Connected sets of positive solutions of elliptic systems in exterior domains

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Abstract

The existence of infinitely many connected sets of positive solutions for a certain elliptic system is investigated in this paper. We consider semilinear equations with perturbed Laplace operators described in an exterior domain. We show that each of these solutions $\mathbf{u} = (u_1, u_2)$ has the minimal asymptotic decay, namely $u_i(x) = O(\|x\|^{2-n})$ as $\|x\| \rightarrow \infty$, $i = 1, 2$, and finite energy in a neighborhood of infinity. Our main tool is the sub and super-solutions theorem which is based on the Sattinger's iteration procedure. We do not need any growth assumptions concerning nonlinearities.

Keywords Evanescent solutions · Finite energy solutions · Sub and supersolutions methods · Exterior domains

Mathematics Subject Classification 35J47 · 35B09 · 35B40

1 Introduction

The purpose of this paper is to formulate conditions which guarantee the existence of continua of positive solutions of the following system involving perturbed Laplace operators

$$\begin{cases} \operatorname{div}(a_1(\|x\|)\nabla u_1(x)) + f_1(x, u_1(x), u_2(x)) + g_1(x)x \cdot \nabla u_1(x) = 0 \\ \operatorname{div}(a_2(\|x\|)\nabla u_2(x)) + f_2(x, u_1(x), u_2(x)) + g_2(x)x \cdot \nabla u_2(x) = 0 \end{cases} \quad (1)$$

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for $x \in G_R = \{x \in \mathbb{R}^n, \|x\| > R\}$ with $n > 2$, $R > 0$ and $\|x\| := \sqrt{\sum_{i=1}^n x_i^2}$. We are interested in global solutions vanishing at infinity, i.e.

$$\lim_{\|x\| \rightarrow \infty} u_i(x) = 0 \quad \text{for } i = 1, 2, \quad (2)$$

which are often called evanescent solutions.

Many problems modeled by similar systems arise in various areas of applied mathematics, in biological, chemical or physical phenomena, for example in pseudoplastic fluids [9], reaction–diffusion processes, chemical heterogeneous catalysts [4] or heat conduction in electrically conducting materials [18]. Recently the existence and multiplicity of solutions for such elliptic systems considered also in unbounded domains has been widely discussed in the literature (see e.g. [5, 11–14, 23–25, 27, 34, 36] and the references therein).

The existence or nonexistence of radially symmetric solutions for the Emden–Fowler system involving p -Laplace operators and some real parameters was discussed e.g. in [8]. There the approach is based on suitable transformations which play a crucial role in the reduction of the main problem to a quadratic system. In [20] and [21] results concerning more general problems associated with the existence of weak solutions of elliptic inequalities can be found. In Covei's paper [13] the following Lane–Emden–Fowler system is investigated

$$\begin{cases} -\Delta u(x) = a_1(x)F_1(x, u(x), v(x)) \\ -\Delta v(x) = a_2(x)F_2(x, u(x), v(x)) \\ u = v = 0 \text{ in } \partial\Omega \end{cases}$$

for bounded domains $\Omega \subset \mathbb{R}^n$ or $\Omega = \mathbb{R}^n$, in the case when nonlinearities F_1, F_2 are positive and satisfy some growth condition with respect to the second and third variables. The multiplicity of solutions for system of similar form and their additional properties were discussed, among others, in [11]. In that paper the variational approach allowed the authors to show the existence of at least nine solutions in the case when Ω is a bounded regular domain in $\mathbb{R}^n$, the right-hand side is a Carathéodory function and satisfies, among others, some growth conditions (see Th.1.1). Precisely, these solutions $U = (u_1, u_2)$, satisfy the following sign conditions: both u_1 and u_2 are strictly positive or negative in the first four solutions; four others are such that one of the two coordinates is of the one sign while the other is of changing sign, and finally both coordinates change their sign in the ninth solution. There are many results concerning the existence of infinitely many solutions for the case when the system does not depend on ∇u_i . For example, vanishing solutions for Schrödinger–Poisson system (also called Schrödinger–Maxwell equations), was discussed in [24], where an approximation methods were employed. Such systems are motivated by many problems arising in quantum mechanic, in plasma physics, semiconductor theory or nonlinear optics (see e.g. [6, 7, 15–17, 30] and the references therein). Based on variational methods Ambrosetti and Ruiz proved the existence of infinitely many radial solutions for the problem

$$\begin{cases} -\Delta u + u + \lambda vu = |u|^{p-1}u & \text{in } \mathbb{R}^3 \\ -\Delta v = K(x)u^2, \end{cases}$$

when $p \in (2, 5)$ and $\lambda > 0$ is sufficiently large (see [2] and [3]). In the case when $p \in (1, 2)$ and $\lambda > 0$ is small enough, the existence of finite number of such solutions was proved there. In 2011 d'Avenia, Pomponio and Vaira constructed infinitely many nonradial solutions of the above system with the first equation with the function $K(x)$ instead of constant parameter λ for $p \in (1, 5)$ and K decaying at infinity. In [10,26] and [33] the following general problem

$$\begin{cases} -\Delta u + \lambda V(x)u + K(x)vu = f(x, u) & \text{in } \mathbb{R}^3, \\ -\Delta v = K(x)u^2 \end{cases}$$

was considered for the positive potential V . Applying the variant of the fountain theorem the authors obtain infinitely many high-energy solutions for superlinear f . In the sublinear case and $K \equiv 1$, Sun constructed infinitely many small-energy solutions for such problem [32]. Similar multiplicity results for sublinear and odd f and positive nonconstant K was proved in [34]. The authors obtained the existence of finite number of nontrivial solution for superlinear case. The existence of a sequence of solutions decaying at infinity of the above system was obtained in [35] in the case when $K \equiv 1$, $\lambda = 1$, f is odd and satisfies particular estimates. We want to join in the discussion concerning the multiplicity of global solutions for (1) under quite mild conditions concerning nonlinearities. We focus on positive solutions with minimal growth and finite energy in a neighborhood of infinity. The aim of this paper is twofold. On the one hand we want to formulate sufficient conditions for the existence of infinitely many unbounded connected sets of global solutions for our problem. Here we have to emphasize that this results will be obtained without assumptions concerning the oddness or growth of nonlinearity f_i with respect to the second and/or third variables. These condition are often met in the literature. On the other hand we describe precisely the asymptotics of solutions and their gradients.

By a solution of our problem we understand a vector function $\mathbf{u} = (u_1, u_2) \in \left(C_{loc}^{2+\alpha}(G_R)\right)^2 := C_{loc}^{2+\alpha}(G_R) \times C_{loc}^{2+\alpha}(G_R)$ satisfying (1) and (2). Moreover, we say that $\mathbf{u} = (u_1, u_2)$ is a finite energy solution (or a solution with finite energy) in a neighborhood of infinity when there exists a nonnegative radial function $\psi := (\psi_1, \psi_2) \in C^1(\Omega_R)$ with $\psi_i(x) \equiv 1$, $i = 1, 2$, for $\|x\|$ sufficiently large and such that $\psi_i u_i \in D_0^{1,2}(\Omega_R)$, where $D_0^{1,2}(\Omega_R)$ denotes the completion of $C_0^\infty(\Omega_R)$ in the norm $\|\varphi\| := \|\nabla \varphi\|_{L^2(\Omega_R)}$ (see, among others, e.g. [27]).

