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Renormalized solutions of semilinear elliptic equations with general measure data

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Abstract
In the paper we first propose a definition of renormalized solution of semilinear elliptic equation involving operator corresponding to a general (possibly nonlocal) symmetric regular Dirichlet form satisfying the so-called absolute continuity condition and general (possibly nonsmooth) measure data. Then we analyze the relationship between our definition and other concepts of solutions considered in the literature (probabilistic solutions, solution defined via the resolvent kernel of the underlying Dirichlet form, Stampacchia's definition by duality). We show that under mild integrability assumption on the data all these concepts coincide.

Keywords Semilinear elliptic equation · Dirichlet form and operator · measure data · renormalized solution

Mathematics Subject Classification Primary: 35D99; Secondary: 35J61 · 60H30

1 Introduction

Let L be the operator associated with a symmetric regular Dirichlet form $(\mathcal{E}, D(\mathcal{E}))$ on $L^2(E;m)$, $f : E \times \mathbb{R} \to \mathbb{R}$ be a measurable function and μ be a bounded signed Borel measure on E. In the paper we consider semilinear equations of the form

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$$-Lu = f(\cdot, u) + \mu \quad \text{in } E. \tag{1.1}$$

One of the important problems that arises when studying such equations is the problem of proper definition of a solution. This problem has been dealt with by many authors. In the present paper we first introduce yet another definition of a solution of (1.1). It is a slight modification of the definition of a renormalized solution introduced in [13] in case μ is smooth. Then we analyze the relationship between this new definition and other concepts of solutions known in the literature.

In case L is a uniformly elliptic divergence form operator and f does not depend on u, some definition, now called Stampacchia's definition by duality, was proposed by Stampacchia [24]. Later on, to deal with equations with more general local operator L, the definitions of entropy solution and renormalized solution were introduced. For a comparison of different forms of these definitions and remarks on other concepts of solutions of equations of the form (1.1) with local operator L and f not depending on u see [6]. Elliptic equations with local operators and nonlinear dependence on general measure data are studied in [7,18].

In case f depends on u most of known results are devoted to the case where μ is smooth. Recall (see [10]) that μ admits a unique decomposition

$$\mu = \mu_d + \mu_c \tag{1.2}$$

into the smooth (diffuse) part μ_d and the concentrated part μ_c, i.e. μ_d is a bounded Borel measure, which is "absolutely continuous" with respect to the capacity Cap determined by $(\mathcal{E}, D(\mathcal{E}))$, and μ_c is a bounded Borel measure which is "singular" with respect to Cap. In case L is local and μ is smooth entropy and renormalized solutions of (1.1) are studied in numerous papers (see, e.g., [1,8] and the references given there). A definition of renormalized solutions applicable to (1.1) with general L associated with a general transient (possibly non-symmetric) Dirichlet form was recently given in [13]. If $(\mathcal{E}, D(\mathcal{E}))$ is symmetric and $f(\cdot, u) \in L^1(E; m)$, renormalized solutions in the sense of [13] coincide with probabilistic solutions of (1.1) defined earlier in [12] (see also [14] for equations with operator L associated with a non-symmetric quasi-regular form and [17] for equations with nonlinear dependence on measure data). Recall that a measurable $u : E \to \mathbb{R}$ is a probabilistic solution of (1.1) in the sense of [12,14] if the following nonlinear Feynman–Kac formula

$$u(x) = E_x\Big(\int_0^\zeta f(X_t, u(X_t))\, dt + \int_0^\zeta dA_t^\mu\Big) \tag{1.3}$$

is satisfied for quasi-every $x \in E$. In (1.3), $\mathbf{M} = (X, P_x)$ is a Markov process with life time ζ associated with $\mathcal{E}$, E_x denotes the expectation with respect to P_x and A^μ is the continuous additive functional of $\mathbf{M}$ associated with μ in the Revuz sense (see Sect. 2). The equivalence between renormalized and probabilistic solutions allows one to use effectively probabilistic methods in the study of renormalized solutions of (1.1). Also note that if $f \in L^1(E; m)$ then renormalized solutions of (1.1) coincide with Stampacchia's solutions by duality defined in [12,14].

The semilinear case with general, possibly nonsmooth bounded measure μ is much more involved. The study of (1.1) with nonsmooth measure was initiated in 1975 by

Brezis and Bénilan in case L is the Laplace operator Δ (see [2,4] and the references given there for results and historical comments). For some existence and uniqueness results in case L is the fractional Laplacian $\Delta^{\alpha/2}$ with $\alpha \in (0,2)$ see Chen and Véron [5]. Very recently, Klimsiak [11] started the study of (1.1) in case L corresponds to a transient symmetric regular Dirichlet form satisfying the following absolute continuity condition:

(ACR) $R_\alpha(x,\cdot)$ is absolutely continuous with respect to m for each $\alpha > 0$ and $x \in E$,

where $R_\alpha(x,dy)$ denotes the resolvent kernel associated with $(\mathcal{E}, D(\mathcal{E}))$ (see Sect. 2.2). Equivalently,

