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Rings of congruence preserving functions

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Abstract Let $C_0(G)$ denote the near-ring of congruence preserving functions of the group G. We investigate the question "When is $C_0(G)$ a ring?". We obtain information externally via the lattice structure of the normal subgroups of G and internally via structural properties of the group G.

Keywords Congruence preserving functions · 1-affine complete · Normal subgroup lattice

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1 Introduction: background and notation

Let $G = \langle G, +, 0\rangle$ be a group, written additively but not necessarily abelian, with neutral element 0. The structure of the near-ring $C_0(G) = \langle C_0(G), +, \circ\rangle$ of zero fixing congruence preserving functions on G has been the topic of several previous

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investigations [1,4]. In this paper we initiate the study of characterizing those groups G such that $C_0(G)$ is a ring.

This investigation also has roots in universal algebra. Recall that a unary polynomial function $p : G \to G$ is a function that can be written in the form $p(x) := a_0 + k_0 x + a_1 + k_1 x + \cdots + a_{n-1} + k_{n-1} x + a_n$, $x \in G$, n a nonnegative integer, $a_0, a_1, \ldots, a_n \in G$, $k_0, \ldots, k_{n-1} \in \mathbb{Z}$. We let $P_0(G) = \langle P_0(G), +, \circ \rangle$ denote the near-ring of zero preserving polynomial functions on G (See [1,11].)

The near-ring $C_0(G)$ is a subnear-ring of the near-ring $M_0(G) := \{f : G \to G \mid f(0) = 0\}$ of zero fixing self maps on G where, as usual, the operations in $M_0(G)$ are pointwise addition of functions and composition of functions. Let Inn (G) denote the semigroup (under composition) of inner automorphisms of the group G and let $I(G)$ denote the subnear-ring of $M_0(G)$ generated additively by Inn (G). From the definitions we see $P_0(G) = I(G)$.

Now let $f \in C_0(G)$. Then by definition, for each congruence, ρ of G, for each $x, y \in G$, $x \rho y$ implies $f(x) \rho f(y)$. As is well-known, there is a lattice isomorphism between the congruence lattice, Con(G), of congruences on G and the lattice, $\eta(G)$, of normal subgroups of G. Thus $f \in M_0(G)$ is congruence preserving if f is compatible with every normal subgroup of G. That is for $x, y \in G$ we have $f \in C_0(G)$ if and only if $x + H = y + H$ implies $f(x) + H = f(y) + H$, for each $H \in \eta(G)$.

Recall for any subgroup H of G the *normal closure* of H in G, denoted by $\overline{H}$, is defined by $\overline{H} = \bigcap\{N \trianglelefteq G \mid H \subseteq N\}$. For $x \in G$ we write $\overline{x}$ for $\overline{\langle x \rangle}$, the principal closure of x. Thus we get the following characterization of congruence preserving functions: Let $f \in M_0(G)$. Then $f \in C_0(G)$ if and only if $f(x) - f(y) \in \overline{x - y}$ for each $x, y \in G$.

Using the above definitions one finds that every zero fixing unary polynomial on G is congruence preserving so $I(G) = P_0(G) \subseteq C_0(G) \subseteq M_0(G)$. If every congruence preserving function is also a polynomial then the group G is said to be *1-affine complete*. Finite abelian 1-affine complete groups have been characterized by Nöbauer [14]. All 1-affine complete groups of order up to 100 can be found in [17]. In relation to our problem under consideration, $C_0(G)$ will be a ring if $P_0(G)$ is a ring and G is 1-affine complete.

It is known when $I(G) = P_0(G)$ is a ring [8]. A group G is said to be a 2-Engel group if $[x, [x, y]] = 0$ for all $x, y \in G$. Equivalently (see [8,12,13]) G is a 2-Engel group if every element of G commutes with all of its conjugates. A 2-Engel group G is nilpotent of class at most 3 and, for finite groups, a 2-Engel group of class 3 must be a group of order 3^n. The smallest 2-Engel group of class 3 is the Burnside group, $B(3, 3)$, of order 3^7.

Theorem 1.1 (Chandy) *The near-ring $I(G)$ is a ring if and only if G is a 2-Engel group. Moreover, $I(G)$ is a commutative ring if and only if G is of class 2.*

In the remainder of this paper we restrict to finite groups G. In the next section we first show when considering $C_0(G)$ we may restrict to p-groups, p a prime. Using results of Nöbauer [14] we completely characterize those finite abelian groups G such that $C_0(G)$ is a ring.

In Sect. 3 we turn to nonabelian groups and find several necessary conditions for $C_0(G)$ to be a ring. In this section we focus on certain properties of the lattice

$\eta(G)$ of normal subgroups. In the final section we consider internal properties of the group G and we conclude with a complete answer to our question for groups of order $p^n,\ 1 \le n \le 5,\ p > 2$.

2 The abelian case

We begin this section by showing that, for finite groups, we can restrict to p-groups, where as usual p denotes a prime integer. We use results of Nöbauer, [14], to obtain this restriction.

