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# On isometric embeddings and continuous maps onto the irrationals

**Elżbieta Pol**[1] · **Roman Pol**[1]

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**Abstract** Let $f : E \to F$ be a continuous map of a complete separable metric space $E$ onto the irrationals. We shall show that if a complete separable metric space $M$ contains isometric copies of every closed relatively discrete set in $E$, then $M$ contains also an isometric copy of some fiber $f^{-1}(y)$. We shall show also that if all fibers of $f$ have positive dimension, then the collection of closed zero-dimensional sets in $E$ is non-analytic in the Wijsman hyperspace of $E$. These results, based on a classical Hurewicz's theorem, refine some results from Pol and Pol (Isr J Math 209:187–197, 2015) and answer a question in Banakh et al. (in: Pearl (ed) Open problems in topology II. Elsevier, Amsterdam, 2007).

**Keywords** Isometric embeddings · Effros Borel spaces · Zero-dimensional sets

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## 1 Introduction

In [13] we proved that each complete separable metric space containing isometric copies of every countable complete metric space contains isometric copies of every separable metric space.

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✉ Elżbieta Pol
E.Pol@mimuw.edu.pl

Roman Pol
R.Pol@mimuw.edu.pl

1 University of Warsaw, Warsaw, Poland

We shall refine this result to the following effect.

**Theorem 1.1** *Let $f : E \to F$ be a continuous map of a complete separable metric space onto a non-$\sigma$-compact metric space. Then there exists a relatively discrete set $S$ in $E$ such that, for any complete separable metric space $M$ containing isometric copies of every subset of $S$ closed in $E$, some fiber $f^{-1}(y)$ embeds isometrically in $M$.*

The result from [13] follows from this theorem, if we consider the restriction map $f : C[0, 1] \to C[\frac{1}{2}, 1]$ (recall that by the Banach–Mazur theorem, cf. [8, Theorem 5.17], the space $(C(I), d_{\sup})$ of all real-valued continuous functions on the interval $I = [0, 1]$, equipped with the metric $d_{\sup}(f, g) = \sup\{| f(t) - g(t) | : t \in I\}$, is isometrically universal for all separable metric spaces).

Also, as in [13], one can replace in this theorem isometries by uniform homeomorphisms.

The proofs will go along the same lines as in [13], and an essential part of the reasonings can be taken directly from [13], cf. Sect. 4.

However, a classical Hurewicz's theorem on non-analyticity of the set of compact subsets of the rationals is applied in a different way than in [13]. We shall prove a result based on the Hurewicz theorem in a slightly more general form than needed for Theorem 1.1 in Sect. 3, to establish a link with some questions concerning the dimension, discussed in Sect. 5.

## 2 Preliminaries

### 2.1 The Effros Borel spaces

Our terminology related to the descriptive set theory follows [7,9]. An analytic space is a metrizable continuous image of the irrationals.

Given an analytic space $E$, we denote by $F(E)$ the space of closed subsets of $E$ and $\mathcal{B}_{F(E)}$—the Effros Borel structure in $F(E)$, is the $\sigma$-algebra in $F(E)$ generated by the sets $\{A \in F(E) : A \cap U \neq \emptyset\}$, where $U$ is open in $E$.

We shall say that $\mathcal{A} \subset F(E)$ is a Souslin set in the Effros Borel space $(F(E), \mathcal{B}_{F(E)})$ if $\mathcal{A}$ is a result of the Souslin operation on sets from $\mathcal{B}_{F(E)}$.

If $X$ is a compact metrizable space, we shall consider the hyperspace $F(X)$ with the Vietoris topology and then $\mathcal{B}_{F(X)}$ coincides with the $\sigma$-algebra of Borel sets in the compact metrizable space $F(X)$.

If $X$ is a compact metrizable extension of an analytic set $E \subset X$, the map $A \to \overline{A}$ (the closure is taken in $X$) from $F(E)$ to $F(X)$ is a Borel isomorphism, with respect to the Effros Borel structures, onto the analytic subspace $\{\overline{A} : A \in F(E)\}$ of the hyperspace $F(X)$ and hence Souslin sets in $F(E)$ are mapped onto analytic sets in $F(X)$, cf. [7, Section 2]. In particular, if $E \subset G \subset X$ and $G$ is analytic, the collection of closures of elements of $F(E)$ in $G$ is a Souslin set in $F(G)$.

### 2.2 The Hurewicz theorem

Let $I = [0, 1]$ and let $\mathbb{Q}$ be the set of rationals in $I$.

The classical Hurewicz theorem asserts that any Souslin set in $F(I)$ containing all compact subsets of $\mathbb{Q}$, contains an element intersecting $I\backslash\mathbb{Q}$.

