Asymmetrizing trees of maximum valence $2^{\aleph_0}$

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Abstract

Let T be a finite or infinite tree and m the minimum number of vertices moved by the non-identity automorphisms of T . We give bounds on the maximum valence d of T that assure the existence of a vertex coloring of T with two colors that is preserved only by the identity automorphism. For finite m we obtain the bound $d \leq 2^{m/2}$ when T is finite, and $d \leq 2^{(m-2)/2} + 2$ when T is infinite. For countably infinite m the bound is $d \leq 2^m$. This relates to a question of Babai, who asked whether there existed a function $f(d)$ such that every connected, locally finite graph G with maximum valence d has a 2-coloring of its vertices that is only preserved by the identity automorphism if the minimum number m of vertices moved by each non-identity automorphisms of G is at least $m \geq f(d)$. Our results give a positive answer for trees. The trees need not be locally finite, their maximal valence can be $2^{\aleph_0}$. For finite m we also extend our results to asymmetrizing trees by more than two colors.

Keywords Automorphisms · Asymmetric colorings · Finite and infinite trees

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Dedicated to Gert Sabidussi on the occasion of his 90th birthday.

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1 Introduction

A vertex coloring of a graph G is *asymmetric* if the only color preserving automorphism of G is the identity. Such colorings go back at least to Babai [2], who showed that every k -regular tree, where $k \geq 2$ is an arbitrary cardinal, has an asymmetric 2-coloring.

Independently Albertson and Collins [1] introduced the term *distinguishing coloring* for an asymmetric coloring. Their paper spawned numerous others on the subject, for example [3,7,9,11]. Many of them relate the existence of asymmetric colorings with a given number of colors to the *motion* $m(G)$ of a graph. It is the minimal number of vertices every non-identity automorphism of G moves.

Albertson and Collins also introduced the term *distinguishing number* for the smallest positive integer d for which an asymmetric coloring of a graph G exists and abbreviated it by $D(G)$. We call it the *asymmetric coloring number*, but use the same notation.

There is a well-known conjecture about infinite, locally finite graphs, that is, about infinite graphs, where every vertex has finite valence. It is the *Infinite Motion Conjecture* (IMC) of Tucker [17], and states that every connected, infinite locally finite graph G has an asymmetric 2-coloring if every nontrivial automorphism of G moves infinitely many vertices. Despite many strong partial results, for example by Lehner [11], it is still open.

In connection with the IMC Babai¹ asked the following question.

Question 1 *Does there exist a function $f(d)$ such that every connected graph G with maximal valence d and motion $m(G) \geq f(d)$ has an asymmetric 2-coloring?*

The example of a star with d arms of length d shows that $f(d)$ is at least $2\lceil \log_2 d \rceil$. For the case of trees we show that $d \leq 2^{m(T)/2}$ (which is equivalent to $m(T) \geq 2\lceil \log_2 d \rceil$) implies that T has an asymmetric 2-coloring. The validity of the IMC for locally finite trees is obtained as an easy corollary.

Besides the paper of Babai [2] there are several other articles about the asymmetrization of trees that predate Albertson and Collins [1]. Notably Polat and Sabidussi [15] and Polat [13,14]. The first considers trees of arbitrary cardinality, but also contains an algorithm that determines the number of inequivalent asymmetrizing sets of finite trees. The other two extended and in some sense complete the investigation. Polat and Sabidussi do not consider motion, which means that we cannot directly use results from that paper. We thus prefer a direct approach.

For trees T that are not locally finite we prove that they have an asymmetric 2-coloring if their maximum valence is at most $2^{\aleph_0}$ and their motion is infinite. It implies that the Infinite Motion Conjecture holds for countable trees even without the assumption of local finiteness.

At the end of Sect. 2 we also derive bounds on the valence of trees with given finite motion that guarantee the existence of asymmetric colorings with more than two colors.

¹ Private communication.

2 Trees with finite valences

We shall now show that Babai's question has a positive answer for trees. That the IMC holds for trees follows as an easy corollary.