(ACT) $p_t(x,\cdot)$ is absolutely continuous with respect to m for each $t > 0$ and $x \in E$,

where $p_t(x,dy)$ is the transition function associated with $(\mathcal{E}, D(\mathcal{E}))$. The above conditions are satisfied for instance if L is a uniformly divergence form operator or $L = \Delta^{\alpha/2}$ with $\alpha \in (0,2)$. If the form is transient, then under (ACR) the resolvent kernel $R_0(x,dy)$ has a density r. In [11] a measurable function u on E is called a solution of (1.1) if

$$u(x) = \int_E r(x,y) f(y,u(y))\,dy + \int_E r(x,y)\,\mu(dy) \tag{1.4}$$

for quasi every $x \in E$. In case $\mu_c = 0$, the above equation reduces to (1.3), so the definition of [11] reduces to the probabilistic definition of a solution given in [12,14]. In [11] also a partly probabilistic interpretation of (1.4) is given. This suggests that solutions defined via the resolvent density, i.e. by (1.4), may be equivalently defined as renormalized solutions in the same manner as in [13]. In the present paper we show that this is indeed possible. The definition of a renormalized solution adopted in the present paper is a minor modification of the definition of [13]. In our opinion, it is natural, especially from the probabilistic point of view. Moreover, in many cases considered so far in the literature (μ is smooth or μ is nonsmooth and $L = \Delta$ or $L = \Delta^{\alpha/2}$, like in [4,5]) the solutions considered there coincide with the renormalized defined in the present paper.

The main result of the paper says that if the form is transient and (ACR) is satisfied then the renormalized solution is a solution in the sense of (1.4), and if u is a solution of (1.1) in the sense of (1.4) and $u \in L^1(E;m)$ then u is a renormalized solution. We find important that, as in the case of smooth measures, this correspondence when combined with probabilistic interpretation of (1.4) given in [11] enables one to study renormalized solutions of (1.1) with the help of probabilistic methods. For results on (1.1) obtained in this way we defer the reader to [11]). Finally, note that at the end of the paper we describe some interesting situations in which solutions of (1.1) in the sense of (1.4) automatically have the property that $f(\cdot,u) \in L^1(E;m)$.

2 Preliminaries

In the paper E is a separable locally compact metric space and m is a Radon measure on E such that supp$[m] = E$. By $\mathcal{B}(E)$ (resp. $\mathcal{B}^+(E)$) we denote the set of all real (resp. nonnegative) Borel measurable functions on E, and by $\mathcal{B}_b(E)$ the subset of $\mathcal{B}(E)$ consisting of all bounded functions.

For $u : E \to \mathbb{R}$ we set $u^+(x) = \max\{u(x), 0\}$, $u^-(x) = \max\{-u(x), 0\}$.

2.1 Dirichlet forms

By $(\mathcal{E}, D(\mathcal{E}))$ we denote a symmetric regular Dirichlet form on $H = L^2(E; m)$ (see [9, Section 1.1] for the definition). In case $(\mathcal{E}, D(\mathcal{E}))$ is transient, by $(D_e(\mathcal{E}), \mathcal{E})$ we denote the extended Dirichlet space of $(\mathcal{E}, D(\mathcal{E}))$ (see [9, Section 1.5]).

In the paper we define capacity Cap as in [9, Section 2.1]. Recall that an increasing sequence $\{F_n\}$ of closed subsets of E is called nest if $\text{Cap}(E \setminus F_n) \to 0$ as $n \to \infty$. A subset $N \subset E$ is called exceptional if $\text{Cap}(N) = 0$. We will say that some property of points in E holds quasi everywhere (q.e. for short) if the set for which it does not hold is exceptional.

We say that a function u on E is quasi-continuous if there exists a nest $\{F_n\}$ such that $u_{|F_n}$ is continuous for every $n \geq 1$. By [9, Theorem 2.1.7], each function $u \in D_e(\mathcal{E})$ has a quasi-continuous m-version.

Let μ be a signed Borel measure on E, and let $|\mu| = \mu^+ + \mu^-$, where μ^+ (resp. μ^-) we denote the positive (resp. negative) part of of μ. We say that μ is smooth if $|\mu|$ does not charge exceptional sets and there exists a nest $\{F_n\}$ such that $|\mu|(F_n) < \infty$, $n \geq 1$. The set of all smooth measures on E will be denoted by S. By $\mathcal{M}_b$ we denote the set of all signed Borel measures on E such that $\|\mu\|_{TV} := |\mu|(E) < \infty$, and by $\mathcal{M}_{0,b}$ the subset of $\mathcal{M}_b$ consisting of all smooth measures. S^+ is the subset of S consisting of nonnegative measures. Similarly we define $\mathcal{M}_b^+$, $\mathcal{M}_{0,b}^+$. By [10, Lemma 2.1], for every $\mu \in \mathcal{M}_b$ there exists a unique pair $(\mu_d, \mu_c) \in \mathcal{M}_b \times \mathcal{M}_b$ such that $\mu_d \in \mathcal{M}_{0,b}$, μ_c is concentrated on some exceptional Borel subset of E and (1.2) is satisfied. If μ is nonnegative, so are μ_d, μ_c. For a complete description of the structure of μ_c see [15].

2.2 Markov processes

Let $E \cup \Delta$ be the one-point compactification of E. When E is already compact, we adjoin Δ to E as an isolated point. We adopt the convention that every function f on E is extended to $E \cup \{\Delta\}$ by setting $f(\Delta) = 0$.