Here we also recall standard definitions of a super-solution and a sub-solution of (1)–(2) (see e.g. [25]). Precisely, by a super-solution of (1)–(2) in G_R we understand a vector function $\bar{\mathbf{u}} = (\bar{u}_1, \bar{u}_2) \in \left(C_{loc}^{2+\alpha}(G_R)\right)^2$ satisfying the following differential inequalities

$$\begin{cases} \operatorname{div}(a_1(\|x\|)\nabla \bar{u}_1(x)) + f_1(x, \bar{u}_1(x), \bar{u}_2(x)) + g_1(x) \cdot \nabla \bar{u}_1(x) \leq 0, & x \in G_R, \\ \operatorname{div}(a_2(\|x\|)\nabla \bar{u}_2(x)) + f_2(x, \bar{u}_1(x), \bar{u}_2(x)) + g_2(x) \cdot \nabla \bar{u}_2(x) \leq 0, & x \in G_R, \\ \lim_{\|x\| \rightarrow \infty} \bar{u}_i(x) = 0 \text{ for } i = 1, 2. \end{cases}$$

Analogously, as for a sub-solution $\underline{\mathbf{u}} = (\underline{u}_1, \underline{u}_2)$ of (1)–(2) in G_R , the sign of the inequality should be reversed.

To show the existence of continua of positive solutions with finite energy we assume the following hypotheses

- (H1) $g_i : G_{l_0} \rightarrow (0, +\infty)$, where $1 < l_0 \leq R$, is locally Hölder continuous with exponent $\alpha \in (0, 1)$, $i = 1, 2$;
 (H2) $a_i : [1, +\infty) \rightarrow (c_1, c_2)$ belongs to $C^{1+\alpha}([1, +\infty))$, where $0 < c_1 < c_2$, $i = 1, 2$;
 (H3) f_1, f_2 are continuously differentiable in $G_R \times (\alpha_1, \beta_1) \times (\alpha_2, \beta_2)$, where $\alpha_i < 0 < \beta_i$, $i = 1, 2$,
 (a) $f_1(x, 0, 0), f_2(x, 0, 0)$ are positive in G_R ,
 (b) f_1 is nondecreasing in u_2 and f_2 is nondecreasing in u_1 ;
 (c) for each $i = 1, 2$, there exists $d_i \in (0, \beta_i)$, $m_i < 4(n-2)d_i c_1^2 c_2^{-1}$ such that for all $r \in [1, +\infty)$,

$$\sup_{(u_1, u_2) \in [0, d_1] \times [0, d_2]} \sup_{\|x\|=r} f_i(x, u_1, u_2) \leq m_i r^{2-2n}$$

and there exist $K_1, K_2 > 1$ such that the following inequality holds

$$\left[\frac{1}{2} \left(1 - K_i^{2-n} \right)^2 \frac{1}{K_i^2} \right]^{-\frac{1}{n-2}} \leq R. \quad (3)$$

It is worth emphasizing that we need the monotonicity and differentiability of f_i only on some right-hand side neighborhood of the origin. Moreover, we can consider both sublinear and superlinear f_i . It is associated with the fact that we have to control only the value of nonlinearities $f_i(x, \cdot, \cdot)$ in the rectangle $[0, d_1] \times [0, d_2]$. Thus we can omit growth conditions concerning variable $\mathbf{u} = (u_1, u_2)$. Our main tool is Theorem 1 (given below) which says that the existence of a sub-solution $\underline{\mathbf{u}} = (\underline{u}_1, \underline{u}_2)$ and a super-solution $\bar{\mathbf{u}} = (\bar{u}_1, \bar{u}_2)$ of our problem such that $0 \leq \underline{u}_i \leq \bar{u}_i \leq d_i$ for $i = 1, 2$, implies the existence of a solution $\mathbf{u} = (u_1, u_2)$ of (1) which is squeezed between them. The proof of Theorem 1 is based on classical ideas associated with the Sattinger's monotone iteration procedure (see [31]) and the approach described by Kawano [25] for the system which does not contain the gradient term. The similar theorem was also proved in the [29] (Th. 1.2) but only for the special case when $a_i \equiv 1$, $i = 1, 2$. Therefore we can not use it directly. For the reader's convenience we present the proof of this result in the "Appendix".

Now we formulate precisely the theorem which will be our main tool in the proof of the existence result. The proof of Theorem 1 is given at the end of the paper (in "Appendix").

Theorem 1 Assume that (H1), (H2), (H3) hold and $\bar{\mathbf{u}} = (\bar{u}_1, \bar{u}_2)$ and $\underline{\mathbf{u}} = (\underline{u}_1, \underline{u}_2)$ are, respectively, a positive super-solution and a nonnegative sub-solution of (1)–(2) such that

$$\underline{u}_i \leq \bar{u}_i \leq d_i \text{ for } i = 1, 2 \text{ and all } x \in \bar{G}_R.$$

Then there exists a vector function $\mathbf{u} = (u_1, u_2) \in \left(C_{loc}^{2+\alpha}(G_R)\right)^2$ satisfying (1)–(2) and such that for $i = 1, 2$, $\underline{u}_i \leq u_i \leq \bar{u}_i$ in $\bar{G}_R$ and $u_i(x) = \bar{u}_i(x)$ on ∂G_R .

This approach allows us to prove two main results: the first one is associated with the existence of sequences of uncountable sets of solutions and the other one gives additional information concerning the asymptotics of solutions and their gradients.

Theorem 2 Suppose that (H1), (H2) and (H3) hold. Then there exist infinitely many connected sets of positive solutions $\{(u_1^{q,l}, u_2^{q,l}), q \in \mathbb{R}\}_{l \in \mathbb{N}}$ of (1)–(2), where $\mathbb{N} := \{1, 2, 3, \dots\}$. For all $q \in \mathbb{R}$, the sequence $\{(u_1^{q,l}, u_2^{q,l})\}_{l \in \mathbb{N}}$ is nondecreasing in G_R and increasing on ∂G_R (with respect to each coordinate). Moreover

1. there exists at least one sequence of continua $\{S_l^0\}_{l \in \mathbb{N}}$ consisting of all solutions $(u_1^{q,l}, u_2^{q,l})$ such that both coordinates $u_1^{q,l}$ and $u_2^{q,l}$ have different value on ∂G_R , i.e.

$$S_l^0 := \{(u_1^{q,l}, u_2^{q,l}), q \in \mathbb{R} \text{ such that } u_1^{q,l} \neq u_2^{q,l} \text{ on } \partial G_R\} \text{ for all } l \in \mathbb{N},$$

2. there exists at least one sequence of continua $\{S_l^1\}_{l \in \mathbb{N}}$ consisting of all solutions $(u_1^{q,l}, u_2^{q,l})$ such that both coordinates $u_1^{q,l}$ and $u_2^{q,l}$ have the same value on ∂G_R , i.e.

$$S_l^1 := \{(u_1^{q,l}, u_2^{q,l}); q \in \mathbb{R} \text{ such that } u_1^{q,l} = u_2^{q,l} \text{ on } \partial G_R\} \text{ for all } l \in \mathbb{N},$$

3. each of such sequences $\{(u_1^{q,l}, u_2^{q,l})\}_{l \in \mathbb{N}}$ generates another solution $(\tilde{u}_1^q, \tilde{u}_2^q)$ of (1)–(2) and, in consequence, $\{(\tilde{u}_1^q, \tilde{u}_2^q), q \in \mathbb{R}\}$ is also a connected set of positive solutions of (1)–(2).