We recall from [14] that a direct sum $G = G_1 \oplus \cdots \oplus G_n$ is said to be *skew-free* if every congruence ρ of G is of the form $\rho = \rho_1 + \cdots + \rho_n$ where the ρ_i are congruences on the $G_i,\ i = 1, 2, \ldots, n$. In particular when G is a finite nilpotent group with direct sum of its unique Sylow subgroups $G = S_{p_1}(G) \oplus \cdots \oplus S_{p_n}(G)$ then G is skew-free. Recall from the introduction that $\mathrm{Con}(G)$ is lattice isomorphic to $\eta(G)$, the lattice of normal subgroups of G.

Theorem 2.1 [14, Satz 1] *Let $G = A \oplus B$ be skew-free. Then the map $\psi : C_0(G) \to C_0(A) + C_0(B)$ given by $\psi(\rho) = (\rho_A, \rho_B)$, $\rho \in C_0(G)$ is a near-ring isomorphism.*

From straight forward calculations one finds the following theorem and corollary.

Theorem 2.2 *Let G be a finite nilpotent group, $G = S_{p_1}(G) \oplus \cdots \oplus S_{p_n}(G)$. Then $C_0(G) \cong C_0(S_{p_1}(G)) \oplus \cdots \oplus C_0(S_{p_n}(G))$.*

Corollary 2.3 *Let G be a finite group with Sylow decomposition as in Theorem 2.2. Then $C_0(G)$ is a ring if and only if $C_0(S_{p_i}(G))$ is a ring for each $i \in \{1, 2, \ldots, n\}$.*

Hence we only need to consider groups, G, of prime power order when investigating the structure of the near-ring, $C_0(G)$, of zero fixing congruence preserving functions on G.

We now turn to abelian groups. We need a further result of Nöbauer [14].

Theorem 2.4 [14, Satz 3,4] *Let p be a prime, $\mathbb{Z}_{p^\alpha}$ be the cyclic group of order p^α and let $A = \mathbb{Z}_{p^{\alpha_1}} \oplus \cdots \oplus \mathbb{Z}_{p^{\alpha_n}}$ with $\alpha_1 \ge \alpha_2 \ge \cdots \ge \alpha_n$. Then A is 1-affine complete if and only if one of the following conditions holds:*

(a) $n > 1,\ \alpha_1 = \alpha_2,\ p$ *arbitrary,*
(b) $n > 1,\ \alpha_1 = \alpha_2 + 1,\ p = 2$ *or*
(c) $n = 1,\ \alpha_1 = 1,\ p = 2.$

Theorem 2.5 *Let A be an abelian group of prime power order. The following are equivalent:*

(1) A is 1-affine complete;
(2) $C_0(A)$ is a ring;
(3) $C_0(A)$ is a commutative ring.

Proof Since A is abelian, A is a 2-Engel group of nilpotency class at most 2 so $I(A) = P_0(A)$ is a commutative ring. If A is 1-affine complete then $C_0(A)$ is a ring. Thus (1) implies (3).

It is clear that (3) implies (2) so it remains to show (2) implies (1). We assume A is not 1-affine complete and show $C_0(A)$ is not a ring. From Theorem 2.4 we know the form A has to be so that A is not 1-affine complete.

Case (i) $|A| = p^m$ for some prime $p > 2$. In this case we must have $A \cong \mathbb{Z}_{p^{\alpha_1}} \oplus \cdots \oplus \mathbb{Z}_{p^{\alpha_n}}$ with $n = 1$ or $\alpha_1 > \alpha_2 \geq \cdots \geq \alpha_n$. We let $\mathbb{Z}_{p^{\alpha_1}} = \langle g \rangle$ and let $D = \{x \in A \mid p^{\alpha_1 - 1}x = 0\} = \{(a_1, \ldots, a_n) \mid a_1 \in \langle pg \rangle\}$.
Define $c : A \to A$ by

$$c(x) = \begin{cases} (0, \ldots, 0), & x \in D \\ (p^{\alpha_1 - 1}g, 0, \ldots, 0), & x \notin D. \end{cases}$$

We show $c \in C_0(A)$. To this end, let $x, y \in A$. If $x, y \in D$ or $x, y \notin D$ then $c(x) - c(y) = (0, \ldots, 0) \in \overline{x - y}$ so we take $x \notin D$, $y \in D$. Thus $c(x) - c(y) = (p^{\alpha - 1}g, 0, \ldots, 0)$. Now $p^{\alpha_1 - 1}(x - y) = (p^{\alpha_1 - 1}kg, 0, \ldots, 0)$, $p \nmid k$ so $x \notin D$. But then $(p^{\alpha_1 - 1}g, 0, \ldots, 0) \in \overline{x - y}$ so $c \in C_0(A)$. Further, $[c(\mathrm{id} + \mathrm{id})]((g, 0, \ldots, 0)) = c(\mathrm{id}((g, 0, \ldots, 0)) + \mathrm{id}((g, 0, \ldots, 0))) = c((2g, 0, \ldots, 0)) = (p^{\alpha_1 - 1}g, 0, \ldots, 0)$ while $[c \circ \mathrm{id} + c \circ \mathrm{id}]((g, 0, \ldots, 0)) = 2c((g, 0, \ldots, 0)) = 2(p^{\alpha_1 - 1}g, 0, \ldots, 0) \neq (p^{\alpha_1 - 1}g, 0, \ldots, 0)$. Hence $C_0(G)$ is not a ring.

