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The independence graph of a finite group

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Abstract
Given a finite group G, we denote by $\Delta(G)$ the graph whose vertices are the elements G and where two vertices x and y are adjacent if there exists a minimal generating set of G containing x and y. We prove that $\Delta(G)$ is connected and classify the groups G for which $\Delta(G)$ is a planar graph.

Keywords Generating sets · Generating graph · Connectivity · Planarity · Soluble groups

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1 Introduction

The generating graph of a finite group G is the graph defined on the elements of G in such a way that two distinct vertices are connected by an edge if and only if they generate G. It was defined by Liebeck and Shalev in [14], and has been further investigated by many authors: see for example [5,7,9,10,13,17,20–22] for some of the range of questions that have been considered. Clearly the generating graph of G is an edgeless graph if G is not 2-generated. We propose and investigate a possible generalization, that gives useful information even when G is not 2-generated.

Let G be a finite group. A generating set X of G is said to be minimal if no proper subset of X generates G. We denote by $\Gamma(G)$ the graph whose vertices are the elements of G and in which two vertices x and y are joined by an edge if and only if $x \neq y$ and there exists a minimal generating set of G containing x and y. Roughly speaking, x and y are adjacent vertices of $\Gamma(G)$ is they are 'independent', so we call $\Gamma(G)$ the *independence graph* of G. We will denote by $V(G)$ the set of the non-isolated vertices

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of $\Gamma(G)$ and by $\Delta(G)$ the subgraph of $\Gamma(G)$ induced by $V(G)$. Our main result is the following.

Theorem 1 *If G is a finite group, then the graph $\Delta(G)$ is connected.*

We prove a stronger result in the case of finite soluble groups. For a positive integer u, we denote by $\Gamma_u(G)$ the subgraph of $\Gamma(G)$ in which x and y are joined by an edge if and only if there exists a minimal generating set of size u containing x and y. As before, we denote by $\Delta_u(G)$ the subgraph of $\Gamma_u(G)$ induced by the set $V_u(G)$ of its non-isolated vertices. Notice that, even when G is u-generated, the set $V_u(G)$ is in general different from $V(G)$. For example if $G = \mathrm{Sym}(4)$, then $\{(1, 2)(3, 4), (1, 2), (1, 2, 3)\}$ is a minimal generating set for G, so $(1, 2)(3, 4) \in V(G)$; however $(1, 2)(3, 4) \notin V_2(G)$. If G is a non-cyclic 2-generated group, then $\Gamma_2(G)$ coincides with the generating graph of G and it follows from [10, Theorem 1] that $\Delta_2(G)$ is a connected graph if G is soluble. We generalize this result in the following way.

Theorem 2 *If $u \in \mathbb{N}$ and G is a finite soluble group, then $\Delta_u(G)$ is connected.*

Recall that a graph is said to be embeddable in the plane, or planar, if it can be drawn in the plane so that its edges intersect only at their ends. The 2-generated finite groups whose generating graph is planar have been classified in [18]. Our next result gives a classification of the finite groups G such that $\Gamma(G)$ is a planar graph.

Theorem 3 *Let G be a finite group. Then $\Gamma(G)$ is planar if and only either $G \in \{C_2 \times C_2, C_2 \times C_4, D_4, Q_8, \mathrm{Sym}(3)\}$ or $G = C_n$ is cyclic of order n and one of the following occurs:*

(1) *n is a prime-power.*
(2) *$n = p \cdot q$, where p and q are distinct primes and $p \leq 3$.*
(3) *$n = 4 \cdot q$, where q is an odd prime.*

Other results, and some related open questions, are presented in Section 5.

2 Proof of Theorem 1

Lemma 4 *Let $g \in G$. Then g is isolated in $\Gamma(G)$ if and only if either $G = \langle g \rangle$ or $g \in \mathrm{Frat}(G)$.*

Proof Suppose $g \notin \mathrm{Frat}(G)$. There exists a maximal subgroup M of G with $g \notin M$. The set $X = \{g\} \cup M$ contains a minimal generating X of G and $g \in X$ (otherwise $G = \langle X \rangle \leq M$). If $X \neq \{g\}$, then g is not isolated, otherwise $\langle g \rangle = G$. □

Proposition 5 *If G is a finite cyclic group, then $\Delta(G)$ is connected.*

Proof Let $|G| = p_1^{a_1} \dots p_t^{a_t}$, where $p_1, \dots, p_t$ are distinct primes. If $t = 1$, then $V(G) = \varnothing$. So assume $t > 1$ and, for $1 \leq i \leq t$, let g_i be an element of G of order $|G|/p_i^{a_i}$. The subset $X = \{g_1, \dots, g_t\}$ induces a complete subgraph of $\Delta(G)$. Now let $x \in V(G)$. Since $x \notin \mathrm{Frat}(G)$, there exists $i \in \{1, \dots, t\}$ such that $p_i^{a_i}$ divides $|x|$ and x is adjacent to g_i. □