Lemma 1 *Let (T, v) be a finite rooted tree with motion m and valence at most $2^{m/2}$. Then (T, v) has at least $2^{m/2}$ asymmetric 2-colorings with v white.*

Proof Let (T, v) be a minimal counterexample with v colored white. The branches at v are all isomorphic to one rooted tree (S, u) ; otherwise we could prune a branch to get a smaller counterexample. Then either (S, u) has motion at least m and valence at most $2^{m/2}$, in which case it has at least $2^{m/2}$ asymmetric colorings (since (T, v) is a minimal counterexample). Or (S, u) is itself asymmetric with n vertices, where $m = 2n$ (since the motion comes from interchanging two branches). In this case, (S, u) also has $2^n = 2^{m/2}$ asymmetric colorings.

Number the branches $1, \dots, k$ where k is at most $2^{m/2}$ (since the valence is at most $2^{m/2}$). There are at least $2^{m/2}$ asymmetric colorings of the first branch. Since there are at most $2^{m/2} - 1$ other branches, we can choose a different asymmetric coloring for each of the other branches giving us at least $2^{m/2}$ colorings of (T, v) with v white, a contradiction of (T, v) being a counterexample. $\square$

Note that these colorings may be equivalent under automorphism of (T, v) .

In the proof of Theorem 1 we need asymmetric 2-colorings of (T, v_0) where all neighbors of v are white. As this is not assured by the bound $d \leq 2^{m/2}$ we need a tighter one.

Corollary 1 *Let (T, v) be a finite rooted tree with motion m and valence $d \leq 2^{(m/2)-1} + 2$. If the valence of v is at most $d - 2$, then (T, v) has an asymmetric 2-coloring where all neighbors of v are white.*

Proof If any of the branches of v in (T, v) has a nontrivial automorphism, then it has nonzero motion m and at least $2^{m/2}$ asymmetric 2-colorings with branch root white, because $d \leq 2^{m/2-1} + 2 \leq 2^{m/2}$. If any of the branches, say (T, v_1) , is asymmetric, and if the number of branches isomorphic to (T, v_1) is k , then we number the branches of v as $(T, v_1), \dots, (T, v_k)$, where $1 \leq k \leq d - 2$ and where $v_1, \dots, v_k$ are the roots of the branches. Clearly every branch (T, v_i) has at least $m/2 - 1$ vertices. Hence every branch has at least $2^{m/2-1} \geq d - 2$ asymmetric 2-colorings where the root is white. As $k \leq d - 2$ we can color the $(T, v_1), \dots, (T, v_k)$ with pairwise different asymmetric 2-colorings, where the $v_1, \dots, v_k$ are white. $\square$

In the sequel *one-sided infinite paths*, also called *rays*, and *two-sided infinite paths*, or *double rays*, will play an important role. It is easy to see that every tree without vertices of valence 1 contains a double ray. On the other hand, it is not so clear that every infinite connected tree contains a ray if it has no vertex of infinite valence, although this assertion is true not only for trees, but also for graphs, see [4, Proposition 8.2.1]. It is a consequence of König's Infinity Lemma from 1927. For a complete statement and proof see König [10, Satz 6 (Unendlichkeitslemma)] or Diestel [4, Lemma 8.1.2].

Furthermore, in a locally finite, infinite tree T each vertex v of valence 1 is the origin of a ray. To see this, let T be such a tree and v a vertex of valence 1. T must have a ray, say R , by [4, Proposition 8.2.1]. Let $R(x, y)$ be the subpath of R from x to y . Then $R \setminus R(x, y) \cup P$ is a ray with origin v .

Whenever we invoke any of the above facts we will refer to them as consequences of König's Infinity Lemma.

Vertices of valence 1 are also called *leaves*. In the proof of the following theorem we use the fact that locally finite trees without leaves have asymmetric 2-colorings. This is well known, and there are many different proofs, compare [18, Theorem 3.1], [7, Theorem 4.2] or [8, Corollary 5 (ii)].

Theorem 1 *Let T be a tree with finite motion m and maximum valence d , where $d \leq 2^{m/2}$ if T is finite or infinite with a double ray, and $d \leq 2^{m/2-1} + 2$ otherwise. Then T has an asymmetric 2-coloring.*

Proof If T is finite, then it has a center consisting of a single vertex or an edge, which is stabilized by each automorphism. If the center is a single vertex, we consider it as the root of T and apply Lemma 1. If it is an edge ab , then we fix a and b by assigning them different colors, and apply Lemma 1 to the two rooted trees that we obtain by removing the edge between a and b .

