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Volterra operators and Hankel forms on Bergman spaces of Dirichlet series

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Abstract
For a Dirichlet series g, we study the Volterra operator $T_g f(s) = -\int_s^{+\infty} f(w)g'(w)\,dw$, acting on a class of weighted Hilbert spaces $\mathcal{H}_w^2$ of Dirichlet series. We obtain sufficient / necessary conditions for T_g to be bounded (resp. compact), involving BMO and Bloch type spaces on some half-plane. We also investigate the membership of T_g in Schatten classes. Moreover, we show that if T_g is bounded, then g is in $\mathcal{H}_w^p$, the L^p-version of $\mathcal{H}_w^2$, for every $0 < p < \infty$. We also relate the boundedness of T_g to the boundedness of a multiplicative Hankel form of symbol g, and the membership of g in the dual of $\mathcal{H}_w^1$.

Keywords Volterra operator · Dirichlet series · Hankel forms

Mathematics Subject Classification Primary 31B10 · 32A36; Secondary 30B50 · 30H20

1 Introduction

Dirichlet series are functions of the form

$$f(s) = \sum_{n=1}^{+\infty} a_n n^{-s}, \quad \text{with } s \in \mathbb{C}. \tag{1.1}$$

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For a real number θ, $\mathbb{C}_\theta$ stands for the half-plane $\{s,\ \Re s > \theta\}$, and $\mathbb{D}$ for the unit disk. $\mathcal{D}$ denotes the class of functions f of the form (1.1) in some half-plane $\mathbb{C}_\theta$, and $\mathcal{P}$ is the space of Dirichlet polynomials.

The increasing sequence of prime numbers will be denoted by $(p_j)_{j\geq 1}$, and the set of all primes by $\mathbb{P}$. Given a positive integer n, $n = p^\kappa$ will stand for the prime number factorization $n = p_1^{\kappa_1} p_2^{\kappa_2} \cdots p_d^{\kappa_d}$, which associates uniquely to n the finite multi-index $\kappa(n) = (\kappa_1, \kappa_2, \ldots, \kappa_d)$. The number of prime factors in n is denoted by $\Omega(n)$ (counting multiplicities), and by $\omega(n)$ (without multiplicities).

The space of eventually zero complex sequences c_{00} consists in all sequences which have only finitely many non zero elements. We set $\mathbb{D}^\infty_{\text{fin}} = \mathbb{D}^\infty \cap c_{00}$ and $\mathbb{N}^\infty_{0,\text{fin}} = \mathbb{N}_0^\infty \cap c_{00}$, where $\mathbb{N}_0 = \mathbb{N} \cup \{0\}$ is the set of non-negative integers.

Let $F : \mathbb{D}^\infty_{\text{fin}} \to \mathbb{C}$ be analytic, i.e. analytic at every point $z \in \mathbb{D}^\infty_{\text{fin}}$ separately with respect to each variable. Then F can be written as a convergent Taylor series

$$F(z) = \sum_{\alpha \in \mathbb{N}^\infty_{0,\text{fin}}} c_\alpha z^\alpha,\ z \in \mathbb{D}^\infty_{\text{fin}}.$$

The truncation $A_m F$ of F onto the first m variables is defined by

$$A_m F(z) = F(z_1, \ldots, z_m, 0, 0, \ldots).$$

For z, χ in $\mathbb{D}^\infty$, we set $z.\chi := (z_1\chi_1, z_2\chi_2, \ldots)$, and $\mathfrak{p}^{\mathbf{x}} := (p_1^x, p_2^x, \ldots)$ for a real number x, .

The Bohr lift [11] of the Dirichlet series $f(s) = \sum_{n=1}^{+\infty} a_n n^{-s}$ is the power series

$$\mathcal{B}f(\chi) = \sum_{n=1}^{+\infty} a_n \chi^{\kappa(n)} = \sum_{\alpha \in \mathbb{N}^\infty_{0,\text{fin}}} \tilde{a}_\alpha \chi^\alpha, \ \text{ where } \tilde{a}_\alpha = a_{p^\alpha}, \chi \in \mathbb{D}^\infty_{\text{fin}},$$

with the multiindex notation $\chi^\alpha = \chi_1^{\alpha_1} \chi_2^{\alpha_2} \cdots$.

Given a sequence of positive numbers $w = (w_n)_n = (w(n))_n$, one considers the Hilbert space (see [21,23])

$$\mathcal{H}^2_w := \left\{ \sum_{n=1}^{+\infty} a_n n^{-s} : \sum_{n=1}^{+\infty} \frac{|a_n|^2}{w_n} < +\infty \right\}.$$

The choice $w_n = 1$ corresponds to the space $\mathcal{H}^2$, introduced in [19].

The weights considered in this article satisfy $w_n = O(n^\epsilon)$ for every $\epsilon > 0$; from the Cauchy-Schwarz inequality, Dirichlet series in $\mathcal{H}^2_w$ absolutely converge in $\mathbb{C}_{1/2}$.

We are interested in the Volterra operator T_g of symbol $g(s) = \sum_{n=1}^{+\infty} b_n n^{-s}$, defined by

$$T_g f(s) := -\int_s^{+\infty} f(w) g'(w)\, dw,\ \Re s > \frac{1}{2}. \tag{1.2}$$

On the unit disk $\mathbb{D}$, the Volterra operator, whose symbol is an analytic function g, is given by

$$J_g f(z) := \int_0^z f(u)g'(u)du, \ z \in \mathbb{D}. \tag{1.3}$$

Pommerenke [26] showed that J_g (1.3) is bounded on the Hardy space $H^2(\mathbb{D})$ if and only if g is in $BMOA(\mathbb{D})$. Let σ be the Haar measure on the unit circle $\mathbb{T}$. Fefferman's duality Theorem states that $BMOA(\mathbb{D})$ is the dual space of $H^1(\mathbb{D})$. Thus the boundedness of J_g is equivalent to the boundedness of the Hankel form

$$H_g(f, h) := \int_{\mathbb{T}} f(u)h(u)\overline{g(u)}d\sigma(u), \ f, h \in H^2(\mathbb{D}). \tag{1.4}$$

Let V be the Lebesgue measure on $\mathbb{C}$, normalized such that $V(\mathbb{D}) = 1$.

Many authors, in particular [2], have studied Volterra operators on Bergman spaces of $\mathbb{D}$. The classical Bergman space $A^2_\gamma(\mathbb{D})$, $\gamma > 0$, is associated to the measure $d\tilde{m}_\gamma(z) := \gamma \left(1 - |z|^2\right)^{\gamma-1} dV(z)$. J_g is bounded on $A^2_\gamma(\mathbb{D})$ if and only if g is in the Bloch space, which is the dual of $A^1_\gamma(\mathbb{D})$.

The Bergman space of the finite polydisk $A^2_\gamma(\mathbb{D}^d)$, $d \geq 1$, corresponds to the measure

$$d\widetilde{\nu}_\gamma(z) := d\tilde{m}_\gamma(z_1) \times \cdots \times d\tilde{m}_\gamma(z_d).$$

The boundedness of the Hankel form

$$H_g(f, h) := \int_{\mathbb{D}^d} f(z)h(z)\overline{g(z)}d\widetilde{\nu}_\gamma(z), \ f, h \in A^2_\gamma(\mathbb{D}^d), \tag{1.5}$$

is equivalent to the membership of g to the Bloch space (see [17]), defined by

$$\text{Bloch}(\mathbb{D}^d) := \left\{ f : \mathbb{D}^d \to \mathbb{C} \text{ holomorphic} : \max_{\kappa \in \mathcal{I}_d} \sup_{z \in \mathbb{D}^d} \left|\partial^\kappa f(\kappa.z)\right| (1 - |z|)^\kappa < +\infty \right\},$$

where $\mathcal{I}_d$ denotes the set of multi-indices $\kappa = (\kappa_1, \dots, \kappa_d)$, with entries in $\{0, 1\}$, and

$$z = (z_1, \dots, z_d), \ \partial^\kappa = \partial^{\kappa_1}_{z_1} \cdots \partial^{\kappa_d}_{z_d}, \ (1 - |z|)^\kappa = (1 - |z_1|)^{\kappa_1} \cdots (1 - |z_d|)^{\kappa_d}.$$

Recall that for $0 < p < \infty$, the Hardy space of Dirichlet series $\mathcal{H}^p$ is the space of Dirichlet series $f \in \mathcal{D}$ such that $\mathcal{B}f$ is in $H^p(\mathbb{D}^\infty)$, endowed with the norm

$$\|f\|_{\mathcal{H}^p} := \|\mathcal{B}f\|_{H^p(\mathbb{D}^\infty)} = \left(\int_{\mathbb{T}^\infty} |\mathcal{B}f(z)|^p \, d\sigma_\infty(z)\right)^{1/p},$$

σ_∞ being the Haar measure of the infinite polytorus $\mathbb{T}^\infty$.

The norm in the space $\mathcal{H}^\infty := H^\infty(\mathbb{C}_0) \cap \mathcal{D}$ is

$$\|f\|_{\mathcal{H}^\infty} = \sup_{s\in\mathbb{C}_0} |f(s)|\,.$$

Let $H^\infty(\mathbb{D}^\infty)$ be the space of series F which are finitely bounded, i.e.

$$\|F\|_{H^\infty(\mathbb{D}^\infty)} = \sup_{m\in\mathbb{N}_0, z\in\mathbb{D}^\infty} |A_m F(z)| < \infty.$$

Via the Bohr isomorphism, we have [16,19]

$$\|f\|_{\mathcal{H}^\infty} = \|\mathcal{B}f\|_{H^\infty(\mathbb{D}^\infty)}\,. \tag{1.6}$$

Several abscissae are related to a function g in $\mathcal{D}$, of the form $g(s) = \sum_{n=1}^{+\infty} b_n n^{-s}$:

the abscissa of convergence $\sigma_c = \inf\left\{\sigma \in \mathbb{R} \,:\, \sum_{n=1}^{+\infty} b_n n^{-\sigma} \text{ converges}\right\}$;
the abscissa of absolute convergence $\sigma_a = \inf\left\{\sigma \in \mathbb{R} \,:\, \sum_{n=1}^{+\infty} |b_n|\, n^{-\sigma} \text{ converges}\right\}$;
the abscissa of uniform convergence
$\sigma_u = \inf\left\{\theta \in \mathbb{R} \,:\, \sum_{n=1}^{+\infty} b_n n^{-s} \text{ converges uniformly in } \mathbb{C}_\theta\right\}$.

The abscissa of regularity and boundedness, denoted by σ_b, is the infimum of those θ such that $g(s)$ has a bounded analytic continuation, to the half-plane $\Re(s) > \theta + \epsilon$, for every $\epsilon > 0$.

We have $-\infty \leq \sigma_c \leq \sigma_u \leq \sigma_a \leq +\infty$, and, if any of the abscissae is finite $\sigma_a - \sigma_c \leq 1$. Moreover, it is known that $\sigma_b = \sigma_u$ [11] , and $\sigma_a - \sigma_u \leq \frac{1}{2}$.

Volterra operators (1.2) on the spaces $\mathcal{H}^p$ have been investigated in [13]. Our aim is to study similar questions for the spaces $\mathcal{H}^2_w$, associated to specific weights w in the class $\mathcal{W}$ defined below.

Definition 1 Let $\beta > 0$. A sequence w belongs to $\mathcal{W}$ if it has one of the following forms:

(1) $w_n = [d(n)]^\beta$, where $d(n)$ is the number of divisors of the integer n. Then $\mathcal{H}^2_w := \mathcal{B}^2_\beta$.
(2) $w_n = d_{\beta+1}(n)$, where $d_\gamma(n)$ are the Dirichlet coefficients of the power of the Riemann zeta function, namely $\zeta^\gamma(s) = \sum_{n=1}^{+\infty} d_\gamma(n) n^{-s}$. Then $\mathcal{H}^2_w := \mathcal{A}^2_\beta$.

As in the case of $\mathcal{H}^2$ [13], we obtain sufficient/necessary conditions for T_g to be bounded on the Hilbert spaces $\mathcal{H}^2_w$. However, due to the lack of information of the behavior of the symbols in the strip $0 < \Re s < 1/2$, it seems difficult to get an " if and only if" condition. In the Hardy space setting, it is shown that T_g is bounded on $\mathcal{H}^2$ provided that g in $BMOA(\mathbb{C}_0)$. Since the spaces $\mathcal{A}^2_\beta$ and $\mathcal{B}^2_\beta$ (see Sect. 2) locally behave like Bergman spaces of the half plane $\mathbb{C}_0$, we would expect that the membership of g in Bloch$(\mathbb{C}_0)$ (resp. Bloch$_0(\mathbb{C}_0)$) would imply the boundedness (resp.

compactness) of T_g on $\mathcal{H}^2_w$. We obtain such a sufficient condition when $\mathcal{B}g$ depends on a finite number of variables $z_1, \dots, z_d$. However, our method specfically uses that d is finite, and we do not know whether the same result holds if $\mathcal{B}g$ is a function of infinitely many variables.

Le $\mathfrak{N}_d$ be the set of positive integers which are multiples of the primes $p_1, \dots, p_d$,

$$\mathcal{D}_d := \left\{ f \in \mathcal{D} \, : \, f(s) = \sum_{n \in \mathfrak{N}_d} a_n n^{-s} \right\}, \quad \text{and } \mathcal{H}^p_{d,w} := \mathcal{H}^p_w \cap \mathcal{D}_d.$$

One of our main results is the following.

Theorem 1 *Let T_g be the operator defined by (1.2) for some Dirichlet series g in $\mathcal{D}$.*

(a) If $g(s) = \sum_{n=2}^{+\infty} b_n n^{-s}$ is in $\mathcal{D}_d \cap Bloch(\mathbb{C}_0)$, then T_g is bounded on $\mathcal{H}^2_w$ and

$$\left\| T_g \right\|_{\mathcal{L}(\mathcal{H}_w)} \lesssim \|g\|_{Bloch(\mathbb{C}_0)} .$$

(b) If g is in $BMOA(\mathbb{C}_0)$, then T_g is bounded on $\mathcal{H}^2_w$ and

$$\left\| T_g \right\|_{\mathcal{L}(\mathcal{H}_w)} \lesssim \|g\|_{BMOA(\mathbb{C}_0)} .$$

(c) If T_g is bounded on $\mathcal{H}^2_w$, then g is in $Bloch(\mathbb{C}_{1/2})$ and

$$\|g\|_{Bloch(\mathbb{C}_{1/2})} \lesssim \left\| T_g \right\|_{\mathcal{L}(\mathcal{H}_w)} .$$

Via the Bohr lift, $\mathcal{H}^2_w$ are L^2-spaces of functions on the polydisk $\mathbb{D}^\infty$. Precisely, there exists a probability measure μ_w on $\mathbb{D}^\infty$ such that

$$\|f\|^2_{\mathcal{H}^2_w} = \int_{\mathbb{D}^\infty} |\mathcal{B}f(z)|^2 \, d\mu_w(z).$$

Analogously to the spaces $\mathcal{H}^p$, we define the space $\mathcal{H}^p_w$, $0 < p < \infty$ (see Sect. 2), as the closure of Dirichlet polynomials under the norm (quasi-norm if $0 < p < 1$)

$$\|f\|_{\mathcal{H}^p_w} = \|\mathcal{B}f\|_{L^p(\mathbb{D}^\infty, \mu_w)} .$$

Let $\mathcal{X}_w = \mathcal{X}(\mathcal{H}^2_w)$ be the space of symbols g giving rise to bounded operators T_g on $\mathcal{H}^2_w$. Our study provides the following strict inclusions:

$$BMOA(\mathbb{C}_0) \cap \mathcal{D} \subset_{\neq} \mathcal{X}_w \subset_{\neq} \cap_{0<p<\infty} \mathcal{H}^p_w.$$

We will also compare $\mathcal{X}_w$ with other spaces of Dirichlet series, in particular with the dual of $\mathcal{H}^1_w$, and the space of symbols g generating a bounded Hankel form

$$H_g(fh) := \langle fh, g \rangle_{\mathcal{H}^2_w}$$

on the weak product $\mathcal{H}_w^2 \odot \mathcal{H}_w^2$. As in the case of $\mathcal{H}^2$ [13], we only get partial results. For Dirichlet series involving d primes, we have

$$\mathcal{D}_d \cap \text{Bloch}(\mathbb{C}_0) \subset \mathcal{D}_d \cap \mathcal{X}_w \subset_{\neq} \mathcal{B}^{-1}\text{Bloch}(\mathbb{D}^d).$$

The paper is organized as follows. Section 2 starts by presenting some properties of the spaces $\mathcal{H}_w^2$. As a space of analytic functions on the half-plane $\mathbb{C}_{1/2}$, $\mathcal{H}_w^2$ is continuously embedded in a space of Bergman type of $\mathbb{C}_{1/2}$. In view of the Bohr lift, the norm of $\mathcal{H}_w^2$ can be expressed in terms of a probability measure μ_w on the polydisk. For $0 < p < \infty$, we consider the Bohr–Bergman space $\mathcal{H}_w^p$, and derive equivalent norms for these spaces.