Lemma 6 *Let N be a normal subgroup of a finite group G. If $Y = \{y_1, \dots, y_t\}$ has the property that $\langle Y, N\rangle = G$, but $\langle Z, N\rangle \neq G$ for every proper subset Z of Y, then there exist $n_1, \dots, n_u \in N$ such that $\{y_1, \dots, y_t, n_1, \dots, n_u\}$ is a minimal generating set of G.*

Proof Since $G = \langle Y, N\rangle$, $Y \cup N$ contains a minimal generating set X of G, and the minimality property of Y implies $Y \subseteq X$. □

Lemma 7 *Let N be a normal subgroup of a finite group G. If x_1N and x_2N are joined by an edge of $\Delta(G/N)$, then x_1n_1 and x_2n_2 are joined by an edge of $\Delta(G)$ for every $n_1, n_2 \in N$.*

Proof Let $\{x_1N, x_2N, x_3N, \dots, x_tN\}$ be a minimal generating set of G/N. By Lemma 6, for every $n_1, n_2 \in N$, there exists $m_1, \dots, m_u \in N$ such that

$$\{x_1n_1, x_2n_2, x_3, \dots, x_t, m_1, \dots, m_u\}$$

is a minimal generating set of G. □

We will write $x_1 \sim_G x_2$ if x_1 and x_2 belong to the same connected component of $\Delta(G)$. The following lemma is an immediate consequence of Lemma 7.

Lemma 8 *Let N be a normal subgroup of a finite group G and let $x, y \in G$. If $xN, yN \in V(G/N)$ and $xN \sim_{G/N} yN$, then $x \sim_G y$.*

Lemma 9 [3, Corollary 1.5] *Let G be a finite group with $S := F^*(G)$ nonabelian simple. If x, y are nontrivial elements of G, then there exists $s \in G$ such that $\langle x, s\rangle$ and $\langle y, s\rangle$ both contain S.*

Lemma 10 *Let G be a finite monolithic primitive group. Assume that $N = \mathrm{soc}\, G$ is non abelian and that $G = \langle x_1, N\rangle = \langle x_2, N\rangle$. Then there exists $m \in N$ such that $\langle x_1, m\rangle = \langle x_2, m\rangle = G$.*

Proof We have $N = S_1 \times \cdots \times S_t$, where $t \in \mathbb{N}$ and $S_i \cong S$ with S a nonabelian simple group. First consider the case $t = 1$. By Lemma 9 there exists $m \in N$ with $\langle x_1, m\rangle = \langle x_2, m\rangle = G$. Assume $t > 1$. We have $G \leq \mathrm{Aut}(S) \wr \mathrm{Sym}(t)$ and it is not restrictive to assume $x_1 = (h_1, \dots, h_t)\sigma$ with $h_1, \dots, h_t \in \mathrm{Aut}(S)$, $\sigma \in \mathrm{Sym}(t)$ and $\sigma(1) = 2$. There exists $u \in \mathbb{Z}$ such that $x_2^u = (h_1^*, \dots, h_2^*)\sigma$, with $h_1^*, \dots, h_t^* \in \mathrm{Aut}(S)$. Set $l_1 := x_1$, $l_2 := x_2^u$, $k_1 := h_1$, $k_2 := h_1^*$. Let w be an element of S of order 2. By Lemma 9, there exists $s \in S$ such that $\langle w^{k_1}, s\rangle = \langle w^{k_2}, s\rangle = S$. For $1 \leq i \leq t$, consider the projection $\pi_i : N \to S_i \cong S$. Let $m = (w, s, 1, \dots, 1) \in N \cong S^t$. For $i \in \{1, 2\}$, the subgroup $R_i := \langle m, x_i\rangle$ contains $\langle m, m^{l_i}\rangle \leq N$. Notice that $S = \langle s, w^{k_i}\rangle \leq \pi_2(\langle m, m^{l_i}\rangle)$, hence $\pi_2(R_i \cap N) \cong S$. Since $R_iN = G$, we deduce that $\pi_j(R_i \cap N) \cong S$ for each $j \in \{1, \dots, t\}$. In particular (see for example [4, Proposition 1.1.39]) either $N \leq R_i$ or there exist $k \in \{1, \dots, t\}$ and $h \in \mathrm{Aut}(S)$ such that $\pi_k(z) = h(\pi_1(z))$ for each $z \in R_i \cap N$. The second possibility cannot occur, since $m = (w, s, 1, \dots, 1) \in R_i \cap N$ and s and w are not conjugate in $\mathrm{Aut}\, S$ ($|w| = 2$, while $|s| \neq 2$, otherwise S would be generated by two involutions). So $N \leq R_i$ and consequently $R_i = G$. □

