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# Note on the semi-simplicity of measure algebras

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**Abstract**
In this paper we prove that the measure algebra of a locally compact abelian group is semi-simple. This result extends the corresponding result of S. A. Amitsur in the discrete group case using a completely different approach.

**Keywords** Measure algebra · Spectral synthesis · Semi-simple

**Mathematics Subject Classification** 16D60 · 28A60

## 1 Introduction

In the sequel $\mathbb{C}$ denotes the set of complex numbers. We recall that the *measure algebra* of a locally compact Abelian group $G$ is the set $\mathcal{M}_c(G)$ of all compactly supported complex Borel measures on $G$, which can be identified with the topological dual space of the topological vector space $\mathcal{C}(G)$ of all continuous complex valued functions on $G$, when the latter is equipped with the topology of uniform convergence on compact sets. The space $\mathcal{M}_c(G)$ turns into a commutative unital involutive complex algebra when equipped with the convolution defined by

$$\langle \mu * \nu, f \rangle = \int_G f(x+y) d\mu(x)\, d\nu(y),$$

and with the involution

$$\langle \mu^*, f \rangle = \langle \mu, f^* \rangle$$

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for each $\mu, \nu$ in $\mathcal{M}_c(G)$ and $f$ in $\mathcal{C}(G)$. Here $f^*(x) = \overline{f(-x)}$ whenever $x$ is in $G$. The unit element of $\mathcal{M}_c(G)$ is $\delta_o$, where, in general, $\delta_x$ denotes the point mass with support set $\{x\}$. We call $\mathcal{M}_c(G)$ the *measure algebra* of the group $G$.

The locally convex topological vector space $\mathcal{C}(G)$ is a topological vector module over the measure algebra when we define

$$\mu * f(x) = \int_G f(x-y)\,d\mu(y)$$

for $f$ in $\mathcal{C}(G)$, $\mu$ in $\mathcal{M}_c(G)$, and $x$ in $G$.

In the special case, when $G$ is a discrete group, the measure algebra is called *group algebra* and is denoted by $\mathbb{C}G$. The algebraic properties of the measure algebra, resp. the group algebra play a basic role in spectral analysis and synthesis on $G$. In particular, if $G$ is a discrete group, then it is proved in [1] that $\mathbb{C}G$ is semisimple. The purpose of the present note is to show that this holds in the non-discrete case as well. Our approach here is completely different from that of [1].

## 2 Exponential maximal ideals

From now on we always denote by $G$ a locally compact commutative topological group. An ideal in $\mathcal{M}_c(G)$ is called *exponential* if the residue algebra is topologically isomorphic to the complex field (see [2]). Clearly, in this case the ideal is weak*-closed and maximal. We will show that the intersection of all exponential ideals is zero. As a consequence we obtain that the Jacobson radical of $\mathcal{M}_c(G)$, i.e. the intersection of all maximal ideals, is zero.

Recall that the nonzero continuous function $m : G \to \mathbb{C}$ is called an *exponential*, if

$$m(x+y) = m(x)m(y)$$

holds for each $x, y$ in $K$. In this case $m(0) = 1$.

We shall use the following lemma.

**Lemma 1** *A necessary and sufficient condition for the ideal $I$ is exponential is that there exists an exponential $m : G \to \mathbb{C}$ such that $\mu$ is in $I$ if and only if $\langle \mu, \check{m} \rangle = 0$.*

In general, we use the notation $\check{f}(x) = f(-x)$ for each $f$ in $\mathcal{C}(G)$ and $x$ in $G$.

***Proof*** First we show the sufficiency. We define $F : \mathcal{M}_c(G) \to \mathbb{C}$ by

$$F(\mu) = \langle \mu, \check{m} \rangle$$

for each $\mu$ in $\mathcal{M}_c(G)$. We show that $F$ is a multiplicative functional of the algebra $\mathcal{M}_c(G)$, i.e. $F$ is a weak*-continuous linear functional satisfying

$$F(\mu * \nu) = F(\mu)F(\nu) \tag{1}$$

for each $\mu, \nu$ in $\mathcal{M}_c(G)$. The linearity and weak*-continuity is obvious, we need to show (1) only. We have

$$F(\mu * \nu) = \langle \mu * \nu, \check{m} \rangle = \int_G \check{m}(x * y)\, d\mu(x)\, d\nu(y)$$
$$= \int_G \check{m}(x)\, d\mu(x) \int_G \check{m}(y)\, d\nu(y) = F(\mu)F(\nu),$$

which proves our statement.