Suppose now that T is infinite and has a double ray. Let Q be the subgraph of T consisting of all double rays of T . We claim that Q is a subtree. As it is acyclic, we only have to show that it is connected. To see this, assume that Q_1 and Q_2 are connected components of Q . Because T is connected, there must be a path P between them, connecting a vertex x in Q_1 with a vertex y in Q_2 . Vertex x is the origin of a ray R_1 in Q_1 , y the origin of a ray R_2 in Q_2 , and $R_1 \cup P \cup R_2$ is a double ray in T . Hence P is part of a double ray and thus in Q . Clearly Q is preserved by all automorphisms of T . Because Q has no leaves, there is an asymmetric 2-coloring c_Q of Q . If $Q = T$, then there is nothing else to show.

Otherwise we form the subgraph S of T consisting of all edges that are not in Q and of the endpoints of those edges. Every connected components of S is a tree with exactly one vertex in Q , and every component of S is uniquely determined by the vertex it shares with Q . We thus introduce the notation (S, v) for the connected component of S containing v , and consider v as the root of (S, v) . If (S, v) is infinite, then it contains a ray R with origin v by König's Infinity Lemma. Because v is a vertex of a two-sided infinite path of Q it is also the origin of a ray, say R' in Q . Hence $R \cup R'$ is a two-sided infinite path of T and R would have to be in Q . Hence all components (S, v) of S are finite. Clearly their motion is at least that of T . As the root v of (S, v) is fixed by the coloring c_Q of Q we can apply Lemma 1 to extend c_Q to an asymmetric 2-coloring of $Q \cup (S, v)$. Applying this procedure to all connected components of S we obtain the claimed asymmetric 2-coloring of T .

If T has no two-sided infinite path, then it must have a leaf, say v_0 . Again, because T is infinite, we infer by König's Infinity Lemma that T has a ray R with origin v_0 . As before we form the subgraph S of T consisting of all edges that are not in R and of the endpoints of those edges. All connected components of S are finite, rooted trees (S, v) , $v \neq v_0$, whose endpoint v is in R . Clearly each v has valence at most $d - 2$ in (S, v) , and, by assumption, $d \leq 2^{m/2-1} + 2$. Thus, by Corollary 1, each (S, v) has an

asymmetric 2-coloring where all neighbors of v in (S, v) are white. We color all (S, v) accordingly. As the roots v can be colored arbitrarily, we color them and v_0 black.

Thus all vertices of R are black and all neighboring vertices of R white. This fixes R , hence all roots of the (S, v) , and thus T . $\square$

The following corollary shows the validity of the IMC for trees.

Corollary 2 *Every locally finite tree with infinite motion has an asymmetric 2-coloring.*

Proof We proceed as in the proof of Theorem 1 with the simplification that we extend asymmetric 2-colorings of Q or R to a coloring c of T by coloring all other vertices white. Clearly c fixes Q , respectively R , and thus the roots of the components (S, v) of S . As before the (S, v) are finite. Because the motion of T is infinite, they all must be asymmetric. Hence c is an asymmetric 2-coloring of T . $\square$

We wish to mention that this also follows from results in [18, Theorem 3.2 and Theorem 4.3]. In the first of the two cited theorems it is shown that the asymmetric coloring number $D(T)$ of a tree with double rays (referred to as multi-ended trees there) is 1 or the maximum of 2 and the supremum of the asymmetric coloring numbers of the rooted trees (S, v) , where $v \in V(Q)$. If we have infinite motion, the trees have to be asymmetric and so $D(T)$ is 1 or 2.

In the second theorem it is shown that infinite, locally finite trees with no double rays (referred to as one-ended trees there) are asymmetric or their asymmetric coloring number is the maximum of the asymmetric coloring numbers the rooted trees (S, v) , where $v \in V(R)$. In the latter case the motion would be finite.

In our terminology this means that infinite, locally finite trees with no double rays have finite motion or are asymmetric.

As motion was not considered in [18] it is not surprising that no reference to the IMC was made.

2.1 General connected graphs

We have answered Babai's question for trees using $f(d) = 2\lceil \log_2 d \rceil$. It is interesting to note that for maximum valence $d = 3$, the same function gives an asymmetric coloring with the exception of the cube and Petersen graph [5, Corollary 3.7].

For vertex transitive graphs of valence 4 Lehner and Verret [12, Theorem 1,2] show that all but five graphs and one infinite family of graphs have an asymmetric 2-coloring. They all have motion 2.