By [9, Theorems 4.2.8, 7.2.3] there exists a unique (up to equivalence) m-symmetric Hunt process $\mathbf{M} = (\Omega, \mathcal{F}, (\mathcal{F}_t)_{t\geq 0}, (X_t)_{t\geq 0}, \zeta, (P_x)_{x\in E\cup\Delta})$ with state space E, life time ζ and cemetery state Δ whose Dirichlet space is $(\mathcal{E}, D(\mathcal{E}))$. This means in particular that for every $\alpha > 0$ and $f \in \mathcal{B}_b(E) \cap H$ the resolvent of $\mathbf{M}$, that is the function

$$R_\alpha f(x) = E_x \int_0^\infty e^{-\alpha t} f(X_t)\, dt, \quad x \in E$$

is a quasi-continuous m-version of $G_\alpha f$.

Let $R_\alpha(x,dy)$ denote the kernel on $(E,\mathcal{B}(E))$ defined as $R_\alpha(x,B)=R_\alpha\mathbf{1}_B(x)$. In the paper we will assume that $\mathbf{M}$ satisfies (ACR) condition formulated in Sect. 1. By [9, Theorem 4.2.4], for symetric forms considered in the present paper (ACR) is equivalent to (ACT). In general, for non-symmetric forms, (ACT) is stronger than (ACR). Also note that in the literature (ACR) is sometimes called Meyer's hypothesis (L) (see [23, Chapter I, Exercise 10.25]

Assume that $(\mathcal{E},D(\mathcal{E}))$ is transient. Then there exists a nonnegative $\mathcal{B}(E)\otimes\mathcal{B}(E)$-measurable function $r:E\times E\to\mathbb{R}$ such that $r(x,y)=r(y,x)$, $x,y\in E$ and for every Borel set $B\subset E$,

$$R(x,B)=\int_B r(x,y)\,m(dy),\quad x\in E.$$

In fact, $r(x,y)=\lim_{\alpha\downarrow 0}r_\alpha(x,y)$, where $r_\alpha(x,y)$ is the density of $R_\alpha(x,dy)$ constructed in [9, Lemma 4.2.4] (see remarks in [3, p. 256]). We call r the resolvent density.

In what follows given a positive Borel measure on E, we write

$$R_\alpha\mu(x)=\int_E r_\alpha(x,y)\,\mu(dy),\qquad R\mu(x)=\int_E r(x,y)\,\mu(dy),\quad x\in E,\quad \alpha>0.$$

For a signed Borel measure μ on E, we set $R\mu(x)=R\mu^+(x)-R\mu^-(x)$, whenever $R\mu^+(x)<+\infty$ or $R\mu^-(x)<+\infty$, and we adopt the convention that $R\mu(x)=+\infty$ if $R\mu^+(x)=R\mu^-(x)=+\infty$.

Proposition 2.1 *Assume that $(\mathcal{E},D(\mathcal{E}))$ is transient and* (ACR) *is satisfied. If $\mu\in\mathcal{M}_b$ then $R|\mu|(x)<+\infty$ for q.e. $x\in E$.*

Proof See [11, Proposition 3.2]. □

Denote by $\mathbb{M}$ the set of all signed Borel measures μ on E such that $R|\mu|(x)<+\infty$ for m-a.e. $x\in E$. By Proposition 2.1, $\mathcal{M}_b\subset\mathbb{M}$. In general, the inclusion is strict (see the remark following [14, Proposition 3.2]).

We define additive functional (AF in abbreviation) and continuous AF of $\mathbf{M}$ as in [9, Sections 5.1]. By [9, Theorem 5.1.4], there is a one to one correspondence (called Revuz correspondence) between the set of smooth measures μ on E and the set of positive continuous AFs A of $\mathbf{M}$. It is given by the relation

$$\lim_{t\to 0^+}\frac{1}{t}E_m\int_0^t f(X_s)\,dA_s=\int_E f(x)\,\mu(dx),\quad f\in\mathcal{B}^+(E),$$

where E_m denotes the expectation with respect to the measure $P_m(\cdot)=\int_E P_x(\cdot)\,m(dx)$. In what follows the positive continuous AF of $\mathbf{M}$ corresponding to a positive $\mu\in S$ will be denoted by A^μ. If μ in S, then $\mu^+,\mu^-\in S$, and we set $A^\mu=A^{\mu^+}-A^{\mu^-}$. Note that if $\mu\in S^+$ then for every $\alpha\ge 0$,

$$R_\alpha\mu(x)=E_x\int_0^\zeta e^{-\alpha t}\,dA^\mu_t=E_x\int_0^\infty e^{-\alpha t}\,dA^\mu_t \tag{2.1}$$

for q.e. $x \in E$. Indeed, if $\alpha > 0$ and μ is a measure of finite 0-order energy integral ($\mu \in S_0^{(0)}$ in notation; see [9, Section 2.2] for the definition), then (2.1) follows from Exercise 4.2.2 and Lemma 5.1.3 in [9]. The general case follows by approximation. We first let $\alpha \downarrow 0$ to get (2.1) for $\alpha \geq 0$ and $\mu \in S_0^{(0)}$, and then we use the 0-order version of [9, Theorem 2.2.4] (see remark following [9, Corollary 2.2.2]) to get (2.1) for any $\alpha \geq 0$ and $\mu \in S^+$.