Proof of Theorem 1 We prove the theorem by induction on the order of G. If can be easily seen that $x \in V(G)$ if and only if $x\,\mathrm{Frat}(G) \in V(G/\mathrm{Frat}(G))$ and that $\Delta(G)$ is connected if and only if $\Delta(G/\mathrm{Frat}(G)$ is connected. So if $\mathrm{Frat}(G) \neq 1$, the conclusion follows by induction. We may so assume $\mathrm{Frat}(G) \neq 1$. Let N be a minimal normal subgroup of G and let $x, y \in V(G)$. If xN and yN are non-isolated vertices of G/N, then by induction $xN \sim_G yN$, so it follows from Lemma 14 that $x \sim_G y$. This means that the set Ω_N of the elements $g \in V(G)$ such that $gN \in V(G/N)$ is contained in a unique connected component, say Γ_N, of $\Delta(G)$. Assume now $g \in V(G) \setminus \Omega_N$. If G/N is non-cyclic, then $gN \in \mathrm{Frat}(G/N)$. In particular a minimal generating set of G containing g must contain also an element z such that $zN \notin \mathrm{Frat}(G/N)$. But then $z \in \Omega_N$ and, since $z \in \Gamma_N$ and $g \sim_G z$, we conclude $g \in \Gamma_N$. In other words, if G/N is cyclic, then $\Gamma_N = V(G)$. So we may assume that G/N is cyclic for every minimal normal subgroup N of G.

This implies that one of the following occur:

(1) G is cyclic;
(2) $G \cong C_p \times C_p$
(3) G has a unique minimal normal subgroup, say N, and N is not central.

If G is cyclic, then the conclusion follows from Proposition 5. If $G \cong C_p \times C_p$, then $\Delta(G)$ is a complete multipartite graph, with $p+1$ parts of size $p-1$. So we may assume that the third case occurs. First assume that N is abelian. In this case N has a cyclic complement, $H = \langle h \rangle$, acting faithfully and irreducibly on N. We have $\langle n, h \rangle = G$ for every non trivial element n of G, and this implies that there exists a unique connected component Λ of $\Delta(G)$ containing all the non trivial elements of N. Let now $g \in G \setminus N$. There is a conjugate h^* of h in G with $g \notin \langle h^* \rangle$. If $1 \neq n \in N$, then $G = \langle n, h^* \rangle = \langle g, h^* \rangle$, so $g \sim_G h^* \sim_G n$, hence $g \in \Lambda$. We remain with the case when N is non-abelian. Let $F/N = \mathrm{Frat}(G/N)$ and set $\Sigma_1 = F \setminus \{1\}$, $\Sigma_2 = \{g \in G \mid \langle g \rangle N = G\}$, $\Sigma_3 = \{g \in G \mid gN \in V(G/N)\}$ (we have $\Sigma_3 = \varnothing$ if and only if $|G/N|$ is a prime power). Notice that $V(G)$ is the disjoint union of Σ_1, Σ_2 and Σ_3. By Lemma 10, all the elements of Σ_2 belong to the same connected component, say Γ, of $\Delta(G)$. Assume $\Sigma_3 \neq \varnothing$. Fix $y \in \Sigma_2$ and choose n such that $G = \langle y, n \rangle$. Let p be a prime divisor of $|G/N|$ and let y_1, y_2 be generators, respectively, of a Sylow p-subgroup and a p-complement of $\langle y \rangle$. Since $\{y_1, y_2, n\}$ is a minimal generating set for G, it follows $y_1, y_2 \in \Sigma_3$ and that $y_1 \sim_G y_2 \sim_G y \sim_G n$. But we noticed in the first part of this proof that all the elements of $\Sigma_3 = \Omega_N$ belong to the same connected component, and so $\Sigma_2 \cup \Sigma_3 \subseteq \Gamma$. Finally let $g \in \Sigma_1$ and let X be a minimal generating set of G containing g. Certainly $X \cap (\Sigma_2 \cup \Sigma_3) \neq \varnothing$, so $g \in \Gamma$. □

3 Soluble groups

Let u be a positive integer and G a finite group. In this section we will use the following notations. We will denote by $\Omega_u(G)$ the set of the minimal generating sets of G of size u, by $\Gamma_n(G)$ the graph whose vertices are the elements of G and in which x_1 and x_2 are adjacent if and only if there exists $X \in \Omega_n(G)$ with $x_1, x_2 \in X$. Moreover we will denote by $V_n(G)$ the set of the non-isolated vertices of $\Gamma_u(G)$ and by $\Delta_u(G)$ the subgraph of $\Gamma_u(G)$ induces by $V_u(G)$. Finally we will write $x_1 \sim_{G,u} x_2$ to indicate that x_1 and x_2 belong to the same connected component of $\Delta_u(G)$.

We will need a series of preliminary results before giving the proof of Theorem 2. The following is immediate.

Lemma 11 *Let G be a finite group. Then $\Delta_u(G)$ is connected if and only if $\Delta_u(G/\operatorname{Frat}(G)$ is connected.*

Given a subset X of a finite group G, we will denote by $d_X(G)$ the smallest cardinality of a set of elements of G generating G together with the elements of X.