In Sect. 3, we present some properties of the Dirichlet series which belong to a BMO or Bloch space of some half-plane $\mathbb{C}_\theta$. In particular, we relate the Carleson measures for both spaces of Dirichlet series and Bergman type spaces.

Section 4 is devoted to the proof of Theorem 1. First we consider the case when g is a function of $p_1^{-s}, \ldots, p_d^{-s}$. To prove (b), we observe that the boundedness of T_g on $\mathcal{H}^2$ implies the boundedness of T_g on $\mathcal{H}_w^2$. On another hand, combining the fact that $\mathcal{H}_w^2$ is embedded in a Bergman type space of the half-plane $\mathbb{C}_{1/2}$ with some characterizations of Carleson measures, we establish that

$$\mathcal{X}_w \subset \text{Bloch}(\mathbb{C}_{1/2}).$$

Compactness and Schatten classes are considered in Sects. 5 and 6.

In Sect. 7, we consider some specific symbols: fractional primitives of translates of a “weighted zeta”-function and homogeneous symbols. These examples will be used in Sect. 8.

In Sect. 8, we investigate the relationship between the boundedness of the Volterra operator T_g, the boundedness of the Hankel form

$$H_g(fh) = \langle fh, g\rangle_{\mathcal{H}_w^2},$$

and the membership of g in the dual of $\mathcal{H}_w^1$. In particular, we study examples of Hankel forms on Bergman spaces of Dirichlet series, which are the counterparts of the Hilbert multiplicative matrix [12].

Additionally, we show the strictness of the inclusions derived previously

$$BMOA(\mathbb{C}_0) \cap \mathcal{D} \subset_{\neq} \mathcal{X}_w \subset_{\neq} \cap_{0<p<\infty}\mathcal{H}_w^p,$$

and compare the space $\mathcal{D}_d \cap \mathcal{X}_w$ with Bloch spaces.

For two functions f, g, the notation $f = O(g)$ or $f \lesssim g$, means that there exists a constant C such that $f \leq Cg$. If $f = O(g)$ and $g = O(f)$, we write $f \asymp g$.

2 The Bohr–Bergman spaces $\mathcal{B}^2_\beta$, $\mathcal{A}^2_\beta$

2.1 The spaces $\mathcal{B}^2_\beta$, $\mathcal{A}^2_\beta$

These spaces are related to number theory. The number of divisors of the integer n, $d(n)$, is $d(n) = (\kappa_1 + 1) \cdots (\kappa_d + 1)$ when $n = p^\kappa$. We consider the following scale of Hilbert spaces

$$\mathcal{B}^2_\beta = \left\{ f(s) = \sum_{n=1}^{+\infty} a_n n^{-s} : \|f\|_{\mathcal{B}^2_\beta} := \left(\sum_{+\infty}^{n=1} \frac{|a_n|^2}{[d(n)]^\beta} \right)^{\frac{1}{2}} < \infty \right\}, \quad \text{for } \beta > 0.$$

The case $\beta = 0$ corresponds to the Hardy space $\mathcal{H}^2$. The reproducing kernels of $\mathcal{B}^2_\beta$ are

$$K^{\mathcal{B}^2_\beta}(s,u) = \zeta_\beta(s + \overline{u}), \text{ where } \zeta_\beta(s) = \sum_{n=1}^{+\infty} [d(n)]^\beta \, n^{-s}.$$

It is shown in [30] that there exists $\phi_\beta(s)$, an Euler product which converges absolutely in $\mathbb{C}_{1/2}$, such that

$$\zeta_\beta(s) = [\zeta(s)]^{2^\beta} \phi_\beta(s), \text{ and } \phi_\beta(1) \neq 0.$$

Another family of spaces arises from the so-called generalized divisor function. For $\gamma > 0$, the numbers $d_\gamma(n)$ are defined by the relation

$$\zeta^\gamma(s) = \sum_{n=1}^{+\infty} d_\gamma(n) n^{-s}.$$

A computation involving Euler products shows that we have

$$d_\gamma(p^r) = \frac{\gamma(\gamma+1)\cdots(\gamma+r-1)}{r!}, \quad \text{for } p \in \mathbb{P}, \quad \text{and any integer } r.$$

From its definition, d_γ is a multiplicative function, i.e. $d_\gamma(kl) = d_\gamma(k)d_\gamma(l)$ if k and l are relatively prime. Thus, $d_\gamma(n)$ can be computed explicitly from the decomposition $n = p^\kappa$.

We define the spaces

$$\mathcal{A}^2_\beta = \left\{ f(s) = \sum_{n=1}^{+\infty} a_n n^{-s} : \|f\|_{\mathcal{A}^2_\beta} := \left(\sum_{+\infty}^{n=1} \frac{|a_n|^2}{d_{\beta+1}(n)} \right)^{\frac{1}{2}} < \infty \right\}, \quad \text{for } \beta > 0,$$

with reproducing kernels $K^{\mathcal{A}^2_\beta}(s,u) = \zeta^{\beta+1}(s+\overline{u})$.

Notice that, in each case, the reproducing kernel has the form

$$K^{\mathcal{H}^2_w}(s,u) = Z_w(s+\overline{u}),$$

where $Z_w(s) := \sum_{n=1}^{+\infty} w_n n^{-s}$ has a singularity at $s=1$, with an estimate of the type

$$Z_w(s) = C_w(s-1)^{-(\delta+1)}\left[1+O(1)\right]. \tag{2.1}$$

2.2 Bohr–Bergman spaces on $\mathbb{D}^\infty$

The Bohr correspondence is an isometry between $\mathcal{H}^2_w$ and the weighted Bergman space of the infinite polydisk

$$H^2_w(\mathbb{D}^\infty) = \left\{ \sum_{\nu\in\mathbb{N}^\infty_{0,\mathrm{fin}}} a_\nu z^\nu \,:\, \sum_\nu \frac{|a_\nu|^2}{w_\nu} < \infty \right\}, \quad \text{where } w_\nu = \prod_j w_{\nu_j}.$$

In particular, the space $\mathcal{H}^2$ is identified with the Hardy space $H^2(\mathbb{T}^\infty)$ [19].
Let us consider the following probability measures on the unit disk $\mathbb{D}$,

$$dm_w(z) := M(|z|^2)dV(z),$$
$$\text{where } M(r) = \begin{cases} \frac{1}{\Gamma(\beta)}\left(\log\frac{1}{r}\right)^{\beta-1}, & \text{if } w_n = [d(n)]^\beta, \\ \beta(1-r)^{\beta-1}, & \text{if } w_n = d_{\beta+1}(n) \end{cases} \quad \beta > 0.$$

On the finite polydisk $\mathbb{D}^d$ $(d\in\mathbb{N})$, the corresponding Bergman spaces $H^2_w(\mathbb{D}^d)$ - specifically $B^2_\beta(\mathbb{D}^d)$ and $A^2_\beta(\mathbb{D}^d)$- are the L^2-closures of polynomials with respect to the norm

$$\|f\|_{H^2_w(\mathbb{D}^d)} := \left(\int_{\mathbb{D}^d} |f(z_1,\ldots,z_d)|^2\, dm_w(z_1)\times\cdots\times dm_w(z_d)\right)^{1/2}$$

If $f(z) = \sum_{n\in\mathbb{N}^d} a_n z^n$ is defined on $\mathbb{D}^d$, we have

$$\|f\|^2_{B^2_\beta(\mathbb{D})} = \sum_{n\in\mathbb{N}} \frac{|a_n|^2}{(n+1)^\beta}$$
$$\text{and } \|f\|^2_{A^2_\beta(\mathbb{D})} = \sum_{n\in\mathbb{N}} |a_n|^2 \frac{n!}{(\beta+1)(\beta+2)\cdots(\beta+n)}. \tag{2.2}$$

When d is finite, the estimate

$$\frac{n!}{(\beta+1)(\beta+2)\cdots(\beta+n)} \asymp (1+n)^{-\beta}$$

yields that, the spaces $B^2_\beta(\mathbb{D}^d)$ and $A^2_\beta(\mathbb{D}^d)$ coincide as sets, with equivalent norms. However, the norms are no longer equivalent in the case of infinitely many variables.

The $\mathcal{H}^2_w$-norm will be computed via the rotation invariant probability measure

$$d\mu_w(\chi) = dm_w(\chi_1) \times dm_w(\chi_2) \times dm_w(\chi_3) \times \cdots \text{ on } \mathbb{D}^\infty.$$

Applying the Bohr lift to a Dirichlet series $f(s) = \sum_{n=1}^{+\infty} a_n n^{-s}$, and using (2.2) for each variable, one obtains the following formula (see [5] in the case of $\mathcal{B}^2_\beta$)

$$\int_{\mathbb{D}^\infty} |\mathcal{B}f(\chi)|^2 \, d\mu_w(\chi) = \sum_{n=1}^{+\infty} \frac{|a_n|^2}{w_n} = \|f\|^2_{\mathcal{H}^2_w}.$$

Definition 2 For $0 < p < \infty$, the Bohr–Bergman spaces of Dirichlet series $\mathcal{B}^p_\beta$ and $\mathcal{A}^p_\beta$ - denoted by $\mathcal{H}^p_w$ - are the completions of the Dirichlet polynomials in the norm (quasi norm when $0 < p < 1$)

$$\|f\|^p_{\mathcal{H}^p_w} := \int_{\mathbb{D}^\infty} |\mathcal{B}f(\chi)|^p \, d\mu_w(\chi).$$

The Kronecker flow of the point $\chi = (\chi_1, \chi_2, \ldots) \in \mathbb{C}^\infty$ is given by

$$\mathcal{T}_t(\chi) = \left(2^{-it}\chi_1, 3^{-it}\chi_2, 5^{-it}\chi_3, \ldots\right), \; t \in \mathbb{R},$$

which defines an ergodic flow on $\mathbb{T}^\infty$ by Kronecker's theorem.

Therefore, it follows from Fubini's Theorem that, for any rotation invariant probability measure $d\nu$ on $\mathbb{D}^\infty$ and any probability measure $d\lambda$ on $\mathbb{R}$, we have

$$\|f\|^p_{L^p(\mathbb{D}^\infty, d\nu)} = \int_{\mathbb{D}^\infty} \int_{\mathbb{R}} |(\mathcal{B}f)(\mathcal{T}_t\chi)|^p \, d\lambda(t) d\nu(\chi). \tag{2.3}$$

2.3 On the half-plane $\mathbb{C}_{1/2}$

For $\theta \in \mathbb{R}$, let τ_θ be the following mapping from $\mathbb{D}$ to $\mathbb{C}_\theta$,

$$\tau_\theta(z) = \theta + \frac{1+z}{1-z}. \tag{2.4}$$

For $\delta > 0$, the conformally invariant Bergman space $A_{i,\delta}\left(\mathbb{C}_{1/2}\right)$ is the space of those functions f which are analytic in $\mathbb{C}_{1/2}$, and such that

$$\|f\|^2_{A_{i,\delta}(\mathbb{C}_{1/2})} := \left\|f \circ \tau_{1/2}\right\|^2_{A^2_\delta(\mathbb{D})} = 4^\delta \delta \int_{\mathbb{C}_{1/2}} |f(s)|^2 \frac{\left(\sigma - \frac{1}{2}\right)^{\delta-1}}{\left|s + \frac{1}{2}\right|^{2\delta+2}} dm(s) < \infty.$$

The weights w of the class $\mathcal{W}$ satisfy a Chebyshev-type estimate

$$\sum_{n\leq x} w_n \asymp x\,(\log x)^{\delta}\,, \quad \text{where } \delta=\delta(w):=\begin{cases}2^{\beta}-1 & \text{if } w_n=[d(n)]^{\beta}\,,\\ \beta & \text{if } w_n=d_{\beta+1}(n).\end{cases} \tag{2.5}$$

For any real number τ, set $S_\tau=\left[\frac{1}{2},1\right]\times[\tau,\tau+1]$. As mentioned in the introduction, the Dirichlet series which belong the $\mathcal{H}^2_w$ absolutely converge in $\mathbb{C}_{1/2}$. The space $\mathcal{H}^2_w$ is locally embedded in $A_{i,\delta(w)}\left(\mathbb{C}_{1/2}\right)$ [23,25], which means

$$\sup_{\tau\in\mathbb{R}}\int_{S_\tau}|f(s)|^2\,\frac{\left(\sigma-\frac{1}{2}\right)^{\delta-1}}{\left|s+\frac{1}{2}\right|^{2\delta+2}}dm(s)\leq c\left(\mathcal{H}^2_w\right)\|f\|^2_{\mathcal{H}^2_w}\,.$$

Since functions in $\mathcal{H}^2_w$ are uniformly bounded in $\mathbb{C}_1$, these embeddings are global (see [5,8]).

Lemma 1 *Let $\delta=\delta(w)$ be defined in (2.5). Then $\mathcal{H}^2_w$ is continuously embedded in $A_{i,\delta}\left(\mathbb{C}_{1/2}\right)$.*

2.4 Generalized vertical limits

Every $\chi=(\chi_1,\chi_2,\ldots)$ in $\mathbb{C}^\infty$ defines a completely multiplicative function by the formula $\chi(n)=\chi^\kappa$, where $n=p^\kappa$. For f of the form (1.1), the twisted Dirichlet series [5,6], is defined by

$$f_\chi(s)=\sum_{n=1}^{+\infty}a_n\chi(n)n^{-s}. \tag{2.6}$$

Notice that if $\chi\in\mathbb{T}^\infty$, f_χ is the vertical limit of f, introduced in [19].

We also consider the translations $f_\delta(s)=f(s+\delta)$, $\delta\in\mathbb{R}$. For those $\chi\in\mathbb{D}^\infty$ and $s=\sigma+it$ for which the series (2.6) converges, we have

$$f_\chi(s)=(\mathcal{B}f_\sigma\mathcal{T}_t)(\chi). \tag{2.7}$$

When f is in $\mathcal{H}^2_w$, the Cauchy-Schwarz inequality implies that (2.7) holds whenever $s\in\mathbb{C}_{1/2}$ and $\chi\in\overline{\mathbb{D}}^\infty$. By the Rademacher-Menchov Theorem (see [22]), (2.7) can be extended in the following way (the argument given in [5] for $\mathcal{B}^2_\beta$ remains true for $\mathcal{A}^2_\beta$).

Lemma 2 *If f is in $\mathcal{H}^2_w$, the Dirichlet series f_χ as defined in (2.6) converges in $\mathbb{C}_0$ for almost every $\chi\in\mathbb{D}^\infty$, with respect to μ_w.*

Recall that τ_θ, $\theta\in\mathbb{R}$, is the conformal mapping defined in (2.4). For $0<p<\infty$, the conformally invariant Hardy space $H^p_i(\mathbb{C}_\theta)$, is the space of those functions f

such that $f \circ \tau_\theta$ is in $H^p(\mathbb{T})$, the usual Hardy space of the unit disk. Setting $d\lambda(t) = \pi^{-1}(1+t^2)^{-1}dt$, we get

$$\|f\|^p_{H_i^p(\mathbb{C}_\theta)} = \int_{\mathbb{R}} |f(\theta+it)|^p \, d\lambda(t) = \frac{1}{2\pi}\int_{-\pi}^{\pi} |f \circ \tau_\theta(u)|^p \, du, \quad \text{for } f \in H_i^p(\mathbb{C}_\theta).$$

Let f be in $\mathcal{H}^p_w$. In view of relation (2.3), and using the same argument as in [6,19], one can prove that for almost all χ, with respect to μ_w, f_χ can be extended analytically on $\mathbb{C}_0$ to an element of $H_i^p(\mathbb{C}_0)$.The norm of f in $\mathcal{H}^p_w$ can be expressed as

$$\|f\|^p_{\mathcal{H}^p_w} = \int_{\mathbb{D}^\infty} \|f_\chi\|^p_{H_i^p(\mathbb{C}_0)} \, d\mu_w(\chi). \tag{2.8}$$

2.5 A Littlewood–Paley formula

We now derive another expression for the norm in $\mathcal{H}^p_w$.

Proposition 1 *Let λ be a probability measure on $\mathbb{R}$, and $p \geq 1$.*

(a) If $f \in \mathcal{H}^p_w$, then $\|f\|^p_{\mathcal{H}^p_w} \asymp I_p(f)$, where

$$I_p(f) := |f(+\infty)|^p + 4\int_{\mathbb{D}^\infty}\int_{\mathbb{R}}\int_0^{+\infty} |f_\chi(y+it)|^{p-2} \left|f'_\chi(y+it)\right|^2 y\,dy\,d\lambda(t)\,d\mu_w(\chi).$$

When $p = 2$, we have $\|f\|^2_{\mathcal{H}^2_w} = I_2(f)$.