Case (ii) $|A| = 2^m$. In this case $A = \mathbb{Z}_{2^{\alpha_1}} \oplus \cdots \oplus \mathbb{Z}_{2^{\alpha_n}}$ with either $n > 1$ and $\alpha_1 - 1 > \alpha_2 \geq \cdots \geq \alpha_n$ or $n = 1$ and $\alpha_1 > 1$. Again we handle both cases together and as above we let $\mathbb{Z}_{2^{\alpha_1}} = \langle g \rangle$. Let $D = \{x \in A \mid 2^{\alpha_1 - 2}x = 0\} = \{(a_1, \ldots, a_n) \mid a_1 \in \langle 4g \rangle\}$ and further let $\bar{D} = D \cup (D + (2g, 0, \ldots, 0))$.
Define $c : A \to A$ by

$$c(x) = \begin{cases} (0, 0, \ldots, 0), & x \in \bar{D} \\ (2^{\alpha_1 - 2}g, 0, \ldots, 0), & x \notin \bar{D}. \end{cases}$$

As above we show $c \in C_0(A)$. Let $x, y \in A$, $x \notin \bar{D}$, $y \in \bar{D}$. Then $c(x) - c(y) = (2^{\alpha_1 - 2}g, 0, \ldots, 0)$. Now since $x \notin \bar{D}$, $x = (kg, 0, \ldots, 0)$ where $k \in \{1, 3\}$. Thus $2^{\alpha_1 - 2}(x - y) = (2^{\alpha_1 - 2}kg, 0, \ldots, 0)$ so $(2^{\alpha_1 - 2}g, 0, \ldots, 0) \in \overline{x - y}$. Using $f = \mathrm{id}$, $h = 3 \cdot \mathrm{id}$ we have $[c(f + g)]((g, 0, \ldots, 0)) = [c(\mathrm{id} + 3 \cdot \mathrm{id})]((g, 0, \ldots, 0)) = c(\mathrm{id}((g, 0, \ldots, 0)) + 3 \cdot \mathrm{id}((g, 0, \ldots, 0))) = c((4g, 0, \ldots, 0)) = (0, \ldots, 0) \neq c((g, 0, \ldots, 0)) + c((3g, 0, \ldots, 0))$. So $C_0(A)$ is not a ring. □

We now have a characterization of those finite abelian groups A for which $C_0(A)$ is a ring.

Corollary 2.6 *Let A be a finite abelian group. The following are equivalent:*

(1) A is 1-affine complete;
(2) $C_0(A)$ is a ring;
(3) $C_0(A)$ is a commutative ring.

Proof As in Theorem 2.5 (1) implies (3) and (3) implies (2). Using the Sylow decomposition of A we see that (2) implies (1) follows from Theorem 2.5 and Nöbauer [14, Lemma 5]. □

In the next section we give several necessary conditions for $C_0(G)$ to be a ring.

3 Lattice conditions

We start with some conditions on the congruence lattice, $\mathrm{Con}(G)$. Since $\mathrm{Con}(G)$ is lattice isomorphic to the normal subgroup lattice, $\eta(G)$, we often state our properties in terms of normal subgroups.

We recall that G must be a 2-Engel group, and thus nilpotent of class at most 3, for $C_0(G)$ to be a ring.

Our first lattice concept is that of splitting pair. This property has been used previously [3,5,15]. Let $D, E \in \eta(G)$, $D \subset G$, $\{0\} \subset E$. The pair (D, E) is called a *splitting pair* if for each $N \in \eta(G)$, $N \subseteq D$ or $N \supseteq E$. If G has a splitting pair then *G splits*.

Now let (D, E) be a splitting pair for G and let $0 \neq b \in E$. Define $f : G \to G$ by

$$f(x) = \begin{cases} 0, & x \in D \\ b, & \text{otherwise.} \end{cases}$$

We show $f \in C_0(G)$. Let $x, y \in G$ and let $H \in \eta(G)$ with $x + H = y + H$. If $E \subseteq H$ then since $f(x) - f(y) \in \{-b, 0, b\} \subseteq E$ we get $f(x) + H = f(y) + H$. If $H \subseteq D$ and $x \in D$ then $x + H \subseteq D$ and so $y + H \subseteq D$ which means $y \in D$. By symmetry if $x \notin D$ then $y \notin D$, hence in both cases $f(x) + H = f(y) + H$. This establishes that $f \in C_0(G)$. Now if $C_0(G)$ is a ring then for $v \notin D$, $[f \circ (\mathrm{id} + \mathrm{id})](v) = [(f \circ \mathrm{id} + f \circ \mathrm{id})](v)$ or $f(2v) = 2f(v)$. If $2v \notin D$ then $b = 2b$, which contradicts $b \neq 0$. Therefore $2v \in D$ and further $0 = 2b$. Since $0 \neq b$ was arbitrary in E we get $E \cong (\mathbb{Z}_2)^n$ for some $n > 0$. This establishes:

Theorem 3.1 *Let G be a finite group such that* $2 \nmid |G|$. *If G splits then* $C_0(G)$ *is not a ring.*

Proof From the above discussion, when G splits then G has a subgroup $E \cong (\mathbb{Z}_2)^n$, $n > 0$ which is a contradiction since $2 \nmid |G|$. □

In particular if G is a finite p-group, $p > 2$, and G splits, then $C_0(G)$ is not a ring. The situation is different in the non-split case as the next examples illustrate. These examples and some of the calculations have been done with GAP using the package Sonata [2].