3 Probabilistic solutions and solutions defined via the resolvent density

We assume that $(\mathcal{E}, D(\mathcal{E}))$ is transient and (ACR) is satisfied. Consider the problem

$$-Lu = f_u + \mu, \tag{3.1}$$

where $f : E \times \mathbb{R} \to \mathbb{R}$ is a measurable function, $f_u = f(\cdot, u)$, $\mu \in \mathbb{M}$ and L is the operator associated with $(\mathcal{E}, D(\mathcal{E}))$, i.e. the nonpositive definite self-adjoint operator on H such that

$$D(L) \subset D(\mathcal{E}), \qquad \mathcal{E}(u, v) = (-Lu, v), \quad u \in D(L), v \in D(\mathcal{E}),$$

where $(\cdot, \cdot)$ denotes the usual inner product in H (see [9, Corollary 1.3.1]).

The following two definitions of solutions of (3.1) were introduced in [11].

Definition 3.1 We say that a measurable function $u : E \to \mathbb{R} \cup \{-\infty, +\infty\}$ is a solution of (1.1) if $f_u \cdot m \in \mathbb{M}$ and (1.4) is satisfied for q.e. $x \in E$.

Definition 3.2 We say that a measurable $u : E \to \mathbb{R} \cup \{-\infty, +\infty\}$ is a probabilistic solution of (1.1) if

(a) $f_u \cdot m \in \mathbb{M}$ and there exists an AF M of $\mathbf{M}$ such that such that for q.e. $x \in E$ the process M is an $(\mathcal{F})_{t \geq 0}$-local martingale under P_x and

$$u(X_t) = u(X_0) - \int_0^t f_u(X_s)\,ds - \int_0^t dA_s^{\mu_d} + \int_0^t dM_s, \quad t \geq 0, \quad P_x\text{-a.s.} \tag{3.2}$$

(b) for every exceptional set $N \subset E$, every stopping time T such that $T \geq \zeta$ and every sequence $\{\tau_k\} \subset \mathcal{T}$ such that $\tau_k \nearrow T$ and $E_x \sup_{t \leq \tau_k} |u(X_t)| < \infty$ for all $x \in E \setminus N$ and $k \geq 1$, we have

$$E_x u(X_{\tau_k}) \to R\mu_c(x), \quad x \in E \setminus N. \tag{3.3}$$

Any sequence $\{\tau_k\}$ with the properties listed in condition (b) will be called the reducing sequence for u, and we will say that $\{\tau_k\}$ reduces u.

Remark 3.3 (i) By [11, Remark 3.10], if $\mu_c = 0$, then the above definition reduces to the definition introduced in [12].

(ii) Assume that u is a probabilistic solution of (1.1). Then for q.e. $x \in E$ we have

$$E_x u^+(X_{\tau_k}) \to R\mu_c^+(x), \qquad E_x u^-(X_{\tau_k}) \to R\mu_c^-(x). \tag{3.4}$$

Indeed, if u is a solution of (1.1) then by [11, Theorem 6.3], $Lu^+ \in \mathbb{M}$. In different words, u^+ is a solution of the equation $Lu^+ = \nu$ with some $\nu \in \mathbb{M}$. Hence, by condition (b) of Definition 3.2, $E_x u^+(X_{\tau_k}) \to R\nu_c(x)$ for q.e. $x \in E$. But by [11, Theorem 6.3], $(Lu^+)_c = (Lu)_c^+$. Hence $\nu_c = (f_u \cdot m + \mu)_c^+ = \mu_c^+$, which proves the first convergence in (3.4). The second convergence follows from the first one and (3.3).

Proposition 3.4 *Let $\mu \in \mathbb{M}$. A measurable $u : E \to \mathbb{R} \cup \{-\infty, +\infty\}$ is a solution of* (1.1) *in the sense of Definition* 3.1 *if and only if it is a solution of* (1.1) *in the sense of Definition* 3.2.

Proof See [11, Proposition 3.12]. □

In what follows for a function u on E and a measure μ on E, we set

$$\langle \mu, u \rangle = \int_E u(x)\mu(dx)$$

whenever the integral is well defined, and for $k \ge 0$, we write

$$T_k u(x) = \max\{\min\{u(x), k\}, -k\}, \quad x \in E.$$

Remark 3.5 (i) By [11, Theorem 3.7], if u is a solution of (1.1) then u is quasi-continuous.

(ii) Let u be a solution of (1.1) with $\mu \in \mathcal{M}_b$. If $f_u \in L^1(E; m)$ then by [11, Theorem 3.3], $T_k u \in D_e(\mathcal{E})$ for every $k \ge 0$. If, in addition, $m(E) < \infty$ or $\mathcal{E}$ satisfies Poincaré type inequality then $T_k u \in D(\mathcal{E})$ for $k \ge 0$ (see [11, Remark 3.4]).

In closing this section we recall yet another concept of solutions introduced in [11].

We say that $u : E \to \mathbb{R} \cup \{-\infty, +\infty\}$ is a solution of (1.1) in the sense of Stampacchia if for every $v \in \mathcal{B}(E)$ such that $\langle |\mu|, R|v| \rangle < \infty$ the integrals (u, v), $f_u \cdot m, Rv)$ are finite and

$$(u, v) = (f_u, Rv) + \langle \mu, Rv \rangle.$$

By [11, Proposition 4.12], if $\mu \in \mathbb{M}$, then u is a solution of (1.1) in the sense of Stampacchia if and only if it is a solution of (1.1) in the sense of Definition 3.1.

4 Renormalized solutions

As in Sect. 3, in this section we assume that $(\mathcal{E}, D(\mathcal{E}))$ is transient and (ACR) is satisfied. As for the right-hand side of (1.1), we restrict our considerations to bounded measures.

The following definition extends [13, Definition 3.1] to possibly nonsmooth measures.