(b) Let $f \in \mathcal{D}$, $f(s) = \sum_{n=1}^{+\infty} a_n n^{-s}$, such that f and f_χ converge on $\mathbb{C}_0$ for a.a. $\chi \in \mathbb{D}^\infty$. If $I_p(f) < \infty$, then $f \in \mathcal{H}^p_w$.

Proof Since the real variable t corresponds to a rotation in each variable of $\mathbb{D}^\infty$, the rotation invariance of μ_w entails that $I_p(f)$ does not depend on the choice of the probability measure λ. For general $p \geq 1$, we prove (a), by using (2.8). We adapt the argument from [10] (for $\mathcal{H}^p$), by integrating over the polydisk $\mathbb{D}^\infty$ instead of the polytorus $\mathbb{T}^\infty$.

Suppose f is in $\mathcal{H}^2_w$, and take $y > 0$. From (2.3) and the rotation invariance, we obtain

$$\begin{aligned}\int_{\mathbb{R}}\int_{\mathbb{D}^\infty} \left|f'_\chi(y+it)\right|^2 d\mu_w(\chi)\,d\lambda(t) &= \int_{\mathbb{D}^\infty} \left|\mathcal{B}f'_y(\chi)\right|^2 d\mu_w(\chi) \\ &= \sum_{n=1}^{+\infty} \frac{|a_n|^2}{w_n}(\log n)^2 n^{-2y}.\end{aligned}$$

Integration against y on $(0, +\infty)$ gives the formula (see details in [7] for the case of $\mathcal{H}^2$).

If f is as in (b), the integrand in $I_p(f)$ is measurable. For $\chi \in \mathbb{D}^\infty$, the change of variables $s = y + it = \omega(z) = 2\frac{1+z}{1-z}$ transfers the Littlewood–Paley formula from $\mathbb{D}$ to $\mathbb{C}_0$,

$$\begin{aligned}
&\int_{\mathbb{R}} \left|f_\chi(it)\right|^p \frac{2}{\pi(2^2+t^2)}dt \\
&\asymp \left|f_\chi(2)\right|^p \\
&\quad + \int_{\mathbb{D}} \left(1-|z|^2\right) \left|f_\chi(\omega(z))\right|^{p-2} \left|f'_\chi(\omega(z))\right|^2 \left|\omega'(z)\right|^2 dV(z) \\
&\asymp \left|f_\chi(2)\right|^p \\
&\quad + \int_0^{+\infty} \int_{\mathbb{R}} \frac{2y}{(y+2)^2+t^2} \left|f_\chi(y+it)\right|^{p-2} \left|f'_\chi(y+it)\right|^2 dtdy \\
&\lesssim \left\|f^*\right\|^p_{L^\infty(\overline{\mathbb{C}_2})} \\
&\quad + \int_0^{+\infty} \int_{\mathbb{R}} \frac{y}{1+t^2} \left|f_\chi(y+it)\right|^{p-2} \left|f'_\chi(y+it)\right|^2 dtdy,
\end{aligned}$$

where $f^*(s) := \sum_{n=1}^{+\infty} |a_n|\, n^{-s}$ is bounded on $\overline{\mathbb{C}_2}$.

Integrating on $\mathbb{D}^\infty$ with respect to μ_w, and using (2.3), we get that

$$\|\mathcal{B}f\|^p_{L^p(\mathbb{D}^\infty,\mu_w)} \lesssim \left\|f^*\right\|^p_{L^\infty(\overline{\mathbb{C}_2})} + I_p(f) < \infty.$$

Therefore, $\mathcal{B}f \in L^p(\mathbb{D}^\infty, \mu_w)$. The martingale $(A_m\mathcal{B}f)_m$ (with respect to the increasing sequence of σ-algebras of the sets $\mathbb{D}^m \times \{0\}$) converges in $L^p(\mathbb{D}^\infty, \mu_w)$ to $\mathcal{B}f$. Polynomial approximation in the Bergman spaces of the finite polydisks $\mathbb{D}^m$ shows that $\mathcal{B}f$ is in $\mathcal{BH}^p_w$.

□

3 Spaces of symbols of Volterra operators in half-planes

If g is in $\mathcal{D}$, the definition (1.2) of T_g shows that we can assume that $g(+\infty) = 0$, i.e.

$$g(s) = \sum_{n=2}^{+\infty} b_n n^{-s}.$$

As in the study of Volterra operators on Bergman spaces the unit disk [2], and on the space of Dirichlet series $\mathcal{H}^2$ [13], the boundedness of T_g on $\mathcal{H}^2_w$ will be related to Carleson measures, and to the membership of g to a BMO space or a Bloch space.

Let Y be either $\mathcal{H}^2_w$ or the Bergman space $A_{i,\delta}\left(\mathbb{C}_{1/2}\right)$, $\delta > 0$. A positive Borel measure μ on $\mathbb{C}_{1/2}$ is called a Carleson measure for Y if there exists a constant C such that,

$$\int_{\mathbb{C}_{1/2}} |f|^2 \, d\mu \le C \, \|f\|_Y^2 \ \text{ for all } f \in Y.$$

The smallest such constant, denoted by $\|\mu\|_{CM(Y)}$, is called the Carleson constant for μ with respect to Y. A Carleson measure μ is a vanishing Carleson measure for Y if we have

$$\lim_{k\to\infty} \int_{\mathbb{C}_{1/2}} |f_k|^2 \, d\mu = 0,$$

for every weakly compact sequence $(f_k)_k$ in Y (which means that $\|f_k\|_Y$ is bounded and $f_k(s) \to 0$ on every compact set of $\mathbb{C}_{1/2}$).

3.1 *BMO* spaces of Dirichlet series

The space $BMOA(\mathbb{C}_\theta)$ consists of holomorphic functions g in the half-plane $\mathbb{C}_\theta$ which satisfy

$$\|g\|_{BMO(\mathbb{C}_\theta)} := \sup_{I\subset\mathbb{R}} \frac{1}{|I|} \int_I \left| g(\theta+it) - \frac{1}{|I|} \int_I g(\theta+i\tau) d\tau \right| dt < \infty.$$

Any g in $\mathcal{D} \cap BMOA(\mathbb{C}_0)$ has an abscissa of boundedness $\sigma_b \le 0$ (Lemma 2.1 of [13]).

The space $VMOA(\mathbb{C}_0)$ consists in those functions g in $BMOA(\mathbb{C}_0)$ such that

$$\lim_{\delta\to 0^+} \sup_{|I|<\delta} \frac{1}{|I|} \int_I \left| f(it) - \frac{1}{|I|} \int_I f(i\tau) d\tau \right| dt = 0.$$

3.2 Bloch spaces of Dirichlet series

The Bloch space $\mathrm{Bloch}(\mathbb{C}_\theta)$ consists of holomorphic functions in the half-plane $\mathbb{C}_\theta$ which satisfy

$$\|g\|_{\mathrm{Bloch}(\mathbb{C}_\theta)} := \sup_{\sigma+it\in\mathbb{C}_\theta} (\sigma-\theta) \left| f'(\sigma+it) \right|.$$

Lemma 3 *If g be in $\mathcal{D} \cap Bloch(\mathbb{C}_0)$.*

(a) Its abscissa of boundedness satifies $\sigma_b \le 0$.
(b) For every $\chi \in \mathbb{D}^\infty$, g_χ is in $Bloch(\mathbb{C}_0)$, and $\|g_\chi\|_{Bloch(\mathbb{C}_0)} \le \|g\|_{Bloch(\mathbb{C}_0)}$.
(c) Suppose that $y_0 > \frac{1}{2}$. Then there exists a constant $C = C(y_0)$, such that,

$$\left| g'_\chi(y+it) \right| \le C 2^{-y} \|g\|_{Bloch(\mathbb{C}_0)}, \ \text{for all } \chi \in \mathbb{D}^\infty, \ t \in \mathbb{R}, \ y \ge y_0.$$

Proof Let $\epsilon > 0$. If $s = \sigma + it$ is in $\mathbb{C}_0$, the definition of the Bloch-norm implies that

$$\epsilon \left|g'(\epsilon + s)\right| \leq (\epsilon + \sigma) \left|g'(\epsilon + s)\right| \leq \|g\|_{\mathrm{Bloch}(\mathbb{C}_0)} .$$

It follows that g', and then g is bounded in $\mathbb{C}_\epsilon$; (a) is proved.

Now fix $\sigma > 0$. Let $m \geq 1$ be an integer, and $z = (z_1, \ldots, z_m, z_{m+1}, \ldots)$, χ in $\mathbb{D}^\infty$. From the properties of $\mathcal{H}^\infty$ and the proof of (a), we have

$$\left|A_m \mathcal{B}(g'_\sigma)_\chi(z)\right| = \left|A_m \mathcal{B} g'_\sigma(z.\chi)\right| \leq \left\|\mathcal{B} g'_\sigma\right\|_{H^\infty(\mathbb{T}^\infty)} = \left\|g'_\sigma\right\|_{\mathcal{H}^\infty} ,$$

and $\left\|(g'_\sigma)_\chi\right\|_{\mathcal{H}^\infty} = \left\|\mathcal{B}(g'_\sigma)_\chi\right\|_{H^\infty(\mathbb{T}^\infty)} \leq \left\|g'_\sigma\right\|_{\mathcal{H}^\infty}$. Therefore, $(g'_\sigma)_\chi$ is in $\mathcal{H}^\infty$; (b) holds, due to

$$\sigma \left|g'_\chi(\sigma + it)\right| \leq \|g\|_{\mathrm{Bloch}(\mathbb{C}_0)} , \quad \text{for all } t \in \mathbb{R}, \chi \in \mathbb{T}^\infty, \sigma > 0.$$

If $0 < \delta < y_0 - \frac{1}{2}$, the Cauchy-Schwarz inequality and Parseval's relation induce that

$$\begin{aligned}\left|g'_\chi(y + it)\right|^2 &\leq \left(\sum_{n=2}^{+\infty} |b_n| (\log n) n^{-y}\right)^2 = \left(\sum_{n=2}^{+\infty} |b_n| (\log n) n^{-\frac{\delta}{2}} n^{-\left(\frac{\delta}{2}+\frac{1}{2}\right)} n^{-\left(y-\frac{1}{2}-\delta\right)}\right)^2 \\ &\lesssim \zeta(1+\delta) 2^{-2y} \left\|\mathcal{B} g'_{\delta/2}\right\|^2_{H^2(\mathbb{T}^\infty)} .\end{aligned}$$

We now get (c) from the chain of inequalities

$$\left\|\mathcal{B} g'_{\delta/2}\right\|_{H^2(\mathbb{T}^\infty)} \leq \left\|\mathcal{B} g'_{\delta/2}\right\|_{H^\infty(\mathbb{T}^\infty)} = \left\|g'_{\delta/2}\right\|_{\mathcal{H}^\infty} \leq \frac{2}{\delta} \|g\|_{\mathrm{Bloch}(\mathbb{C}_0)} ,$$

□

Now, recall several characterizations of Bloch functions, which are extracted from [2,18].

Lemma 4 *Assume $\delta > 0$. For g holomorphic in $\mathbb{C}_\theta$, the following are equivalent:*

(a) *$g \in Bloch(\mathbb{C}_\theta)$;*
(b) *$h = g \circ \tau_\theta \in Bloch(\mathbb{D})$;*
(c) *The measure $d\mu_{\mathbb{C}_\theta, g}(s) = \left|g'(\sigma + it)\right|^2 \frac{(\sigma-\theta)^{\delta+1}}{|s-\theta+1|^{2\delta+2}} d\sigma dt$ is a Carleson measure for $A_{i,\delta}(\mathbb{C}_\theta)$;*
(d) *The measure $d\mu_{\mathbb{D},h}(z) = \left|h'(z)\right|^2 \left(1 - |z|^2\right)^{\delta+1} dm_1(z)$ is a Carleson measure for $A^2_\delta(\mathbb{D})$;*
(e) *The operator J_h, given by*

$$J_h f(z) = \int_0^z f(t) h'(t) dt,$$

is bounded on $A^2_\delta(\mathbb{D})$.

Moreover, the quantities

$$\|g\|_{Bloch(\mathbb{C}_\theta)}\,,\ \left\|\mu_{\mathbb{C}_\theta,g}\right\|_{CM(\mathbb{C}_\theta)}\,,\ \left\|J_g\right\|_{\mathcal{L}\left(A^2_\delta(\mathbb{D})\right)}$$

are comparable.

The little Bloch space is the space

$$\mathrm{Bloch}_0(\mathbb{C}_\theta)=\left\{f\in\mathrm{Bloch}(\mathbb{C}_\theta)\ :\ \lim_{\sigma\to\theta}(\sigma-\theta)\left|g'(s)\right|=0\right\}.$$

The membership in $\mathrm{Bloch}_0(\mathbb{C}_\theta)$ is characterized by a little oh version of Lemma 4, involving vanishing Carleson measures.

We show that Dirichlet polynomials are dense in $\mathcal{D}\cap\mathrm{Bloch}_0(\mathbb{C}_0)$. For $g(s)=\sum_{n\geq1}b_n n^{-s}$, the partial sum operator is defined by $S_N g(s)=\sum_{n=1}^N b_n n^{-s}$.

Proposition 2 *Let g be in $Bloch_0(\mathbb{C}_0)\cap\mathcal{D}$, and $\epsilon>0$. Then there exists P in $\mathcal{P}$ such that*

$$\|g-P\|_{Bloch(\mathbb{C}_0)}\leq\epsilon.$$

If in addition g is in $\mathcal{D}_d$, P can be chosen in $\mathcal{D}_d$.

Proof For every $\delta>0$, $g_\delta=g(\delta+.)$ is also in $\mathrm{Bloch}_0(\mathbb{C}_0)$. As δ tends to 0, $(g_\delta)_\delta$ converges to g uniformly on compact sets of $\mathbb{C}_0$, and $\lim_{\sigma\to0^+}\sigma\left|g'_\delta(s)\right|=0$, uniformly with respect to $\delta\in(0,1)$. It then follows from [3] that $\lim_{\delta\to0^+}\|g-g_\delta\|_{\mathrm{Bloch}(\mathbb{C}_0)}=0$. Thus, we can choose $\delta>0$ such that $\|g-g_\delta\|_{\mathrm{Bloch}(\mathbb{C}_0)}\leq\frac{\epsilon}{2}$. Since $\sigma_b(g)=\sigma_u(g)\leq0$, the partial sums $(S_N g)_N$ converge uniformly to g in $\overline{\mathbb{C}_\delta}$, $\lim_{N\to+\infty}\|S_N g_\delta-g_\delta\|_{\mathcal{H}^\infty}=0$. For large N, the triangle inequality implies that

$$\begin{aligned}\|g-S_N g_\delta\|_{\mathrm{Bloch}(\mathbb{C}_0)}&\leq\|g-g_\delta\|_{\mathrm{Bloch}(\mathbb{C}_0)}+\|g_\delta-S_N g_\delta\|_{\mathrm{Bloch}(\mathbb{C}_0)}\\&\leq\frac{\epsilon}{2}+2\,\|S_N g_\delta-g_\delta\|_{\mathcal{H}^\infty}\leq\epsilon.\end{aligned}$$

□

3.3 Carleson measures on the half-plane $\mathbb{C}_{1/2}$

On $\mathbb{C}_{1/2}$, we consider Carleson squares

$$Q(s_0)=\left(\frac{1}{2},\sigma_0\right]\times\left[t_0-\frac{\epsilon}{2},t_0+\frac{\epsilon}{2}\right],\ \text{where } s_0=\sigma_0+it_0\in\mathbb{C}_{1/2}$$

is the midpoint of the right edge of the square and $\epsilon=\sigma_0-\frac{1}{2}$.

We need the following property (see Section 7.2 in [31]).

Lemma 5 *Let $\delta > 0$ and let μ be a Borel measure on $\mathbb{C}_{1/2}$. Then μ is a Carleson measure for $A_{i,\delta}\left(\mathbb{C}_{1/2}\right)$ if and only if, for every square $Q(s_0)$, with $s_0 = \sigma_0 + it_0$, we have*

$$\mu\left(Q(s_0)\right) = O\left((2\sigma_0 - 1)^{\delta+1}\right) \text{ as } \sigma_0 \to \left(\frac{1}{2}\right)^+.$$

In addition, μ is a vanishing Carleson measure for $A_{i,\delta}\left(\mathbb{C}_{1/2}\right)$ if and only if, uniformly for t_0 in $\mathbb{R}$,

$$\mu\left(Q(s_0)\right) = o\left((2\sigma_0 - 1)^{\delta+1}\right) \text{ as } \sigma_0 \to \left(\frac{1}{2}\right)^+.$$

By Lemma 1, $\mathcal{H}_w^2$ is embedded in the Bergman-type space $A_{i,\delta}\left(\mathbb{C}_{1/2}\right)$, the exponent $\delta = \delta(w)$ being defined in (2.5). Bounded Carleson measures for both spaces $\mathcal{H}_w^2$ and $A_{i,\delta}\left(\mathbb{C}_{1/2}\right)$ have been compared in [8,23,24]. We extend their results.