Example 3.2 (1) Group with GAP index $3^7/6010$. $G = \langle e_1, e_2, e_3, e_4, c_1, c_2, c_3\rangle$, $3e_i = 3c_j = 0$, $[e_i, c_j] = [c_k, c_j] = 0$, $[e_1, e_2] = c_1$, $[e_1, e_3] = c_2$, $[e_2, e_3] = c_3$ otherwise $[e_l, e_m] = 0$, $i = 1, 2, 3, 4$, $j, k = 1, 2, 3$. Thus G is a group of exponent 3, nilpotent of class 2 with $G' = \langle c_1, c_2, c_3\rangle \subseteq Z(G)$ [9]. From GAP, G does not split but is 1-affine complete so $C_0(G)$ is a ring since nilpotent of class 2 means G is 2-Engel.

(2) GAP index $3^7/6576$. $G = \langle e_1, e_2, e_3, e_4, c_1, c_2, c_3\rangle$, $3e_i = 3c_i = 0$, $[e_i, c_j] = [c_k, c_j] = 0$, $i = 1, 2, 3, 4$, $j, k = 1, 2, 3$ with $[e_1, e_2] = c_1$, $[e_1, e_3] = [e_2, e_4] = c_2$, $[e_3, e_4] = c_3$, otherwise $[e_m, e_l] = 0$. Again G is of exponent 3, nilpotent of class 2, with $G' = \langle c_1, c_2, c_3\rangle \subseteq Z(G)$ [9]. Using GAP, G does not split and is not 1-affine complete. We show $C_0(G)$ is not a ring. For $x \in G$, $x = \alpha e_1 + \beta e_2 + \gamma e_3 + \delta e_4 + d$ where $d \in G'$. Define $f : G \to G$ by

$$f(x) = \begin{cases} 2(c_1 + c_2 + c_3), & \text{if } \beta + \gamma = 3,\\ 2c_3, & \text{if } \beta = 0, \gamma \neq 0,\\ 2c_1, & \text{if } \beta \neq 0, \gamma = 0,\\ 2c_1 + c_2 + 2c_3, & \text{if } \beta = \gamma \neq 0,\\ 0, & \text{otherwise.} \end{cases}$$

Calculations show that $f \in C_0(G)$. Now $f \circ (\mathrm{id} + \mathrm{id})(e_3) = f(2e_3) = 2c_3$ while $(f \circ \mathrm{id} + f \circ \mathrm{id})(e_3) = 2c_3 + 2c_3 = c_3 \neq 2c_3$. Thus $C_0(G)$ is not a ring.

Above we denoted the normal closure of $x \in G$ by $\overline{x}$. For $x \in G$, let $P_0(G)x = \{p(x) \mid p \in P_0(G)\}$ and $C_0(G)x = \{c(x) \mid c \in C_0(G)\}$.

Lemma 3.3 *Let G be a group and let $x \in G$.*

(1) $P_0(G)x = \overline{x}$.
(2) $P_0(G)x = C_0(G)x$.
If further G is 2-Engel then
(3) $\overline{x}$ is abelian;
(4) $\langle C_0(G), +\rangle$ is an abelian group.
(5) If G is nilpotent of class at most 2 then $\overline{x} = \langle x\rangle + [x, G]$.

Proof (1) Clearly $P_0(G)x \subseteq \overline{x}$. On the other hand, $P_0(G)x$ is a normal subgroup of G containing x, so $\overline{x} \subseteq P_0(G)x$.
(2) One has $P_0(G)x \subseteq C_0(G)x$. For $c \in C_0(G)$, $c(x) - c(0) \in \overline{x - 0}$ so $c(x) \in \overline{x}$. Thus $C_0(G)x \subseteq \overline{x} = P_0(G)x$.
(3) When G is 2-Engel, $P_0(G)$ is a ring so with 1) we get that $P_0(G)x$ is an abelian group.
(4) Follows from 2) since G is 2-Engel.
(5) In [10] Ecker shows $p \in P_0(G)$ has the form $p(x) = kx + [x, g]$ for some integer k and $g \in G$ when G is nilpotent of class at most 2. Thus $P_0(G)x \subseteq \langle x\rangle + [x, G]$. But $\langle x\rangle + [x, G] \subseteq \overline{x} = P_0(G)x$. □

We next give a characterization of those groups G for which $C_0(G)$ is a ring. The usefulness of this result is somewhat limited since it requires knowledge of all $c \in C_0(G)$.

Theorem 3.4 *Let G be a finite 2-Engel group. Then $C_0(G)$ is a ring if and only if $c|_{\overline{x}} \in End(\overline{x})$ for each $c \in C_0(G)$ and $x \in G$.*

Proof Let c, f, g be arbitrary in $C_0(G)$ and let $x \in G$. From Lemma 3.3 $\langle C_0(G), +\rangle$ is an abelian group. Suppose $c|_{\overline{x}} \in \mathrm{End}(\overline{x})$ for each $x \in G$. Then $c \circ (f + g)(x) =$

$c(f(x)+g(x)) = c(f(x)) + c(g(x)) = (c \circ f + c \circ g)(x)$, since $f(x), g(x) \in \bar{x}$ by (1) and (2) of Lemma 3.3. Thus $C_0(G)$ is a ring.