Lemma 12 [10, Lemma 2] *Let X be a subset of G and N a normal subgroup of G and suppose that $\langle g_1, \dots, g_r, X, N\rangle = G$. If $r \geq d_X(G)$, we can find $n_1, \dots, n_r \in N$ so that $\langle g_1n_1, \dots, g_rn_r, X\rangle = G$.*

Lemma 13 *Let N be a normal subgroup of a finite group group G and consider the projection $\pi : G \to G/N$. Suppose $A \in \Omega_u(G/N)$ and $b \in V_u(G)$ with $bN \in A$. Then there exists $B \in \Omega_u(G)$ such that $b \in B$ and $A = \pi(B)$.*

Proof Let $A = \{bN, z_1N, \dots, z_{u-1}N\}$ and $t = d_{\{b\}}(G)$. Since $b \in V_u(G)$, $t \leq u-1$. By Lemma 12, there exist $n_1, \dots, n_{u-1} \in N$ such that $\langle b, z_1n_1, \dots, z_{u-1}n_{u-1}\rangle = G$. The set $B := \{b, z_1n_1, \dots, z_{u-1}n_{u-1}\}$ satisfies the requests of the statement. □

Lemma 14 *Let N be a normal subgroup of a finite group G and let $x, y \in V_u(G)$. If $xN, yN \in V_u(G/N)$ and $xN \sim_{G/N,u} yN$, then there exists $n \in N$ such that $x \sim_{G,u} yn$.*

Proof Since $xN \sim_{G/N,u} yN$, there exists a sequence $A_1, \dots, A_t$ of elements of $\Omega_u(G/N)$ such that $xN \in A_1$, $yN \in A_t$ and $A_i \cap A_{i+1} \neq \varnothing$ for $1 \leq i \leq t-1$. We claim that there exists a sequence $B_1, \dots, B_t$ of minimal generating sets of G such that $x \in B_1$, $\pi(B_i) = A_i$ for $1 \leq i \leq t$ and $B_i \cap B_{i+1} \neq \varnothing$ for $1 \leq i \leq t-1$. By Lemma 13, there exists a minimal generating set B_1 of G with $A_1 = \pi(B_1)$ and $x \in B_1$. Suppose that $B_1, \dots, B_j$ have been constructed for $j < t$. There exists $g \in B_j$ such that $gN \in A_j \cap A_{j+1}$. Again by Lemma 13, there exists a minimal generating set B_{j+1} of G with $A_{j+1} = \pi(B_{j+1})$ and $g \in B_{j+1}$. □

Denote by $d(G)$ and $m(G)$, respectively, the smallest and the largest cardinality of a minimal generating set of G. A nice result in universal algebra, due to Tarski and known with the name of Tarski irredundant basis theorem (see for example [6, Theorem 4.4]) implies that, for every positive integer k with $d(G) \leq k \leq m(G)$, G contains an independent generating set of cardinality k. The proof of this theorem relies on a clever but elementary counting argument which implies also the following result:

Lemma 15 *For every k with $d(G) \leq k < m(G)$ there exists a minimal generating set $\{g_1, \dots, g_k\}$ with the property that there are $1 \leq i \leq k$ and x_1, x_2 in G such that $\{g_1, \dots, g_{i-1}, x_1, x_2, g_{i+1}, \dots, g_k\}$ is again a minimal generating set of G. Moreover x_1, x_2 can be chosen with the extra property that $g_i = x_1x_2$.*

Recall that for a d-generator finite group G, the swap graph $\Sigma_d(G)$ is the graph in which the vertices are the ordered generating d-tuples and in which two vertices $(x_1, \dots, x_d)$ and $(y_1, \dots, y_d)$ are adjacent if and only if they differ only by one entry.

Proposition 16 *Let G be a finite soluble group. Then $\Delta_{d(G)}(G)$ is connected.*

Proof Let $d = d(G)$. If G is cyclic, then $\Delta_d(G)$ is a null graph, and there is nothing to prove. Assume $d \geq 2$ and let $x, y \in V_d(G)$. Let $X, Y \in \Omega_d(G)$ with $x \in X$ and $y \in Y$. By [12], the swap graph $\Sigma_d(G)$ is connected, so there exists a path in $\Sigma_d(G)$ joining X to Y. Notice that if A, B are adjacent vertices of $\Sigma_d(G)$, then there exists two connected components Γ_A and Γ_B of $\Delta_d(G)$ containing, respectively, A and B. On the other hand $A \cap B \neq \varnothing$, by the way in which the swap graph is defined. Thus $\Gamma_A \cap \Gamma_B \neq \varnothing$ and consequently $\Gamma_A = \Gamma_B$ and all the elements of $A \cup B$ belong to the same connected component. This implies in particular that if $A_1 = X, A_2, \dots, A_{t-1}, A_t = Y$ is a path joining X and Y, then all the elements of $\cup_{1 \leq i \leq t} A_i$ belong to the same connected component. □

Proof of Theorem 2 We may assume $d(G) \leq u \leq m(G)$, otherwise $V_u(G)$ is empty. If $u = d(G)$, then the results follows from Proposition 16. So we assume $u > d(G)$. We prove the statement by induction on $|G|$. By Lemma 11, we may assume $\mathrm{Frat}(G) = 1$.