Lemma 6 *Let μ be a positive Borel measure on $\mathbb{C}_{1/2}$.*

(1) *If μ is a Carleson measure (resp. vanishing Carleson measure) for $\mathcal{H}_w^2$, then μ is a Carleson measure (resp. vanishing Carleson measure) for $A_{i,\delta}\left(\mathbb{C}_{1/2}\right)$ and*

$$\|\mu\|_{CM(A_{i,\delta}(\mathbb{C}_{1/2}))} \lesssim \|\mu\|_{CM(\mathcal{H}_w^2)}.$$

(2) *Assume that μ has bounded support. If μ is a Carleson measure (resp. vanishing Carleson measure) for $A_{i,\delta}\left(\mathbb{C}_{1/2}\right)$, then μ is a Carleson measure (resp. vanishing Carleson measure) for $\mathcal{H}_w^2$ and*

$$\|\mu\|_{CM(\mathcal{H}_w^2)} \lesssim \|\mu\|_{CM(A_{i,\delta}(\mathbb{C}_{1/2}))}.$$

Proof Suppose that μ is a Carleson measure for $\mathcal{H}_w^2$, and let $Q(s_0)$ be a small Carleson square in $\mathbb{C}_{1/2}$. For the test function $f_{s_0}(s) = K^{\mathcal{H}_w^2}(s, s_0)$, we have

$$\int_{Q(s_0)} \left|f_{s_0}\right|^2 d\mu \le \int_{\mathbb{C}_{1/2}} \left|f_{s_0}\right|^2 d\mu \le C(\mu) \left\|K^{\mathcal{H}_w^2}(., s_0)\right\|_{\mathcal{H}_w^2}^2 \lesssim Z_w\left(\Re s_0\right).$$

From the estimate of Z_w (2.1) and Lemma 5, μ is a Carleson measure for $A_{i,\delta}\left(\mathbb{C}_{1/2}\right)$, since

$$\left(\Re s_0 - \frac{1}{2}\right)^{-2(\delta+1)} \mu\left(Q(s_0)\right) \lesssim \left(\Re s_0 - \frac{1}{2}\right)^{-(\delta+1)}.$$

For μ a Carleson measure for $A_{i,\delta}\left(\mathbb{C}_{1/2}\right)$ with bounded support, (2) holds [23,24].

As for vanishing Carleson measures, the reasoning used in [8] for $\mathcal{B}^2_\beta$ can be transfered to the spaces $\mathcal{A}^2_\beta$, with the test functions

$$f_k(s) = \frac{K^{\mathcal{H}^2_w}(s, s_k)}{\left\| K^{\mathcal{H}^2_w}(., s_k) \right\|_{\mathcal{H}^2_w}},$$

where $s_k = 1/2 + \epsilon_k + i\tau_k$ is a sequence in $\mathbb{C}_{1/2}$ such that $\epsilon_k \to 0$. □

We also require an equivalent norm for $A_{i,\delta}\left(\mathbb{C}_{1/2}\right)$, when $\delta > 0$. For Bergman spaces of the unit disk, recall the following consequence of Stanton's formula [28,29]:

$$\|h\|^2_{A_\delta(\mathbb{D})} \asymp |h(0)|^2 + \int_{\mathbb{D}} \left|h'(z)\right|^2 \left(1 - |z|^2\right)^{\delta+1} dV(z), \quad \text{for } h \text{ holomorphic on } \mathbb{D}.$$

Via the mapping $\tau_{1/2}$, we obtain that, for any f holomorphic on $\mathbb{C}_{1/2}$,

$$\|f\|^2_{A_{i,\delta}(\mathbb{C}_{1/2})} \asymp \left|f(\frac{3}{2})\right|^2 + \int_{\mathbb{C}_{1/2}} \left|f'(s)\right|^2 \frac{\left(\sigma - \frac{1}{2}\right)^{\delta+1}}{\left|s + \frac{1}{2}\right|^{2\delta+2}} dV(s). \tag{3.1}$$

4 Boundedness of T_g

In this section, we characterize functions in $\mathcal{X}_w$, and prove Theorem 1.

4.1 Carleson measure characterization

The boundedness of T_g on $\mathcal{H}^2_w$ can be described in terms of Carleson measures. This generalizes the setting of the Hardy space $\mathcal{H}^2$ [13].

Recall that $\mathcal{H}^2_w$ is associated to the probability measure μ_w on the polydisk $\mathbb{D}^\infty$.

Proposition 3 *T_g is bounded on $\mathcal{H}^2_w$ if and only if there exists a constant $C = C(g)$ such that*

$$\begin{aligned} \left\|T_g f\right\|^2_{\mathcal{H}^2_w} &\asymp \int_{\mathbb{D}^\infty} \int_{\mathbb{R}} \int_0^{+\infty} \left|f_\chi(\sigma + it)\right|^2 \left|g'_\chi(\sigma + it)\right|^2 \frac{\sigma d\sigma dt}{1 + t^2} d\mu_w(\chi) \\ &\leq C^2 \left\|f\right\|^2_{\mathcal{H}^2_w}, \end{aligned} \tag{4.1}$$

or, equivalently

$$\int_{\mathbb{D}^\infty} \int_0^{+\infty} \left|f_\chi(\sigma)\right|^2 \left|g'_\chi(\sigma)\right|^2 \sigma d\sigma d\mu_w(\chi) \leq C^2 \left\|f\right\|^2_{\mathcal{H}^2_w}. \tag{4.2}$$

The smallest constant C satisfying (4.1) is such that $C \asymp \left\|T_g\right\|_{\mathcal{L}(\mathcal{H}^2_w)}$.

Proof Applying the Littlewood–Paley formula (Proposition 1) to the measure $d\lambda(t) = \pi^{-1}(1+t^2)^{-1}dt$ and the function $T_g f$, we get (4.1).

The rotation invariance of the measure $d\mu_w(\chi)$ gives (4.2).

□

4.2 Proof of Theorem 1 (a): $\mathcal{B}g$ depends on a finite number of variables

For $1 \le q$ and $d \ge 1$, recall that $f \in \mathcal{H}^q_{d,w}$ if and only if f is in $\mathcal{H}^q_w$ and $\mathcal{B}f$ is a function of $z_1, \ldots, z_d$.

When needed, we shall identify $z = (z_1, \ldots, z_d) \in \mathbb{D}^d$ with $(z, 0) \in \mathbb{D}^d \times \{0\}$.

If $g(s) = \sum_{n=2}^{+\infty} b_n n^{-s}$ is in $\mathcal{H}^2_{d,w}$, we observe that for $z \in \mathbb{D}^d$,

$$\mathcal{B}g'(z) = \sum_{j=1}^{d} \log p_j \sum_{\alpha \in \mathbb{N}^d} \tilde{b}_\alpha \alpha_j z^\alpha = R\mathcal{B}g(z),$$

where R is the operator

$$RG(z_1, \ldots, z_d) = \sum_{j=1}^{d} (\log p_j) z_j \partial_j G(z_1, \ldots, z_d).$$

We define the set

$$\Delta_\epsilon := \left\{ z = (z_1, \ldots, z_d) \in \mathbb{D}^d, \ \forall j, \ |z_j| < p_j^{-\epsilon} \right\}, \quad \text{for } \epsilon > 0.$$

Take $x > 0$, $t \in \mathbb{R}$, and $z \in \mathbb{D}^d$. By construction, $z \in \overline{\Delta_{\sigma(z)}}$ and $\sigma(\mathfrak{p}^{-\mathbf{x}}.z) \ge \sigma(z) + x\frac{\log p_1}{\log p_d}$.

For $g \in \mathcal{D}_d$, we write $g_z(x) = g_{(z,0)}(x) = \mathcal{B}g_x(z)$. Since g is in Bloch($\mathbb{C}_0$), we apply (1.6) to g'_x, and get

$$\begin{aligned} \left|g'_z(x+it)\right| &= \left|\mathcal{B}g'_x(\mathcal{T}_t z)\right| \le \sup_{\zeta \in \overline{\Delta_{\sigma(\mathfrak{p}^{-\mathbf{x}}.z)}}} \left|\mathcal{B}g'(\zeta)\right| \\ &= \sup_{s \in \overline{\mathbb{C}_{\sigma(\mathfrak{p}^{-\mathbf{x}}.z)}}} \left|g'(s)\right| \le \frac{\log p_d}{\log p_1} \frac{\|g\|_{\mathrm{Bloch}(\mathbb{C}_0)}}{x + \sigma(z)}, \end{aligned} \tag{4.3}$$

Proof of Theorem 1(a) Let $f(s) = \sum_{n \ge 1} a_n n^{-s}$ be in $\mathcal{H}^2_w$, and, for $\chi = (z, z') \in \mathbb{D}^d \times \mathbb{D}^\infty$,

$$\mathcal{B}f(\chi) = \sum_{(\alpha,\alpha') \in \mathbb{N}^d \times \mathbb{N}^\infty_{0,\mathrm{fin}}} c_{\alpha,\alpha'} z^\alpha z'^{\alpha'} = \sum_{\alpha \in \mathbb{N}^d} c'_\alpha(z') z^\alpha, \text{ where } c'_\alpha(z') = \sum_{\alpha' \in \mathbb{N}^\infty_{0,\mathrm{fin}}} c_{\alpha,\alpha'} z'^{\alpha'}.$$

In view of Proposition 3, we aim to estimate $\|T_g f\|^2_{\mathcal{H}^2_w} \asymp \mathcal{I}_1 + \mathcal{I}_2$, where

$$\mathcal{I}_1 := \int_{\mathbb{D}^\infty} \int_0^1 \left|f_\chi(x)\right|^2 \left|g'_\chi(x)\right|^2 x dx d\mu_w(\chi),$$
$$\text{and } \mathcal{I}_2 := \int_{\mathbb{D}^\infty} \int_1^{+\infty} \left|f_\chi(x)\right|^2 \left|g'_\chi(x)\right|^2 x dx d\mu_w(\chi).$$

By (4.3), the rotation invariance and Fubini's Theorem, we have

$$\begin{aligned}\mathcal{I}_1 &\lesssim \|g\|^2_{\mathrm{Bloch}(\mathbb{C}_0)} \int_0^1 x \int_{\mathbb{D}^\infty} \int_{\mathbb{D}^d} \frac{1}{[x+\sigma(z)]^2} \\ &\quad \left| \sum_{\alpha\in\mathbb{N}^d} c'_\alpha(\mathfrak{p}'^{-\mathbf{x}}.z') \left(z_1 p_1^{-x}\right)^{\alpha_1} \cdots \left(z_d p_d^{-x}\right)^{\alpha_d} \right|^2 d\mu_w(z,z')dx \\ &\lesssim \|g\|^2_{\mathrm{Bloch}(\mathbb{C}_0)} \int_{\mathbb{D}^\infty} \int_0^1 x \sum_{\alpha\in\mathbb{N}^d} \left|c'_\alpha(\mathfrak{p}'^{-\mathbf{x}}.z')\right|^2 I_\alpha(x) dx d\mu_w(z'),\end{aligned}$$

where

$$I_\alpha(x) := \int_{\mathbb{D}^d} \frac{1}{[x+\sigma(z)]^2} \left|z_1 p_1^{-x}\right|^{2\alpha_1} \cdots \left|z_d p_d^{-x}\right|^{2\alpha_d} d\mu_w(z).$$

Using the rotation invariance again as well as the fact that $p_j \geq 1$, and setting $\mathcal{J}_\alpha := \int_0^1 x I_\alpha(x) dx$, we get

$$\begin{aligned}\mathcal{I}_1 &\lesssim \|g\|^2_{\mathrm{Bloch}(\mathbb{C}_0)} \sum_{\alpha\in\mathbb{N}^d} \int_0^1 x I_\alpha(x) \left(\int_{\mathbb{D}^\infty} \left| \sum_{\alpha'} c_{\alpha,\alpha'} (\mathfrak{p}'^{-\mathbf{x}}.z')^{\alpha'} \right|^2 d\mu_w(z') \right) dx \\ &\lesssim \|g\|^2_{\mathrm{Bloch}(\mathbb{C}_0)} \sum_{\alpha,\alpha'} \left|c_{\alpha,\alpha'}\right|^2 \mathcal{J}_\alpha \left(\int_{\mathbb{D}^\infty} \left|z'^{\alpha'}\right|^2 d\mu_w(z') \right) \\ &\lesssim \|g\|^2_{\mathrm{Bloch}(\mathbb{C}_0)} \sum_{\alpha,\alpha'} \frac{\left|c_{\alpha,\alpha'}\right|^2 \mathcal{J}_\alpha}{w\left(p_{d+1}^{\alpha_{d+1}}\right) \cdots w\left(p_r^{\alpha_r}\right)}.\end{aligned}$$

For the moment, we admit that $\mathcal{J}_\alpha \leq C(d,w) \left[\prod_{j=1}^d w(p_j^{\alpha_j})\right]^{-1}$, which will be proved in Lemma 7. Hence,

$$\mathcal{I}_1 \lesssim \|g\|^2_{\mathrm{Bloch}(\mathbb{C}_0)} \sum_{\alpha,\alpha'} \frac{\left|c_{\alpha,\alpha'}\right|^2}{w(p^{(\alpha,\alpha')})} \lesssim \|g\|^2_{\mathrm{Bloch}(\mathbb{C}_0)} \|f\|^2_{\mathcal{H}^2_w}.$$

Combining Lemma 3 with the following observation,

$$\int_{\mathbb{D}^\infty} \left|f_\chi(x)\right|^2 d\mu_w(\chi) = \int_{\mathbb{D}^\infty} \left|\sum_{n=p^\alpha} a_n n^{-x} \chi^\alpha\right|^2 d\mu_w(\chi)$$
$$= \sum_{n\geq 1} \frac{|a_n|^2 n^{-2x}}{w_n} \leq \|f\|^2_{\mathcal{H}^2_w},$$

we estimate $\mathcal{I}_2$,

$$\mathcal{I}_2 \lesssim \int_1^{+\infty} x \int_{\mathbb{D}^\infty} \|g\|^2_{\mathrm{Bloch}(\mathbb{C}_0)} 4^{-x} \left|f_\chi(x)\right|^2 d\mu_w(\chi)dx \lesssim \|g\|^2 \,|_{\mathrm{Bloch}(\mathbb{C}_0)} \|f\|^2_{\mathcal{H}^2_w}.$$

□

Recall that

$$I_\alpha(x) = \int_{\mathbb{D}^d} \frac{1}{[x+\sigma(z)]^2} \left|z_1 p_1^{-x}\right|^{2\alpha_1} \cdots \left|z_d p_d^{-x}\right|^{2\alpha_d} d\mu_w(z),\ \alpha \in \mathbb{N}^d,\ 0 < x < 1.$$

Lemma 7 *There exists a constant $C = C(w, d)$, such that*

$$\mathcal{J}_\alpha := \int_0^1 x I_\alpha(x)dx \leq C \prod_{j=1}^d \frac{1}{w\left(p_j^{\alpha_j}\right)}.$$

The proof of Lemma 7 relies on technical computations (Lemma 8).

Lemma 8 *For $0 < T < 1$, and a real number $p \geq 2$, set $L := -\frac{\log T}{2\log p}$ and $K = \min(1, L)$. There exists a constant $C = C(p, w) > 0$, such that*

$$J(p,T) := (\log T)^{-2} \int_0^K x M\left(Tp^{2x}\right) dx$$
$$\lesssim C \begin{cases} M(T) & \text{if } \beta \geq 1 \text{ or } (\beta < 1, p^{-2} < T < 1), \\ M\left(Tp^2\right) & \text{if } \beta < 1, 0 < T \leq p^{-2}. \end{cases}$$

Proof When $p^{-2} < T < 1$, the change of variables $u = Tp^{2x}$ gives

$$J(p,T) = (\log T)^{-2} \frac{1}{(2\log p)^2} \int_T^1 \log\frac{u}{T} M(u)\frac{du}{u}.$$

Since $\log\frac{u}{T} \leq \log\frac{1}{T}$ and $1 \leq \frac{1}{u} \leq \frac{1}{T} < p^2$,

$$J(p,T) \leq (\log T)^{-2}\left(\frac{1}{2\log p}\right)^2 \int_T^1 \log\frac{1}{T} M(u)\frac{1}{u}du \lesssim M(T).$$

Next suppose that $0 < T \leq p^{-2}$. Since $(\log T)^2 \geq 4(\log p)^2$, we notice that

$$J(p,T) \lesssim \int_0^1 xM(Tp^{2x})dx \lesssim \begin{cases} \int_0^1 M(T)dx \text{ if } \beta \geq 1, \\ \int_0^1 M(Tp^2)dx \text{ if } \beta < 1 \end{cases}.$$

□

Proof of Lemma 7 Resorting to polar coordinates, and using changes of variables, we have

$$\mathcal{J}_\alpha \leq \int_Q \frac{xt^\alpha}{\left[x+\sigma\left(p_1^x\sqrt{t_1},\dots,p_1^x\sqrt{t_d}\right)\right]^2}\left(\prod_{k=1}^d M\left(p_k^{2x}t_k\right)p_k^{2x}\right)dxdt_1\cdots dt_d,$$

where $Q = \left\{(x,t) \in (0,1)\times(0,1)^d,\ \forall k = 1..d, 0 < t_k < p_k^{-2x}\right\}$.

