^{B^2}(f)T_n^{B^2}(g))(e_\alpha^{B^2}) = 0$. This can occur for at most $(|m_1 + m_2 - k_1 - k_2| + |n_1 + n_2 - i_1 - i_2| + 2)n$ values of α .

It is clear that in the matrix representation of $T_n^{B^2}(fg) - T_n^{B^2}(f)T_n^{B^2}(g)$, each row and each column will have at most one non zero entry and that is either K_α or A_α^{fg} for some α . For each n , the matrix is of order n^2 . We can fix any order for the basis $\{e_\alpha^{B^2}\}_{0 \leq \alpha_1, \alpha_2 \leq n-1}$. From the above three cases it is clear that the entries of $T_n^{B^2}(fg) - T_n^{B^2}(f)T_n^{B^2}(g)$ and $\mathbb{P}_n(T_{fg}^{B^2} - T_f^{B^2}T_g^{B^2})\mathbb{P}_n$ are the same except for those corresponding to case 2 and case 3. In case 2, the entry K_α in $\mathbb{P}_n(T_{fg}^{B^2} - T_f^{B^2}T_g^{B^2})\mathbb{P}_n$ is replaced by A_α^{fg} in $T_n^{B^2}(fg) - T_n^{B^2}(f)T_n^{B^2}(g)$. In case 3, K_α is replaced by 0. So except at most $M_n = (|m_1 + m_2 - k_1 - k_2| + |n_1 + n_2 - i_1 - i_2| + 2 + |m_2 - k_2| + |n_2 - i_2| + 2)n$ entries, all other entries are the same and $M_n = o(n^2)$. $\square$

Lemma 4.10 $\{T_n^{B^2}(fg) - T_n^{B^2}(f)T_n^{B^2}(g)\}$ converges to $\{O_{n^2}\}$ in Type 2 weak cluster sense for $f, g \in C(\mathbb{B}^2)$.

Proof Let $f = z_1^{m_1} z_2^{n_1} \bar{z}_1^{k_1} \bar{z}_2^{i_1}$ and $g = z_1^{m_2} z_2^{n_2} \bar{z}_1^{k_2} \bar{z}_2^{i_2}$. Since $T_{fg}^{B^2} - T_f^{B^2}T_g^{B^2}$ is compact, $\{\mathbb{P}_n(T_{fg}^{B^2} - T_f^{B^2}T_g^{B^2})\mathbb{P}_n\}$ converges to $\{O_{n^2}\}$ in Type 2 strong cluster sense. So for $\epsilon > 0$, there exist two positive integers $N_{1,\epsilon}$ and $N_{2,\epsilon}$ ($N_{1,\epsilon}$ is independent of n) such that for $n > N_{2,\epsilon}$ $\mathbb{P}_n(T_{fg}^{B^2} - T_f^{B^2}T_g^{B^2})\mathbb{P}_n = R_{n^2} + N_{n^2}$, where rank of $R_{n^2} \leq N_{1,\epsilon}$ and $\|N_{n^2}\| < \epsilon$. Note that each row and column of both R_{n^2} and N_{n^2} will contain at most one non zero entry (we split $\mathbb{P}_n(T_{fg}^{B^2} - T_f^{B^2}T_g^{B^2})\mathbb{P}_n$ into R_{n^2} and N_{n^2} such that the non zero entries of R_{n^2} are K_α with $|K_\alpha| \geq \epsilon$ and that of N_{n^2} are K_α with $|K_\alpha| < \epsilon$). Likewise for all $n > N_{2,\epsilon}$, we can split $T_n^{B^2}(fg) - T_n^{B^2}(f)T_n^{B^2}(g) = R'_{n^2} + N'_{n^2}$ with non zero entries of N'_{n^2} are those of $T_n^{B^2}(fg) - T_n^{B^2}(f)T_n^{B^2}(g)$ whose modulus value is less than ϵ and that of R'_{n^2} are that of $T_n^{B^2}(fg) - T_n^{B^2}(f)T_n^{B^2}(g)$ whose modulus value is greater than or equal to ϵ . So rank of $R'_{n^2} \leq N'_{1,\epsilon} \leq N_{1,\epsilon} + M_n$ ($N'_{1,\epsilon}$ is the number of non-zero entries of R'_{n^2}). Thus $N'_{1,\epsilon} = o(n^2)$. Similarly, $\{T_n^{B^2}(fg) - T_n^{B^2}(f)T_n^{B^2}(g)\}$ converges to $\{O_{n^2}\}$ in Type 2 weak cluster sense for f, g in the $*$ -algebra generated by $\{z_1^m z_2^n \bar{z}_1^k \bar{z}_2^i : m, n, i, k \in \mathbb{Z}_+\}$. Hence for any $f, g \in C(\mathbb{B}^2)$, the C^* -algebra generated by $\{z_1^m z_2^n \bar{z}_1^k \bar{z}_2^i : m, n, i, k \in \mathbb{Z}_+\}$, $\{T_n^{B^2}(fg) - T_n^{B^2}(f)T_n^{B^2}(g)\}$ converges to $\{O_{n^2}\}$ in Type 2 weak cluster sense. $\square$