We shall derive from this theorem the following observation, which we shall use in the next section.

Let us arrange points of $\mathbb{Q}$ into a sequence $q_1, q_2, \ldots$ (without repetitions), let

$$D = \left\{ \left( q_n, \frac{1}{m} \right) : n = 1, 2, \ldots, \ m \geq n \right\}, \ L = (I\backslash\mathbb{Q}) \times \{0\}, \tag{2.1}$$

let

$$T = L \cup D \tag{2.2}$$

be the subspace of the plane (notice that $D$ is relatively discrete in $T$), and let

$$\mathcal{D} = \{A \subset D : A \text{ is closed in } T\}. \tag{2.3}$$

**Lemma 2.2.1** *For any Souslin set $\mathcal{E}$ in $F(T)$ containing $\mathcal{D}$, some element of $\mathcal{E}$ intersects $L$.*

*Proof* For $A \in F(T)$, $\overline{A}$ will denote the closure in the plane. As was recalled in 2.1, the set $\{\overline{A} : A \in \mathcal{E}\}$ is analytic in $F(\overline{T})$ (notice that $\overline{T} = (I \times \{0\}) \cup D$), hence the set $\{(K, \overline{A}) \in F(I) \times F(\overline{T}) : A \in \mathcal{E} \text{ and } K \times \{0\} \subset \overline{A}\}$ is analytic in the product of the hyperspaces, and so is its projection onto $F(I)$,

$$\mathcal{E}^\star = \{K \in F(I) : K \times \{0\} \subset \overline{A} \text{ for some } A \in \mathcal{E}\}. \tag{2.4}$$

If $K \subset \mathbb{Q}$ is compact, $A = D \cap (K \times I)$ is closed in $T$ and $K \times \{0\} \subset \overline{A}$, hence $K \in \mathcal{E}^\star$, cf. (2.4). By the Hurewicz theorem, there is $A \in \mathcal{E}$ such that $\overline{A}$ intersects $L$, cf. (2.1) and (2.4), and since $A$ is closed in $T$, $A$ intersects $L$. □

### 2.3 A remark on continuous maps onto the irrationals

We shall need the following observation. This is close to some well-known results, but for readers convenience, we shall provide a brief justification.

**Lemma 2.3.1** *Let $f : E \to F$ be a continuous map of an analytic space onto a non-$\sigma$-compact metrizable space. There is a closed copy of the irrationals $P$ in $F$ and continuous maps $g_n : P \to E$ such that, for each $t \in P$, $\{g_n(t) : n = 1, 2, \ldots\}$ is a dense subset of $f^{-1}(t)$.*

*Proof* Let $p : \mathbb{N}^\mathbb{N} \to E$ be a continuous surjection of the irrationals onto the analytic space $E$.

Then $u = f \circ p : \mathbb{N}^\mathbb{N} \to F$ is a continuous surjection onto a non-$\sigma$-compact metrizable space and one can find a closed copy of the irrationals $P$ in $F$ such that the restriction map $u \mid u^{-1}(P) : u^{-1}(P) \to P$ is open, cf. [14, proof of Theorem 3.1].

By a selection theorem of Michael [10], one can define a sequence of continuous selections $w_n : P \to u^{-1}(P)$ for the lower-semicontinuous multifunction $t \to u^{-1}(t)$ such that, for each $t \in P$, the set $\{w_n(t) : n = 1, 2, \ldots\}$ is dense in $u^{-1}(t)$.

Then the functions $g_n = p \circ w_n : P \to f^{-1}(P)$ satisfy the assertion. □

## 3 An application of the Hurewicz theorem

The following proposition strengthens a known fact that, for the irrationals $\mathbb{N}^{\mathbb{N}}$, any Souslin set in $F(\mathbb{N}^{\mathbb{N}})$ containing all countable closed sets in $\mathbb{N}^{\mathbb{N}}$, contains also a non-$\sigma$-compact set (this is stated in [9, Exercises 27.8, 27.9]; to derive this fact from the proposition, notice that $\mathbb{N}^{\mathbb{N}}$ is homeomorphic to $\mathbb{N}^{\mathbb{N}} \times \mathbb{N}^{\mathbb{N}}$ and consider the projection $\mathbb{N}^{\mathbb{N}} \times \mathbb{N}^{\mathbb{N}} \to \mathbb{N}^{\mathbb{N}}$).

The setting is a bit more general than needed for Theorem 1.1, but it is useful to establish connections with some topics in the dimension theory, discussed in Sect. 5.