Let N be a minimal normal subgroup of G. Let K be a complement of N in G. We have $d(K) \leq d(G) < u$ and $m(K) = m(G/N) = m(G) - 1 \geq u - 1$ (see [16, Theorem 2]). By the Tarski irredundant basis theorem, K has a minimal generating set $\{k_1, \dots, k_{u-1}\}$ of size $u - 1$ and $\{k_1, \dots, k_{u-1}, m\}$ is a minimal generating set of G for every $m \neq 1$. This implies that all the non-trivial elements of N belong to the same connected component, say Γ, of $\Delta(G)$.

In order to complete our proof, we are going to show that $X \cap \Gamma \neq \varnothing$, for every minimal generating set $X = \{x_1, \dots, x_u\}$ of G. For each $i \in \{1, \dots, u\}$, there exists $k_i \in K$ and $n_i \in N$ such that $x_i = k_i n_i$. We may order the indices in such a way that $Y = \{k_1, \dots, k_t\}$ is a minimal generating set for K.

We distinguish two cases.

a) $t < u$. Let $H = \langle x_1, \dots, x_t \rangle$. Since $G = \langle Y \rangle N = HN$ and $H \neq G$, we deduce that H is a complement for N in G and $\langle H, x_{t+1} \rangle = G$. In particular $t = u - 1$ and $\{x_1, \dots, x_t, m\} \in \Omega_u(G)$ for every $1 \neq m \in G$. This implies $\{x_1, \dots, x_t\} \subseteq \Gamma \cap X$.

b) $t = u$. Since $d(K) \leq d(G) < u$ and $m(K) \geq u$, by Lemma 15 there exists $\{z_1, \dots, z_u\} \in \Omega_u(K)$ with the property that $\{z_1 z_2, \dots, z_u\} \in \Omega_{u-1}(K)$. We first want to prove that if $n \in N$ and $\tilde{z} := z_u n \in V_n(G)$, then $\tilde{z} \in \Gamma$. First suppose that there exists a complement H on N in G containing $\tilde{z}$. There exist $m_1, \dots, m_{u-1} \in N$ such that $z_i m_i \in H$ for $1 \leq i \leq u - 1$. This implies

$$H = \langle z_1 m_1 z_2 m_2, z_3 m_3, \dots, z_{u-1} m_{u-1}, \tilde{z} \rangle,$$

but then $\{z_1 m_1 z_2 m_2, z_3 m_3, \dots, z_{u-1} m_{u-1}, \tilde{z}, m\} \in \Omega_u(G)$ for every $1 \neq m \in M$. Thus $\tilde{z}$ and m are adjacent vertices of $\Delta_u(G)$ and consequently $\tilde{z} \in \Gamma$. Now assume that no complement of N in G contains $\tilde{z}$. If $1 \neq m \in N$, then $\langle z_1 z_2, z_3, \dots, z_u, m \rangle = G$, hence $d_{\{z_u\}}(G) \leq u - 1$. Since $G = \langle z_1, z_2, \dots, z_u, N \rangle$, by Lemma 12 there exist $m_1, \dots, m_{u-1} \in M$ such that $\langle z_1 m_1, \dots, z_{u-1} m_{u-1}, z_u \rangle = G$. As before, this implies

$z_u \in \Gamma$ and consequently $\{z_1 m_1, \ldots, z_{u-1} m_{u-1}\} \subseteq \Gamma$. On the other hand, since no complement for N in G contains $\tilde{z}$, it must be $\langle z_1 m_1, \ldots, z_{u-1} m_{u-1}, \tilde{z} \rangle = G$. So $\tilde{z}$ is adjacent to the vertices $z_1 m_1, \ldots, z_{u-1} m_{u-1}$ of $\Delta_u(G)$ and consequently $\tilde{z} \in \Gamma$. Now we can conclude our proof. Since $\{x_1 N, \ldots x_u N\}, \{z_1 N, \ldots z_u N\} \in \Omega_u(G/N)$, we have $x_1 N, z_u N \in V_n(G/N)$, so by induction $x_1 N \sim_{G/N,u} z_u N$. By Lemma 14 there exists $n \in N$ such that $x_1 \sim_{G,u} z_u n$. But we proved before that $z_u n \in \Gamma$, and this implies $x_1 \in \Gamma \cap X$. □

4 Planar graphs

Lemma 17 *Let N be a normal subgroup of a finite group G. If $\Gamma(G)$ is planar, then either G/N is cyclic of prime-power order or $|N| \leq 2$.*

Proof Assume that G/N is not a cyclic group of prime-power order. Then $\Delta(G/N)$ is not a null-graph. In particular there exist x and y in G such that xN and yN are joined by an edge of $\Gamma(G/N)$. By Lemma 7, the subgraph of $\Gamma(G)$ induced by $xN \cup yN$ is isomorphic to the complete bipartite graph $K_{a,a}$, with $|a| = N$. If $\Gamma(G)$ is planar, then $K_{a,a}$ is planar, and this implies $a \leq 2$. □