For the converse let $a, b \in \overline{x}$. Thus by 1) and 2) of Lemma 3.3 there exist $h, l \in C_0(G)$, $a = h(x)$, $b = l(x)$. Now let $c \in C_0(G)$. It is clear that $c(\overline{x}) \subseteq \overline{x}$. Moreover since $C_0(G)$ is a ring we have $c(a+b) = c(h(x)+l(x)) = c(h(x)) + c(l(x)) = c(a) + c(b)$ which shows $c \in \mathrm{End}(\overline{x})$. □

Therefore if one can construct a congruence preserving function that is not linear on some normal closure $\overline{x}$, $x \in G$, where G is 2-Engel, then $C_0(G)$ is not a ring. In the next example, using Theorem 3.4, we show that the result of Theorem 3.1 on groups which split is not true for $p = 2$.

Example 3.5 Let G be a semidirect product of $\mathbb{Z}_4$ and $\mathbb{Z}_4$: $G = \langle x, y \mid 4x = 4y = 0,\ y + x = 3x + y\rangle$. We have $Z(G) = \langle 2x, 2y\rangle$ and one verifies that $D = Z(G)$ and $E = \langle 2x\rangle$ is a splitting pair for $\eta(G)$. Define $c : G \to G$ by

$$c(w) = \begin{cases} 0, & w \in Z(G), \\ 2x, & w \notin Z(G). \end{cases}$$

Let $u, v \in G$. If $u, v \in Z(G)$ or $u, v \notin Z(G)$ then $c(u) - c(v) = 0 \in \overline{u-v}$. If $u \notin Z(G)$ and $v \in Z(G)$ then $u - v \notin Z(G)$ and so $\overline{u-v} \supseteq \langle 2x\rangle$ since $(Z(G), \langle 2x\rangle)$ is a splitting pair. Thus $c(u) - c(x) = 2x \in \overline{u-v}$ so $c \in C_0(G)$ but $c \notin P_0(G)$ since $c(x) = 2x$ while $c(y) = 2x \neq 2y$.

Using GAP one finds $|P_0(G)| = 16$ and $|C_0(G)| = 32$ so we have $C_0(G) = P_0(G) + \langle c\rangle = \{p + lc \mid l \in \{0, 1\},\ p \in P_0(G)\}$. For $w \notin Z(G)$, calculations show that c is linear on $\overline{w}$. Thus for all $w \in G$, $c|_{\overline{w}} \in \mathrm{End}(\overline{w})$. Thus for each $p \in P_0(G)$, $p + c$ is linear on each $\overline{w}$ so from Theorem 3.4 $C_0(G)$ is a ring.

We turn to another lattice condition, a particular case of a splitting pair. If (D, E) is a splitting pair for $\eta(G)$ and $D = E$ we say D is a *cutting element* and G *cuts*.

Lemma 3.6 *Let G be a finite p-group of nilpotency class at most 2 such that I is a cutting element for $\eta(G)$. Then $I \subseteq Z(G)$.*

Proof Let T be the maximal cutting element for $\eta(G)$ which exists since G is finite and cutting elements form a chain in $\eta(G)$. We have $T \supseteq I$. If G is abelian then $I \subseteq Z(G) = G$ so we take G of class 2, hence $G' \subseteq Z(G)$. If T is also a maximal element in $\eta(G)$ then G has a unique maximal normal subgroup. Thus from [16], G is cyclic, contrary to G being of class 2. Thus we suppose T is not a maximal element in $\eta(G)$. If $T \subseteq G'$ then T, hence I, is contained in $Z(G)$. To complete the proof we show $G' \subset T$ cannot occur. Suppose $G' \subset T$ and let $N \in \eta(G)$ be maximal with $G' \subseteq N \subset T$. Since G/G' is abelian, G/N is also abelian. Therefore G/N has a unique minimal normal subgroup T/N. But this means that G/N is subdirectly irreducible and (from [7] p. 64) G/N is cyclic. However this contradicts the fact that T is the unique maximal cutting element but not a maximal element in $\eta(G)$. Thus we have $I \subseteq Z(G)$. □

Theorem 3.7 *Let G be a finite nonabelian p-group such that G cuts. Then $C_0(G)$ is not a ring.*

Proof Let I be a cutting element. If G is of nilpotency class greater than 3 then G is not 2-Engel, hence $C_0(G)$ is not a ring. Further if $p > 2$ then from Theorem 3.1, $C_0(G)$ is not a ring. Therefore $p = 2$ and G is nilpotent of class 2, $|G| = 2^n$, $n \geq 3$. From Lemma 3.6 we get $I \subseteq Z(G)$.