Definition 4.1 Let $\mu \in \mathcal{M}_b(E)$. We say that $u : E \to \mathbb{R} \cup \{-\infty, +\infty\}$ is a renormalized solution of (1.1) if

(a) u is quasi-continuous, $f_u \in L^1(E; m)$ and $T_k u \in D_e(\mathcal{E})$ for every $k \geq 0$,
(b) there exists a sequence $\{\nu_k\} \subset \mathcal{M}_{0,b}(E)$ such that $R\nu_k \to R\mu_c$ q.e. as $k \to \infty$, and for every $k \in \mathbb{N}$ and every bounded $v \in D_e(\mathcal{E})$,

$$\mathcal{E}(T_k u, v) = \langle f_u \cdot m + \mu_d, \tilde{v} \rangle + \langle \nu_k, \tilde{v} \rangle. \tag{4.1}$$

Note that in the case of local operators, the above definition is essentially [6, Definition 2.29]. A similar in spirit definition of renormalized solutions of parabolic equations with local Leray–Lions type operators is considered in [19, Definition 4.1] (in case $\mu_c = 0$) and [20, Definition 3] (in the case of general bounded measures).

In case $\mu_c = 0$, Definition 4.1 reduces to [13, Definition 3.1] with the exception that in [13] in condition (b) it is required that $\|\nu_k\|_{TV} \to 0$. Note that in the case where $\mu_c \neq 0$ the condition $R\nu_k \to R\mu_c$ q.e. cannot be replaced by the condition $\|\nu_k - \mu_c\|_{TV} \to 0$ because the limit, in the total variation norm, of diffuse measures is diffuse. Also, if $\mu_c \neq 0$, then $\|\nu_k\|_{TV} \nrightarrow 0$, because by [16, Lemma 2.5], if $\|\nu_k\|_{TV} \to 0$, then there is a subsequence $\{\nu_{k'}\}$ such that $R\nu_{k'} \to 0$ q.e. We see that the difference between the case $\mu_c = 0$ and $\mu_c \neq 0$ is quite similar to that for parabolic equations considered in [19,20] (cf. [19, Definition 4.1] and [20, Definition 3]).

Remark 4.2 (i) Let $E \subset \mathbb{R}^d$ be a bounded domain, and let L be the Laplace operator Δ on E with zero boundary conditions. By [11, Remark 4.15], if u is a renormalized solution of (1.1), then u is a weak solution in the sense of [4].
(ii) Let $\alpha \in (0, 2]$, $E \subset \mathbb{R}^d$ be a bounded domain, and let L be the fractional Laplacian $\Delta^{\alpha/2}$ on E with zero boundary conditions. By [11, Remark 4.13], if u is a renormalized solution of (1.1), then u is a solution of (1.1) in the sense of [5, Definition 1.1].

The following lemma is a modification of [12, Lemma 5.4]. As compared with [12, Lemma 5.4], we do not assume that μ is smooth, but we additionally require that the form satisfies (ACT).

Lemma 4.3 *Assume that $\nu \in \mathbb{M} \cap S^+$, $\mu \in \mathcal{M}_b^+$. If $R\nu \leq R\mu$ m-a.e. then $\nu \in \mathcal{M}_{0,b}^+$. In fact, $\|\nu\|_{TV} \leq \|\mu\|_{TV}$.*

Proof Set $g_n = n(1 - nR_n 1)$. Then by the resolvent identity,

$$Rg_n = nR_n 1 \leq 1, \quad n \geq 1.$$

Since by [3, Chapter II, Proposition (2.2)] the constant function 1 is excessive relative to $\mathbf{M}$, $g_n \geq 0$ and, by [3, Chapter II, Proposition (2.3)], $Rg_n \nearrow 1$. Since the resolvent density r is symmetric, applying Fubini's theorem we get

$$\begin{aligned}\langle \mu, Rg_n\rangle &= \int_E\Big(\int_E r(x,y)g_n(y)\,dy\Big)\,\mu(dx)\\ &= \int_E\Big(\int_E r(y,x)\,\mu(dx)\Big)\,g_n(y)\,dy = \langle g_n, R\mu\rangle.\end{aligned}$$

Likewise, $\langle \nu, Rg_n\rangle = \langle g_n, R\nu\rangle$. Since $R\nu \le R\mu$ m-a.e., it follows from the above that

$$\langle \mu, Rg_n\rangle \ge \langle \nu, Rg_n\rangle, \quad n \ge 1.$$

Therefore

$$\|\nu\|_{TV} = \lim_{n\to\infty}\langle Rg_n, \nu\rangle \le \lim_{n\to\infty}\langle Rg_n, \mu\rangle = \|\mu\|_{TV},$$

which proves the lemma. □

Theorem 4.4 *Let $\mu \in \mathcal{M}_b$.*

(i) *If u is a probabilistic solution of* (1.1) *and $f_u \in L^1(E;m)$ then u is a renormalized solution of* (1.1).
(ii) *If u is a renormalized solution of* (1.1) *then u is a probabilistic solution of* (1.1).