Proposition 18 $\Gamma(C_n)$ *is planar if and only if one of the following occurs:*

(1) *n is a prime-power.*
(2) *$n = p \cdot q$, where p and q are distinct primes and $p \leq 3$.*
(3) *$n = 4 \cdot q$, where q is an odd prime.*

Proof If $n = p^a$ is a prime power, then $\Gamma(C_n)$ is an edgeless graph, and consequently it is planar. Assume that n is not a prime power and let $p < q$ be the two smaller prime divisors of n. We have that C_n contains a normal subgroup N such that $G/N \cong C_{p \cdot q}$, and it follows from Lemma 17 that $|N| \leq 2$. If $|N| = 1$, then $\Delta(C_n) \cong K_{p-1,q-1}$, and consequently $\Gamma(C_n)$ is planar if and only if $p \leq 3$. If $|N| = 2$, then $p = 2$ and $\Delta(G) \cong K_{2,2(q-1)}$, which is a planar graph. □

Lemma 19 *Let G be a finite group. If G is not cyclic, then there exists a normal subgroup N of G with the property that $d(G/N) = 2$ but G/M is cyclic for every normal subgroup M of G with $N < M$.*

Proof Let $\mathcal{M}$ be the set of the normal subgroups M of G with the property that $d(G/M) = 2$. We claim that if G is not cyclic, then $\mathcal{M} \neq \varnothing$. Indeed let

$$1 = N_t < \cdots < N_0 = G$$

be a chief series of G and let j be the smallest positive integer with the property that G/N_j is not cyclic. By [15, Theorem 1.3], $d(G/N_j) = 2$. Once we know that $\mathcal{M}$ is not empty, any subgroup in $\mathcal{M}$ which is maximal with respect to the inclusion satisfies the requests of the statement. □

Proposition 20 *Let G be a finite, non-cyclic group. Then $\Gamma(G)$ is planar if and only if $G \in \{C_2 \times C_2, C_2 \times C_4, D_4, Q_8, \mathrm{Sym}(3)\}$*

Proof Let G be a non-cyclic group. Choose a normal subgroup N of G as described in Lemma 19. It follows from Lemma 7 that $\Gamma(G)$ contains a subgraph isomorphic to $\Delta(G/N)$. So if $\Gamma(G)$ is planar, then $\Gamma(G/N)$ is planar and $|N| \leq 2$. By [18] either $G/N \cong C_2 \times C_2$ or $G/N \cong \mathrm{Sym}(3)$. If $G/N \cong C_2 \times C_2$ then either $d(G) = m(G)$ and $G \in \{C_2 \times C_2, C_2 \times C_4, D_4, Q_8\}$, or $d(G) = m(G) = 3$ and $G \cong C_2 \times C_2 \times C_2$. In the last case $\Delta(G) \cong K_7$ is not planar. In the other cases, $\Gamma(G)$ coincides with the generating graph of G and it is planar. If $G/N \cong \mathrm{Sym}(3)$, then $G \cong \mathrm{Sym}(3)$, $G \cong D_6$ or $G \cong C_3 \rtimes C_4$. If $G \cong S_3$ then $\Gamma(G)$ coincides with the generating graph and it is planar. If $G \cong D_6$, then the six non-central involutions induces a complete subgraph, so $\Gamma(G)$ is not planar. If $G \cong C_3 \rtimes C_4$, then the subset $A \cup B$, where A is the set of the six elements of order 4 and B is the set of the four elements with order divisible by 3, induces a non planar graph containing an isomorphic copy of $K_{6,4}$. □

5 Examples and questions

The minimal generating sets for Sym(4) are described in [8]. We have that $d(\mathrm{Sym}(4)) = 2$ and $m(\mathrm{Sym}(4)) = 3$ and the three graphs $\Gamma_2(\mathrm{Sym}(4))$, $\Gamma_3(\mathrm{Sym}(4))$ and $\Gamma(\mathrm{Sym}(4))$ are described in the following tables, where the first column contains a representative x of a conjugacy class of Sym(4), the second column describes the set of the elements of Sym(4) adjacent to x in the graph and the third columns gives the degree of x in the graph. We denote by X_i the set of i-cycles (for $2 \leq i \leq 4$) in Sym(4) and by Y the set of the double transpositions.