Next we give an example of f and g for which $\{T_n^{B^2}(fg) - T_n^{B^2}(f)T_n^{B^2}(g)\}$ converges to $\{O_{n^2}\}$ in Type 2 weak cluster sense but not in Type 2 strong cluster sense.

Example 4.11 Let $g = z_2$ and $f = \bar{z}_2$. Fix a positive integer n and also consider $\alpha = (\alpha_1, \alpha_2)$ where $0 \leq \alpha_1, \alpha_2 \leq n-1$ and consider the matrix $T_n^{B^2}(fg) - T_n^{B^2}(f)T_n^{B^2}(g)$.

$$(T_n^{B^2}(fg) - T_n^{B^2}(f)T_n^{B^2}(g))(e_\alpha^{B^2}) = (T_{fg}^{B^2} - T_f^{B^2}T_g^{B^2})(e_\alpha^{B^2}) = 0,$$

if $\alpha_2 \neq n - 1$ and

$$(T_n^{B^2}(fg) - T_n^{B^2}(f)T_n^{B^2}(g))(e_\alpha^{B^2}) = T_{fg}^{B^2}(e_\alpha^{B^2}) = \frac{n}{\alpha_1 + n + 2} e_\alpha^{B^2},$$

if $\alpha_2 = n - 1$.

So with a suitable order for $\{\alpha \in \mathbb{Z}_+ \times \mathbb{Z}_+ : |\alpha| \leq n - 1\}$, $T_n^{B^2}(fg) - T_n^{B^2}(f)T_n^{B^2}(g)$ is a diagonal matrix such that except n diagonal entries, all other diagonal entries are 0. Rest of the n diagonal entries are $\frac{n}{\alpha_1 + n + 2}$, for $\alpha_1 = 0, 1, \dots, n - 1$. In this case, $\{T_n^{B^2}(fg) - T_n^{B^2}(f)T_n^{B^2}(g)\}$ converges to $\{O_{n^2}\}$ in Type 2 weak cluster sense, but not in Type 2 strong cluster sense.

Now we give a non-trivial example of f and g for which $\{T_n^{B^2}(fg) - T_n^{B^2}(f)T_n^{B^2}(g)\}$ converges to $\{O_{n^2}\}$ in Type 2 strong cluster sense

Example 4.12 Let $f = z_2$ and $g = \bar{z}_2$. Fix a positive integer n and also consider $\alpha = (\alpha_1, \alpha_2)$ where $0 \leq \alpha_1, \alpha_2 \leq n - 1$ and consider the matrix $T_n^{B^2}(fg) - T_n^{B^2}(f)T_n^{B^2}(g)$.

$$\begin{aligned} & (T_n^{B^2}(fg) - T_n^{B^2}(f)T_n^{B^2}(g))(e_\alpha^{B^2}) \\ &= \begin{cases} (T_{fg}^{B^2} - T_f^{B^2}T_g^{B^2})(e_\alpha^{B^2}) = \left(\frac{\alpha_2+1}{\alpha_1+\alpha_2+3} - \frac{\alpha_2}{\alpha_1+\alpha_2+2}\right)e_\alpha^{B^2}, & \text{if } \alpha_2 \neq 0 \\ T_{fg}^{B^2}(e_\alpha^{B^2}) = (T_{fg}^{B^2} - T_f^{B^2}T_g^{B^2})(e_\alpha^{B^2}) = \frac{1}{\alpha_1+3}e_\alpha^{B^2}, & \text{if } \alpha_2 = 0. \end{cases} \end{aligned}$$