Theorem 3 Under hypotheses (H1), (H2) and (H3), solutions $(u_1^{q,l}, u_2^{q,l})$, where $q \in \mathbb{R}$ and $l \in \mathbb{N}$, described in Theorem 2, satisfy the following conditions:

1. $(u_1^{q,l}, u_2^{q,l})$ has finite energy in neighborhood of infinity,
2. for $i \in \{1, 2\}$,

$$u_i^{q,l}(x) = O\left(\frac{1}{\|x\|^{n-2}}\right) \text{ as } \|x\| \rightarrow +\infty \quad (4)$$

(which means that $(u_1^{q,l}, u_2^{q,l})$ has the minimal growth) and for all $\phi \in C^1(1, +\infty)$ such that $\lim_{r \rightarrow +\infty} \phi(r) = 0$ and $\lim_{r \rightarrow +\infty} \phi'(r)r^{n-1} = +\infty$,

$$u_i^{q,l}(x) = o(\phi(\|x\|)) \text{ as } \|x\| \rightarrow +\infty. \quad (5)$$

2 Existence of continua of super-solutions

Taking into account the fact that $f(x, 0, 0) \geq 0$ in Ω_R we obtain the trivial sub-solution $\underline{\mathbf{u}} \equiv (0, 0)$ of (1)–(2). Therefore it suffices to construct an increasing sequences of

positive super-solutions for (1)–(2). To this effect we consider a sequence of auxiliary linear problems. It appears that their radial positive solutions give us a sequence of super-solutions for our main problem. Let $q \in \mathbb{R}$ and $l \in \mathbb{N}$ be fixed. We consider the following two independent linear problems

$$\begin{cases} -\operatorname{div}(a_i(|x|))\nabla \bar{u}_i(x) = \tilde{f}_i^{q,l}(|x|), \text{ for } x \in \Omega_1, \\ \bar{u}_i(x) = 0 \text{ for } |x| = 1, \quad \lim_{|x| \rightarrow \infty} \bar{u}_i(x) = 0, \quad i = 1, 2, \end{cases} \quad (6)$$

where $i = 1, 2$,

$$\tilde{f}_i^{q,l}(r) := \frac{1}{r^{2n}} \left(r^2 m_i + p_i^{q,l} \right)$$

and

$$p_i^{q,l} = \left(4(n-2)d_i c_1^2 c_2^{-1} - \frac{m_i}{n-2} \right) \left(1 - \frac{q}{q+1} \frac{1}{l} \right)$$

for all $l \in \mathbb{N}$, $i = 1, 2$. Now we describe some properties of $\tilde{f}_i^{q,l}$ and $p_i^{q,l}$.

Remark 1 For each $q \in \mathbb{R}$ and $i = 1, 2$,

1. $\{p_i^{q,l}\}_{l \in \mathbb{N}}$ is an increasing sequence of positive numbers,
2. for all $l \in \mathbb{N}$, the following inequality holds

$$\int_1^\infty r^{n-1} \tilde{f}_i^{q,l}(r) dr < 4(n-2)d_i c_1^2 c_2^{-1}, \quad i = 1, 2, \quad (7)$$

3. there exists $K_i > 1$ such that for all $l \in \mathbb{N}$, the following inequality holds

$$R^{2-n} \leq \frac{1}{2} \left(1 - K_i^{2-n} \right)^2 \left[\frac{\inf_{r \in [1, K_i]} r^{2n-2} \tilde{f}_i^{q,l}(r)}{\sup_{r \in [1, +\infty)} r^{2n-2} \tilde{f}_i^{q,l}(r)} \right]. \quad (8)$$

Proof Taking into account the definition of $p_i^{q,l}$ we get that $\{p_i^{q,l}\}_{l \in \mathbb{N}} \subset (0, +\infty)$ and it is an increasing sequence. Moreover, (H3) part (c) implies for all $i \in \{1, 2\}$, the following chain of inequalities

$$\int_1^\infty r^{n-1} \tilde{f}_i^{q,k}(r) dr < \frac{m_i}{n-2} + 4(n-2)d_i c_1^2 c_2^{-1} - \frac{m_i}{n-2} = 4(n-2)d_i c_1^2 c_2^{-1}.$$

Coming to the last part let us note that

$$\sup_{r \in [1, +\infty)} \left(r^{2n-2} \tilde{f}_i^{q,l}(r) \right) = \sup_{r \in [1, +\infty)} \left(m_i + \frac{p_i^{q,l}}{r^2} \right) = m_i + p_i^{q,l}$$

and

$$\inf_{r \in [1, K_i]} \left(r^{2n-2} \tilde{f}_i^{q,k}(r) \right) = \inf_{r \in [1, K_i]} \left(m_i + \frac{p_i^{q,l}}{r^2} \right) = m_i + \frac{p_i^{q,l}}{K_i^2}.$$

Thus, by **(H3)** part **(c)**, $\left[\frac{1}{2} \left(1 - K_i^{2-n} \right)^2 \frac{1}{K_i^2} \right] \geq R^{2-n}$ and further

$$\begin{aligned} R^{2-n} \sup_{r \in [1, +\infty)} \left(r^{2n-2} \tilde{f}_i^{q,l}(r) \right) &\leq \frac{1}{2} \left(1 - K_i^{2-n} \right)^2 \frac{1}{K_i^2} \left(m_i + p_i^{q,l} \right) \\ &\leq \frac{1}{2} \left(1 - K_i^{2-n} \right)^2 \inf_{r \in [1, K_i]} \left(r^{2n-2} \tilde{f}_i^{q,k}(r) \right) \end{aligned}$$

which gives (8). $\square$

Applying the transformation given by diffeomorphism $\psi : [0, 1) \rightarrow [1, +\infty)$, where $\psi(t) = (1 - t)^{\frac{1}{2-n}}$, we use the well-known fact that the solution $(\bar{u}_1^{q,l}, \bar{u}_2^{q,l})$ of (6) can be described as $\bar{u}_i^{q,l}(x) = z_i^{q,l}(\psi^{-1}(|x|))$ for $i = 1, 2$, namely $\bar{u}_i^{q,k}(x) = z_i^{q,k}(1 - |x|^{2-n})$, where $z_i^{q,l}, i = 1, 2$, are solutions of the following two independent Dirichlet problems with singularity at the end-point 1

$$\begin{cases} -(\tilde{a}_i(t)z_i'(t))' = h_i^{q,l}(t), & \text{in } (0, 1) \\ z_i(0) = z_i(1) = 0, \end{cases} \quad (9)$$

with

$$\begin{aligned} h_i^{q,l}(t) &= \frac{1}{(n-2)^2} (1-t)^{\frac{2n-2}{2-n}} \tilde{f}_i^{q,l}((1-t)^{\frac{1}{2-n}}), \\ \tilde{a}_i(t) &= a_i((1-t)^{\frac{1}{2-n}}). \end{aligned}$$

Remark 2 Taking into account assumptions made on f_i we state that for each $i = 1, 2$, $h_i^{q,l}$ satisfies the following conditions

1. for all $l \in \mathbb{N}$ and $q \in \mathbb{R}$, $h_i^{q,l}$ is continuous, $h_i^{q,l}(\cdot) > 0$, $h_i^{q,l} < h_i^{q,l+1}$ and

$$\int_0^1 h_i^{q,l}(r) dr \leq 4d_i c_1^2 c_2^{-1}, \quad (10)$$

2. there exists $0 < \varepsilon_i < 1$ such that for all $l \in \mathbb{N}$ and $q \in \mathbb{R}$

$$(1 - \bar{t}) \left(\sup_{t \in [0, 1)} h_i^{q,l}(t) \right) \leq \frac{1}{2} \varepsilon_i^2 \left(\inf_{t \in [0, \varepsilon_i]} h_i^{q,l}(t) \right), \quad (11)$$

where $\bar{t} := 1 - R^{2-n} \in (0, 1)$.