Calling graphs without asymmetric 2-colorings *exceptional*, we are curious whether the bound $f(d) = 2\lceil \log_2 d \rceil$ holds in general, with a reasonably small class of exceptional graphs, and ask whether this is at least true for valence 4.

Question 2 *For 4-valent graphs that are not vertex transitive, is there a natural small family of 4-valent graphs such that all but finitely many 4-valent connected exceptional graphs of motion at least 4 are contained in that family?*

2.2 Asymmetric colorings with more than two colors

The topic of finding asymmetric colorings of trees by $c > 2$ colors was treated in [6]. (We will use c for the number of colors rather than for a coloring in the remainder of this section.) For us [6, Proposition 3] is of interest. If we rewrite part of it in our terminology we obtain the following proposition.

Proposition 1 *Let $T = (T, v_0)$ be a rooted finite tree of maximal valence d and $c \geq 2$ be a natural number. Then there exists a number $r = r(c, d)$ and a c -coloring of (T, v_0) that fixes all vertices $u \in V(T)$, for which there exists a leaf w in (T, u) such that $d(u, w) \geq r$.*

It means that to each T that satisfies the assumptions of Proposition 1 there is a c -coloring that fixes all vertices with the exception of the non-root vertices of the subtrees (T, u) of (T, v_0) of height at most $r(c, d)$. If we require that the motion m of T is at least $\frac{1}{2}d^r$, then all (T, u) of height at most $r(c, d)$ have to be asymmetric, because their number of vertices is at most d^r and hence no automorphism of (T, v_0) (it must fix v_0) can move at least $\frac{1}{2}d^r$ vertices. Furthermore, if the valence of v_0 is $d - 2$, we can always find an asymmetric c -coloring of (T, v_0) where the neighbors of v_0 in (T, v_0) are colored with at most $c - 1$ colors. By [6, Proposition 3] we infer that $r(c, d) < \log_c d$ for $d \geq 4$ and $2 < c < d - 1$.

With the same arguments as before we thus arrive at the following theorem.

Theorem 2 *Let T be a finite or infinite tree with motion m and maximum valence $d \geq 4$, where $2 < c < d - 1$. If $d^{\log_c d} \leq 2m$, then T has an asymmetric c -coloring.*

Note, if the motion is at least three, then we can find an asymmetric 2-coloring when $d = 3$, and an asymmetric c -coloring when $c = d - 1$.

We did not spell out the bound $d^{\log_2 d}$ for m in the case when $c = 2$, because the bound is $2\lceil \log_2 d \rceil$ is better by far for larger values of d , but wish to mention that the motivation for the first part of this paper was the improvement of the bound $d^{\log_2 d}$.

Question 3 *Does the condition $d \leq c^{m/2}$ ensure an asymmetric c -coloring when $c > 2$?*

3 Trees whose order is the cardinality of the continuum

In this section we show that trees of motion $\aleph_0$ and maximum valence $2^{\aleph_0}$ have asymmetric 2-colorings. In our proof infinite trees without rays play an important role. We shall make use of the fact that every rayless tree has a center consisting of a single vertex or an edge, see Sabidussi and Polat [16].

First some notation. Let T be a tree with root v_0 and $v \in V(T)$. Then the unique neighbor of v that is on the shortest path from v to v_0 is its *down-neighbor*, all other neighbors are *up-neighbors*. The set of all neighbors of v is the *neighborhood* of v and will be denoted by $N(v)$.

Lemma 2 *Asymmetric rooted trees of finite height are finite. For every height there are only finitely many asymmetric rooted trees of that height.*

Proof We proceed by induction. Let $T(k)$ be the number of asymmetric rooted trees of bounded, finite height. We wish to show that $T(k)$ is finite. Clearly $T(0) = 1$. Suppose the assertion is true for trees of height at most k , and let (T, v_0) be a rooted tree of height $k + 1$. Then (T, v) is asymmetric for every neighbor v of v_0 . The height of these trees is at most k , and no two of them can be isomorphic. By the induction hypothesis their number is finite, hence $T(k + 1)$ is also finite. Clearly these trees are all finite. $\square$

Theorem 3 *Each infinite rooted tree of finite height has nontrivial automorphism group and finite motion.*

Proof Suppose this is not true. Then there is a counterexample of smallest height, say T with root v_0 .