So with a suitable order for $\{\alpha \in \mathbb{Z}_+ \times \mathbb{Z}_+ : |\alpha| \leq n - 1\}$ $T_n^{B^2}(fg) - T_n^{B^2}(f)T_n^{B^2}(g)$ is a diagonal matrix and it is equal to $\mathbb{P}_n(T_{fg}^{B^2} - T_f^{B^2}T_g^{B^2})\mathbb{P}_n$. $\{\mathbb{P}_n(T_{fg}^{B^2} - T_f^{B^2}T_g^{B^2})\mathbb{P}_n\}$ converges to $\{O_{n^2}\}$ in Type 2 strong cluster sense as $T_{fg}^{B^2} - T_f^{B^2}T_g^{B^2}$ is compact.

Now we shall state the Korovkin-type theorem for Toeplitz operators on $A^2(\mathbb{B}^2)$. By adopting the same proof technique as that of previous Korovkin-type theorems, we obtain:

Theorem 4.13 Let $\{g_1, g_2, \dots, g_m\} \subseteq C(\overline{\mathbb{B}^2})$ such that $\{P_{U_n}(T_n^{B^2}(g)) - T_n^{B^2}(g)\}$ converges to $\{O_{n^2}\}$ in Type 2 weak cluster sense for every g in the set $\{g_1, g_2, \dots, g_m, \sum_{k=1}^m g_k g_k^*\}$. Then $\{P_{U_n}(T_n^{B^2}(g)) - T_n^{B^2}(g)\}$ converges to $\{O_{n^2}\}$ in weak cluster sense (Type 2) for every g in the C^* -algebra generated by $\{g_1, g_2, \dots, g_m\}$. Also, $\{P_{U_n}(T_n) - T_n\}$ converges to $\{O_{n^2}\}$ in Type 2 weak cluster sense for all T in the C^* -algebra generated by $\{T_g^{B^2} : g \in \{g_1, \dots, g_m\}\}$.

5 An independent approach to Toeplitz operators on Bergman space

We give an independent proof of Theorem 3.9 without using Theorem 2.1. Here, we give an outline of the proof. The following lemma is required.

Lemma 5.1 Let C be the C^* -algebra generated by $\{T_g^B : g \in \{g_1, \dots, g_m\} \subseteq C(\overline{\mathbb{D}})\}$. Then every element of C will be of the form $T_g^B + K$, $K \in \mathcal{K}(A^2(\mathbb{D}))$ and g is a continuous extension to $\mathbb{D}$ of $\hat{g}$ belonging to the C^* -algebra generated by $\{g_1|_{\partial\mathbb{D}}, \dots, g_m|_{\partial\mathbb{D}}\}$.

Proof Let T be in the C^* -algebra generated by $\{T_g^B : g \in \{g_1, \dots, g_m\} \subseteq C(\overline{\mathbb{D}})\}$. Then there exists a sequence $\{A_l\}$ in the $*$ -algebra generated by $\{T_g^B : g \in \{g_1, \dots, g_m\} \subseteq C(\overline{\mathbb{D}})\}$ such that $A_l \rightarrow T$ as $l \rightarrow \infty$. Since for any $f, g \in C(\overline{\mathbb{D}})$, $T_{fg}^B - T_f^B T_g^B$ is compact, A_l will be of the form $T_{g_l}^B + K_l$, for some compact operator K_l and g_l is a function in the $*$ -algebra generated by $\{g_1, g_2, \dots, g_m\}$. Hence $(T_{g_l}^B + K_l)_{l=1}^\infty \rightarrow T$. The Berezin transform $B(K) \in C_0(\mathbb{D})$ for compact K .