Lemma 4 For each $q \in \mathbb{R}$ and $l \in \mathbb{N}$ there exists a super-solution $(\bar{u}_1^{q,l}, \bar{u}_2^{q,l})$ of our problem such that

$$\bar{u}_i^{q,l} < \bar{u}_i^{q,l+1} \text{ in } G_R, \quad i = 1, 2 \quad (12)$$

and for all $l \in \mathbb{N}$ and $q_1, q_2 \in \mathbb{R}$

$$\begin{aligned} &\text{if } q_1 > q_2 \text{ then } \bar{u}_i^{q_1, l} < \bar{u}_i^{q_2, l}, \text{ in } G_R, \quad i = 1, 2, \\ &\text{if } q_1 < q_2 \text{ then } \bar{u}_i^{q_1, l} > \bar{u}_i^{q_2, l}, \text{ in } G_R, \quad i = 1, 2. \end{aligned} \quad (13)$$

Precisely, for each $q \in R$ there exist two types of sequences of super-solutions $\{(\bar{u}_1^{q, l}, \bar{u}_2^{q, l})\}_{l \in \mathbb{N}}$: in the first one for all $l \in \mathbb{N}$, $\bar{u}_1^{q, l} \equiv \bar{u}_2^{q, l}$ in G_R , in the other one for all $l \in \mathbb{N}$, $\bar{u}_1^{q, l} \neq \bar{u}_2^{q, l}$.

Moreover for each $q \in R$ and $l \in \mathbb{N}$ the following assertions hold

$$\bar{u}_i^{q, l}(x) = O\left(\frac{1}{\|x\|^{n-2}}\right) \text{ as } \|x\| \rightarrow +\infty, \quad i = 1, 2 \quad (14)$$

and

$$\bar{u}_i^{q, l}(x) = o(\phi(\|x\|)) \text{ as } \|x\| \rightarrow +\infty, \quad i = 1, 2 \quad (15)$$

for all $\phi \in C^1(1, +\infty)$ such that $\lim_{r \rightarrow +\infty} \phi(r) = 0$ and $\lim_{r \rightarrow +\infty} \phi'(r)r^{n-1} = +\infty$.

Proof We start with the solutions of the sequence of Dirichlet problems (9). To this end we fix $q \in \mathbb{R}$ and $l \in \mathbb{N}$. It is obvious that taking

$$z_i^{q, l}(t) = \int_0^1 \mathbf{G}_i(s, t) h_i^{q, l}(s) ds, \quad (16)$$

where

$$\mathbf{G}_i(s, t) := \frac{1}{c_i} \begin{cases} \int_0^s \frac{1}{\bar{a}_i(r)} dr \int_t^1 \frac{1}{\bar{a}_i(r)} dr & \text{for } 0 \leq s \leq t \\ \int_t^1 \frac{1}{\bar{a}_i(r)} dr \int_s^1 \frac{1}{\bar{a}_i(r)} dr & \text{for } t < s \leq 1, \end{cases}$$

with $c_i := \int_0^1 \frac{1}{\bar{a}_i(s)} dr$, we obtain the solution of (9). By the definition of $\mathbf{G}_i$, we get for all $t \in [0, 1]$,

$$0 \leq z_i^{q, l}(t) = \int_0^1 \mathbf{G}_i(s, t) h_i^{q, l}(s) ds \leq \frac{1}{c_i} \frac{1}{c_1^2} t(1-t) \int_0^1 h_i^{q, l}(s) ds \leq d_i.$$

Let us note that for all $l \in \mathbb{N}$, $i = 1, 2$, $h_i^{q, l}(t) < h_i^{q, l+1}(t)$ for all $t \in (0, 1)$, which implies

$$-\left[a_i(t) \left(\left(z_i^{q, l+1} \right)'(t) - \left(z_i^{q, l} \right)'(t) \right) \right]' = h_i^{q, l+1}(t) - h_i^{q, l}(t) > 0.$$

The above assertion and the boundary condition $z_i^{q,l}(0) = z_i^{q,l+1}(0) = z_i^{q,l}(1) = z_i^{q,l+1}(1) = 0$ lead to the inequality

$$z_i^{q,l}(t) < z_i^{q,l+1}(t) \text{ in } (0, 1). \quad (17)$$

Applying the same reasoning we obtain for all $l \in \mathbb{N}$ and $q_1, q_2 \in \mathbb{R}$,

$$\begin{aligned} \text{if } q_1 > q_2 \text{ then } z_i^{q_1,l} &< z_i^{q_2,l}, \quad i = 1, 2, \text{ in } G_R \\ \text{if } q_1 < q_2 \text{ then } z_i^{q_1,l} &> z_i^{q_2,l}, \quad i = 1, 2, \text{ in } G_R. \end{aligned} \quad (18)$$

Taking into account the facts that the unique solution $z_i^{q,l}$ of (9) is nontrivial and concave, we can state that $z_i^{q,l}$ is positive in $(0, 1)$. Moreover we get

$$(z_i^{q,l})'(t) \leq 0 \text{ for all } t \in [\bar{t}, 1). \quad (19)$$

Indeed, owing to (11) we obtain the following chain of assertions

$$\begin{aligned} (z_i^{q,l})'(\bar{t}) &\leq - \left(\inf_{s \in (0, \varepsilon_i)} h_i^{q,l}(s) \right) \int_0^{\varepsilon_i} s ds + \int_{\bar{t}}^1 \left(\sup_{s \in (0, 1)} h_i^{q,l}(s) \right) ds \\ &= -\frac{1}{2} \varepsilon_i^2 \left(\inf_{s \in (0, \varepsilon_i)} h_i^{q,l}(s) \right) + (1 - \bar{t}) \left(\sup_{s \in (0, 1)} h_i^{q,l}(s) \right) \leq 0. \end{aligned}$$

Moreover, we can describe the behavior of $z_i^{q,l}$ as $t \rightarrow 1^-$. Since $(z_i^{q,l})'(t) = -\int_0^1 s h_i^{q,l}(s) ds + \int_t^1 h_i^{q,l}(s) ds$ we get

$$(0, +\infty) \ni M_i^{q,l} := \int_0^1 s h_i^{q,l}(s) ds = -\lim_{t \rightarrow 1^-} (z_i^{q,l})'(t) = \lim_{t \rightarrow 1^-} \frac{z_i^{q,l}(t)}{(1-t)} \quad (20)$$

and further

$$z_i^{q,l}(t) = O(1-t) \text{ for } t \rightarrow 1^-. \quad (21)$$

Now we consider the vector function $(\bar{u}_1^{q,l}, \bar{u}_2^{q,l})$, where $\bar{u}_i^{q,l}(x) = z_i^{q,l}(1 - \|x\|^{2-n})$ in G_{R_i} which is a positive radial solution of (6). Taking into account (20) we obtain

$$\lim_{\|x\| \rightarrow \infty} \frac{\bar{u}_i^{q,l}(x)}{\|x\|^{2-n}} = M_i^{q,l} \in (0, +\infty)$$

which implies (14). To prove (15) we take an arbitrary $\phi \in C^1(1, +\infty)$ such that $\lim_{r \rightarrow +\infty} \phi(r) = 0$ and $\lim_{r \rightarrow +\infty} \phi'(r)r^{n-1} = +\infty$. Then function $\psi(t) = \phi((1-t)^{\frac{1}{2-n}})$ satisfies conditions: $\lim_{t \rightarrow 1^-} \psi(t) = 0$ and $\lim_{t \rightarrow 1^-} \psi'(t) = +\infty$. It is clear that $\lim_{t \rightarrow 1^-} \frac{(z_i^{q,l})'(t)}{\psi'(t)} = 0$. Thus, applying again the de L'Hospital's rule, we obtain $\lim_{t \rightarrow 1^-} \frac{z_i^{q,l}(t)}{\psi(t)} = 0$ and further

$$\begin{aligned}
\lim_{||x|| \rightarrow +\infty} \frac{\bar{u}_i^{q,l}(x)}{\phi(||x||)} &= \lim_{||x|| \rightarrow +\infty} \frac{z_i^{q,l}(1-||x||^{2-n})}{\phi(||x||)} = \lim_{t \rightarrow 1^-} \frac{z_i^{q,l}(t)}{\phi((1-t)^{\frac{1}{2-n}})} \\
&= \lim_{t \rightarrow 1^-} \frac{z_i^{q,l}(t)}{\psi(t)} = 0
\end{aligned}$$

for $i = 1, 2$, which gives (15). $\square$

3 Proof of the main results

Now we can prove our main result concerning the existence of sequences of connected sets of positive evanescent solutions of (1). The proof of this theorem is based on the sub-solution and super-solutions method described in Theorem 1 and the iteration process applied in [28].