For $t = (t_1,\dots,t_d) \in (0,1)^d$, set

$$l_k(t) := -\frac{\log t_k}{2\log p_k},\ K_k := \min(1,l_k),\ 1 \leq k \leq d,$$
$$l(t) := \min_{1\leq k\leq d} l_k(t),\ K := \min(1,l).$$

We observe that $Q = \left\{(x,t) \in (0,1)\times(0,1)^d,\ 0 < x < K(t)\right\}$. Now, for $1 \leq k \leq d$, we set $Q_k := \left\{(x,t),\ t \in (0,1)^d,\ l(t) = l_k(t),\ 0 < x < K_k(t)\right\}$.

Let (x,t) be in Q_k. We have

$$0 < t_l \leq T_{k,l} := t_k^{\frac{\log p_l}{\log p_k}} < 1, \quad \text{for } 1 \leq l \leq d. \tag{4.4}$$

In addition, since $0 < x < l_k(t)$, (4.4) implies $p_l^x\sqrt{t_l} < p_l^{l_k(t)}\sqrt{t_l} \leq 1$, and we see that $\frac{1}{\sqrt{t_l}p_l^x} \geq p_l^{l_k(t)-x} \geq p_1^{l_k(t)-x}$. Thus

$$(\log p_d)\sigma\left(p_1^x\sqrt{t_1},\dots,p_d^x\sqrt{t_d}\right) = \log\min_{1\leq l\leq d}\left(\frac{1}{\sqrt{t_l}p_l^x}\right) \geq \log p_1\left(l_k(t)-x\right),$$

and $x + \sigma\left(p_1^x\sqrt{t_1},\dots,p_1^x\sqrt{t_d}\right) \gtrsim -\log t_k$.

Set $d\widehat{t_k} = dt_1\cdots dt_{k-1}dt_{k+1}\cdots dt_d$, and

$$\tilde{Q}_k := \left\{(x,t),\ 0 < t_k < 1,\ 0 < t_l < T_{k,l} \text{ for } l \neq k,\ 0 < x < K_k(t)\right\}.$$

It follows that $\mathcal{J}_\alpha \lesssim \sum_{k=1}^d \mathcal{J}_{\alpha,k}$, where

$$\mathcal{J}_{\alpha,k} = \int_{\tilde{Q}_k} \frac{xt^\alpha}{\left[x+\sigma\left(p_1^x\sqrt{t_1},\dots,p_1^x\sqrt{t_d}\right)\right]^2}\left(\prod_{l=1}^d M\left(p_l^{2x}t_l\right)\right)dxdt.$$

We will obtain the Lemma by showing that

$$\mathcal{J}_{\alpha,k} \lesssim \prod_{l=1}^{d} \left[w\left(p_l^{\alpha_l}\right)\right]^{-1}. \tag{4.5}$$

When $\beta \geq 1$, we use that, for $(x,t) \in \tilde{Q}_k$, and $l \neq k$, $M\left(p_l^{2x} t_l\right) \leq M(t_l)$, altogether with Lemma 8. We derive (4.5) from

$$\begin{aligned}\mathcal{J}_{\alpha,k} &\lesssim \int_{0<t_k<1} \left(\int_{\prod_{j\neq k}(0,T_{k,j})} t^{\alpha} \int_0^{K_k(t)} x\,(\log t_k)^{-2} M\left(p_k^{2x} t_k\right) dx \prod_{l\neq k} M(t_l) d\widehat{t_k} \right) dt_k \\ &\lesssim \int_{0<t_k<1} t_k^{\alpha_k} M(t_k) \left(\prod_{j\neq k} \int_0^{T_{k,j}} t_j^{\alpha_j} M\left(t_j\right) dt_j \right) dt_k \lesssim \prod_{j=1}^{d} \int_0^1 t_j^{\alpha_j} M\left(t_j\right) dt_j.\end{aligned}$$

Next, suppose $0 < \beta < 1$. If $(x,t) \in \tilde{Q}_k$, notice that, for $l \neq k$, $t_l p_l^{2x} \leq t_l p_l^{2l_k(t)} \leq 1$; this shows that $M\left(p_l^{2x} t_l\right) \leq M\left(p_l^{2l_k(t)} t_l\right)$. Hence, we see that $\mathcal{J}_{\alpha,k} \lesssim J_1 + J_2$, where, by Lemma 8 and the relation $p_l^{2l_k(t)} = T_{k,l}^{-1}$,

$$\begin{aligned}J_1 &\lesssim \int_{0<t_k<p_k^{-2}} t_k^{\alpha_k} M(p_k^2 t_k) \left(\prod_{j\neq k} \int_0^{T_{k,j}} t_j^{\alpha_j} M\left(t_j T_{k,j}^{-1}\right) dt_j \right) dt_k, \\ J_2 &\lesssim \int_{p_k^{-2}<t_k<1} t_k^{\alpha_k} M(t_k) \left(\prod_{j\neq k} \int_0^{T_{k,j}} t_j^{\alpha_j} M\left(t_j T_{k,j}^{-1}\right) dt_j \right) dt_k.\end{aligned}$$

A change of variables provides the desired estimate. □

4.3 Proof of Theorem 1(b) and (c)

If $f(s) = \sum_{n=1}^{+\infty} a_n n^{-s}$ and $g(s) = \sum_{n=1}^{+\infty} b_n n^{-s}$, we have

$$T_g f(s) = \sum_{n=2}^{\infty} \frac{1}{\log n} \left(\sum_{k|n, k<n} a_k b_{n/k} \right) n^{-s}.$$

As in the case of $\mathcal{H}^2$, the operator

$$a_1 + \sum_{n=2}^{\infty} a_n n^{-s} \mapsto a_1 + \sum_{n=2}^{\infty} a_n (\log n)^{-1} n^{-s}$$

is compact on $\mathcal{H}_w$. Thus, set $b_1 = 1$, and our study will be unchanged if we replace T_g by

$$\tilde{T}_g f(s) = \sum_{n=2}^{\infty} \frac{1}{\log n} \left(\sum_{k|n} a_k b_{n/k} \right) n^{-s}.$$

Lemma 9 *If T_g is bounded on $\mathcal{H}^2$, then g is in $\mathcal{X}_w$, and the operator norms satisfy*

$$\|T_g\|_{\mathcal{L}(\mathcal{H}^2_w)} \le \|T_g\|_{\mathcal{L}(\mathcal{H}^2)}.$$

Proof If $f(s) = \sum_{n=1}^{+\infty} a_n n^{-s}$ is in $\mathcal{H}^2_w$, the function $\tilde{f}(s) = \sum_{n=1}^{+\infty} a_n w_n^{-1/2} n^{-s}$ is in $\mathcal{H}^2$ and $\|f\|_{\mathcal{H}^2_w} = \left\|\tilde{f}\right\|_{\mathcal{H}^2}$. Since $w_k \le w_{kl}$ for any integers k, l, the Lemma is proven by the inequality

$$\|T_g f\|^2_{\mathcal{H}^2_w} \le \sum_{n=2}^{\infty} (\log n)^{-2} \left| \sum_{k|n, k<n} w_k^{-1/2} a_k b_{n/k} \right|^2 = \left\|T_g \tilde{f}\right\|^2_{\mathcal{H}^2}.$$

□

We will also use the sufficient condition proved in Theorem 2.3 in [13], stating that if g is in $BMOA(\mathbb{C}_0) \cap \mathcal{D}$, then T_g is bounded on $\mathcal{H}^2$, with

$$\|T_g\|_{\mathcal{H}^2} \lesssim \|g\|_{BMOA(\mathbb{C}_0)}. \tag{4.6}$$

Proof of Theorem 1(b) and (c) If g is in $BMOA(\mathbb{C}_0)$, T_g is bounded on $\mathcal{H}^2$, and (b) is a consequence of (4.6) and Lemma 9.

To prove (c), we use that $(T_g f)' = fg'$, and that $\mathcal{H}^2_w$ is embedded in $A_{i,\delta}\left(\mathbb{C}_{1/2}\right)$, with $\delta = \delta(w) > 0$. We set

$$d\nu_g(s) = \left|g'(s)\right|^2 \frac{\left(\sigma - \frac{1}{2}\right)^{\delta+1}}{\left|s + \frac{1}{2}\right|^{2(\delta+1)}} dV(s).$$

Now formula (3.1), the boundedness of T_g on $\mathcal{H}^2_w$ and Lemma 1 induce that

$$\int_{\mathbb{C}_{1/2}} |f(s)|^2 d\nu_g(s) \lesssim \|T_g f\|^2_{A_{i,\delta}(\mathbb{C}_{1/2})} \le c(w) \|T_g f\|^2_{\mathcal{H}^2_w} \le c(w) \|T_g\|^2_{\mathcal{L}(\mathcal{H}^2_w)} \|f\|^2_{\mathcal{H}^2_w},$$

Thus, ν_g is a Carleson measure for $\mathcal{H}^2_w$ and $\|\nu_g\|_{CM(\mathcal{H}^2_w)} \lesssim \|T_g\|^2_{\mathcal{L}(\mathcal{H}^2_w)}$. By Lemma 6, ν_g is also a Carleson measure for $A_{i,\delta}\left(\mathbb{C}_{1/2}\right)$ and

$$\|\nu_g\|_{CM(A_{i,\delta}(\mathbb{C}_{1/2}))} \lesssim \|T_g\|^2_{\mathcal{L}(\mathcal{H}^2_w)}.$$

We conclude by the characterization of the Bloch space given in Lemma 4.

□

We get a result which is in agreement with the situation for Hardy spaces [15], Bergman spaces [2] or the Hardy space of Dirichlet series $\mathcal{H}^2$ [13], with the same proof.

Corollary 1 *If g is in $\mathcal{X}_w$, then g is in $\cap_{0<p<\infty}\mathcal{H}^p_w$, and there exists $c>0$, such that the function $e^{c|\mathcal{B}g|}$ is integrable on $\mathbb{D}^\infty$, with respect to $d\mu_w$.*

5 Compactness

We now present a little oh version of Theorem 1.

If the symbol is a vector of the standard orthonormal basis of $\mathcal{H}^2_w$, that is

$$g(s) = e_{w,n}(s) := w_n^{1/2} n^{-s},$$

the operator $T_g^* T_g$ is diagonal, and its eigenvalues

$$\lambda^2_{n,k} = \frac{w_n w_k}{w_{nk}} \left(\frac{\log n}{\log n + \log k} \right)^2$$

tend to 0 as $k \to +\infty$. Thus T_g is compact. It follows that every Dirichlet polynomial generates a compact Volterra operator on $\mathcal{H}^2_w$.

5.1 Case when $\mathcal{B}g$ depends on a finite number of variables

We approximate a symbol g which is in $\mathrm{Bloch}_0(\mathbb{C}_0) \cap \mathcal{D}_d$ by a Dirichlet polynomial P in the $\mathrm{Bloch}(\mathbb{C}_0)$-norm. From Theorem 1(a), T_g is approximated in the operator norm by the compact operator T_P.

Theorem 2 *If g is in $\mathrm{Bloch}_0(\mathbb{C}_0) \cap \mathcal{D}_d$, then T_g is compact on $\mathcal{H}^2_w$.*

5.2 Sufficient/necessary conditions for compactness

In general, if the symbol $g(s) = \sum_{n\geq 2} b_n n^{-s}$ satisfies an inequality of the form $\|T_g\|^2_{\mathcal{L}(\mathcal{H}^2_w)} \leq \sum_{n\geq 2} |b_n|^2 W(n) < \infty$, we approximate T_g in the operator norm by the compact operator $T_{S_N g}$. Therefore, T_g is compact (see [13]).

The little oh version of Theorem 1 is related to the properties of $VMOA(\mathbb{C}_0) \cap \mathcal{D}$, and with the concept of vanishing Carleson measures.

Theorem 3 *Let g be in $\mathcal{D}$.*

(1) If g is in $VMOA(\mathbb{C}_0) \cap \mathcal{D}$, then T_g is compact on $\mathcal{H}^2_w$.
(2) If T_g is compact on $\mathcal{H}^2_w$, then g is in $\mathrm{Bloch}_0(\mathbb{C}_{1/2})$.

Proof In order to prove (1), we use that $VMOA(\mathbb{C}_0) \cap \mathcal{D}$ is the closure of Dirichlet polynomials in $BMOA(\mathbb{C}_0)$ (see [13]), and that, from Theorem 1, we have $\|T_g\|_{\mathcal{L}(\mathcal{H}^2_w)} \lesssim \|g\|_{BMOA(\mathbb{C}_0)}$.

Recall that $\mathcal{H}^2_w$ is embedded in $A_{i,\delta}(\mathbb{C}_{1/2})$, $\delta = \delta(w)$ being defined in (2.5). Assume that T_g is compact on $\mathcal{H}^2_w$, and consider the measure

$$d\nu_g(s) = \left|g'(s)\right|^2 \frac{\left(\sigma - \frac{1}{2}\right)^{\delta+1}}{\left|s + \frac{1}{2}\right|^{2(\delta+1)}} dV(s).$$

Let $(f_k)_k$ be a weakly compact sequence in $\mathcal{H}^2_w$. Formula (3.1), and Lemma 1 imply that

$$\int_{\mathbb{C}_{1/2}} |f_k(s)|^2 \, d\nu_g(s) \asymp \|T_g f_k\|^2_{A_{i,\delta}(\mathbb{C}_{1/2})} \lesssim \|T_g f_k\|^2_{\mathcal{H}^2_w}.$$

By the compactness of T_g, ν_g is a vanishing Carleson measure for $A_{i,\delta}(\mathbb{C}_{1/2})$, with

$$\lim_{k\to\infty} \int_{\mathbb{C}_{1/2}} |f_k(s)|^2 \, d\nu_g(s) = 0.$$

Now, g is in $\text{Bloch}_0(\mathbb{C}_{1/2})$, by the characterization of vanishing Carleson measures (Lemma 5).

□

6 Membership in Schatten classes

Let g be a non constant symbol. As in the case of $\mathcal{H}^2$, the Volterra operator T_g on $\mathcal{H}^2_w$ does not belong to any Schatten class.

Theorem 4 *If the Dirichlet series $g(s) = \sum_{n\geq 2} b_n n^{-s}$ is not 0, then $T_g : \mathcal{H}^2_w \to \mathcal{H}^2_w$ is not in the Schatten class $\mathcal{S}_p$, for any $0 < p < \infty$.*

Proof Recall that $(e_{w,n})_n$ is an orthonormal basis of $\mathcal{H}^2_w$. We follow the reasoning of Theorem 7.2 [13]. Using that $w_{Nn} \lesssim w_N w_n$, we see that, for $N \geq n$,

$$\|T_g e_{w,n}\|^2_{\mathcal{H}^2_w} = \sum_{k=2}^{+\infty} \frac{|b_k|^2 (\log k)^2}{(\log(kn))^2} \frac{w_n}{w_{kn}} \geq \frac{|b_N|^2 (\log N)^2}{(\log(Nn))^2} \frac{w_n}{w_{Nn}} \gtrsim \frac{|b_N|^2 (\log N)^2}{(2\log n)^2} \frac{1}{w_N}.$$

For $p \geq 2$, we obtain

$$\|T_g\|^p_{\mathcal{S}_p} \geq \sum_{n=N}^{+\infty} \|T_g e_{w,n}\|^p_{\mathcal{H}^2_w} = +\infty.$$

Therefore T_g is not in $\mathcal{S}_p$ for $p \geq 2$, neither for $0 < p < \infty$. □

7 Examples

In this section, we study the boundedness of T_g on $\mathcal{H}^2_w$, for specific symbols g. We consider fractional primitives of translates of the weighted Zeta function Z_w and homogeneous symbols, which are the counterparts of the symbols presented in [13] in the $\mathcal{H}^2$ setting. The techniques of proof, as well as the results are similar to theirs, and we omit the details.

7.1 Fractional primitives of translates of Z_w

Proposition 4 *With the notation of (2.5), take $1/2 \le a < 1$, $2\gamma > \delta(w) - 1$. If*

$$g(s) = \sum_{n=2}^{\infty} w_n \frac{n^{-a}}{(\log n)^{\gamma+1}} n^{-s},$$

then T_g is unbounded on $\mathcal{H}^2_w$.