$\Gamma_2(\mathrm{Sym}(4))$		
(1,2)(2,3)	$\varnothing$	0
(1,2)	$\{(2,3,4)^{\pm 1}, (1,3,4)^{\pm 1}, (1,2,3,4)^{\pm 1}, (1,2,4,3)^{\pm 1}\}$	8
(1,2,3)	$X_4 \cup \{(1,4), (2,4), (3,4)\}$	9
(1,2,3,4)	$X_3 \cup \{(1,2), (1,4), (2,3), (3,4), (1,3,2,4)^{\pm 1}, (1,2,4,3)^{\pm 1}\}$	16

$\Gamma_3(\mathrm{Sym}(4))$		
(1,2)(3,4)	$X_2 \cup X_3$	14
(1,2)	$Y \cup \{(1,2,3)^{\pm 1}, (1,2,4)^{\pm 1}, (1,3), (1,4), (2,3), (2,4), (3,4)\}$	12
(1,2,3)	$Y \cup \{(1,2), (1,3), (2,3), (1,2,4)^{\pm 1}, (1,3,4)^{\pm 1}, (2,3,4)^{\pm 1}\}$	12
(1,2,3,4)	$\varnothing$	0

Denote by $\omega(\Gamma)$ the clique number of a graph Γ. By [20, Theorem 1.1], we have $\omega(\Gamma_2(\mathrm{Sym}(4)) = 4$ and a maximal clique is $\{(1,2,3,4), (1,2,4,3), (1,3,2,4), (1,2,3)\}$; $\omega(\Gamma_3(\mathrm{Sym}(4))) = 7$ and a maximal clique is $X_2 \cup \{(1,2)(3,4)\}$; $\omega(\Gamma(\mathrm{Sym}(4))) = 11$ and a maximal clique is $X_2 \cup \{(1,2)(3,4), (1,2,3), (1,2,4), (1,3,4), (2,3,4)\}$. However it is not in general true that $\omega(\Gamma_2(\mathrm{Sym}(n)) \leq$

$\Gamma(\mathrm{Sym}(4))$		
(1,2)(3,4)	$X_2 \cup X_3$	14
(1,2)	$Y \cup X_3 \cup \{(1,3),(1,4),(2,3),(2,4),(3,4),(1,2,3,4)^{\pm 1},(1,2,4,3)^{\pm 1}\}$	20
(1,2,3)	$Y \cup X_2 \cup X_4 \cup \{(1,2,4)^{\pm 1},(1,3,4)^{\pm 1},(2,3,4)^{\pm 1}\}$	21
(1,2,3,4)	$X_3 \cup \{(1,2),(1,4),(2,3),(3,4),(1,3,2,4)^{\pm 1},(1,2,4,3)^{\pm 1}\}$	16

$\omega(\Gamma_{n-1}(\mathrm{Sym}(n))$. Indeed let n be a sufficiently large odd integer. By [2, Theorem 1], $\omega(\Gamma_2(\mathrm{Sym}(n)) = 2^{n-1}$ while by [8, Theorem 2.1] a non-isolated vertex of $\omega(\Gamma_{n-1}(\mathrm{Sym}(n))$ is either a transposition or a 3-cycle or a double transposition, so $\omega(\Gamma_{n-1}(\mathrm{Sym}(n)) \le \binom{n}{2} + 2\cdot\binom{n}{3} + 3\cdot\binom{n}{4}$. The independence number of $\Gamma_2(\mathrm{Sym}(4))$ is 12 and a maximal independent set is $X_3 \cup Y \cup \{id\}$. The independence number of $\Delta_3(\mathrm{Sym}(4))$ is 8 and a maximal independent set is $X_4 \cup \{(1,2),(1,3,4),(1,4,3),id\}$. The independence number of $\Delta(\mathrm{Sym}(4))$ is 6 and a maximal independent set is $Y \cup \{(1,2,3,4),(1,4,3,2),id\}$. For $u \in \{1,2\}$, the degree of the vertex of $\Gamma_u(\mathrm{Sym}(4))$ corresponding to the element g is divisible by the order of g. When $u = 2$, this follows from a more general result. Indeed, by [19, Proposition 2.2], if G is a 2-generated group and $g \in G$, then $|g|$ divides the degree of g in the generating graph of G. However this cannot be generalized to $\Gamma_u(G)$ for arbitrary values of u. For example, consider the dihedral group $G = \langle a, b \mid a^6, b^2, (ab)^3\rangle$ of degree 6. Then $\{a^2, a^3, a^i b\} \in \Omega_3(G)$ for $0 \le i \le 5$ and there are precisely 7 elements adjacent to a^2 in $\Gamma_3(G)$: a^3 and $a^i b$ for $0 \le i \le 5$. We propose the following question.

Question 21 *Let G be a finite group and $g \in G$. Does $|g|$ divide the degree of g in $\Gamma_{d(G)}(G)$?*

For a finite group G, let

$$W(G) = \bigcap_{d(G)\le u\le m(G)} V_u(G).$$

We have seen that $W(\mathrm{Sym}(4)) = X_2 \cup X_3 \ne V(\mathrm{Sym}(4))$. If $d(G) = m(G)$, then $V(G) = W(G)$ by definition. One may ask whether the converse is true.

Question 22 *Does $V(G) = W(G)$ imply $d(G) = m(G)$?*

The answer is positive in the soluble case.