Proof (of Theorem 2) Suppose that (H1), (H2), (H3) hold. Let us fix $q \in \mathbb{R}$. Lemma 4 leads to the existence of a nondecreasing (with respect to each coordinate) sequence $\{(\bar{u}_1^{q,l}, \bar{u}_2^{q,l})\}_{l \in \mathbb{N}}$ of super-solutions of (1)–(2). Let us consider trivial vector function $(0, 0)$ being a sub-solution of (1)–(2) and its super-solution $(\bar{u}_1^{q,1}, \bar{u}_2^{q,1})$. Theorem 1 gives the existence of solution $(u_1^{q,1}, u_2^{q,1})$ for our problem such that for $i = 1, 2$,

$$0 \leq u_i^{q,1} \leq \bar{u}_i^{q,1} \leq d_i \text{ in } G_R \text{ and } u_i^{q,1} = \bar{u}_i^{q,1} \text{ on } \partial G_R.$$

Now we treat the above solution $(u_1^{q,1}, u_2^{q,1})$ as a sub-solution of our problem. Since we have super-solution $(\bar{u}_1^{q,2}, \bar{u}_2^{q,2})$ satisfying the following chain of inequalities for $i = 1, 2$,

$$u_i^{q,1} \leq \bar{u}_i^{q,1} < \bar{u}_i^{q,2} \text{ in } G_R,$$

we can apply again Theorem 1 and derive the existence of a solution $(u_1^{q,2}, u_2^{q,2})$ of (1)–(2) such that for $i = 1, 2$,

$$\begin{aligned}
u_i^{q,1} &\leq u_i^{q,2} \leq \bar{u}_i^{q,2} \leq d_i \text{ in } G_R, \\
u_i^{q,2} &= \bar{u}_i^{q,2} \text{ on } \partial G_R.
\end{aligned}$$

The last equality and the fact that $\bar{u}_i^{q,1} < \bar{u}_i^{q,2}$ on ∂G_R imply that for $i = 1, 2$, $u_i^{q,1} \neq u_i^{q,2}$. Iterating this process and having constructed solution $(u_1^{q,l}, u_2^{q,l})$ of (1)–(2) we obtain the existence of another solution $(u_1^{q,l+1}, u_2^{q,l+1})$ such that

$$\begin{aligned}
u_i^{q,l} &\leq u_i^{q,l+1} \leq \bar{u}_i^{q,l+1} \leq d_i \text{ in } G_R, \\
u_i^{q,l+1} &= \bar{u}_i^{q,l+1} \text{ on } \partial G_R
\end{aligned}$$

for $i = 1, 2$. Since $\bar{u}_i^{q,l} < \bar{u}_i^{q,l+1}$ on ∂G_R we obtain for $i = 1, 2$, $u_i^{q,l} \neq u_i^{q,l+1}$. Finally we obtain a sequence of bounded solutions $\{(u_1^{q,l}, u_2^{q,l})\}_{l \in \mathbb{N}}$ which is nonde-

creasing in G_R and increasing on ∂G_R . Since for all $q \in \mathbb{R}$, $l \in \mathbb{N}$, and $i = 1, 2$, $\lim_{\|x\| \rightarrow \infty} u_i^{q,l}(x) = 0$ we state that we get evanescent solutions.

Let us consider the case when for all $l \in \mathbb{N}$, we take $\tilde{f}_1^{q,l} \neq \tilde{f}_2^{q,l}$, which implies $\bar{u}_1^{q,l} \neq \bar{u}_2^{q,l}$. Then $u_1^{q,l} \neq u_2^{q,l}$ on ∂G_R and we can obtain the first sequence of continua $\{S_l^0\}_{l \in \mathbb{N}}$, where

$$S_l^0 := \{(u_1^{q,l}, u_2^{q,l}), q \in \mathbb{R} \text{ such that } u_1^{q,l} \neq u_2^{q,l} \text{ on } \partial G_R\}$$

for all $l \in \mathbb{N}$. In the case when we consider $\tilde{f}_1^{q,l} \equiv \tilde{f}_2^{q,l}$ for all $l \in \mathbb{N}$, we get $\bar{u}_1^{q,l} \equiv \bar{u}_2^{q,l}$ in G_R and further we obtain $u_1^{q,l} = u_2^{q,l}$ on ∂G_R which allows us to construct the other sequence of continua $\{S_l^1\}_{l \in \mathbb{N}}$, where

$$S_l^1 := \{(u_1^{q,l}, u_2^{q,l}); q \in \mathbb{R} \text{ such that } u_1^{q,l} = u_2^{q,l} \text{ on } \partial G_R\} \text{ for all } l \in \mathbb{N}.$$

Coming to the proof of the last part of Theorem 2 we fix $q \in \mathbb{R}$ and consider the sequence $\{(u_1^{q,l}, u_2^{q,l})\}_{l \in \mathbb{N}}$ of solutions. Based on the classical estimate for solutions of elliptic PDE (see e.g. [27]-Lemma 3.2) in the following annulus $\Omega_{j,R} := \{x \in \mathbb{R}^n, R + \frac{1}{2j} < \|x\| < R + j\}$, $j \in \mathbb{N}$, we state the existence of $D > 0$ independent of l such that

$$\|u_i^{q,l}\|_{C^{2,\alpha}(\bar{\Omega}_{j,R})} \leq D \text{ for all } j \geq 1.$$

The compactness of the injection $C^{2,\alpha}(\bar{\Omega}_{1,R}) \rightarrow C^2(\bar{\Omega}_{1,R})$, gives the existence of a subsequence $\{(u_1^{q,l}, u_2^{q,l})\}_{l \in \mathbb{N}}$ which tends to $(\tilde{u}_1^{1,q}, \tilde{u}_2^{1,q})$ in $\bar{\Omega}_{1,R}$ in the $(C^2(\bar{\Omega}_{1,R}))^2$ norm. Therefore $(\tilde{u}_1^{1,q}, \tilde{u}_2^{1,q})$ is also a solution of (1) in $\bar{\Omega}_{1,R}$. It is clear that the subsequence $\{(u_1^{q,l_k}, u_2^{q,l_k})\}_{k \in \mathbb{N}}$, also satisfies the above estimate, and consequently there exists a subsequence of $\{(u_1^{q,l_k}, u_2^{q,l_k})\}_{k \in \mathbb{N}}$ which converges in the $(C^2(\bar{\Omega}_{2,R}))^2$ norm to $(\tilde{u}_1^{2,q}, \tilde{u}_2^{2,q})$, which is a solution of (1) in $\bar{\Omega}_{2,R}$ and for $i = 1, 2$, $\tilde{u}_i^{2,q} = \tilde{u}_i^{1,q}$ in $\bar{\Omega}_{1,R}$. Iterating this schema, we can construct inductively a sequence $\{(\tilde{u}_1^{j,q}, \tilde{u}_2^{j,q})\}_{j \in \mathbb{N}}$ of solutions of (1) in $\bar{\Omega}_{j,R}$ such that $\tilde{u}_i^{j+1,q} = \tilde{u}_i^{j,q}$ in $\bar{\Omega}_{j,R}$. This property allows us to consider a vector function $(\tilde{u}_1^q, \tilde{u}_2^q)$ given as follows