Proof Abel summation and the Chebyshev estimate induce that g is in $\mathcal{H}^2_w$. If $f(s) = \sum_{n=1}^{\infty} a_n n^{-s}$, and $g(s) = \sum_{n=2}^{\infty} \frac{b_n}{\log n} n^{-s}$, we set $A_n = \sum_{k|n} a_{n/k} b_k$, so that

$$\left\| \tilde{T}_g f \right\|^2_{\mathcal{H}^2_w} = \sum_{n=2}^{\infty} \frac{1}{(w_n \log n)^2} A_n^2.$$

We adapt the test functions of [13], and take $f_J(s) = \prod_{j=1}^{J} \left(1 + w_2^{1/2} p_j^{-s}\right)$, for $J \ge 1$. By construction, it satisfies $\| f_J \|_{\mathcal{H}^2_w} \asymp 2^{J/2}$. Now, for $\mathcal{J}$ a non-empty subset of $\{1, \ldots, J\}$, we set $n_{\mathcal{J}} = \prod_{j \in \mathcal{J}} p_j$, and observe that

$$A_{n_{\mathcal{J}}} = \sum_{1 \le k \le |\mathcal{J}|, \{p_{j_1}, \ldots, p_{j_k}\} \subset \mathcal{J}} w_2^{\frac{|\mathcal{J}|-k}{2}} \left[\log\left(p_{j_1} \cdots p_{j_k}\right)\right]^{-\gamma} w_2^k \left(p_{j_1} \cdots p_{j_k}\right)^{-a} + w_2^{\frac{|\mathcal{J}|}{2}}.$$

First assume that $\gamma \ge 0$. From the prime number Theorem, we obtain that

$$A_{n_{\mathcal{J}}} \gtrsim w_2^{\frac{|\mathcal{J}|}{2}} \left[J \log J\right]^{-\gamma} \left[1 + \sum_{1 \le k \le |\mathcal{J}|, \{p_{j_1}, \ldots, p_{j_k}\} \subset \mathcal{J}} w_2^{k/2} \left(p_{j_1} \cdots p_{j_k}\right)^{-a}\right].$$

Therefore, it follows again from the prime number Theorem that

$$\begin{aligned} \left\| \tilde{T}_g f_J \right\|^2_{\mathcal{H}^2_w} &\gtrsim \sum_{\mathcal{J} \subset \{1, \ldots, J\}, |\mathcal{J}| \ge J/2} \frac{1}{\left(\log n_{\mathcal{J}}\right)^2} \left[J \log J\right]^{-2\gamma} \prod_{j \in \mathcal{J}} \left(1 + w_2^{1/2} p_j^{-a}\right)^2 \\ &\gtrsim 2^{J-1} \left[J \log J\right]^{-2\gamma} \min_{|\mathcal{J}| \ge J/2} \frac{1}{\left(\log n_{\mathcal{J}}\right)^2} \prod_{j \in \mathcal{J}} \left(1 + w_2^{1/2} p_j^{-a}\right)^2 \end{aligned}$$

$$\gtrsim e^{cJ^{1-a}(\log J)^{-a}} \|f_J\|^2_{\mathcal{H}^2_w},$$

for some constant $c > 0$, and T_g is unbounded. The case when $\gamma < 0$ is similar. □

7.2 Homogeneous symbols

An m-homogeneous Dirichlet series has the form

$$g(s) = \sum_{\Omega(n)=m} b_n n^{-s}.$$

We extend Theorem 4.2 in [13] to the spaces $\mathcal{H}^2_w$.

Proposition 5 *There exist weights $W_m(n)$ such that for $g(s) = \sum_{\Omega(n)=m} b_n n^{-s}$,*

$$\|T_g\|_{\mathcal{L}(\mathcal{H}^2_w)} \le \left(\sum_{\Omega(n)=m} |b_n|^2 \, W_m(n)\right)^{1/2}. \tag{7.1}$$

Precisely, there exist absolute constants C_m for which

$$W_m(n) = \begin{cases} C_1 & \text{for } m = 1, \\ C_2 \frac{\log n}{\log_2 n} & \text{for } m = 2, \\ C_m \frac{n^{\frac{m-2}{m}}}{(\log n)^{m-2}} & \text{for } m \ge 3. \end{cases}$$

Moreover, when $m = 2$, $\log_2 n$ cannot be replaced in (7.1) by $\left(\log_2 n\right)^{1+\varepsilon}$ for any $\varepsilon > 0$.

Proof If a linear symbol ($m = 1$) $g(s) = \sum_{p\in\mathbb{P}} b_p p^{-s}$ belongs to $\mathcal{H}^2$, we observe that $\|g\|^2_{\mathcal{H}^2} = 2^\beta \|g\|^2_{\mathcal{B}^2_\beta} = (\beta+1)\|g\|^2_{\mathcal{A}^2_\beta}$. Hence, it follows from Theorem 4.1 in [13] and Lemma 9 that T_g is bounded on $\mathcal{H}^2_w$ and $\|T_g\|_{\mathcal{L}(\mathcal{H}^2_w)} \le \|T_g\|_{\mathcal{L}(\mathcal{H}^2)}$. One can choose $C_1 = \max\left((\beta+1)^{-1}, 2^{-\beta}\right)$.

(7.1) is a consequence of Theorem 4.2 in [13] and Lemma 9. We now prove the sharpness of the factor $\log_2 n$. We assume that for some $\varepsilon > 0$, every 2-homogeneous Dirichlet series g satisfies

$$\|T_g\|_{\mathcal{L}(\mathcal{H}^2_w)} \le C_2 \left(\sum_{\Omega(n)=m} |b_n|^2 \frac{\log n}{\left(\log_2 n\right)^{1+\varepsilon}}\right)^{1/2}. \tag{7.2}$$

For x a large real number, and $q \sim e^x$ a prime number, the symbol considered in [13] is

$$g_x(s) = \sum_{x/2<p\le x} \frac{\left(\log_2(pq)\right)^{1+\varepsilon/2}}{p} (pq)^{-s}.$$

We take as test functions

$$f_x(s) = \sum_{n=1}^{+\infty} a_n n^{-s} = \prod_{x/2<p\le x} \left(1 + w_2^{1/2} p^{-s}\right).$$

If S_x denotes the set of square-free integers generated by the primes $x/2 < p \le x$, we have $\|f_x\|^2_{\mathcal{H}^2_w} \asymp |S_x| = 2^{N(x)}$, where $N(x) := \pi(x) - \pi(x/2)$. Now,

$$\frac{\|T_{g_x} f_x\|^2_{\mathcal{H}^2_w}}{\|f_x\|^2_{\mathcal{H}^2_w}} \gtrsim \frac{1}{|S_x|} \sum_{n\in S_x} w_{nq}^{-1} (\log(nq))^{-2} \left| \sum_{pq|nq} \log(pq) \frac{\left(\log_2(pq)\right)^{1+\varepsilon/2}}{p} a_{n/p} \right|^2.$$

If $n \in S_x$, and $p|n$, we have $a_{n/p} = w_2^{\frac{1}{2}[\omega(n)-1]}$, $w_n = w_2^{\omega(n)}$, and $w_{nq} = w_n w_q$. Thus,

$$\frac{\|T_{g_x} f_x\|^2_{\mathcal{H}^2_w}}{\|f_x\|^2_{\mathcal{H}^2_w}} \gtrsim \frac{1}{|S_x|} \frac{(\log x)^{2+\varepsilon}}{x^2} \sum_{n\in S_x} \omega(n)^2.$$

Now $\sum_{n\in S_x} \omega(n)^2 = \sum_{k=1}^{N(x)} \binom{N(x)}{k} k^2 \asymp N(x)^2 2^{N(x)}$, and (7.2) does not hold, due to

$$\frac{\|T_{g_x} f_x\|_{\mathcal{H}^2_w}}{\|f_x\|_{\mathcal{H}^2_w}} \gtrsim (\log x)^{\varepsilon}.$$

□

We will exhibit an homogeneous symbol g which is in $\mathcal{H}^2_w \cap \mathrm{Bloch}_0(\mathbb{C}_{1/2})$, but not in $\mathcal{X}_w$. In fact, we observe that g is in every $\mathcal{H}^p_w$.

Lemma 10 *If g is an m-homogeneous Dirichlet series in $\mathcal{H}^2_w$, then g is in $\cap_{0<p<\infty} \mathcal{H}^p_w$ and, for any $0 < p < \infty$, there exists $c = c(m, p)$ such that*

$$\|g\|_{\mathcal{H}^p_w} \le c \, \|g\|_{\mathcal{H}^2_w}. \tag{7.3}$$

Proof It is enough to consider the case $p \ge 2$. We first prove the inequality for $p = 2^k$, k being a positive integer, by an induction argument.

Obviously, it holds for $k = 1$.

Our proof is inspired of Lemma 8 in [27]. For any integer m, there exists a constant $C(m)$, such that max $(w_n, d(n)) \leq C(m)$, whenever $\Omega(n) = m$.

If $f(s) = \sum_n a_n n^{-s}$ is m-homogeneous, then $f^2(s) = \sum_n b_n n^{-s}$ is $2m$-homogeneous, and $|b_n|^2 \leq d(n) \sum_{k|n} |a_k|^2 \left|a_{n/k}\right|^2$. Since $w_n \geq \sqrt{w_k}\sqrt{w_{n/k}}$,

$$\begin{aligned} \|f\|^4_{\mathcal{H}^4_w} = \left\|f^2\right\|^2_{\mathcal{H}^2_w} &\leq \sum_{\Omega(n)=2m} d(n) w_n^{-1} \left(\sum_{k|n} |a_k|^2 \left|a_{n/k}\right|^2\right) \\ &\leq C(2m) \sum_{\Omega(n)=2m} \left(\sum_{k|n} \frac{|a_k|^2}{\sqrt{w_k}} \frac{\left|a_{n/k}\right|^2}{\sqrt{w_{n/k}}}\right) \\ &= C(2m) \left(\sum_k \frac{|a_k|^2}{\sqrt{w_k}}\right)^2 \leq C(2m)C(m) \|f\|^4_{\mathcal{H}^2_w}. \end{aligned}$$

Now, suppose that, for some k, an m-homogeneous Dirichlet series h satisfies

$$\|h\|^{2^k}_{\mathcal{H}^{2^k}_w} \leq K(m,k) \|h\|^{2^k}_{\mathcal{H}^2_w} \text{ for any } m.$$

We obtain that

$$\begin{aligned} \|f\|^{2^{k+1}}_{\mathcal{H}^{2^{k+1}}_w} = \left\|f^2\right\|^{2^k}_{\mathcal{H}^{2^k}_w} &\leq K(2m,k) \left\|f^2\right\|^{2^k}_{\mathcal{H}^2_w} = K(2m,k) \|f\|^{2^{k+1}}_{\mathcal{H}^4_w} \\ &\leq K(2m,k) \left[C(2m)C(m) \|f\|^4_{\mathcal{H}^2_w}\right]^{2^{k-1}}. \end{aligned}$$

For general p, (7.3) is a consequence of Hölder's inequality. □

For our construction, we need two technical Lemmas.

Lemma 11 *Assume that* $0 < \delta < 1$ *and* $0 < \eta$. *For* $j = 1..3$, *we set* $h_j(s) = \sum_{p \geq 3} \alpha_{j,p} p^{-s}$, *where*

$$\alpha_{1,p} = \left(\log_2 p\right)^{-\delta}, \ \alpha_{2,p} = \log_2 p, \ \alpha_{3,p} = \log p \left(\log_2 p\right)^{-\eta}.$$

For a real number $\sigma > 1$, *set* $\sigma' := \frac{1}{\sigma - 1}$. *Then we have*

$$h_1(\sigma) \asymp \left(\log \sigma'\right)^{1-\delta}; \ h_2(\sigma) \asymp \log_2\left(\sigma'\right); \ h_3(\sigma) \asymp \sigma' \left(\log \sigma'\right)^{-\eta}, \ \ \textit{as } \sigma \to 1^+. \tag{7.4}$$

Proof These asymptotics will follow from computations inspired by [4,20]. Recall that

$$A_1(t) := \sum_{3 \leq p \leq t} \frac{1}{p} = \log_2 t + O(1). \tag{7.5}$$

Setting $f_1(t) = \frac{t^{-(\sigma-1)}}{(\log_2 t)^\delta}$, we have

$$h_1(\sigma) = \sum_{p\geq 3} \frac{p^{-(\sigma-1)}}{p\left(\log_2 p\right)^\delta} = -\int_3^{+\infty} A_1(t) f_1'(t)dt + O(1)$$
$$\asymp (\sigma-1)\int_3^{+\infty} \left(\log_2 t\right)^{1-\delta} t^{-\sigma} dt$$
$$= (\sigma-1)\left(\int_{\log 3}^{\sigma'} + \int_{\sigma'}^{+\infty}\right)(\log x)^{1-\delta} e^{-(\sigma-1)x} dx.$$

Using integration by parts (for the first integral), and a change of variable (for the second one), we obtain

$$h_1(\sigma) \asymp (\sigma-1)\int_{\log 3}^{\sigma'} (\log x)^{1-\delta} dx + \int_1^{+\infty} \left(\log y + \log\sigma'\right)^{1-\delta} e^{-y} dy$$
$$\asymp (\sigma-1)\left[x(\log x)^{1-\delta}\right]_{x=\log 3}^{x=\sigma'} + \int_1^{+\infty}\left[(\log y)^{1-\delta} + \left(\log\sigma'\right)^{1-\delta}\right] e^{-y}dy$$
$$\asymp \left(\log\sigma'\right)^{1-\delta}.$$

The functions h_2 and h_3 are handled similarly. For $x \geq 3$, summation by parts and (7.5) induce that,

$$A_2(x) := \sum_{3\leq p\leq x} \frac{1}{p\log_2 p} = \frac{A_1(x)}{\log_2 x} + \int_3^x \frac{A_1(t)}{t\log t(\log_2 t)^2}dt + O(1) \asymp \log_3 x.$$

Set $f_2(t) := t^{-(\sigma-1)}$. Then,

$$h_2(\sigma) \asymp -\int_3^{+\infty} A_2(t) f_2'(t)dt + O(1) \asymp (\sigma-1)\int_3^{+\infty}(\log_3 t)t^{-\sigma}dt$$
$$= (\sigma-1)\left(\int_{\log 3}^{e\sigma'} + \int_{e\sigma'}^{+\infty}\right)(\log_2 x)e^{-(\sigma-1)x}dx.$$

Now

$$(\sigma-1)\int_{\log 3}^{e\sigma'}(\log_2 x)e^{-(\sigma-1)x}dx \asymp (\sigma-1)\int_{\log 3}^{e\sigma'}(\log_2 x)dx$$
$$\leq (\sigma-1)e\sigma'\left(\log_2\left(e\sigma'\right)\right) \lesssim \log_2\sigma'.$$

We perform a change of variable in the integral over $[e\sigma', +\infty)$.

$$I_{2,2} := (\sigma-1)\int_{e\sigma'}^{+\infty}(\log_2 x)e^{-(\sigma-1)x}dx = \int_e^{+\infty}\left[\log\left(\log y + \log\sigma'\right)\right]e^{-y}dy$$

$$\geq (\log_2 \sigma') \int_e^{+\infty} e^{-y} dy \gtrsim \log_2 \sigma'.$$

Since $\log(a+b) \leq \log a \log b + 1$, for $a \geq e$ and $b \geq e$, we obtain

$$I_{2,2} \leq \int_e^{+\infty} \left[(\log_2 y)(\log_2 \sigma') + 1\right] e^{-y} dy \lesssim \log_2 \sigma',$$

and $I_{2,2} \asymp \log_2 \sigma'$. It follows that $h_2(\sigma) \asymp \log_2 \sigma'$.