Let $h_l = B(T_{g_l}^B + K_l)$ and note that $h_l \in C(\overline{\mathbb{D}})$ (see [21], $f \in C(\overline{\mathbb{D}})$ implies $B(f) \in C(\overline{\mathbb{D}})$ and $B(f)|_{\partial\mathbb{D}} = f|_{\partial\mathbb{D}}$). Also, $h_l|_{\partial\mathbb{D}} = g_l|_{\partial\mathbb{D}}$. Since

$$|h_l(z) - h_k(z)| \leq \|(T_{g_l}^B + K_l) - (T_{g_k}^B + K_k)\|, \text{ for all } z \in \mathbb{D}$$

it follows that $\{h_l\}$ is a uniform Cauchy sequence in $C(\overline{\mathbb{D}})$. Hence $h_l \rightarrow h_0$ uniformly as $l \rightarrow \infty$, for some $h_0 \in C(\overline{\mathbb{D}})$. By using Cauchy Schwarz inequality, $B(T) - B(T_{g_l}^B + K_l) \rightarrow 0$ uniformly as $l \rightarrow \infty$. Now consider the equation

$$B(T - T_{h_0}^B)(z) = [B(T)(z) - B(T_{g_l}^B + K_l)(z)] + [B(T_{g_l}^B + K_l)(z) - B(T_{h_0}^B)(z)].$$

Choose l such that $\|B(T) - B(T_{g_l}^B + K_l)\|_\infty < \frac{\epsilon}{3}$ and $\|\mathcal{E}_l\|_\infty < \frac{\epsilon}{3}$, where $\mathcal{E}_l = h_l - h_0$. The sum in the second bracket is $h_l(z) - B(h_0)(z) = h_0(z) - B(h_0)(z) + \mathcal{E}_l(z)$. Also note that $h_0(z) = B(h_0)(z)$ on $\partial\mathbb{D}$. Since h_0 and $B(h_0) \in C(\overline{\mathbb{D}})$, $|h_0(z) - B(h_0)(z)| \rightarrow 0$ as $|z| \rightarrow 1^-$.

$$|B(T - T_{h_0}^B)(z)| \leq \frac{\epsilon}{3} + |h_0(z) - B(h_0)(z)| + \frac{\epsilon}{3}.$$

Choose $\delta > 0$ such that whenever $1 - \delta < |z| < 1$, $|h_0(z) - B(h_0)(z)| < \frac{\epsilon}{3}$. Therefore, $|B(T - T_{h_0}^B)(z)| < \epsilon$ whenever $1 - \delta < |z| < 1$. Hence $B(T - T_{h_0}^B)(z) \rightarrow 0$ as $|z| \rightarrow 1^-$. Hence, $T - T_{h_0}^B$ is compact (A is a compact operator on $A^2(\mathbb{D})$ if and only if A is in the C^* -algebra generated by Toeplitz operators with symbol functions from $L^\infty(\mathbb{D})$ and Berezin transform vanishes on the boundary of the disk [16]). $\square$

The independent proof:

By Corollary 3.7, $\{P_{U_n}(K_n) - K_n\}$ converges to $\{O_n\}$ in Type 2 strong cluster sense whenever K is compact. Now we may proceed as follows.

Let $g \in \{g_1, \dots, g_m, \sum_{k=1}^m g_k g_k^*\} \subseteq C(\overline{\mathbb{D}})$. Then for a some compact operator K ,

$$\begin{aligned} P_{U_n}(T_n^B(g)) - T_n^B(g) &= [P_{U_n}(V_n^* T_n^H(g') V_n) - V_n^* T_n^H(g') V_n] + [P_{U_n}(K_n) - K_n] \\ &= V_n^* [P_{V_n U_n}(T_n^H(g')) - T_n^H(g')] V_n + [P_{U_n}(K_n) - K_n]. \end{aligned}$$

If $\{P_{U_n}(T_n^B(g)) - T_n^B(g)\}$ converges to $\{O_n\}$ in Type 2 strong cluster sense, then $\{[P_{V_n U_n}(T_n^H(g')) - T_n^H(g')]\}$ converges to $\{O_n\}$ in Type 2 strong cluster sense. Therefore for every $g' \in \{g_1|_{\partial\mathbb{D}}, \dots, g_m|_{\partial\mathbb{D}}, (\sum_{k=1}^m g_k g_k^*)|_{\partial\mathbb{D}}\}$,

$\{[P_{V_n U_n}(T_n^H(g')) - T_n^H(g')]\}$ converges to $\{O_n\}$ in Type 2 strong cluster sense. By Theorem 1.2, we obtain that $\{P_{V_n U_n}(T_n^H(g')) - T_n^H(g')\}$ converges to $\{O_n\}$ in Type 2 strong cluster sense for g' in the C^* -algebra generated by $\{g_1|_{\partial\mathbb{D}}, \dots, g_m|_{\partial\mathbb{D}}\}$.