Proof (i) Let $Y_t = u(X_t)$, $t \ge 0$. By (3.2), for q.e. $x \in E$,

$$Y_t = Y_0 - \int_0^t f_u(X_s)\,ds - \int_0^t dA_s^{\mu_d} + \int_0^t dM_s, \quad t \ge 0, \quad P_x\text{-a.s.} \tag{4.2}$$

By Itô's formula for convex functions (see, e.g., [22, Theorem IV.66]),

$$u^+(X_t) - u^+(X_0) = \int_0^t \mathbf{1}_{\{Y_{s-}>0\}}\,dY_s + A_t^1, \quad t \ge 0, \tag{4.3}$$

$$u^-(X_t) - u^-(X_0) = -\int_0^t \mathbf{1}_{\{Y_{s-}\le 0\}}\,dY_s + A_t^2, \quad t \ge 0 \tag{4.4}$$

for some increasing processes A^1, A^2. By [11, Remark 3.10], there is a reducing sequence $\{\tau_k\}$ for u. Since M is a local martingale under P_x for q.e. $x \in E$, for q.e. $x \in E$ there exists a sequence of stopping times $\{\sigma_n\}$ (possibly depending on x) such that $E_x\int_0^{t\wedge\sigma_n}\mathbf{1}_{\{Y_{s-}\le 0\}}\,dM_s = 0$, $t \ge 0$, $n \ge 1$. Therefore, by (4.2) and (4.3),

$$E_x A^1_{\tau_k\wedge\sigma_n} = E_x u^+(X_{\tau_k\wedge\sigma_n}) - u^+(x) + E_x\int_0^{\tau_k\wedge\sigma_n}\mathbf{1}_{\{Y_{s-}>0\}}(f_u(X_s)\,ds + dA_s^{\mu_d})$$

for all $k, n \ge 1$. Letting $n \to \infty$ we get

$$E_x A^1_{\tau_k} = E_x u^+(X_{\tau_k}) - u^+(x) + E_x\int_0^{\tau_k}\mathbf{1}_{\{Y_{s-}>0\}}(f_u(X_s)\,ds + dA_s^{\mu_d}).$$

Similarly, by (4.2) and (4.4),

$$E_xA^2_{\tau_k}=E_xu^-(X_{\tau_k})-u^-(x)-E_x\int_0^{\tau_k}\mathbf{1}_{\{Y_{s-}\le 0\}}(f_u(X_s)\,ds+dA^{\mu_d}_s).$$

Letting $k\to\infty$ in the above two equalities and using (3.4) shows that for q.e. $x\in E$,

$$\begin{aligned}E_xA^1_\zeta&\le R\mu_c^+(x)+E_x\int_0^\zeta(|f_u(X_t)|\,ds+dA^{|\mu_d|}_t)\\&=R\mu_c^+(x)+R(|f_u|\cdot m+|\mu_d|)(x),\\E_xA^2_\zeta&\le R\mu_c^-(x)+E_x\int_0^\zeta(|f_u(X_t)|\,ds+dA^{|\mu_d|}_t)\\&=R\mu_c^-(x)+R(|f_u|\cdot m+|\mu_d|)(x).\end{aligned}$$

By this and Proposition 2.1, $E_x(A^1_\zeta+A^2_\zeta)<+\infty$ for q.e. $x\in E$. Therefore by [9, Theorem A.3.16] there exists positive AFs of B^1,B^2 of $\mathbf{M}$ such that B^i, $i=1,2$, is a compensator of A^i under P_x for q.e. $x\in E$. The processes B^1,B^2 are increasing, because A^1 and A^2 are increasing. Since by [9, Theorem A.3.2] the process X has no predictable jumps, it follows from [9, Theorem A.3.5] that B^1,B^2 are continuous. Thus B^1,B^2 are increasing continuous AFs of $\mathbf{M}$ such that A^i-B^i, $i=1,2$, is a martingale under P_x for q.e. $x\in E$. Let $b^i\in S$, $i=1,2$, denote the measure corresponding to B^i in the Revuz sense. Then, by (2.1),

$$Rb^i(x)=E_xB^i_\zeta=E_xA^i_\zeta<+\infty,\quad i=1,2,$$

for q.e. $x\in E$. From this and Lemma 4.3 it follows that $b^1,b^2\in\mathcal{M}_{0,b}$. By Itô's formula, for $k>0$ we have

$$(u^+\wedge k)(X_t)-(u^+\wedge k)(X_0)=\int_0^t\mathbf{1}_{\{u^+(X_{s-})\le k\}}\,du^+(X_s)-A^{1,k}_t,\quad t\ge 0,\tag{4.5}$$

$$(u^-\wedge k)(X_t)-(u^-\wedge k)(X_0)=\int_0^t\mathbf{1}_{\{u^-(X_{s-})\le k\}}\,du^-(X_s)-A^{2,k}_t,\quad t\ge 0,\tag{4.6}$$

for some increasing processes $A^{1,k},A^{2,k}$. By (4.3) and (4.5),

$$E_xA^{1,k}_t\le u^+(x)\wedge k+E_x\int_0^t\mathbf{1}_{\{u^+(X_{s-})\le k\}}\mathbf{1}_{\{Y_{s-}>0\}}\,dY_s+E_x\int_0^t\mathbf{1}_{\{u^+(X_{s-})\le k\}}\,dA^1_s$$

whereas by (4.4) and (4.6),

$$E_xA^{2,k}_t\le u^-(x)\wedge k-E_x\int_0^t\mathbf{1}_{\{u^-(X_{s-})\le k\}}\mathbf{1}_{\{Y_{s-}\le 0\}}\,dY_s+E_x\int_0^t\mathbf{1}_{\{u^-(X_{s-})\le k\}}\,dA^2_s.$$