Let T_1 be a transversal of G/I with $0 \in T_1$. Let $t \in T_1 - \{0\}$ and define $T_2 = (T_1 \backslash (t + t + I)) \cup \{t + t\}$. We note T_2 is a transversal of G/I with $\{t, t + t\} \subseteq T_2$. Suppose first that $t + t \notin I$ so $0 \in T_2$. Let $0 \neq e$ be in I and define $h : G \to G$ by

$$h(x) = \begin{cases} e, & x \in t + t + I, \\ 0, & \text{otherwise.} \end{cases}$$

We note that $h(0) = 0$. To show $h \in C_0(G)$ it suffices to show for $r \in T_2 - \{t + t\}$, $d_1, d_2 \in I$ that $h(t + t + d_1) - h(r + d_2) \in \overline{t + t + d_1 - d_2 - r}$, that is, we must show $e \in \overline{t + t + d_1 - d_2 - r}$. We first observe that $t + t + d_1 - d_2 - r = t + t - r + d_1 - d_2 \notin I$ since $r \in T_2 - \{t + t\}$. Therefore $\overline{t + t + r + d_1 - d_2} \nsubseteq I$ and, since I cuts $\eta(G)$, $I \subseteq \overline{t + t + r + d_1 - d_2}$ giving the desired result that $h \in C_0(G)$. But $h(t + t) = e \neq h(t) + h(t)$. Since h is not linear on $\bar{t}$, $C_0(G)$ is not a ring.

Suppose next we have $0 \neq t + t \in I$. Using T_1 we define $f : G \to G$ by $f(x) = j$ where $x = r + j$, $r \in T_1$, $j \in I$. Since $0 \in T_1$, $f(0) = 0$. For $x = r_1 + j_1$, $y = r_2 + j_2$, $f(x) - f(y) = j_1 - j_2$. If $r_1 = r_2$, $j_1 - j_2 \in \overline{r_1 - r_2 + j_1 - j_2}$. If $r_1 \neq r_2$ then $r_1 - r_2 + j_1 - j_2 \notin I$ so $I \subseteq \overline{r_1 - r_2 + j_1 - j_2}$, hence $f(x) - f(y) = j_1 - j_2 \in \overline{x - y}$. Thus $f \in C_0(G)$. Since $t + t \in I$, $t + t = 0 + t + t$ which means $f(t + t) = t + t$. But $t = t + 0$ so $f(t) = 0 = f(t) + f(t) \neq f(t + t)$. This shows that $C_0(G)$ is not a ring.

For the final case we have $t + t \in I$ and $t + t = 0$. Define $l : G \to G$ by

$$l(x) = \begin{cases} 0, & x \in I, \\ j, & x \notin I \text{ and } x = r + j,\ r \in T_2 = T_1. \end{cases}$$

For $x \notin I$, $y \in I$ say $x = r + j$, $r \in T_2$, $j \in I$ we have $l(x) - l(y) = j$. Moreover $I \subseteq \overline{r + j - y}$ since $r + j - y \notin I$. Thus $l \in C_0(G)$. Since $t + t \in I$ for any $0 \neq i \in I$, $l(t + t + i) = 0$. Further since $t \notin I$, $I \subseteq \bar{t}$, hence $t + i \in \bar{t}$. Now $l(t) + l(t + i) = 0 + i \neq 0$ so l is not linear on $\bar{t}$. Thus in all cases we have found when G is cut, $C_0(G)$ is not a ring. □

4 Structural conditions

In this section we focus on group theoretical properties of a group G to determine when $C_0(G)$ is a ring. We restrict to nilpotency class 2 and p-groups $p \geq 3$.

Theorem 4.1 *Let G be a nonabelian p-group, $p > 2$ such that G' is cyclic. Then $C_0(G)$ is not a ring.*

Proof Let $x \in G - Z(G)$. Thus $\{0\} \neq [x, G] \subseteq [G, G]$. By hypothesis, $G' = \langle r \rangle$ for some $r \in G$ so we have $\overline{r} = \langle r \rangle = G'$.

Let $\langle r' \rangle$ be the unique subgroup of order p in G'. Since $[x, G]$ is a nonzero cyclic subgroup of G', $[x, G']$ contains a cyclic subgroup of order p of G', hence we have $\langle r' \rangle$ is a subgroup of $[x, G']$. Moreover, for each $g \in G$, $p(-g + r' + g) = 0$ so $-g + r' + g$ is in $\langle r' \rangle$ which in turn leads to the fact that $\langle r' \rangle$ is a normal subgroup of $[x, G]$. Therefore, for $x \in G - Z(G)$, $\overline{r'} = \langle r' \rangle \subseteq [x, G] \subseteq \overline{x}$.

Define $h : G \to G$ by

$$h(x) = \begin{cases} r', & x \notin Z(G), \\ 0, & x \in Z(G). \end{cases}$$

We show $h \in C_0(G)$. To this end let $u \notin Z(G)$, $v \in Z(G)$. Then $h(u) - h(v) = r'$ which is in $\overline{u - v}$ since $u - v \notin Z(G)$. This gives $h \in C_0(G)$. For $w \notin Z(G)$, $2w = w + w \notin Z(G)$ since $p > 2$ so if $2w \in Z(G)$, we would have $w \in Z(G)$, a contradiction. From this observation, $h(w+w) = r' \neq 2r' = h(w)+h(w)$. Therefore $C_0(G)$ is not a ring. □

We actually have a little more.