We now focus on h_3. By Mertens' first Theorem, $A_3(x) := \sum_{3\leq p\leq x} \frac{\log p}{p} = \log x + O(1)$, and putting $f_3(t) := t^{-(\sigma-1)} \left(\log_2 t\right)^{-\eta}$, we see that

$$\begin{aligned} h_3(\sigma) &= -\int_3^{+\infty} A_3(t) f_3'(t) dt + O(1) \\ &\asymp (\sigma - 1) \int_3^{+\infty} (\log t)\, t^{-\sigma} \left(\log_2 t\right)^{-\eta} dt \\ &\asymp (\sigma - 1) \left(\int_{\log 3}^{\sigma'} + \int_{\sigma'}^{+\infty} \right) x e^{-(\sigma-1)x} (\log x)^{-\eta}\, dx. \end{aligned}$$

Integration by parts gives that

$$\begin{aligned} I_{3,1} &:= (\sigma - 1) \int_{\log 3}^{\sigma'} x e^{-(\sigma-1)x} (\log x)^{-\eta}\, dx \\ &\asymp (\sigma - 1) \int_{\log 3}^{\sigma'} x\, (\log x)^{-\eta}\, dx \asymp \sigma' \left(\log \sigma'\right)^{-\eta}. \end{aligned}$$

Next, (7.4) is a consequence of

$$\begin{aligned} I_{3,2} &:= (\sigma - 1) \int_{\sigma'}^{+\infty} x e^{-(\sigma-1)x} (\log x)^{-\eta}\, dx \\ &= \frac{1}{\sigma - 1} \int_1^{+\infty} y e^{-y} \left(\log y + \log \sigma'\right)^{-\eta} dy \\ &\lesssim \sigma' \int_1^{+\infty} \frac{y e^{-y}}{(\log \sigma')^{\eta}} dy. \end{aligned}$$

□

Lemma 12 *If* $2\eta > 1$ *and* $\delta + \eta > 1$, *we have*

$$S := \sum_{p_1, p_2, p_3 \in \mathbb{P},\, p_j \geq 3} \frac{1}{p_1 p_2 p_3 \left(\log_2 p_1\right)^{2\delta} \left(\log_2 p_2\right)^{2}} \times \frac{(\log p_3)^2}{\left(\log_2 p_3\right)^{2\eta} (\log(p_1 p_2 p_3))^2} < \infty.$$

Proof For $p_1, p_2 \geq 3$, we set $L := \log(p_1 p_2)$ and $S_3(p_1, p_2) := \sum_{p_3} \frac{(\log p_3)^2}{p_3 \left(\log_2 p_3\right)^{2\eta} (\log p_3 + L)^2}$. Then, we have

$$S = \sum_{p_1, p_2, p_3} \frac{1}{p_1 p_2 \left(\log_2 p_1\right)^{2\delta} \left(\log_2 p_2\right)^2} S_3(p_1, p_2).$$

Under the condition $2\eta > 1$, the sum $S_3(p_1, p_2)$ converges, and

$$\begin{aligned} S_3(p_1, p_2) &= -\int_3^{+\infty} A_1(t) \frac{d}{dt} \left[\frac{(\log t)^2}{\left(\log_2 t\right)^{2\eta} (\log t + L)^2} \right] dt + \frac{O(1)}{L^2} \\ &\lesssim \frac{O(1)}{L^2} + \int_3^{+\infty} \frac{\log t}{t (\log_2 t)^{2\eta} (\log t + L)^2} dt \\ &= \frac{O(1)}{L^2} + \left(\int_{\log 3}^{L} + \int_L^{+\infty} \right) \frac{x dx}{(\log x)^{2\eta} (x + L)^2}. \end{aligned}$$

Integration by parts gives

$$I_{3,1} := \int_{\log 3}^{L} \frac{x dx}{(\log x)^{2\eta} (x + L)^2} \asymp \frac{1}{L^2} \int_{\log 3}^{L} \frac{x dx}{(\log x)^{2\eta}} \asymp (\log L)^{-2\eta}.$$

We handle the second integral via a change of variable:

$$\begin{aligned} I_{3,2} &:= \int_L^{+\infty} \frac{x dx}{(\log x)^{2\eta} (x + L)^2} = \left(\int_1^L + \int_L^{+\infty} \right) \frac{y dy}{(1 + y)^2 (\log y + \log L)^{2\eta}} \\ &\lesssim \frac{1}{(\log L)^{2\eta}} \int_1^L \frac{dy}{y} + \int_L^{+\infty} \frac{dy}{y (\log y)^{2\eta}} \asymp (\log L)^{1-2\eta}. \end{aligned}$$

Therefore

$$S_3(p_1, p_2) \lesssim (\log L)^{1-2\eta}, \ L = \log(p_1 p_2).$$

We next put $M = \log p_1$, and deal with

$$S_2(p_1) := \sum_{p_2} \frac{1}{p_2 (\log_2 p_2)^2} S_3(p_1, p_2) \lesssim \sum_p \frac{1}{p (\log_2 p)^2 \left[\log(\log p + M)\right]^{2\eta - 1}}.$$

With the notation $f_2(t) := \left[\left(\log_2 t\right)^2 \left[\log(\log t + M)\right]^{2\eta-1} \right]^{-1}$, we obtain that

$$S_2(p_1) = \frac{O(1)}{(\log M)^{2\eta-1}} - \int_3^{+\infty} A_1(t) f_2'(t) dt \lesssim \frac{O(1)}{(\log M)^{2\eta-1}} + I_{2,1} + I_{2,2},$$

where

$$I_{2,1} := \int_3^{+\infty} \frac{dt}{t \log t \left(\log_2 t\right)^2 \left[\log\left(\log t + M\right)\right]^{2\eta-1}};$$
$$I_{2,2} := \int_3^{+\infty} \frac{dt}{t \left(\log_2 t\right) \left(\log t + M\right) \left[\log\left(\log t + M\right)\right]^{2\eta}}.$$

We derive

$$\begin{aligned} I_{2,1} &= \left(\int_{\log 3}^{M} + \int_{M}^{+\infty}\right) \frac{dx}{x\,(\log x)^2 \left[\log\left(x+M\right)\right]^{2\eta-1}} \\ &\lesssim \frac{1}{\left[\log M\right]^{2\eta-1}} \int_{\log 3}^{M} \frac{dx}{x\,(\log x)^2} \\ &\quad + (\log M)^{1-2\eta} \int_{M}^{+\infty} \frac{dx}{x\,(\log x)^2} \lesssim (\log M)^{1-2\eta}. \end{aligned}$$

The second integral is estimated in the same way:

$$\begin{aligned} I_{2,2} &= \left(\int_{\log 3}^{M} + \int_{M}^{+\infty}\right) \frac{dx}{(x+M)(\log x)\left[\log(x+M)\right]^{2\eta}} \\ &\lesssim \frac{1}{M(\log M)^{2\eta}} \int_{\log 3}^{M} \frac{dx}{\log x} + \frac{1}{(\log M)^{2\eta-1}} \int_{M}^{+\infty} \frac{dx}{x(\log x)^2} \\ &\asymp \frac{1}{M(\log M)^{2\eta}} \left(\left[\frac{x}{\log x}\right]_{x=\log 3}^{x=M} + \int_{\log 3}^{M} \frac{x^2}{2} \frac{(\log x)^{-2}}{x} dx\right) \\ &\quad + \frac{1}{(\log M)^{2\eta}} \asymp \frac{1}{(\log M)^{2\eta}}. \end{aligned}$$

Therefore, we have

$$S_2(p_1) \lesssim \frac{1}{(\log M)^{2\eta-1}}, \; M = \log p_1.$$

It follows that

$$S \lesssim \sum_{p_1} \frac{1}{p_1(\log_2 p_1)^{2\delta}} S_2(p_1) \lesssim \sum_{p \geq 3} \frac{1}{p(\log_2 p)^{\varepsilon}}, \; \varepsilon := 2\delta + 2\eta - 1.$$

Again, partial summation gives that when $\varepsilon > 1$,

$$\sum_{3\le p} \frac{1}{p(\log_2 p)^{\varepsilon}} \asymp \varepsilon \int_3^{+\infty} \frac{\log_2 t}{t(\log t)(\log_2 t)^{\varepsilon+1}} dt < \infty.$$

□

Proposition 6 *There exists a* 3*-homogeneous function* g *which is in* $\left(\cap_{0<p<\infty}\mathcal{H}_w^p\right)\cap$ *Bloch*$_0(\mathbb{C}_{1/2})$*, such that* T_g *is unbounded on* $\mathcal{H}_w^2$*.*

Proof Using Lemma 11, we see that, if $g' = -(h_1h_2h_3)_{\frac{1}{2}}$, g' is convergent on $\mathbb{C}_{1/2}$, and its estimate near the line $\Re s = \frac{1}{2}$ is determined by the behavior of the functions h_j near the line $\Re s = 1$. Then g is in $\text{Bloch}_0(\mathbb{C}_{1/2})$, because of

$$\left|g'(\sigma)\right| \asymp \frac{1}{\sigma - \frac{1}{2}}\left(\log\frac{1}{\sigma-\frac{1}{2}}\right)^{1-\delta-\eta}\left(\log_2\frac{1}{\sigma-\frac{1}{2}}\right), \quad \text{as } \sigma \to 1/2^+.$$

On another hand, the 3-homogeneous function

$$g(s) = \sum_n b_n n^{-s} = \sum_{p_1,p_2,p_3} \frac{\alpha_{1,p_1}\alpha_{2,p_2}\alpha_{3,p_3}}{\log(p_1p_2p_3)}(p_1p_2p_3)^{-s}$$

is in $\mathcal{H}_w^2$ by Lemma 12, since $\|g\|^2_{\mathcal{H}_w^2} = \sum_n |b_n|^2 w_n^{-1} \asymp \sum_n |b_n|^2 \asymp S < \infty$.

By Lemma 10, g is in $\cap_{0<p<\infty}\mathcal{H}_w^p$.

It remains to prove that T_g is unbounded on $\mathcal{H}_w^2$. We again choose as test functions (cf the proof of Proposition 5)

$$f_x(s) := \prod_{\frac{x}{2}<p\le x}\left(1 + w_2^{1/2}p^{-s}\right) = \sum_{n\ge 1} a_n n^{-s}.$$

S_x is the set of square free integers generated by $\frac{x}{2} < p \le x$. Set $V_x = \left\{n \in S_x,\ \omega(n) \ge \frac{N(x)}{2}\right\}$.

For $n \in V_x$, set

$$A_n := \sum_{p_1p_2p_3|n} b_{p_1p_2p_3}\left(\log(p_1p_2p_3)\right) a_{\frac{n}{p_1p_2p_3}}$$

The coefficients in A_n satisfy

$$b_{p_1p_2p_3}\left(\log(p_1p_2p_3)\right) \gtrsim \frac{\log x}{x^{3/2}\left(\log_2 x\right)^{\eta+\delta+1}}.$$

Since $\|f_x\|^2_{\mathcal{H}_w^2} \asymp |V_x|$, we see that

$$\left\|T_g f_x\right\|^2_{\mathcal{H}_w^2} \ge \sum_{n\in V_x} w_n^{-1}(\log n)^{-2} A_n^2$$

$$\gtrsim \sum_{n \in V_x} w_2^{-\omega(n)} (\omega(n) \log x)^{-2} \times$$

$$\left[\frac{\log x}{x^{3/2} (\log_2 x)^{\eta+\delta+1}} \binom{\omega(n)}{3} \left(w_2^{1/2} \right)^{\omega(n)-3} \right]^2$$

$$\gtrsim \|f_x\|^2_{\mathcal{H}^2_w} \left(\frac{x}{\log x} \right)^4 \frac{1}{x^3 (\log_2 x)^{2(\delta+\eta+1)}},$$

and the proof is complete. □

8 Comparison of $\mathcal{X}_w$ with other spaces of Dirichlet series

The previous results enable us to derive some inclusions involving $\mathcal{X}_w$.

In the context of the unit disk, the space of symbols g for which the Volterra operator J_g (1.3) is bounded on $A^2_\alpha(\mathbb{D})$ is Bloch($\mathbb{D}$). It coincides with the space of holomorphic g such that the Hankel form (1.5) is bounded, and with the dual space of $A^1_\alpha(\mathbb{D})$.

We shall study the counterparts of these facts for $\mathcal{X}_w$.

8.1 Bounded Hankel forms

The Hilbert space $\mathcal{H}^2_w$ is equipped with the inner product $\langle .,. \rangle_{\mathcal{H}^2_w}$. The Hankel form of symbol $g \in \mathcal{D}$ is defined on $\mathcal{H}^2_w$ by

$$H_g(fh) := \langle fh, g \rangle_{\mathcal{H}^2_w}. \tag{8.1}$$

We say that H_g is bounded on $\mathcal{H}^2_w \times \mathcal{H}^2_w$ if there is a constant C such that

$$\left| H_g(fh) \right| \leq C \|f\|_{\mathcal{H}^2_w} \|h\|_{\mathcal{H}^2_w} \quad \text{for } f, h \in \mathcal{H}^2_w.$$

The weak product $\mathcal{H}^2_w \odot \mathcal{H}^2_w$ is the Banach space defined as the closure of all finite sums $F = \sum_k f_k h_k$, where $f_k, h_k \in \mathcal{H}^2_w$, under the norm

$$\|F\|_{\mathcal{H}^2_w \odot \mathcal{H}^2_w} := \inf \sum_k \|f_k\|_{\mathcal{H}^2_w} \|h_k\|_{\mathcal{H}^2_w}.$$

Here the infimum is taken over all finite representations of F as $F = \sum_k f_k h_k$.

Let $\mathcal{Y}$ be a Banach space of Dirichlet series in which the space of Dirichlet polynomials $\mathcal{P}$ is dense. We say that a Dirichlet series ϕ is in the dual space $\mathcal{Y}^*$ if the linear functional induced by ϕ via the $\mathcal{H}^2_w$-pairing is bounded. In other words, $\phi \in \mathcal{Y}^*$ if and only if

$$v_\phi(f) = \langle f, \phi \rangle_{\mathcal{H}^2_w}, \quad f \in \mathcal{P},$$

extends to a bounded functional on $\mathcal{Y}$.

From its definition, H_g (8.1) is bounded on $\mathcal{H}^2_w$ if and only if $g \in \left(\mathcal{H}^2_w \odot \mathcal{H}^2_w\right)^*$.

We aim to relate Hankel forms and Volterra operators. The primitive of $f \in \mathcal{D}$ with constant term 0 is denoted by

$$\partial^{-1} f(s) := -\int_s^{+\infty} f(u)du,$$

We observe that

$$H_g(fh) = f(+\infty)\, h(+\infty)\, g(+\infty) + \left\langle \partial^{-1}(f'h), g \right\rangle_{\mathcal{H}^2_w} + \left\langle \partial^{-1}(fh'), g \right\rangle_{\mathcal{H}^2_w}.$$

The Banach space $\partial^{-1}\left(\partial\mathcal{H}^2_w \odot \mathcal{H}^2_w\right)$ is the completion of the space of Dirichlet series F whose derivatives have a finite sum representation $F' = \sum_k f_k h_k'$, under the norm

$$\|F\|_{\partial^{-1}(\partial\mathcal{H}^2_w \odot \mathcal{H}^2_w)} := |F(+\infty)| + \sum_k \|f_k\|_{\mathcal{H}^2_w} \|h_k\|_{\mathcal{H}^2_w},$$

where the infimum is taken over all finite representations. The product rule $(fg)' = f'g + fg'$ implies that

$$\mathcal{H}^2_w \odot \mathcal{H}^2_w \subset \partial^{-1}\left(\partial\mathcal{H}^2_w \odot \mathcal{H}^2_w\right),$$

and then

$$\left(\partial^{-1}\left(\partial\mathcal{H}^2_w \odot \mathcal{H}^2_w\right)\right)^* \subset \left(\mathcal{H}^2_w \odot \mathcal{H}^2_w\right)^*. \tag{8.2}$$

It has been shown in [14] that, for the space $\mathcal{H}^2_0 = \left\{f \in \mathcal{H}^2 \,:\, f(+\infty) = 0\right\}$, the inclusion $\left(\partial^{-1}\left(\partial\mathcal{H}^2_0 \odot \mathcal{H}^2_0\right)\right)^* \subset \left(\mathcal{H}^2_0 \odot \mathcal{H}^2_0\right)^*$ is strict. As for the space $\mathcal{H}^2_w$, the question whether the inclusion is strict remains open.

The membership of g in $\left(\partial^{-1}\left(\partial\mathcal{H}^2_w \odot \mathcal{H}^2_w\right)\right)^*$ is equivalent to the boundedness of the so-called "half-Hankel form"

$$(f, h) \mapsto \left\langle \partial^{-1}(f'h), g \right\rangle_{\mathcal{H}^2_w}. \tag{8.3}$$

As in the case of $\mathcal{H}^2$, the boundedness of T_g implies the boundedness of H_g.

Theorem 5 *If the Volterra operator T_g is bounded on $\mathcal{H}^2_w$, then the Hankel form H_g is bounded.*

Proof We adapt the proof of Corollary 6.2 in [13] to the framework of the polydisk $\mathbb{D}^\infty$. Polarizing the Littlewood–Paley formula (1), we get

$$\langle f, g\rangle_{\mathcal{H}^2_w} = f(+\infty)g(+\infty) + 4\int_{\mathbb{D}^\infty}\int_{\mathbb{R}}\int_0^{+\infty} f'_\chi(\sigma+it)\overline{g'_\chi(\sigma+it)}\sigma d\sigma \frac{dt}{1+t^2} d\mu_w(\chi).$$

Then, we derive an expression of the half-Hankel form

$$\left\langle \partial^{-1}(f'h), g\right\rangle_{\mathcal{H}^2_w} = 4\int_{\mathbb{D}^\infty}\int_{\mathbb{R}}\int_0^{+\infty} f'_\chi(\sigma+it)h_\chi(\sigma+it)\overline{g'_\chi(\sigma+it)}\sigma d\sigma \frac{dt}{1+t^2}d\mu_w(\chi).$$

Since T_g is bounded on $\mathcal{H}^2_w$, the Carleson measure characterization (4.1) induces that the form (8.3) is also bounded. Then H_g is bounded on $\mathcal{H}^2_w \odot \mathcal{H}^2_w$ by the inclusion (8.2). □

The previous Theorem states that we have

$$\mathcal{X}_w \subset \left(\mathcal{H}^2_w \odot \mathcal{H}^2_w\right)^*.$$

The rest of the section is devoted to study the reverse inclusion.