Proposition 23 *Let G be a finite soluble group. If $V(G) = W(G)$, then $d(G) = m(G)$.*

Proof Let $d = d(G)$, $m = m(G)$. Since $V_u(G/\mathrm{Frat}(G)) = V_u(G)\,\mathrm{Frat}(G)/\mathrm{Frat}(G)$, we may assume $\mathrm{Frat}(G) = 1$. First assume that G is cyclic. Then $|G| = p_1 \dots p_m$, with $p_1, \dots, p_m$ distinct primes. Notice that $V_1(G) = \varnothing$, so $V(G) = W(G) \subseteq V_1(G)$ implies $V(G) = \varnothing$ and this is possible only if $m = 1$. Now assume that G is not cyclic. By assumption $V_d(G) = V(G) = G \setminus \{1\}$. This is equivalent to say that $d_{\{g\}}(G) = d - 1$ for any $1 \ne g \in G$. By [9, Corollary 2.20, Theorem 2.21] either G

is an elementary abelian p-group or there exist a finite vector space V, a nontrivial irreducible soluble subgroup H of $\mathrm{Aut}(V)$ and an integer $d > d(H)$ such that

$$G \cong V^{r(d-2)+1} \rtimes H,$$

where r is the dimension of V over $\mathrm{End}_H(V)$ and H acts in the same way on each of the $r(d-2)+1$ factors. In the first case $d = m$, and we are done. In the second case, by [16, Theorem 2], $m = r(d-2)+1+m(H)$. If $d = 2$, then $H = \langle h \rangle$ is a cyclic group and $G = V \rtimes H$. Since H is a maximal subgroup, if $h \in \Omega_u(G)$, then $u = 2$. On the other hand, by assumption, $h \in V_m(G)$, and therefore $m = 2$. Assume $d > 2$. This implies $t = r(d-2)+1 \geq 2$. We are going to prove that $r = 1$. If $r \neq 1$, then there exist $v_1, v_2 \in V$ that are $\mathrm{End}_G(V)$-linearly independent. This implies that the H-submodule W of V^t generated by $w = (v_1, v_2, 1, \ldots, 1)$ is H-isomorphic to V^2. As a consequence, if $w \in \Omega_u(G)$, then $u-1 \leq m(G/W) = m-2$. But then $w \notin \Omega_m(G)$, against the assumption $V_m(G) = V(G)$. So $r = 1$, and this implies that H is isomorphic to a subgroup of the multiplicative group of $\mathrm{End}_G(V)$, and consequently it is cyclic. Moreover $t = d-1$ and $m(G) = m(H)+d-1$. Let h be a generator for of H. Notice that $h \notin \Omega_u(G)$ if $u-1 > t = d-1$. Since, by assumption, $h \in \Omega_u(G)$, it must be $m-1 \leq d-1$, and consequently $m = d$. □

The finite groups with $d(G) = m(G)$ are described in [1]. All the finite groups with this property are soluble. So Question 22 is equivalent to the following.

Question 24 *Does there exist an unsoluble group G with $V(G) = W(G)$?*

Another question that we propose is the following.

Question 25 *Let G be a finite non-cyclic group. Is the graph $\Delta(G)$ Hamiltonian?*

Notice that if G is cyclic, then $\Delta(G)$ is not necessarily Hamiltonian. For example, if $G \cong C_{2 \cdot p}$, with p and odd prime, then $\Delta(G) \cong K_{1,p-1}$. In the case of $\mathrm{Sym}(4)$ the affirmative answer to the previous question follows from the Dirac's Theorem, stating that an n-vertex graph in which each vertex has degree at least $n/2$ must have a Hamiltonian cycle. However it is not in general true that any vertex of $\Delta(G)$ has degree at least $|V(G)|/2$. Consider for example $G = \langle a, b \mid a^5, b^4, b^{-1}aba^3 \rangle$. The graph $\Delta(G)$ has 19 vertices, and the degree of b^2 in this graph is 8. In any case, we may use Dirac's Theorem in the case of finite nilpotent groups.

Theorem 26 *If G is a finite non-cyclic nilpotent group, then $\Delta(G)$ is Hamiltonian.*

Proof Let $g \in V(G)$ and let $H = \langle g \rangle \mathrm{Frat}(G)$. Let $n = |V(G)|$ and d the degree of $g \in \Delta(G)$. Since $G/\mathrm{Frat}(G)$ is a direct product of elementary abelian p-groups, any element of $G \setminus H$ is adjacent to g in $\Delta(G)$. Since G is not cyclic, H is a proper subgroup of G, hence $|G| \geq 2|H|$ and therefore

$$d = |G| - |H| \geq \frac{|G| - |\mathrm{Frat}(G)|}{2} = \frac{n}{2}.$$

So the conclusion follows from Dirac's theorem. □

Finally, a question that remains open is whether Theorem 2 remains true is the solubility assumption is removed.

Question 27 *Let G be a finite group and* $u \in \mathbb{N}$. *Is* $\Delta_u(G)$ *a connected graph?*

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