By assumption, the ideal $I$ coincides with the kernel of $F$: $I = \text{Ker}\, F$, and $\mathcal{M}_c(G)/\text{Ker}\, F \cong \mathbb{C}$, hence $I$ is an exponential ideal.

To prove the converse, let $I$ be an exponential ideal, then $I$ is maximal and $\mathcal{M}_c(G)/I \cong \mathbb{C}$. Let $F : \mathcal{M}_c(G) \to \mathbb{C}$ be the natural homomorphism and we define

$$m(x) = F(\delta_{-x})$$

for $x$ in $G$. Then we have

$$m(x + y) = F(\delta_{-x-y}) = F(\delta_{-x} * \delta_{-y}) = F(\delta_{-x})F(\delta_{-y}) = m(x)m(y).$$

Using the fact that finitely supported measures in $\mathcal{M}_c(G)$ form a weak*-dense subspace, we have

$$\langle \mu, \check{m} \rangle = F(\mu)$$

for each $\mu$ in $\mathcal{M}_c(G)$. As $m(0) = 1$ and $m$ is clearly continuous, we have that $m$ is an exponential. If $\mu$ is in $I$, then $\mu$ is in Ker $F$, hence

$$\langle \mu, \check{m} \rangle = F(\mu) = 0.$$

Conversely, if $\langle \mu, \check{m} \rangle = 0$, then $F(\mu) = 0$, hence $\mu$ is in Ker $F = I$. The theorem is proved. □

Closed submodules of the module $\mathcal{C}(G)$ are called *varieties*.

The *orthogonal complement* $X^\perp$ of a subset $X$ in $\mathcal{C}(G)$ is defined as

$$X^\perp = \{\mu \in \mathcal{M}_c(G) : \langle \mu, f \rangle = 0 \text{ for each } f \in X\}.$$

Similarly, the *orthogonal complement* $Y^\perp$ of a subset $Y$ in $\mathcal{M}_c(G)$ is defined as

$$Y^\perp = \{f \in \mathcal{C}(G) : \langle \mu, f \rangle = 0 \text{ for each } \mu \in Y\}.$$

A standard application of the Hahn–Banach Theorem gives the relations

$$V^{\perp\perp} = V, \quad I^{\perp\perp} = I$$

for each variety $V$ in $\mathcal{C}(G)$ and weak*-closed ideal $I$ in $\mathcal{M}_c(G)$.

Also, the following relations are important, and can be proved easily (see [2]):

**Theorem 1** *For each family* $(V_i)$ *of varieties and* $(I_i)$ *of weak*-closed ideals we have*

$$\Big(\sum V_i\Big)^{\perp} = \bigcap V_i^{\perp}, \quad \Big(\sum I_i\Big)^{\perp} = \bigcap I_i^{\perp},$$
$$\Big(\bigcap V_i\Big)^{\perp} = \sum V_i^{\perp}, \quad \Big(\bigcap I_i\Big)^{\perp} = \sum I_i^{\perp}.$$

## 3 The main result

**Theorem 2** *Let G be a commutative locally compact topological group. Then the measure algebra* $\mathcal{M}_c(G)$ *is semi-simple.*

***Proof*** We show that the intersection of all exponential maximal ideals in $\mathcal{M}_c(G)$ is zero. By Theorem 1, this is equivalent to the relation

$$\sum I_i^{\perp} = \mathcal{C}(G), \tag{2}$$

where $I_i$ runs through all exponential ideals in $\mathcal{M}_c(G)$. By Lemma 1, $I_i^{\perp}$ is the one dimensional space in $\mathcal{C}(G)$ generated by an exponential $m_i$, hence Eq. (2) states that the finite linear combinations of all exponentials on $G$ form a dense subspace in $\mathcal{C}(G)$. To prove this we use the Stone–Weierstrass Theorem. Indeed, for a given compact set $C$ in $G$ let $\mathcal{A}_C$ denote the set of the restrictions of all finite linear combinations of exponentials on $G$ to $C$. Clearly, $\mathcal{A}_C$ is a complex linear space in $\mathcal{C}(C)$. Moreover, $\mathcal{A}_C$ is a unital algebra: indeed, the product of two exponentials is an exponential, and 1 is an exponential. Also, $\mathcal{A}_C$ is closed under complex conjugation as the complex conjugate of an exponential is an exponential again. Finally, $\mathcal{A}_C$ is a separating family: indeed, for any two elements $x \neq y$ in $C$ there exists an exponential $m$ with $m(x) \neq m(y)$. It follows that $\mathcal{A}_C$ is uniformly dense in $\mathcal{C}(C)$, which implies that the finite linear combinations of all exponentials on $G$ form a dense subspace in $\mathcal{C}(G)$. The theorem is proved. □

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