Now by Lemma 5.1, we know that every operator T in the C^* -algebra generated by $\{T_g^B : g \in \{g_1, \dots, g_m\}\}$ is of the form $T_g^B + K$, where g is a continuous extension of g' in the C^* -algebra generated by $\{g_1|_{\partial\mathbb{D}}, \dots, g_m|_{\partial\mathbb{D}}\}$ onto the disk $\overline{\mathbb{D}}$. Also, $T_g^B = V^* T_{g'}^H V + K_g$, where K_g is a compact operator on $A^2(\mathbb{D})$. Then we have,

$$\begin{aligned} P_{U_n}(T_n) - T_n &= P_{U_n}(T_n^B(g) + K_n) - (T_n^B(g) + K_n) \\ &= [V_n^* P_{V_n U_n}(T_n^H(g')) V_n - V_n^* T_n^H(g') V_n] + [P_{U_n}(K'_n) - K'_n], \end{aligned}$$

where $K' = K + K_g$. Since, $\{P_{V_n U_n}(T_n^H(g')) - T_n^H(g')\}$ converges to $\{O_n\}$ in Type 2 strong cluster sense, $P_{U_n}(T_n) - T_n$ converges to O_n in Type 2 strong cluster sense.

6 Concluding remarks and future problems

1. The convergence in Type 2 and Type 1 strong/weak cluster sense is closely related to the preconditioning problem of large linear systems. For if $\{A_n\}$ and $\{B_n\}$ are two sequences of Hermitian matrices such that $\{A_n B_n\}$ converges to $\{O_n\}$ in Type 2 (or Type 1) strong cluster sense, where $\gamma I_n \geq B_n \geq \delta I_n$, for all n , then B_n satisfies the criteria to become a good preconditioner of A_n (see [11] for details). These facts were also established by some numerical experiments in the case of Toeplitz operators on Hardy space [6]. There is scope for such experiments in the setting of various function spaces considered in this article.
2. In the case of the Bergman space $A^2(\mathbb{D})$, we obtained Korovkin-type results for a C^* -algebra inside $VM O \cap L^\infty(\mathbb{D})$ which properly contains $C(\overline{\mathbb{D}})$ ($\mathbb{D}$ denotes the open unit disc in $\mathbb{C}$). The major achievement is that we could go beyond the class of continuous functions. The C^* -subalgebra $\mathcal{D}$ will contain functions which are nowhere continuous in $\mathbb{D}$. For example,

$$f(z) = \begin{cases} 1 - |z|, & \text{if } |z| \text{ is rational} \\ 0, & \text{if } |z| \text{ is irrational.} \end{cases}$$

It is easy to see that f can be approximated by a sequence of compactly supported functions in $\mathbb{D}$ and hence $f \in \mathcal{D}$. Whether Korovkin-type theorems hold for the full algebra $VM O \cap L^\infty(\mathbb{D})$ is a problem of interest.

3. We obtained Korovkin-type theorems for Toeplitz operators on functions spaces on domains of dimension $n = 2$ like $H^2(\mathbb{T}^2)$, $A^2(\mathbb{B}^2)$ and $F^2(\mathbb{C}^2)$. Here, we obtain the results with respect to convergence in weak cluster sense. In a similar way, we can obtain results in the case of $n \geq 3$.

Acknowledgements The authors thank M.+N.N. Namboodiri for fruitful discussions.

Funding Open Access funding enabled and organized by Projekt DEAL. Wolfram Bauer wishes to thank Kerala State Higher Education Council for supporting the research visit to Cochin under the Erudite scheme. Rahul Rajan is thankful to IP@Leibniz of Leibniz University of Hannover, Germany for supporting the research visit to Hannover. Rahul Rajan is also supported by UGC-SRF.

Declarations

Conflict of interest The authors declare that they have no conflict of interest.

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