**Proposition 3.1** *Let $f : E \to F$ be a continuous map of an analytic space onto a non-$\sigma$-compact metrizable space. Then there exists a relatively discrete set $S$ in $E$ such that for any Souslin set $\mathcal{A}$ in $F(E)$ containing all subsets of $S$ closed in $E$, there are $A \in \mathcal{A}$ and $y \in F$ with $f^{-1}(y) \subset A$.*

*Proof* Let $P$ be a closed copy of the irrationals in $F$ and $g_n : P \to E$ continuous maps described in Lemma 2.3.1, and let $T = L \cup D$ be the subspace of the plane defined in (2.1) and (2.2).

Since $T$ is a zero-dimensional $G_\delta$-subset of the plane, there is a homeomorphic embedding

$$h : T \to P, h(T) \text{ closed in } P. \tag{3.1}$$

Let us arrange points of $D$ into a sequence without repetitions

$$D = \{d_1, d_2, \ldots\} \text{ and } c_n = h(d_n). \tag{3.2}$$

We shall check that, cf. (3.2),

$$S = \{g_m(c_n) : n = 1, 2, \ldots, \ m \le n\} \subset E \tag{3.3}$$

satisfies the assertion of the proposition.

Since $g_m(c_n) \in f^{-1}(c_n)$, $f(S) = h(D)$ is relatively discrete and $S$ intersects each fiber of $f$ in at most finite set, cf. (3.3). Therefore $S$ is relatively discrete.

Let, for $X \in F(T)$,

$$\varphi(X) = f^{-1}(h(X \cap L)) \cup (S \cap f^{-1}(h(X \cap D))). \tag{3.4}$$

Since all accumulation points of $S$ in $E$ are in $f^{-1}(h(L))$ and $h(X)$ is closed in $F$, cf. (3.1), we have $\varphi(X) \in F(E)$.

We shall check that

$$\varphi : F(T) \to F(E) \text{ is Borel,} \tag{3.5}$$

with respect to the Effros Borel structure.

To that end, let us fix an open set $U$ in $E$, and let

$$\mathcal{U} = \{X \in F(T) : \varphi(X) \cap U \neq \emptyset\}. \tag{3.6}$$

Let $X \in \mathcal{U}$. If for some $m \leq n$, $d_n \in X$ and $g_m(c_n) \in U$, cf. (3.2), (3.3), (3.4), the element $\{Y \in F(T) : d_n \in Y\}$ of $\mathcal{B}_{F(T)}$ contains $X$ and is contained in $\mathcal{U}$.

Let $a \in X \cap L$ and $f^{-1}(h(a)) \cap U \neq \emptyset$. Since the points $g_m(h(a))$ are dense in $f^{-1}(h(a))$, there is $m$ such that $g_m(h(a)) \in U$. Let $V$ be a neighbourhood of $h(a)$ in $F$ such that $g_m(V) \subset U$, and let us pick a rectangle $J = (r, s) \times [0, \frac{1}{p})$ disjoint from $\{d_1, \ldots, d_m\}$ with $r, s \in \mathbb{Q}$, containing $a$, such that $h(J \cap T) \subset V$. If $Y \in F(T)$ hits $J$, there is either $b \in Y \cap L$ with $h(b) \in V$ and then $f^{-1}(h(b)) \subset \varphi(Y)$ intersects $U$, or there is $d_n \in Y \cap J$ with $n > m$ and then, cf. (3.3), (3.4), $g_m(c_n) \in \varphi(Y) \cap U$.

Therefore the element $\{Y \in F(T) : Y \cap J \neq \emptyset\}$ of $\mathcal{B}_{F(T)}$ contains $X$ and is contained in $\mathcal{U}$.

We demonstrated that $\mathcal{U}$ is a countable union of elements of $\mathcal{B}_{F(T)}$, hence belongs to the Effros Borel structure of $F(T)$.

Having checked (3.5), let us consider the set

$$\mathcal{S} = \{A \subset S : A \in F(E)\} \tag{3.7}$$

and let

$$\mathcal{S} \subset \mathcal{A}, \ \mathcal{A} \text{ is Souslin in } (F(E), \mathcal{B}_{F(E)}). \tag{3.8}$$

By (3.5),

$$\mathcal{E} = \varphi^{-1}(\mathcal{A}) \text{ is Souslin in } (F(T), \mathcal{B}_{F(T)}). \tag{3.9}$$

If $X \subset D$ is closed in $T$, $h(X)$ is closed in $F$ and $\varphi(X)$ is closed in $E$, cf. (3.4), hence $\varphi(X) \in \mathcal{S}$, cf. (3.7). Therefore, by (3.8), for the set $\mathcal{D}$ defined in (2.3), we have $\mathcal{D} \subset \mathcal{E}$ and Lemma 2.2.1 provides $X \in \mathcal{E}$ and a point $a \in X \cap L$. By (3.4) and (3.9) we get $A = \varphi(X) \in \mathcal{A}$ and $f^{-1}(h(a)) \subset A$. □

## 4 Proof of Theorem 1.1

We shall recall briefly some reasonings from [13] to derive this theorem from Proposition 3.1.