By the above two inequalities,

$$E_x(A^{1,k}_\zeta + A^{2,k}_\zeta) \le u^+(x)\wedge k + u^-(x)\wedge k + R(|f_u|\cdot m + |\mu_d|)(x) + R(b^1+b^2)(x).$$

Hence $E_x(A^{1,k}_\zeta + A^{2,k}_\zeta) < +\infty$ for q.e. $x \in E$. Let $B^{1,k}$, $B^{2,k}$ be positive AFs of $\mathbf{M}$ such that $B^{i,k}$, $i = 1, 2$, is a compensator of $A^{i,k}$ under P_x for q.e. $x \in E$. As in case of B^1, B^2, we show that $B^{1,k}$, $B^{2,k}$ increasing continuous AFs of $\mathbf{M}$ such that $A^{i,k} - B^{i,k}$, $i = 1, 2$, is a martingale under P_x for q.e. $x \in E$. Let $b^{i,k} \in S$, $i = 1, 2$, denote the measure corresponding to $B^{i,k}$ in the Revuz sense. Then $R(b^{1,k}+b^{2,k})(x) = E_x(A^{1,k}_\zeta + A^{2,k}_\zeta) < +\infty$ for q.e. $x \in E$, and hence, by Lemma 4.3, that $b^{1,k}, b^{2,k} \in \mathcal{M}_{0,b}$. Let $Y^k_t = T_k u(X_t)$. Since $T_k u = (u^+\wedge k) - (u^-\wedge k)$, from (4.2)–(4.6) we get

$$\begin{aligned} Y^k_t - Y^k_0 = &-\int_0^t \mathbf{1}_{\{-k\le Y_{s-}\le k\}}(f_u(X_s)\,ds + dA^{\mu_d}_s) - B^{1,k}_t \\ &+ \int_0^t \mathbf{1}_{\{u^+(X_s)\le k\}}\,dB^1_s + B^{2,k}_t - \int_0^t \mathbf{1}_{\{u^-(X_s)\le k\}}\,dB^2_s + M^k_t, \end{aligned} \tag{4.7}$$

where

$$\begin{aligned} M^k_t = &\int_0^t \mathbf{1}_{\{-k\le Y_{s-}\le k\}}\,dM_s - (A^{1,k}_t - B^{1,k}_t) + (A^{2,k}_t - B^{2,k}_t) \\ &+ \int_0^t \mathbf{1}_{\{u^+(X_{s-})\le k\}}\,d(A^1_s - B^1_s) - \int_0^t \mathbf{1}_{\{u^-(X_{s-})\le k\}}\,d(A^2_s - B^2_s). \end{aligned}$$

Since M^k is a martingale under P_x for q.e. $x \in E$, from (4.7) it follows that for q.e. $x \in E$,

$$\begin{aligned} T_k u(x) = &E_x T_k(X_t) + E_x \int_0^t \mathbf{1}_{\{-k\le Y_{s-}\le k\}}(f_u(X_s)\,ds + dA^{\mu_d}_s) \\ &+ E_x B^{1,k}_t - E_x \int_0^t \mathbf{1}_{\{u^+(X_s)\le k\}}\,dB^1_s - E_x B^{2,k}_t + E_x \int_0^t \mathbf{1}_{\{u^-(X_s)\le k\}}\,dB^2_s. \end{aligned}$$

Since $T_k u(X_t) \to 0$ P_x-a.s. as $t \to \infty$, $E_x T_k u(X_t) \to 0$ by the Lebesgue dominated convergence theorem. Therefore from the above equality it follows that

$$T_k u(x) = R(\mathbf{1}_{\{-k\le u\le k\}}(f_u\cdot m + \mu_d)) + R(b^{1,k} - \mathbf{1}_{\{u^+\le k\}}b^1) - R(b^{2,k} - \mathbf{1}_{\{u^-\le k\}}b^2).$$

Set

$$\nu_k = \mathbf{1}_{\{u\notin[-k,k]\}}(f_u\cdot m + \mu_d) + b^{1,k} - \mathbf{1}_{\{u^+\le k\}}b^1 - b^{2,k} + \mathbf{1}_{\{u^-\le k\}}b^2.$$

Then $\nu_k \in \mathcal{M}_{0,b}$ and for q.e. $x \in E$,

$$T_k u(x) = R(f_u \cdot m + \mu_d)(x) + R\nu_k(x). \tag{4.8}$$

On the other hand, by Proposition 3.4, $u(x) = R(f_u \cdot m + \mu_d)(x) + R\mu_c(x)$ for q.e. $x \in E$. Hence $R\nu_k(x) \to R\mu_c(x)$ for q.e. $x \in E$. By Remark 3.5(ii), $T_k u \in D_e(\mathcal{E})$. Finally, since $T_k u = R\lambda_k$ with $\lambda_k = f_u \cdot m + \mu_d + \nu_k \in \mathcal{M}_{0,b}$, repeating step by step the reasoning following [13, (3.14)] shows that $T_k u$ satisfies (4.1), which completes the proof of (i).