Corollary 4.2 *If G is a nonabelian p-group, $p > 2$, such that G' is cyclic, then $\eta(G)$ splits.*

Proof From the proof of Theorem 4.1 we get if $N \trianglelefteq G$ and $N \nsubseteq Z(G)$ then for $x \in N - Z(G)$, $N \supseteq \overline{x} \supseteq \overline{r'}$. Thus $\langle Z(G), \overline{r'} \rangle$ is a splitting pair for $\eta(G)$. □

Corollary 4.3 *Let G be a p-group, $p > 2$, of nilpotency class 2. If G is 2-generated (generated by 2 elements) then $C_0(G)$ is not a ring.*

Proof If $G = \langle x, y \rangle$ then one finds $G' = \langle [x, y] \rangle$ so G' is cyclic [6]. The result now follows from the above theorem. □

We remark that Example 3.5 shows that Corollary 4.3 does not hold for $p = 2$.

Recall that a group G is abelian by cyclic, or G is said to be an extension of an abelian group by a cyclic group if there exists an abelian normal subgroup A of G such that G/A is cyclic. For finite G one may always take A to be a maximal abelian normal subgroup.

Theorem 4.4 *Let G be a nonabelian p-group, $p > 2$, of nilpotency class 2 which is abelian by cyclic. Then $C_0(G)$ is not a ring.*

Proof We let A be a maximal abelian normal subgroup such that $G/A \cong \mathbb{Z}_{p^k}$, k a positive integer. Let $G/A = \langle b + A \rangle$ so $G = \langle A, b \rangle$. Since G is nonabelian there exists $a_1 \in A$ such that $[a_1, b] \neq 0$. Every $x \in G$ can be decomposed into a sum of the form $x = a + \beta b + c$, with $a \in A$, $\beta \in \mathbb{Z}$, $c \in G'$. (Recall the basic assumption that G is of class 2 so $[G, G] \subseteq Z(G)$.)

Using this decomposition we define $f : G \to G$ by

$$f(x) = \begin{cases} [b, a_1], & p \nmid \beta, \\ 0, & p \mid \beta. \end{cases}$$

Let $x \in G$. We note $[x, a_1] = \beta[b, a_1]$ so when $p \nmid \beta$, $[b, a_1] \in \overline{x}$. Also, $f \in C_0(G)$. For if $u, v \in G$, $u = a + \beta b + c$, $v = a' + \beta' b + c'$, $a, a' \in A$, $\beta, \beta' \in \mathbb{Z}$, $c, c' \in [G, G]$ with $p \nmid \beta$ and $p \mid \beta'$ then $p \nmid (\beta - \beta')$ so $[b, a_1] \in \overline{u - v}$. Thus $f(u) - f(v) = [b, a_1] \in \overline{u - v}$. For $u = a + \beta b + c$ with $p \nmid \beta$, $f(u) + f(u) = [b, a_1] + [b, a_1]$ while $f(u + u) = f(2u) = [b, a_1]$. Using Theorem 3.4, $C_o(G)$ is not a ring. □

Corollary 4.5 *Let G be a nonabelian p-group, $p > 2$, of class 2 such that there exists $g \in G$ with $G/C_G(g)$ cyclic. Then $C_0(G)$ is not a ring.*

Proof Let $G/C_G(g) = \langle b + C_G(g)\rangle$ so $G = \langle C_G(g), b\rangle$. For $x \in G$, $x = w + \beta b + c$, $w \in C_G(g)$, $\beta \in \mathbb{Z}$, $c \in G'$ and since $b \notin C_G(g)$, $[b, g] \neq 0$. Now $[x, g] = \beta[b, g]$. The remainder of the proof is as above and is omitted. □

As we did following Theorem 4.1, we again show that under the hypothesis of Theorem 4.4, $\eta(G)$ splits.

Theorem 4.6 *Let G be a p-group, nilpotent of class 2, which is abelian by cyclic. Then $\eta(G)$ splits.*

Proof As above we let A be a maximal abelian normal subgroup with $G/A = \langle b + A\rangle$. Then $G = \langle A, b\rangle$ and $b \notin A$ so there exists $a_1 \in A$, $[a_1, b] \neq 0$. Let $A = \langle a_1, a_2, \ldots, a_n\rangle$ so $G = \langle a_1, \ldots, a_n, b\rangle$. Let $A_0 = \langle a_1, \ldots, a_n, pb\rangle$. If $pb = 0$ then $A_0 = A$ and the same type of argument works. We first show that A_0 is a normal subgroup of G. Let $g = a + \beta b + c \in G$, $a \in A$, $\beta \in \mathbb{Z}$, $c \in G'$. It suffices to show $-g + pb + g \in A_0$. To this end, $-g + pb + g = -c - \beta b - a + pb + a + \beta b + c = -\beta b - a + pb + a + \beta b = -\beta b + pb + [pb, a] + \beta b = pb + [pb, a] \in A_0$. Therefore $A_0 \trianglelefteq G$.

Further $A_0 \subset G$. For if $A_0 = G$ then $b \in A_0 = \langle a_1, a_2, \ldots, a_n, pb\rangle$. From this, $b = a + \alpha(pb) + c$ where $a \in A$ and c is a sum of commutators. Since A is a maximal abelian normal subgroup we have $Z(G) \subseteq A$ and since G is nilpotent of class 2, $[G, G] \subseteq Z(G) \subseteq A$. Thus $c \in A$. Hence $(1 - \alpha p)b \in A$ which implies $b \in A$ since $1 - \alpha p$ is invertible modulo p, a contradiction. Thus $A_0 \neq G$.