Given $f : E \to F$ as in this theorem, let us pick $S$ satisfying the assertion of Proposition 3.1.

Let $e$ be the complete metric on $E$ and let $(M, d)$ be any complete separable metric space, containing isometric copies of every subset of $S$ closed in $E$. Let

$$\mathcal{H} = \{T \in F(E \times M) : \text{ for every } (x_1, y_1), (x_2, y_2) \in T,\ e(x_1, x_2) = d(y_1, y_2)\}.$$

One checks, cf. [13, p. 193], that $\mathcal{H}$ is in $\mathcal{B}_{F(E\times M)}$ and the map $T \to \pi(T)$ associating to $T \in \mathcal{H}$ its projection onto $E$ is a Borel map $\pi : \mathcal{H} \to F(E)$.

Therefore $\mathcal{A} = \pi(\mathcal{H})$ is a Souslin set in $(F(E), \mathcal{B}_{F(E)})$. If $X \subset S$ is closed in $E$, there is an isometry $f : X \to f(X) \subset M$ and the graph of $f$ is an element of $\mathcal{H}$.

It follows that the Souslin set $\mathcal{A}$ contains all subsets of $S$ closed in $E$, and by the choice of $S$, some $A \in \mathcal{A}$ contains a fiber $f^{-1}(y)$.

Now, $A = \pi(T)$ and $T$ is the graph of an isometry that embeds $A$ in $M$. In effect, $f^{-1}(y)$ embeds isometrically in $M$.

## 5 The collections of zero-dimensional sets in Effros Borel spaces

Our terminology concerning the dimension theory follows [15].

Given an analytic space, we shall write

$$F_0(E) = \{A \in F(E) : \dim A = 0\}. \tag{5.1}$$

We shall derive from Proposition 3.1 the following result.

**Proposition 5.1** *Let $E$ be an analytic space that admits a continuous map $f : E \to F$ onto a non-$\sigma$-compact metrizable space such that all fibers $f^{-1}(y)$ have positive dimension. Then for any analytic extension $G$ of $E$ with* $\dim(G\backslash E) \le 0$*, the set $F_0(G)$ is not Souslin in the Effros Borel space $(F(G), \mathcal{B}_{F(G)})$.*

*Proof* By Proposition 3.1, there is a relatively discrete set $S$ in $E$ such that for any Souslin set $\mathcal{A}$ in $F(E)$ containing $\mathcal{S} = \{A \in F(E) : \emptyset \ne A \subset S\}$, some element of the set $\mathcal{A}$ contains a fiber of $f$ and hence has positive dimension.

Now, consider an analytic extension $G$ of $E$ with $\dim(G\backslash E) \le 0$ and, aiming at a contradiction assume that $F_0(G)$ is Souslin in $F(G)$. As was recalled in Sect. 2.1, the map $A \to \overline{A}$ from $F(E)$ to $F(G)$ is Borel, and hence we would get that the set $\mathcal{A} = \{A \in F(E) :\ \dim\overline{A} \le 0\}$ is Souslin in $F(E)$.

If $A \in \mathcal{S}$, then $A$ is a relatively discrete closed set in $E$, and hence $\overline{A}\backslash A$ is a closed subset of $G$ contained in $G\backslash E$. This implies that $\dim\overline{A} = 0$, i.e., $\mathcal{S} \subset \mathcal{A}$. However, all members of $\mathcal{A}$ are zero-dimensional, which contradicts properties of $\mathcal{S}$. □

In particular, if $\mathbb{P}$ is the set of irrationals in $I = [0, 1]$,

$$F_0(\mathbb{P} \times I) \text{ is not Souslin in } F(\mathbb{P} \times I) \tag{5.2}$$

(this rectifies a remark in [5, §3.A]).

Banakh et al. [1, Question 9.12], asked about the Borel type of the collection $F_0(E)$ in the space $CL(E) = F(E)\backslash\{\emptyset\}$, when $E$ is a completely metrizable separable space, and $CL(E)$ is considered with the Wijsman topology $\tau_W$, determined by some metric $d$ generating the topology of $E$ (i.e., $\tau_W$ is the weakest topology making all functionals $A \to \mathrm{dist}(z, A)$, $z \in E$, continuous), cf. [2,4].