$$\tilde{u}_i^q := \tilde{u}_i^{j,q} \text{ in } \Omega_{j,R} \text{ for all } j \geq 1, i = 1, 2$$

and state that it satisfies (1). Let us consider an arbitrary bounded set $\bar{M} \subset \Omega_R$. It is clear that there exists $j \in \mathbb{N}$ such that $\bar{M} \subset \Omega_{j,R}$. Applying the above results for $\Omega_{j,R}$ we derive that $(\tilde{u}_1^q, \tilde{u}_2^q) \in (C^2(\bar{M}))^2$. Now, the regularity arguments associated with the Schauder's estimates imply that $(\tilde{u}_1^q, \tilde{u}_2^q) \in (C^{2,\alpha}(\bar{M}))^2$, consequently, $(\tilde{u}_1^q, \tilde{u}_2^q) \in (C_{loc}^{2,\alpha}(\Omega_R))^2$ and satisfies (1)–(2). Finally, the properties of $(u_1^{q,l}, u_2^{q,l})$ allowed us to prove that $(\tilde{u}_1^q, \tilde{u}_2^q)$ is a positive evanescent solution of (1). $\square$

Our task is now to prove the other result concerning the asymptotics of solutions and their gradients. Here we employee the ideas described in [19,27,28].

Proof (of Theorem 3) Suppose that **(H1)**, **(H2)**, **(H3)** hold. Let us fix $q \in \mathbb{R}, l \in \mathbb{N}$, and consider the solution $(u_1^{q,l}, u_2^{q,l})$ given in Theorem 2. We prove that each $(u_1^{q,l}, u_2^{q,l})$ has finite energy applying the standard approach (see e.g. [27]). To this end we consider $x \in \Omega_R$ such that $\|x\| \geq 2R$, and a ball $B(x; r/2)$ of center x and radius $r/2$, where $r = \|x\|$. Then, applying (14) and the estimates for solutions of elliptic problems ([22], Th. 6.2), we obtain the existence of $L_i^{q,l} > R$, $C_i^{q,l} > 0$ such that for all $x \in \mathbb{R}^n$, $\|x\| > L_i^{q,l}$,

$$\begin{aligned} \frac{r}{2} |\nabla u_i^{q,l}(x)| &\leq C_i^{q,l} \left(\|u_i^{q,l}\|_{C(B(x;r/2))} + 3 \left(\frac{r}{2}\right)^2 \|f_i^{q,l}\|_{C(B(x;r/2))} \right) \\ &\leq C_i^{q,l} \left(\|u_i^{q,l}\|_{C(B(x;r/2))} + \frac{3m}{4} \right) r^{2-n}, \end{aligned}$$

where $m := \max\{m_1, m_2\}$. Finally, taking into account (14), we obtain the existence of $M_i^{q,l} > 0$ and $L_i^{q,l} > 1$ such that for all $x \in \mathbb{R}^n$, $\|x\| > L_i^{q,l}$,

$$|\nabla u_i^{q,l}(x)| \leq C_i^{q,l} \left(\frac{3M_i^{q,l}}{2} + \frac{3m}{4} \right) \|x\|^{1-n}.$$

According to the definition given in the first section, this assertion implies that $(u_1^{q,l}, u_2^{q,l})$ has finite energy in a neighborhood of infinity.

Assertions (4) and (5) are a simple consequence of (14) and (15). $\square$

4 Final remarks and examples

Employing the Kawano's approach (see [25]) we can obtain the same conclusion as in Theorem 2 also in the case when condition (b) in **(H3)** is replaced by the following condition

(b') f_1 is nonincreasing in u_2 and f_2 is nonincreasing in u_1 in $G_R \times [0, d_1] \times [0, d_2]$.

Then in the proof of Theorem 1 we have to choose different starting point of the monotone procedure and consider a super-subsolution $(\bar{u}_1, \bar{u}_2)$ of our problem instead of super-solution. In this case we can also prove the existence of sequences of connected sets of solutions.

Now two natural questions appear. The first one concerns the existence of functions for which **(H3)** holds. It turns out that it is easy to find many examples of f_1, f_2 satisfying **(H3)** among functions of the form $f_i(x, u_1, u_2) = \bar{f}_i(u_1, u_2) (\|x\|^q + k(x))^{-1}$, where $q > 2n - 2$, k is positive and sufficiently smooth and $\bar{f}_i$ is a polynomial, exponential or rational function or their combinations, e.g. $\bar{f}_i(u_1, u_2) = c(u_1^5 + u_1^4 + (u_1 + u_2)^2 + 1)$ or $\bar{f}_i(u_1, u_2) = c(e^{u_1+u_2} + \frac{u_1^3+u_2^3}{(4-u_1)(5-u_2)})$. With help of our approach we can also investigate problems of the Emden-Fowler type when $\bar{f}_i(u_1, u_2) = c(u_1^\alpha + u_2^\beta + M)$ with $\alpha, \beta, M > 0$. At the end of this paper an example of the problem with superlinear $f_i(x, \cdot, \cdot)$ will be discussed more precisely.

The other question is associated with R . Because of condition (3), the radius R has to be sufficiently large. For $n = 3$ we can consider Ω_R being the complement of large balls because in this case R has to be greater than or equal to 32. But for $n = 4$, we can consider Ω_R with all $R \geq 3.6742$. It is easy to note that for high dimension n , radius R can be close to 1. This fact is associated with the behavior of each function

$t_n(k) = \left[\frac{2k^{2n-2}}{(k^{n-2}-1)^2} \right]^{\frac{1}{n-2}}$ in $(1, +\infty)$, where $n \in \mathbb{N}$. We can note that for each $n \in \mathbb{N}$ the minimal value of $t_n(\cdot)$ in $(1, +\infty)$ is attained at $k_n = \sqrt[n-2]{n-1}$ and then

$$\lim_{n \rightarrow \infty} t_n(k_n) = \lim_{n \rightarrow \infty} \left[\frac{2(n-1)(n-1)^{\frac{2}{n-2}}}{(n-2)^2} \right]^{\frac{1}{n-2}} = 1.$$

Condition (H3) part (c) can be rewritten as follows $R > t_n(k_n)$. Below is the table with values $t_n(k_n)$ for higher n

n	5	10	50	100	200	400
$t_n(k_n)$	2.0770	1.2029	1.0189	1.0083	1.0038	1.0018

Example 1 Let us consider the following system

$$\begin{cases} \operatorname{div}(a_1(|x|)u_1(x)) + \frac{1}{2}(2^{u_1(x)+u_2(x)} + u_1^2(x) + u_2^2(x))(|x|^6 + 1)^{-1} \\ \quad + g_1(x)x \cdot \nabla u_1(x) = 0, \\ \operatorname{div}(a_2(|x|)u_2(x)) + \frac{u_1^5(x)+u_2^3(x)+1}{(3-u_1(x))(7-u_2(x))}(|x|^8 + 1)^{-1} + g_2(x)x \cdot \nabla u_2(x) = 0 \\ \lim_{|x| \rightarrow \infty} u_i(x) = 0 \text{ for } i = 1, 2, \end{cases} \quad (22)$$

for $x \in G_R$, where $G_R = \{x \in \mathbb{R}^3, |x| > R\}$ and for $i = 1, 2$, $g_i(x) = \frac{1}{|x| - \frac{1}{2} \sin x_i}$, $a_i(|x|) = 1 + \frac{|x|^4}{x_i^2 + |x|^4}$. We show that Theorems 2 and 3 can be applied for the system given above. To this end we note that for g_i and a_i , $i = 1, 2$, (H1) and (H2) hold with $c_1 = 1$, $c_2 = 2$. Since, in our case

$$f_1(x, u_1, u_2) = \frac{1}{2}(2^{u_1+u_2} + u_1^2 + u_2^2) \left(|x|^6 + 1 \right)^{-1}$$

and

$$f_2(x, u_1, u_2) = \frac{1 + u_1^5 + u_2^3}{(3 - u_1)(7 - u_2)} \left(|x|^8 + 1 \right)^{-1}$$

we see that for $d_1 = d_2 = 1$, simple calculations allows us to obtain for $r \geq 1$,

$$\sup_{(u_1, u_2) \in [0, d_1] \times [0, d_2]} \sup_{||x||=r} f_1(x, u_1, u_2) \leq 2r^{-6}$$

and

$$\sup_{(u_1, u_2) \in [0, d_1] \times [0, d_2]} \sup_{||x||=r} f_2(x, u_1, u_2) \leq r^{-8}.$$