(ii) Assume that u is a renormalized solution of (1.1). Then $T_k u$ is a solution in the sense of duality of the linear equation

$$-L(T_k u) = f_u + \mu_d + \nu_k,$$

and hence $T_k u$ is a probabilistic solution of the above equation (see the arguments in [13, p. 1924]). Hence

$$T_k u(x) = E_x\Big(\int_0^\zeta (f_u(X_t)\,dt + dA_t^{\mu_d}) + \int_0^\zeta dA_t^{\nu_k}\Big) = R(f_u \cdot m + \mu_d)(x) + R\nu_k(x)$$

for q.e. $x \in E$. Since $R\nu_k \to R\mu_c$ q.e., letting $k \to \infty$ in the above equation we see that (1.4) is satisfied for q.e. $x \in E$, i.e. u is a solution of (1.1) in the sense of Definition 3.1. By this and Proposition 3.4, u is a probabilistic solution of (1.1). □

Note that by Proposition 3.4, in the formulation of Theorem 4.4 we may replace "probabilistic solution" by "solution in the sense of Definition 3.1", while by [11, Proposition 4.12] we may replace "probabilistic solution" by "solutions in the sense of Stampacchia".

By Theorem 4.4, a probabilistic solution u is a renormalized solution once we know that $f_u \in L^1(E; m)$. We close this section with describing some interesting situations in which this condition holds true.

Proposition 4.5 *Let $\mu \in \mathcal{M}_b$ and let $f : E \times \mathbb{R} \to \mathbb{R}$ be a measurable function such that $f(\cdot, 0) \in L^1(E; m)$ and for every $x \in E$ the mapping $\mathbb{R} \ni y \mapsto f(x, y)$ is continuous and nonincreasing. If u is a probabilistic solution of* (1.1) *then $f_u \in L^1(E; m)$.*

Proof See [11, Proposition 4.8]. □

Following [4,11] we call $\mu \in \mathbb{M}$ a good measure (relative to L and f) if there exists a probabilistic solution of (1.1).

Proposition 4.6 *Assume that f satisfies the assumptions of Proposition 4.5 and $\mu \in \mathbb{M}$ is good relative to L and f. Then there exists a unique renormalized solution of* (1.1). *Moreover, for every $k \geq 0$,*

$$\mathcal{E}(T_k u, T_k u) \leq k(\|\mu\|_{TV} + \|f_u\|_{L^1(E;m)}), \tag{4.9}$$

$$\|f_u\|_{L^1(E;m)} \leq 2\|f(\cdot, 0)\|_{L^1(E;m)} + \|\mu\|_{TV}. \tag{4.10}$$

Proof The existence of a solution follows immediately from Theorem 4.4(i) and Proposition 4.5. Uniqueness follows from Theorem 4.4(ii) and [11, Corollary 4.3]. Estimate (4.9) follows from [11, Theorem 3.3], whereas (4.10) from [11, Proposition 4.8]. □

The following remark shows that the monotonicity assumption imposed on f in Propositions 4.5 and 4.6 can be relaxed in case μ is nonnegative.

Remark 4.7 (i) Assume that $\mu \in \mathbb{M}$ is nonnegative and f satisfies the following "sign condition": for every $x \in E$,

$$yf(x, y) \leq 0, \quad y \in \mathbb{R}. \tag{4.11}$$

Then if u is a probabilistic solution of (1.1), then $u \geq 0$ q.e. To see this, let us consider a reducing sequence $\{\tau_k\}$ for u. Then by (4.2), (4.4) and Itô's formula for convex functions (see [22, Theorem IV.66]), for q.e. $x \in E$ we have

$$u^-(x) = E_x u^-(X_{\tau_k}) - \int_0^{\tau_k} \mathbf{1}_{\{Y_{s-} \leq 0\}} f(X_s, Y_s)\, ds - \int_0^{\tau_k} \mathbf{1}_{\{Y_{s-} \leq 0\}}\, dA_s^{\mu_d} - E_x A_{\tau_k}^2.$$

Since $\mu \geq 0$, $\mu_d \geq 0$ and $\mu_c \geq 0$. In particular, A^{μ_d} is increasing. Since A^2 is also increasing and f satisfies (4.11), it follows that $u^-(x) \leq E_x u^-(X_{\tau_k})$. By this and (3.4), $u(x) \leq \limsup_{k\to\infty} E_x u^-(X_{\tau_k}) = R\mu_c^-(x) = 0$ for q.e. $x \in E$.

(ii) Obviously (4.11) is satisfied if $f(x, 0) = 0$ and f is nonincreasing. Therefore if μ in Proposition 4.5 is nonnegative, then without loss of generality we may assume that $f(\cdot, y) = 0$ for $y \leq 0$, i.e. f satisfies the condition imposed on f in [4] (see [4, Remark 1]) and in [11, Section 5].

(iii) If f satisfies (4.11) and $\mu \in \mathcal{M}_b^+$ is good (relative to L and f), then $f_u \in L^1(E; m)$, and hence there exists a renormalized solution of (1.1). Indeed, if $\mu \geq 0$ then by part (i), $u \geq 0$ q.e., and consequently $Rf_u + R\mu \geq 0$ q.e. and $f_u \leq 0$. Hence $0 \leq R(-f_u) = -Rf_u \leq R\mu$ q.e. By this and Lemma 4.3, $-f_u \cdot m \in \mathcal{M}_b^+$, so $f_u \in L^1(E; m)$.

The problem of existence of solutions of (1.1) for f satisfying the assumptions of Proposition 4.5 [or more general "sign condition" (4.11)] and the related problem of characterizing the set of good measures are very subtle, and are beyond the scope of the present paper. For many positive results in this direction in the case where A is the Laplace operator we defer the reader to [4,21]. Interesting existence and uniqueness results for equations involving the fractional Laplace operator are to be found in [5,11].

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