Consider the subtrees (T, v) for all neighbors v of v_0 . Suppose they are all finite. If any of them has a nontrivial symmetry, then it has finite motion, and thus also T . Hence all of them must be asymmetric. Because T is infinite, there must be infinitely many such subtrees (T, v) . They cannot be pairwise nonisomorphic, because their height is at most $k - 1$, and because there are only finitely many asymmetric rooted trees of each height. Hence there is a pair of isomorphic subtrees (T, u) , (T, w) of T , where $u, w \in N(v_0)$. The automorphism of T that interchanges them has motion $2|V(T)|$, and thus the motion of T is finite.

This implies that at least one (T, v) must be infinite. Its height is less than that of T . Therefore it has nontrivial automorphism group and finite motion, and thus also T . $\square$

We now show that the bound on the valences in the IMC can be relaxed for rayless infinite trees.

Theorem 4 *Let T be a rayless tree of infinite motion and maximum valence 2^{n_0} . Then T has an asymmetric 2-coloring.*

Proof Polat and Sabidussi [16] proved that any rayless tree T has a center consisting of a single vertex v_0 or an edge $v_0 w_0$. In the first case it suffices to prove the theorem for the rooted tree (T, v_0) .

In the second we assign different colors to the endpoints and remove the edge $v_0 w_0$, but not v_0 or w_0 . We obtain two rooted trees, say (S, v_0) and (S', w_0) , where the roots are fixed. Both are rayless and have maximum valence 2^{n_0} . Unless they are asymmetric, they must have infinite motion. If they are both asymmetric, then they must be isomorphic, otherwise T could not have infinite motion. If only one is asymmetric, we assign the color of its root to all of its vertices, which fixes it as a subtree of T . Hence we are left with the case, where one or both of the rooted trees (S, v_0) and (S', w_0) are rayless with infinite motion and maximum valence 2^{n_0} . This reduces the problem to the consideration of the first case.

Suppose (T, v_0) is a rooted tree with infinite motion, then it must have unbounded height by Lemma 2.

We show first that (T, v_0) must have asymmetric subtrees T_v of unbounded height. To see this, consider all branches (T, u) where u is a neighbor of v_0 . If (T, u) has finite

height, then it has finite motion or is finite and asymmetric graph. If it has finite motion, then so does (T, v_0) , contrary to assumption. Hence (T, u) is a finite asymmetric tree. If two such (T, u) are isomorphic, then (T, v_0) has finite motion, so any two such branches (T, u) of finite height are non-isomorphic asymmetric trees. By Lemma 2 there are only finitely many such trees to every height. Hence, if all (T, u) have finite height, then their number is at most countable, but as their union, together with v_0 , contains all vertices of T , this is not possible, because $|V(T)| = 2^{\aleph_0}$.

Hence, there must be at least one (T, u) , say (T, v_1) , of unbounded height. If it has a symmetry, i.e. a nontrivial automorphism, then it has infinite motion. By the previous argument it must have an up-neighbor v_2 of v_1 such that (T, v_2) has unbounded height. We cannot continue this process indefinitely, because (T, v_0) is rayless. Suppose it stops at v_i . Then (T, v_i) has unbounded height and is asymmetric.

This means that every ascending chain $v_1, v_2, \dots$, where the (T, v_i) have unbounded height and infinite motion, ends in a (T, w) that is asymmetric and has unbounded height or is finite and asymmetric.

We remark that to least two of the (T, w) that are asymmetric and have unbounded height there must be a root-preserving automorphism that maps one into the other, otherwise (T, v_0) could not have infinite motion.

The finite asymmetric (T, v) -s come in sets S with a common neighbor, say z , to their roots. If the heights of the trees in the set S are unbounded, and if z has no other up-neighbors than the roots of the elements in S , then (T, z) is asymmetric of unbounded height. Otherwise, the vertex z must have an up-neighbor z^* , where (T, z^*) has unbounded height and infinite motion. But then there is an asymmetric tree (T, w^*) of unbounded height such that z^* is on the shortest path from w^* to v_0 .

We have thus shown that each vertex v of (T, v_0) for which (T, v) is infinite, is either a vertex of an infinite asymmetric subtree of unbounded height, or on the path from v_0 to the root v of an infinite asymmetric subtree (T, v) of unbounded height.