Now let $N \in \eta(G)$ such that $N \nsubseteq A_0$. For $n \in N - A_0$, $n = a + \delta b + c$, $a \in A$, $p \nmid \delta$, $c \in G'$, hence $[n, a_1] = \delta[b, a_1]$ and since $p \nmid \delta$, $0 \neq [b, a_1] \in \overline{n} \subseteq N$. From this, $\langle [b, a_1]\rangle \subseteq N$ for each $N \in \eta(G)$ such $N \nsubseteq A_0$. This shows that $(A_0, \langle [b, a_1]\rangle)$ is a splitting pair for $\eta(G)$. □

In Theorems 4.6 and 4.1 one has the situation where G has a partition $G = X \cup (G - X)$ with the property that $\bigcap\{\overline{u} \mid u \in X\} \neq \{0\}$ and, for each $u \in X$, for each $v \in G - X$, $u - v \in X$. It is an open question if this condition implies the splitting of $\eta(G)$.

We apply the above results to p-groups, $p > 2$, of small order. Let G be a group of order p^n, $p > 2$, $1 \leq n \leq 5$. When $n = 1$ or $n = 2$, G is abelian so $C_0(G)$ is a ring if and only if $G \cong \mathbb{Z}_p + \mathbb{Z}_p$. Thus we take $n \geq 3$ and since the abelian case is known from Theorems 2.4 and 2.5 we restrict to nonabelian groups.

Theorem 4.7 *Let G be a nonabelian p-group, $p > 2$ of order p^n, $3 \leq n \leq 5$ such that G is nilpotent of class 2. Then $C_0(G)$ is not a ring.*

Proof (i) $n = 3$. Since g is nonabelian we have $|Z(G)| = p$. Thus $\{0\} \neq G' \subseteq Z(G)$, hence G' is cyclic and the result follows from Theorem 4.1.

(ii) $n = 4$. Let $\Phi(G)$ denote the Frattini subgroup of G. We know $G' \subseteq \Phi(G)$ and if $|G/\Phi(G)| = p^k$ then G is generated by k elements [16]. If $|\Phi(G)| = p$ then G' is cyclic while if $|\Phi(G)| = p^2$ then G is generated by 2 elements. Using Theorem 4.1 and Corollary 4.3 we see that $C_0(G)$ is not a ring. If $|\Phi(G)| = p^3$ then G has a unique maximal normal subgroup which cuts G. The result now follows from Theorem 3.7.

(iii) $n = 5$. If $|\Phi(G)| = p$ or p^3 or p^4 then as in the above case we have $C_0(G)$ is not a ring. It remains to consider $|\Phi(G)| = p^2$. We must have $G' = \Phi(G)$ for if $G' \neq \Phi(G)$ then G' is cyclic and we are finished. Thus we have $G' = \Phi(G) \subseteq Z(G)$ since G is of class 2. For $x \in G - Z(G)$ define $\varphi_x : G \to G$ by $\varphi_x(w) = [x, w]$. Since G is of class 2, φ_x is an endomorphism of G and $\ker \varphi_x = C_G(x) \supseteq \langle x \rangle + Z(G)$ while $Im\varphi_x = [x, G] \subseteq \overline{x}$. We have $|\langle x \rangle + Z(G)| = p|Z(G)|$ so if $G' \subset Z(G)$ then $|Z(G)| = p^3$ and $|\ker \varphi_x| = |C_G(x)| = |\langle x \rangle + Z(G)| = p^4$. But this means G is abelian by cyclic so the result follows from Theorem 4.4. Thus we take $|G'| = |\Phi(G)| = |Z(G)| = p^2$. Thus $|\ker \varphi_x| = |C_G(x)| \geq |\langle x \rangle + Z(G)| = p^3$. If $|C_G(x)| = p^4$ then the result follows from Corllary 4.5, so we take $|\ker \varphi_x| = |C_G(x)| = p^3$. But then $|Im\varphi_x| = |G/\ker \varphi_x| = p^2$. Thus $|[x, G]| = p^2$, so $[x, G] = G'$ which means $G' \subseteq \overline{x}$ for each $x \notin Z(G)$. Thus, if $N \trianglelefteq G$ and $N \nsubseteq Z(G)$ then $N \supseteq G'$. This shows that $Z(G)$ cuts $\eta(G)$ and so $C_0(G)$ is not a ring. □

□

In conclusion we have found that when G is a finite abelian p-group then $C_0(G)$ is a ring if and only if G is 1-affine complete. For nonabelian p-groups, $p = 2$, we have seen that $C_0(G)$ can be a ring properly containing $P_0(G)$. For $p > 2$ we have several classes for which $C_0(G)$ is not a ring. In fact, for $p > 2$ the authors have no example of a nonabelian p-group for which $C_0(G)$ is a ring unless G is 1-affine complete. We thus close with the following.

Conjecture For finite nonabelian p-groups G, $p > 2$, $C_0(G)$ is a ring if and only if G is 1-affine complete.

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