Therefore, it suffices to take $m_1 = 2$ and $m_2 := 1$. It is clear that $m_i \leq 2 = 4(n-2)d_i c_1^2 c_2^{-1}$. Then we can state that (H3) also holds. Finally, Theorem 2 leads to the existence of sequences of connected sets of positive solutions (22) with asymptotics described in Theorem 3.

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5 Appendix

Now we prove the main tool which allowed us to obtain Theorem 2.

Proof (of Theorem 1) As in [25], we start with the existence of solutions on bounded domains and next we construct a solution on Ω_R .

PART 1: Our first aim is to obtain a sequence of solutions (u_1^k, u_2^k) for following auxiliary system

$$\begin{cases} \operatorname{div}(a_1(||x||)\nabla u_1(x)) + f_1(x, u_1(x), u_2(x)) + g_1(x)x \cdot \nabla u_1(x) = 0 & \text{in } \Omega_k \\ \operatorname{div}(a_2(||x||)\nabla u_2(x)) + f_2(x, u_1(x), u_2(x)) + g_2(x)x \cdot \nabla u_2(x) = 0 & \text{in } \Omega_k \\ u_1 = \bar{u}_1 \text{ and } u_2 = \bar{u}_2 & \text{on } \partial\Omega_k, \end{cases} \quad (23)$$

where $\Omega_k := \{x \in \mathbb{R}^n, R < ||x|| < R + k\}$, with $k \in \mathbb{N}$. Taking into account (H3) we derive the existence of positive constants K_1, K_2 such that $\frac{\partial f_1(x, u_1, u_2)}{\partial u_1} + K_1 \geq 0$ and $\frac{\partial f_2(x, u_1, u_2)}{\partial u_2} + K_2 \geq 0$ for all $(x, u_1, u_2) \in \Omega_k \times [0, d_1] \times [0, d_2]$. Now we apply the Sattinger's monotone iteration schema with the starting point $\mathbf{u}_0 = (u_{0,1}, u_{0,2}) = (\bar{u}_1, \bar{u}_2)$. Let us consider the following problem in Ω_k

$$\begin{cases} \operatorname{div}(a_1(||x||)\nabla u_1(x)) + g_1(x)x \cdot \nabla u_1(x) - K_1 u_1(x) \\ \quad = -(f_1(x, u_{0,1}(x), u_{0,2}(x)) + K_1 u_{0,1}(x)), \\ \operatorname{div}(a_2(||x||)\nabla u_2(x)) + g_2(x)x \cdot \nabla u_2(x) - K_2 u_2(x) \\ \quad = -(f_2(x, u_{0,1}(x), u_{0,2}(x)) + K_2 u_{0,2}(x)), \\ u_1 = \bar{u}_1 \text{ and } u_2 = \bar{u}_2 & \text{on } \partial\Omega_k \end{cases} \quad (24)$$

which contains two independent linear elliptic equations. Applying results described in [22] (Chapter 6), we have the existence of a classical solution $\mathbf{u}_1 = (u_{1,1}, u_{1,2})$ of (24). After simple calculations we state

$$\begin{aligned} & \operatorname{div}(a_1(|x|)|\nabla(u_{1,1}(x) - u_{0,1}(x)) + g_1(x)x \cdot \nabla(u_{1,1}(x) - u_{0,1}(x)) \\ & \quad - K_1(u_{1,1}(x) - u_{0,1}(x)) \\ & = -f_1(x, \bar{u}_1(x), \bar{u}_2(x)) - \operatorname{div}(a_1(|x|)|\nabla\bar{u}_1(x)) - g_1(x)x \cdot \nabla\bar{u}_1(x) \geq 0 \end{aligned}$$

where the last inequality follows from the fact that $(\bar{u}_1, \bar{u}_2)$ is a super-solution of (1)–(2). Since $u_{1,1}(x) = \bar{u}_1(x) = u_{0,1}(x)$ on $\partial\Omega_k$, we state, by the maximum principle, $u_{1,1}(x) \leq \bar{u}_1(x) = u_{0,1}(x)$. Analogously we obtain $u_{1,2}(x) \leq \bar{u}_2(x)$ in Ω_k . Having constructed a pair $\mathbf{u}_{m-1} = (u_{m-1,1}, u_{m-1,2}) \in (C^{2+\alpha}(\bar{\Omega}_k))^2$ we can iterate this process and obtain $\mathbf{u}_m = (u_{m,1}, u_{m,2}) \in (C^{2+\alpha}(\bar{\Omega}_k))^2$ as a classical solution of the linear problem in Ω_k

$$\begin{cases} \operatorname{div}(a_1(|x|)|\nabla u_1(x)) + g_1(x)x \cdot \nabla u_1(x) - K_1 u_1(x) \\ \quad = -(f_1(x, u_{m-1,1}(x), u_{m-1,2}(x)) + K_1 u_{m-1,1}(x)), \\ \operatorname{div}(a_2(|x|)|\nabla u_2(x)) + g_2(x)x \cdot \nabla u_2(x) - K_2 u_2(x) \\ \quad = -(f_2(x, u_{m-1,1}(x), u_{m-1,2}(x)) + K_2 u_{m-1,2}(x)) \\ \quad u_1 = \bar{u}_1 \text{ and } u_2 = \bar{u}_2 \text{ on } \partial\Omega_k. \end{cases}$$

We show inductively that for all $m \in \mathbb{N}$,

$$u_{m,1}(x) \leq u_{m-1,1}(x) \text{ and } u_{m,2}(x) \leq u_{m-1,2}(x) \text{ in } \bar{\Omega}_k. \quad (25)$$

Let us note that (25) is proved above for $m = 1$. Fix arbitrary integer $m \geq 1$ and assume that $u_{m,1}(x) \leq u_{m-1,1}(x)$ and $u_{m,2}(x) \leq u_{m-1,2}(x)$ in Ω_k . Then properties of f_1 and f_2 give the following assertion: for all $x \in \bar{\Omega}_k$,

$$\begin{aligned} & \operatorname{div}(a_1(|x|)|\nabla(u_{m+1,1}(x) - u_{m,1}(x)) + g_1(x)x \cdot \nabla(u_{m+1,1}(x) - u_{m,1}(x)) \\ & \quad - K_1(u_{m+1,1}(x) - u_{m,1}(x)) \\ & \geq f_1(x, u_{m-1,1}(x), u_{m,2}(x)) + K_1 u_{m-1,1}(x) \\ & \quad - f_1(x, u_{m,1}(x), u_{m,2}(x)) - K_1 u_{m,1}(x) \\ & \geq f_1(x, u_{m,1}(x), u_{m,2}(x)) + K_1 u_{m,1}(x) \\ & \quad - f_1(x, u_{m,1}(x), u_{m,2}(x)) - K_1 u_{m,1}(x) = 0. \end{aligned}$$

The same reasoning allows us to state the inequality

$$\begin{aligned} & \operatorname{div}(a_1(|x|)|\nabla(u_{m+1,2}(x) - u_{m,2}(x)) + g_1(x)x \cdot \nabla(u_{m+1,2}(x) - u_{m,2}(x)) \\ & \quad - K_1(u_{m+1,2}(x) - u_{m,2}(x)) \geq 0. \end{aligned}$$

Since $u_{m+1,1} = u_{m,1}$ and $u_{m+1,2} = u_{m,2}(x)$ in $\partial\Omega_k$, the maximum principle implies that $u_{m+1,1}(x) \leq u_{m,1}(x)$ and $u_{m+1,2}(x) \leq u_{m,2}(x)$ in $\bar{\Omega}_k$. Finally, the induction principle implies that for all $m \in \mathbb{N}$, (25) holds.