The Wijsman hyperspace $(CL(E), \tau_W)$ is completely metrizable, separable, cf. [6], and the Borel sets with respect to $\tau_W$ coincide with the members of the Effros Borel structure in $CL(E)$. Therefore,

$$F_0(\mathbb{P} \times I) \text{ is not a Borel (or even Souslin) set in CL } (\mathbb{P} \times I). \tag{5.3}$$

One can check that its complement $F(\mathbb{P} \times I)\backslash F_0(\mathbb{P} \times I)$ is Souslin. Let us consider, however, the subspace $I^2\backslash\mathbb{Q}^2$ of the square, $\mathbb{Q} = I\backslash\mathbb{P}$. Since $(I^2\backslash\mathbb{Q}^2)\backslash(\mathbb{P}\times I) = \mathbb{Q}\times\mathbb{P}$ is zero-dimensional, also $F_0(I^2\backslash\mathbb{Q}^2)$ is not Souslin in $F(I^2\backslash\mathbb{Q}^2)$, by Proposition 5.1. But it is not clear to us whether $F(I^2\backslash\mathbb{Q}^2)\backslash F_0(I^2\backslash\mathbb{Q}^2)$ is Souslin.

In fact, we do not know an answer to the following general question.

**Question 5.2** *Does there exist an analytic space $E$ such that $F(E)\backslash F_0(E)$ is not Souslin in the Effros Borel structure?*

This question is related to the following problem, asked in [11], where countable-dimensional spaces are countable unions of zero-dimensional spaces.

**Problem 5.3** *Is the collection $\mathcal{C}$ of all countable-dimensional compact sets in the Hilbert cube $I^{\mathbb{N}}$ coanalytic in the hyperspace $F(I^{\mathbb{N}})$ equipped with the Vietoris topology?*

To see the link between these two questions, let us consider a Borel set $E \subset I^{\mathbb{N}}$ such that $I^{\mathbb{N}}\backslash E$ is countable-dimensional and all countable-dimensional subsets of $E$ are at most zero-dimensional, cf. [12]. We shall assume in addition that $E$ is disjoint from the set $\Sigma$ consisting of points in $I^{\mathbb{N}}$ with all but finitely many coordinates zero.

By [3, Ch.V, §5], there is a homeomorphism $h : I^{\mathbb{N}}\backslash\Sigma \to \mathbb{R}^{\mathbb{N}} \times \mathbb{R}^{\mathbb{N}}$ (where $\mathbb{R}$ is the real line), let $p : \mathbb{R}^{\mathbb{N}} \times \mathbb{R}^{\mathbb{N}} \to \mathbb{R}^{\mathbb{N}}$ be the projection and let $f = p \circ h \mid E : E \to \mathbb{R}^{\mathbb{N}}$. Then $f$ is a continuous surjection whose all fibers are uncountable-dimensional. Therefore, by Corollary 5.1, $F_0(E)$ is not Souslin in the Effros Borel space.

We do not know if the set $\mathcal{E} = F(E)\backslash F_0(E)$ is Souslin. Let us show, however, that if this is the case, $\mathcal{C}$ in Problem 5.3 is coanalytic.

Suppose that $\mathcal{E}$ is Souslin in $(F(E), \mathcal{B}_{F(E)})$. Then, as was noticed in Sect. 2.1, the collection $\mathcal{E}^\star = \{\overline{A} : A \in \mathcal{E}\}$ of the closures in $I^{\mathbb{N}}$ is analytic in the hyperspace $F(I^{\mathbb{N}})$. Now, $K \in F(I^{\mathbb{N}})\backslash\mathcal{C}$ if and only if $K \cap E$ is uncountable-dimensional, which is equivalent to $K \cap E \notin F_0(E)$. Therefore, $F(I^{\mathbb{N}})\backslash\mathcal{C}$ is the projection of the analytic set $\{(K, L) \in F(I^{\mathbb{N}}) \times F(I^{\mathbb{N}}) : L \subset K \text{ and } L \in \mathcal{E}^\star\}$, hence it is analytic.

*Added in the revision* Concerning Question 5.2, Debs and Saint Raymond gave in a recent paper "The descriptive complexity of the set of all closed zero-dimensional subsets of a Polish space" a subtle construction of a $G_\delta$-set $E$ in $I^3$ such that $F_0(E)$ is not even a $C$-set in $F(E)$ (in particular, $F(E)\backslash F_0(E)$ is not Souslin). The question concerning $F_0(I^2\backslash\mathbb{Q}^2)$ remains open.

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