Let l^2_w denote the Hilbert space of complex sequences $a = (a_n)_n$ such that

$$\|a\|_{l^2_w} := \left(\sum_{n\geq 1} \frac{|a_n|^2}{w_n}\right)^{1/2} < \infty.$$

A sequence $(\rho_n)_n$ generates the following multiplicative Hankel form

$$\rho(a,b) := \sum_{n=1}^{+\infty}\sum_{m=1}^{+\infty} a_m b_n \frac{\rho_{mn}}{w_{mn}}, \ \ a, b \in l^2_w. \tag{8.4}$$

The symbol of the form is the Dirichlet series $g(s) = \sum_{n\geq 1} \overline{\rho_n} n^{-s}$. The form ρ is said to be bounded if there is a constant C such that

$$|\rho(a,b)| \leq C \|a\|_{l^2_w} \|b\|_{l^2_w}.$$

If f and h are Dirichlet series with coefficients a and b, respectively, we have

$$H_g(fh) = \langle fh, g\rangle_{\mathcal{H}^2_w} = \rho(a,b).$$

When the symbol g has non negative coefficients, there is equivalence between the boundedness of H_g and the half-Hankel form (8.3). In fact, the proof given for $\mathcal{H}^2$ in [14] is valid for the spaces $\mathcal{H}^2_w$.

Proposition 7 *Let $g(s) = \sum_{n\geq 1} \overline{\rho_n} n^{-s}$ be in $\mathcal{H}^2_w$. The linear functional defined on $\mathcal{H}^2_w$*

$$v_g(f) := \langle f, g\rangle_{\mathcal{H}^2_w}$$

is bounded on $\partial^{-1}\left(\partial\mathcal{H}_w^2 \odot \mathcal{H}_w^2\right)$ *if and only if the weighted form*

$$J_g(a,b) = \sum_{n=1}^{+\infty}\sum_{m=1}^{+\infty} a_m b_n \frac{\log n}{\log m + \log n}\frac{\rho_{mn}}{w_{mn}},$$

(where it is understood that for $m = n = 1$*, the summand is* 0*) is bounded on* $l_w^2 \odot l_w^2$*. The norms are equivalent, i.e.*

$$\|g\|_{(\partial^{-1}(\partial\mathcal{H}_w^2\odot\mathcal{H}_w^2))^*} \asymp \left\|v_g\right\| \asymp |\rho_1| + \left\|J_g\right\|.$$

If $\rho_k \geq 0$ *for all* k*, then* $g \in \left(\partial^{-1}\left(\partial\mathcal{H}_w^2 \odot \mathcal{H}_w^2\right)\right)^*$ *if and only if* $g \in \left(\mathcal{H}_w^2 \odot \mathcal{H}_w^2\right)^*$*, with equivalent norms.*

Proposition 7 will enable us to provide examples of symbols g for which the Hankel form H_g and the half-Hankel form (8.3) are bounded, but the Volterra operator T_g is unbounded (see the proof of Proposition 9). This differs from the case of weighted Dirichlet spaces on the unit disk, for which the boundedness of H_g, the form (8.3) and T_g are equivalent [1].

For convergence reasons, we will consider Hankel forms defined on Dirichlet series without constant term. So we will work on the space

$$\mathcal{H}_{w,0}^2 = \left\{f \in \mathcal{H}_w^2 \ : \ f(+\infty) = 0\right\}.$$

We have seen in Lemma 1 that the space $\mathcal{H}_w^2$ is embedded in a Bergman space of the form $A_{i,\delta}\left(\mathbb{C}_{1/2}\right)$. For $\delta > 0$, it is thus natural to define the Hankel form

$$H^{(\delta)}(fh) := \int_{1/2}^{+\infty} f(\sigma)h(\sigma)\left(\sigma - \frac{1}{2}\right)^{\delta} d\sigma, \ \ f, h \in \mathcal{H}_{w,0}^2. \tag{8.5}$$

Such multiplicative forms have been considered in the context of $\mathcal{H}^2$ [12] and on $\mathcal{A}_1^2$ [9].

Since $K^{\mathcal{H}_w^2}(s,u) - 1 = \sum_{n\geq 2} w_n n^{-\overline{u}} n^{-s}$ is the reproducing kernel of $\mathcal{H}_{w,0}^2$, we see that $H^{(\delta)}(fh) = \langle fh, \phi_\delta\rangle_{\mathcal{H}_w^2}$, where

$$\phi_\delta(s) = \int_{1/2}^{+\infty}\left[K^{\mathcal{H}_w^2}(s,\sigma) - 1\right]\left(\sigma - \frac{1}{2}\right)^{\delta} d\sigma = \sum_{n=2}^{+\infty}\frac{w_n}{\sqrt{n}\,(\log n)^{\delta+1}} n^{-s}.$$

Proposition 8 *Let* $\delta > 0$ *as in* (2.5)*. Then* $H^{(\delta)}$ *defined in* (8.5) *is a multiplicative Hankel form with symbol* ϕ_δ*, which is bounded on* $\mathcal{H}_{w,0}^2 \odot \mathcal{H}_{w,0}^2$*.*

Proof The proof is similar to that of Theorem 13 in [9]. The Cauchy-Schwarz inequality ensures that

$$\left|H^{(\delta)}(fh)\right| \leq \left(\int_{1/2}^{+\infty} |f(\sigma)|^2 \left(\sigma - \frac{1}{2}\right)^\delta d\sigma\right)^{1/2} \left(\int_{1/2}^{+\infty} |h(\sigma)|^2 \left(\sigma - \frac{1}{2}\right)^\delta d\sigma\right)^{1/2}.$$

If $f(s) = \sum_{n=2}^{+\infty} a_n n^{-s}$, notice the pointwise estimate

$$|f(\sigma)|^2 \leq \|f\|^2_{\mathcal{H}^2_w} \left(\sum_{n=2}^{+\infty} w_n n^{-2\sigma}\right) \lesssim \|f\|^2_{\mathcal{H}^2_w} 4^{-\sigma}, \quad \text{for } \sigma \geq 1.$$

Since the bounded measure $d\mu(\sigma + it) = \chi_{)1/2,1]}(\sigma) \left(\sigma - \frac{1}{2}\right)^\delta d\sigma$, supported on the real line, is Carleson for $A_{i,\delta}\left(\mathbb{C}_{1/2}\right)$, μ is Carleson for $\mathcal{H}^2_w$ by Lemma 6, and

$$\int_{1/2}^{+\infty} |f(\sigma)|^2 \left(\sigma - \frac{1}{2}\right)^\delta d\sigma = \left(\int_{1/2}^{1} + \int_{1}^{+\infty}\right) |f(\sigma)|^2 \left(\sigma - \frac{1}{2}\right)^\delta d\sigma \lesssim \|f\|^2_{\mathcal{H}^2_w}.$$

□

We next exhibit symbols giving rise to bounded Hankel forms and bounded half-Hankel forms, though the associated Volterra operator is unbounded.

Proposition 9 *We have the strict inclusions*

$$\mathcal{X}(\mathcal{H}^2_{w,0}) \subset_{\neq} \left(\mathcal{H}^2_{w,0} \odot \mathcal{H}^2_{w,0}\right)^*;$$
$$\mathcal{X}_w \subset_{\neq} \left(\mathcal{H}^2_w \odot \mathcal{H}^2_w\right)^*.$$

Proof It just remains to check the strictness of the inclusions. For the exponent $\delta = \delta(w)$ and $\frac{1}{2} \leq a < 1$, consider the symbol in $\mathcal{H}^2_{w,0}$

$$g(s) = \sum_{n=2}^{+\infty} \frac{w_n}{n^a (\log n)^{\delta+1}} n^{-s}.$$

From Proposition 8 and the fact that the coefficients are positive, g is in $\left(\mathcal{H}^2_{w,0} \otimes \mathcal{H}^2_{w,0}\right)^*$ for any $\frac{1}{2} \leq a < 1$. In fact, the half Hankel form corresponding to g is bounded. We have seen in Proposition 4 that T_g is not bounded on $\mathcal{H}^2_w$. Since $T_g 1 = g$, g does not belong to $\mathcal{X}(\mathcal{H}^2_{w,0})$.

In order to prove that $g \in \left(\mathcal{H}^2_w \odot \mathcal{H}^2_w\right)^*$, we consider the associated multiplicative form ρ (8.4). Let f, h be Dirichlet series with coefficients a, b, belonging to $\mathcal{H}^2_w$. Since

$$\rho(a,b) = \sum_{m,n\geq 2} a_m b_n \frac{\rho_{mn}}{w_{mn}} + a_1 \sum_{n=1}^{+\infty} b_n \frac{\rho_n}{w_n} + b_1 \sum_{m=1}^{+\infty} a_m \frac{\rho_m}{w_m}$$
$$= H_g\left((f - f(\infty))(g - g(\infty))\right) + f(\infty)\langle h, g\rangle_{\mathcal{H}^2_w} + g(\infty)\langle f, g\rangle_{\mathcal{H}^2_w},$$

the first part of the proof entails that H_g is bounded on $\mathcal{H}^2_w \odot \mathcal{H}^2_w$. □

8.2 $\mathcal{X}_w$ and the dual of $\mathcal{H}^1_w$

Keeping in mind the results known for Bergman spaces of the unit disk, it is natural to compare $\mathcal{X}_w$ and $\left(\mathcal{H}^1_w\right)^*$.

In general, the dual of $\mathcal{H}^1_w$ is not known. However, it is shown in [9] that

$$\mathcal{K} \subset \left(\mathcal{A}^1_1\right)^*, \tag{8.6}$$

where $\mathcal{K}$ is the space of Dirichlet series $f(s) = \sum_{n=1}^{+\infty} a_n n^{-s}$ such that

$$\sum_{n=1}^{+\infty} \frac{d_4(n)}{[d(n)]^2} |a_n|^2 < \infty.$$

The following consequence of this inclusion will stress upon the difference between the finite and infinite dimensional setting.

Proposition 10 *$\left(\mathcal{A}^1_1\right)^*$ is not contained in $\mathcal{X}\left(\mathcal{A}^2_1\right)$.*

Proof By Abel summation and the Chebyshev estimate, the symbol

$$g(s) = \sum_{n=2}^{+\infty} \frac{d(n)}{n^a (\log n)^2} n^{-s}, \text{ for } \frac{1}{2} < a < 1,$$

is in $\mathcal{K}$, and thus in $\left(\mathcal{A}^1_1\right)^*$. However, T_g is unbounded on $\mathcal{A}^2_1$ (Proposition 4). □

8.3 $\mathcal{X}_w$ and the spaces $\mathcal{H}^p_w$

It has been shown in [13] that $BMOA(\mathbb{C}_0) \cap \mathcal{D} \subset_{\neq} \mathcal{X}(\mathcal{H}^2) \subset_{\neq} \cap_{0<p<\infty} \mathcal{H}^p$. We have an analogue for Bergman spaces of Dirichlet series.

Theorem 6 *We have the strict inclusions*

$$BMOA(\mathbb{C}_0) \cap \mathcal{D} \subset_{\neq} \mathcal{X}_w \subset_{\neq} \cap_{0<p<\infty} \mathcal{H}^p_w.$$

Proof The inclusions have been proved in Theorem 1 and Corollary 1. As observed in [13], the symbols $g(s) = \sum_{n=2}^{+\infty} \frac{\psi(n)}{\log n} n^{-s}$, where ψ is the completely multiplicative function defined on the primes by $\psi(p) := \lambda p^{-1} \log p$, $0 < \lambda \leq 1$, are in $\mathcal{X}(\mathcal{H}^2)$, and satisfy

$$\sum_{n=1}^{+\infty} \psi(n) n^{-\sigma} \asymp \exp\left(\lambda \sum_p \frac{\log p}{p^{1+\sigma}}\right) \asymp \exp\left(\lambda \frac{1}{\sigma}\right), \ \sigma > 0.$$

Hence, they are not in $BMOA(\mathbb{C}_0)$, though they belong to $\mathcal{X}_w$ (Lemma 9).

The second inclusion is strict by Proposition 6. □

With the method of Proposition 4, one can show that $g(s) = \sum_{n\geq 2} \frac{n^{-a}}{\log n} n^{-s}$, $1/2 \leq a < 1$, is not in $\mathcal{X}_w$, though it belongs to $BMOA(\mathbb{C}_{1-a})$ [13]. Therefore, we have the strict inclusion

$$\mathcal{X}_w \subset_{\neq} \text{Bloch}(\mathbb{C}_{1/2}).$$

8.4 $\mathcal{X}_w \cap \mathcal{D}_d$ and Bloch spaces

Theorem 7 *Let d be a positive integer. The following inclusions hold*

$$\mathcal{D}_d \cap Bloch(\mathbb{C}_0) \subset \mathcal{D}_d \cap \mathcal{X}_w \subset_{\neq} \mathcal{B}^{-1} Bloch(\mathbb{D}^d).$$

Proof The first inclusion has been shown in Theorem 1(a).

If g is in $\mathcal{D}_d \cap \mathcal{X}_w$, Theorem 5 implies that H_g is bounded on $\mathcal{H}^2_w$. Therefore, the form $H_{\mathcal{B}g}$ (1.4) is bounded on the Bergman space $H^2_w(\mathbb{D}^d)$. From [17], $\mathcal{B}g$ is in $\text{Bloch}(\mathbb{D}^d)$.

Here is a function g which is not in $\mathcal{X}_w$, such that $\mathcal{B}g$ is in $\text{Bloch}(\mathbb{D}^2)$. Suppose that

$$g'(s) = \frac{1}{1-2^{-s}} \log\left(\frac{1}{1-3^{-s}}\right), \ s \in \mathbb{C}_0.$$

Straightforward computations show that $\mathcal{B}g \in \text{Bloch}(\mathbb{D}^2)$. The norms $\|.\|_{A^2_\beta(\mathbb{D}^2)}$ and $\|.\|_{B^2_\beta(\mathbb{D}^2)}$ being equivalent, our setting will be the space $A^2_\beta(\mathbb{D}^2)$. Now, for

$$F(z) = \sum_{n=1}^{\infty} \frac{(n+1)^{\frac{\beta-1}{2}}}{\log(n+1)} z^n = \sum_{n=0}^{\infty} a_n z^n, \ z \in \mathbb{D},$$

define $f(s) = F(2^{-s})F(3^{-s})$, for $s \in \mathbb{C}_0$. We have

$$\|f\|^2_{\mathcal{H}^2_w} = \|F\|^4_{A^2_\beta(\mathbb{D})} \asymp \left(\sum_{n=1}^{\infty} \frac{1}{(n+1)\left(\log(n+1)\right)^2}\right)^2 < \infty.$$

Putting

$$h_1(z_1) = F(z_1)\frac{1}{1-z_1} = \sum_{m=0}^{\infty} A_m z_1^m, \ z_1 \in \mathbb{D},$$

$$h_2(z_2) = F(z_2)\log\left(\frac{1}{1-z_2}\right) = \sum_{n=0}^{\infty} B_n z_2^n,\ z_2 \in \mathbb{D},$$

we have $A_m \gtrsim \frac{(m+1)^{\frac{\beta+1}{2}}}{\log(m+1)}$ and $B_n \gtrsim (n+1)^{\frac{\beta-1}{2}}$. Therefore,

$$\left\|T_g f\right\|^2_{\mathcal{H}^2_w} = \left\|R^{-1}(h_1 h_2)\right\|^2_{A^2_\beta(\mathbb{D}^2)} \asymp \sum_{m,n\geq 1} \frac{|A_m|^2\,|B_n|^2}{(m+n+1)^2(m+1)^\beta(n+1)^\beta}$$
$$\gtrsim \sum_{m\geq 1} \frac{m+1}{(\log(m+1))^2}\,\frac{\log(m+1)}{(m+1)^2} = \sum_{m\geq 1}\frac{1}{(m+1)\log(m+1)} = +\infty,$$

which proves the claim. □

A consequence of Theorems 1 and 6 is that

$$\mathrm{Bloch}(\mathbb{C}_0) \cap \mathcal{D}_d \subset \cap_{0<p<\infty}\mathcal{H}^p_{d,w}.$$

This inclusion can be viewed as a counterpart of the situation of the disk, where $\mathrm{Bloch}(\mathbb{D}) \subset \cap_{0<p<\infty} A^p_\beta(\mathbb{D})$.

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