To sum up, we have constructed monotonic (with respect to each coordinate) and bounded sequence $\{(u_{m,1}, u_{m,2})\}_{m \in \mathbb{N}}$, which is pointwisely convergent in Ω_k to some

vector functions (u_1^k, u_2^k) . We will show that (u_1^k, u_2^k) is a solutions of (23). To this effect we apply the standard reasoning, based on the L^p -estimates of Agmon-Douglis-Nirenberg ([1]), which leads to the existence of $C > 0$ such that for all $m \in \mathbb{N}$,

$$\|u_{m,1}\|_{C^{2+\alpha}(\Omega_k)} \leq C \text{ and } \|u_{m,2}\|_{C^{2+\alpha}(\Omega_k)} \leq C$$

(see [25] or [27] for details). The compact embedding $C^{2+\alpha}(\Omega_k) \times C^{2+\alpha}(\Omega_k) \hookrightarrow C^2(\Omega_k) \times C^2(\Omega_k)$, implies that $\{(u_{m,1}, u_{m,2})\}_{m \in \mathbb{N}}$ tends to (u_1^k, u_2^k) in $(C^2(\Omega_k))^2$, and further (u_1^k, u_2^k) is a classical solution of (23).

PART 2: Now we fix $k_0 > 0$ and consider the sequence $\{(u_1^k, u_2^k)\}_{k > k_0}$. Since $\Omega_{k_0} \subset \Omega_k$ for all $k > k_0$, we can consider the sequence $\{(u_1^k, u_2^k)\}_{k > k_0}$ in Ω_{k_0} . Applying similar reasoning as in the first part we state the existence of a positive constant D such that for all $k > k_0$,

$$\|u_1^k\|_{C^{2+\alpha}(\Omega_{k_0})} \leq D \text{ and } \|u_2^k\|_{C^{2+\alpha}(\Omega_{k_0})} \leq D.$$

Take $k_0 = 1$. Then we have for all $k > 1$, $\|u_1^k\|_{C^{2+\alpha}(\Omega_1)} \leq D_1$ and $\|u_2^k\|_{C^{2+\alpha}(\Omega_1)} \leq D_1$. The compact embedding $(C^{2+\alpha}(\Omega_1))^2$ into $(C^2(\Omega_1))^2$, again allows us to state the existence of a subsequence $\{(u_1^{k_{l_1}}, u_2^{k_{l_1}})\}_{k_{l_1} > 1}$ of $\{(u_1^k, u_2^k)\}_{k > k_0}$ which tends to a certain $(\hat{u}_1^1, \hat{u}_2^1)$ in $(C^2(\Omega_1))^2$. Thus $(\hat{u}_1^1, \hat{u}_2^1)$ satisfies (1) in Ω_1 and $\underline{u}_1 \leq \hat{u}_1^1 \leq \bar{u}_1$ and $\underline{u}_2 \leq \hat{u}_2^1 \leq \bar{u}_2$ in $\bar{\Omega}_1$ and $\hat{u}_1^1 = \bar{u}_1$ and $\hat{u}_2^1 = \bar{u}_2$ on $\{x \in \mathbb{R}^n, \|x\| = R\}$. Proceeding this iteration schema we consider a subsequence $\{(u_1^{k_{l_1}}, u_2^{k_{l_1}})\}_{k_{l_1} > 2}$ on Ω_2 and obtain the subsequence $\{(u_1^{k_{l_2}}, u_2^{k_{l_2}})\}_{k_{l_2} > 2}$ of $\{(u_1^{k_{l_1}}, u_2^{k_{l_1}})\}_{k_{l_1} > 1}$ such that $\{(u_1^{k_{l_2}}, u_2^{k_{l_2}})\}_{k_{l_2} > 2}$ converges in $(C^2(\Omega_2))^2$ to $(\hat{u}_1^2, \hat{u}_2^2)$ being a solution of (1) in Ω_2 and such that $\underline{u}_1 \leq \hat{u}_1^2 \leq \bar{u}_1$, $\underline{u}_2 \leq \hat{u}_2^2 \leq \bar{u}_2$ in $\bar{\Omega}_2$ and $\hat{u}_1^2 = \bar{u}_1$ and $\hat{u}_2^2 = \bar{u}_2$ on $\{x \in \mathbb{R}^n, \|x\| = R\}$. Iterating this process for each $m \in \mathbb{N}$, we construct a sequence $\{(u_1^{k_{l_m}}, u_2^{k_{l_m}})\}_{k_{l_m} > m}$ which is convergent in $(C^2(\Omega_m))^2$ and such that $\{(u_1^{k_{l_m}}, u_2^{k_{l_m}})\}_{k_{l_m} > m}$ is a subsequence of $\{(u_1^{k_{l_{m-1}}}, u_2^{k_{l_{m-1}}})\}_{k_{l_{m-1}} > m-1}$.

To sum up, we can put $\hat{u}_1^m(x) = \lim_{l_m \rightarrow \infty} u_1^{k_{l_m}}(x)$ and $\hat{u}_2^m(x) = \lim_{l_m \rightarrow \infty} u_2^{k_{l_m}}(x)$ in Ω_m and derive that $(\hat{u}_1^m, \hat{u}_2^m)$ is a solution of (1) in Ω_m , $\underline{u}_1 \leq \hat{u}_1^m \leq \bar{u}_1$ and $\underline{u}_2 \leq \hat{u}_2^m \leq \bar{u}_2$ in $\bar{\Omega}_m$ and $\hat{u}_1^m = \bar{u}_1$ and $\hat{u}_2^m = \bar{u}_2$ on $\{x \in \mathbb{R}^n, \|x\| = R\}$. By construction, $\hat{u}_1^m|_{\Omega_{m-1}} = \hat{u}_1^{m-1}$ and $\hat{u}_2^m|_{\Omega_{m-1}} = \hat{u}_2^{m-1}$. This fact implies that we can consider functions $U_1, U_2 : G_R \rightarrow R$ given as follows

$$U_1(x) := \hat{u}_1^m(x) \text{ and } U_2(x) := \hat{u}_2^m(x) \text{ in } \Omega_m$$

and state that the pair $(U_1, U_2) \in (C_{loc}^{2+\alpha}(G_R))^2$ and satisfies (1) in G_R . By properties of the vector functions $(\hat{u}_1^m, \hat{u}_2^m)$ we derive that $\underline{u}_1 \leq U_1 \leq \bar{u}_1$, $\underline{u}_2 \leq U_2 \leq \bar{u}_2$ in $\bar{\Omega}_m$, $U_1 = \bar{u}_1$ and $U_2 = \bar{u}_2$ on $\{x \in \mathbb{R}^n, \|x\| = R\}$. Moreover asymptotics of $\bar{u}_1$

and $\bar{u}_2$ gives the required conditions at infinity: $\lim_{||x|| \rightarrow \infty} U_1(x) = 0$, $\lim_{||x|| \rightarrow \infty} U_2(x) = 0$. $\square$

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