Every infinite asymmetric subtree (T, v) of unbounded height is contained in a maximal asymmetric subtree (T, w) . These subtrees $(T, w_\kappa), \kappa \in K$, are mutually disjoint and their number, i.e. the cardinality of K , is at most the maximal cardinality of the vertex set of (T, v_0) . Because $2^{\aleph_0}$ is the maximal valence of (T, v_0) we conclude that $|K| \leq |(T, v_0)| \leq 2^{\aleph_0}$.

Each $(T, w_\kappa), \kappa \in K$, is infinite and asymmetric, and thus has $2^{\aleph_0}$ inequivalent 2-colorings. We assign 2-colorings to the (T, w_κ) such that no two of them can be mapped into each other by a color preserving automorphism of (T, v_0) and color all other vertices white. Because every vertex of (T, v_0) that is not a vertex of some (T, w_κ) is on the shortest path of some w_κ to v_0 this coloring fixes (T, v_0) . $\square$

Theorem 5 *Let T be a tree of countably infinite motion m and maximal valence $d \leq 2^m$. Then T has an asymmetric 2-coloring.*

Proof By Theorem 4 we can restrict attention to trees T with rays. For such trees the situation is similar to that in Corollary 2. The main difference is that Q can be uncountable and that the components of $S = T \setminus E(Q)$, resp. $S = T \setminus E(R)$, can either be finite asymmetric trees or rayless trees with infinite motion.

We begin with the case when T has double rays. As in the proof of Corollary 2 we first need an asymmetric 2-coloring of Q . To this end we recall a result of Imrich,

Klavžar and Trofimov [7, Theorem 4.2], which asserts that every tree-like graph of maximal valence $\leq 2^{\aleph_0}$ has an asymmetric two coloring. Because each infinite tree without leaves is tree-like, we infer the existence of an asymmetric 2-coloring of Q and use it to color Q .

Q is invariant under all automorphisms of T , hence this coloring assures that all vertices of Q are fixed, no matter how the other vertices are colored. To color them we consider the connected components of $S = T \setminus E(Q)$. They are rooted trees (S, v) , which share only the root v with Q . If such an (S, v) is finite, then it has to be asymmetric, because T has infinite motion, and so we can color all vertices of (S, v) that are different from v white. If (S, v) is infinite, then it must be rayless. If it is asymmetric, then we color all of its vertices but v white. If it has symmetries, then it must have infinite motion, because T has infinite motion, and thus an asymmetric 2-coloring by Theorem 4. By switching colors if needed we can assure that v has the same color as in Q . Applying this to all (S, v) we thus extend the coloring of Q to an asymmetric 2-coloring of T .

It remains to consider the case when T has a ray, but no double ray. Let R be a ray in T . We color it black, and all vertices adjacent to it white. No matter how we color the remaining vertices, this ensures that R is fixed. Now we recall from the proof of Theorem 1 that all components of $S = T \setminus E(R)$ are rayless. If such a component (S, v) is finite, then it must be asymmetric, we color all of its hitherto uncolored vertices white. If (S, v) is infinite, but asymmetric, we do the same.

Now suppose that (S, v) is infinite but is not asymmetric. It must have infinite motion and thus an asymmetric 2-coloring, but this coloring may not color all neighbors of v white. We thus have to consider all lobes (S, u) of (S, v) , where u is a neighbor of v . For the finite lobes it is easily seen that no two of them can be isomorphic, because of the motion assumption, and that they must be asymmetric for the same reason. It remains to consider the infinite (S, u) with a nontrivial automorphism. They are rayless and of infinite motion, which means that they have asymmetric 2-colorings (where u is white) by Theorem 4.

Their number is bounded by the valence of v , that is $2^{\aleph_0}$, but they all may be isomorphic. To solve this we invoke a theorem of Polat and Sabidussi [15, Corollary 4.3], who showed that for infinite trees without leaves the number of asymmetric 2-colorings that are inequivalent under automorphisms is 0 or $2^{|V(T)|}$. $\square$

An immediate consequence is the following corollary.

Corollary 3 *The Infinite Motion Conjecture holds for countable trees even without the assumption of local finiteness.*

Note the similarity between Theorems 1 and 5, which says that T has an asymmetric 2-coloring if it has finite motion m and maximum valence $\leq 2^{(m/2)-1} + 2$.

Question 4 *Let m be an infinite cardinal. Does every tree of motion m and maximum valence $\leq 2^m$ have an asymmetric 2-coloring?*

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